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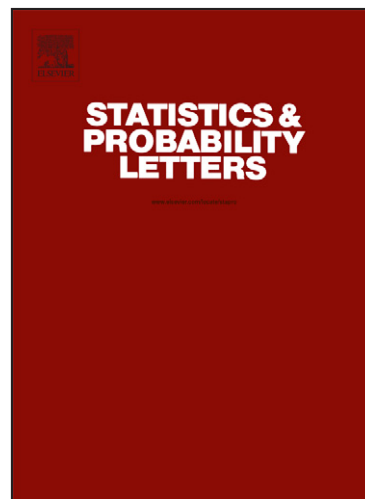
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CORRECTION TO THE PAPER "ON UNIFORM INTEGRABILITY OF RANDOM VARIABLES"

WIOLETTA GRZENDA, DARIUSZ MAJEREK, AND WIESŁAW ZIĘBA

ABSTRACT. In [2] conditions for conditional uniform integrability are provided. Professor A. Paszkiewicz pointed out that assumption of almost sure convergence of sequences $\{X_n, n \geq 1\}$ is necessary. In this paper we will present corrected version of this theorem.

Let (Ω, \mathcal{A}, P) be a probability space, and let \mathcal{F} be a sub- σ -field contained in \mathcal{A} .

Definition 0.1. [4] A sequence $\{X_n, n \geq 1\}$ is uniformly \mathcal{F} -integrable if for every \mathcal{F} -measurable random variable $\varepsilon > 0$ a.s. there exists \mathcal{F} -measurable random variable $\lambda > 0$ a.s. such that

$$\sup E^{\mathcal{F}} |X_n| I_{[|X_n| > \lambda]} < \varepsilon \text{ a.s.}$$

One can prove the following theorem.

Theorem 0.2. [5] A sequence $\{X_n, n \geq 1\}$ is uniformly \mathcal{F} -integrable if and only if

- (1) $\sup E^{\mathcal{F}} |X_n| < \infty$ a.s.
- (2) for every \mathcal{F} -measurable random variable $\varepsilon > 0$ a.s. there exists an \mathcal{F} -measurable random variable $\delta > 0$ a.s. such that

$$E^{\mathcal{F}} I_A < \delta \Rightarrow \sup E^{\mathcal{F}} |X_n| I_A < \varepsilon \text{ a.s.}$$

It is known (Fatou Lemma) that if $\{X_n, n \geq 1\}$ is a nonnegative sequence of random variables ($X_n \geq 0$ a.s. for $n \geq 1$), then

$$(1) \quad E^{\mathcal{F}} \liminf X_n \leq \liminf E^{\mathcal{F}} X_n \text{ a.s.}$$

In [4] it has been shown that if we replace uniform integrability by uniform \mathcal{F} -integrability, then

$$\limsup E^{\mathcal{F}} X_n \leq E^{\mathcal{F}} \limsup X_n \text{ a.s.,}$$

We show that if a sequence $\{X_n, n \geq 1\}$ is almost sure convergent, then the conditional uniform integrability is necessary.

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Theorem 0.3. *Let $\{X_n, n \geq 1\}$ be a sequence of \mathcal{F} -integrable random variables ($X_n \in L^1(\mathcal{F})$, i.e. $E^{\mathcal{F}}|X_n| < \infty$ for $n = 1, 2, \dots$), and $0 \leq X_n \xrightarrow{a.s.} X \in L^1(\mathcal{F})$, $n \rightarrow \infty$. If*

$$\limsup E^{\mathcal{F}} X_n \leq E^{\mathcal{F}} \limsup X_n \text{ a.s.},$$

then $\{X_n, n \geq 1\}$ is uniformly \mathcal{F} -integrable.

Proof.

Let ε be an arbitrary \mathcal{F} -measurable random variable ($\varepsilon > 0$ a.s.). Since $X \in L^1(\mathcal{F})$, there exists a random variable $\delta > 0$ a.s. such that

$$E^{\mathcal{F}} I_A < \delta \implies E^{\mathcal{F}} |X| I_A < \varepsilon \text{ a.s.}$$

