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► **To cite this version:**

Matthias Löwe, Franck Vermet. The capacity of q -state potts neural networks with parallel retrieval dynamics. *Statistics and Probability Letters*, 2009, 77 (14), pp.1505. 10.1016/j.spl.2007.03.030 . hal-00580094

HAL Id: hal-00580094

<https://hal.science/hal-00580094>

Submitted on 26 Mar 2011

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Author's Accepted Manuscript

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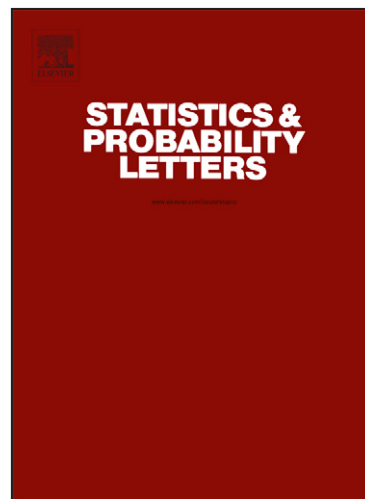
PII: S0167-7152(07)00122-8
DOI: doi:10.1016/j.spl.2007.03.030
Reference: STAPRO 4631

To appear in: *Statistics & Probability Letters*

Received date: 22 September 2005
Revised date: 11 July 2006
Accepted date: 20 March 2007

Cite this article as: Matthias Löwe and Franck Vermet, The capacity of q -state potts neural networks with parallel retrieval dynamics, *Statistics & Probability Letters* (2007), doi:10.1016/j.spl.2007.03.030

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THE CAPACITY OF q -STATE POTTS NEURAL NETWORKS WITH PARALLEL RETRIEVAL DYNAMICS

MATTHIAS LÖWE AND FRANCK VERMET

ABSTRACT. We define a Potts version of neural networks with q states. We give upper and lower bounds for the storage capacity of this model of associative memory in the sense of exact retrieval of the stored information. The critical capacity is of the order $c \frac{N}{\log N}$ where N is the number of neurons and the constant c increases quadratically with q .

1. INTRODUCTION

A Potts neural network is a variant of the well known Hopfield model [7], [1] to the case of $q \geq 2$ states. These can be thought of as colors, for instance. It was first introduced by Kanter [9], and later studied by Ferrari and Picco [5], Gayraud [6]. Recently it has been rediscovered in the context of inverse problems with bio-inspired models by Quenet et al. [13]. The Potts neural network is related to the mean field Potts model described by Wu [15] in the same way as the Hopfield model is related to the Curie-Weiss model (see e.g. [4]). Formally, the q -states Potts neural network consists of a state space S with $|S| = q$ and synaptic efficacies $(W_{ij})_{i,j=1}^N$, N being the size of the network. These efficacies induce a dynamics T on S^N by

$$T_i(\sigma) = \pi \left(\sum_{j=1}^N W_{ij} \sigma_j \right), \quad i = 1, \dots, N \quad (1)$$

where π is a projection operator onto S . One step of the parallel dynamics $T = (T_i)_{i=1}^N$ sends $\sigma \in S^N$ to $(T_i(\sigma))_{i=1}^N$. Suppose we want to store patterns $\xi^1, \dots, \xi^M \in S^N$ in this network in such a way that whenever we start not too far away from one of the ξ^μ the dynamics will lead to the stored pattern. Then the following setup appears naturally. Choose

$$S = \{v_i := e_i - 1/q \mathbf{1}, i = 1, \dots, q\},$$

where e_i is the i 'th unit vector in \mathbb{R}^q and $\mathbf{1} \in \mathbb{R}^q$ is the vector with a 1 in each component. (Other choices of S would lead to a more complicated definition of the synaptic efficacies, see e.g. [5]). For a given set of vectors $\xi^1, \dots, \xi^M \in S^N$ we define

$$W_{ij} := \sum_{\mu=1}^M \xi_i^\mu (\xi_j^\mu)^t$$

where the upper index t indicates transposition. Note that each W_{ij} is itself a $q \times q$ matrix. Also note that the W_{ij} are a generalization of Hebb's learning rule to our state space S . With these definitions for each $\sigma \in S^N$ the dynamics $T = (T_i)_{i=1}^N$ is given by (1) where $W_{ij} \sigma_j$ is to be understood in the sense of matrix multiplication and π is the L^2 -projection onto S . The choice $q = 2$ gives an equivalent formulation of the standard Hopfield model. For the rest of this note we assume that the $\xi^1, \dots, \xi^M \in S^N$ are taken i.i.d. with uniform probability from the set S^N , i.e. all ξ^μ are independent and all its coordinates are

Date: April 6, 2007.

1991 Mathematics Subject Classification. 82C32, 82B44, 60K35.

Key words and phrases. Potts network, neural networks, storage capacity, Hopfield model.

selected independently with uniform probability $1/q$ from the set S . We study the following questions:

- (1) How large can we choose M to ensure that the ξ^μ are fixed points of the dynamic T (with high probability)?
- (2) How large can M be chosen to ensure that corrupted patterns are attracted to their original versions (by one or several steps of the dynamics, with high probability)?

2. RESULTS

The answers to the above questions are given in four theorems below. First we give a lower bound on the storage capacity.

Theorem 2.1. Write $M = c \frac{N}{\log N}$. Then

- (1) If