If

$$E^{\mathcal{F}} X = E^{\mathcal{F}} \lim X_n \leq \liminf E^{\mathcal{F}} X_n \leq \limsup E^{\mathcal{F}} X_n \leq E^{\mathcal{F}} \lim X_n = E^{\mathcal{F}} X \text{ a.s.},$$

then

$$\lim E^{\mathcal{F}} X_n = E^{\mathcal{F}} X \text{ a.s.}$$

and by (1) we have

$$\begin{aligned} E^{\mathcal{F}} 2X &= E \liminf (X_n + X - |X_n - X|) \\ &\leq \liminf (E^{\mathcal{F}} X_n + E^{\mathcal{F}} X - E^{\mathcal{F}} |X_n - X|), \end{aligned}$$

hence

$$\limsup E^{\mathcal{F}} |X_n - X| \leq 0.$$

Let $B_n(\varepsilon) = \{E^{\mathcal{F}} |X_k - X| < \varepsilon, k \geq n\}$ then for $n \geq 1$, $B_n(\varepsilon) \in \mathcal{F}$, $B_n(\varepsilon) \subset B_{n+1}(\varepsilon)$ and $\lim P(B_n(\varepsilon)) = 1$. $X \in L^1(\mathcal{F})$ and $X_k \in L^1(\mathcal{F})$, $k \geq 1$, thus for every n exists any \mathcal{F} measurable random variable $\delta_n > 0$ a.s. such that

$$E^{\mathcal{F}} I_A < \delta_n \implies E^{\mathcal{F}} |X_k - X| I_A < \varepsilon \text{ a.s., } k = 1, 2, \dots, n$$

We can assume that $\delta_n \geq \delta_{n+1}$ a.s. Define

$$\delta^* = \delta_n, \text{ on } C_n(\varepsilon) = B_n(\varepsilon) \setminus B_{n-1}(\varepsilon), B_0(\varepsilon) = \emptyset,$$

If $\delta' = \min\{\delta, \delta^*\}$ and $E I_A < \delta'$, then

$$\begin{aligned} E^{\mathcal{F}} |X_k| I_A &\leq E^{\mathcal{F}} |X + X_k - X| I_A \leq E^{\mathcal{F}} |X| I_A + E^{\mathcal{F}} |X_k - X| I_A I_{(\cup_{n=1}^{\infty} C_n(\varepsilon))} \\ &\leq E^{\mathcal{F}} |X| I_A + E^{\mathcal{F}} |X_k - X| I_{B_k(\varepsilon)} + \sum_{n=k+1}^{\infty} E^{\mathcal{F}} |X_k - X| I_{C_n(\varepsilon)} \\ &\leq \varepsilon + \varepsilon I_{B_k(\varepsilon)} + \varepsilon I_{B_k^c(\varepsilon)} = 2\varepsilon \text{ a.s.} \end{aligned}$$

The proof is complete.

The following example shows that without the assumption that a sequence is almost sure convergent, the theorem 0.3 will not be true.

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Example 1. Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{B}, \mu)$, where μ denotes Lebesgue measure and $\mathcal{F} = \sigma(\Omega, \emptyset)$. For $n = 2^k$, we have

$$X_{2^k}(\omega) = \begin{cases} 2^k, & \omega \in [\frac{0}{2^k}, \frac{1}{2^k}), \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 2^k + s$, $1 \leq s < 2^k$, we have

$$X_{2^k+s}(\omega) = \begin{cases} 2^k, & \omega \in [\frac{0}{2^k}, \frac{1}{2^k}), \\ 2, & \omega \in [\frac{s}{2^k}, \frac{s+1}{2^k}), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\limsup EX_n = 1$, $\limsup X_n = 2$ a.s., thus $\limsup E^{\mathcal{F}} X_n \leq E^{\mathcal{F}} \limsup X_n$ a.s., but the sequence $\{X_n, n \geq 1\}$ is not uniformly \mathcal{F} -integrable, since $\sup E|X_n|I_{[|X_n|>\lambda]} = 1$ for any λ .

The assumption $X_n \geq 0$ is necessary.

Example 2. On $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{B}, \mu)$ we define

$$X_n(\omega) = \begin{cases} n^2, & 0 \leq \omega < \frac{1}{n}, \\ -n^2, & \frac{1}{n} \leq \omega \leq \frac{2}{n}, \\ 0, & \frac{2}{n} < \omega \leq 1, \end{cases}$$

and $X(\omega) \equiv 0$, then $X, X_n \in L^1$, $n \geq 1$ and $X_n \xrightarrow{a.s.} X$,

$$\limsup EX_n \leq E \limsup X_n,$$

but for $n^2 > \lambda$, $E|X_n|I_{[|X_n|>\lambda]} = 2n \rightarrow \infty$.

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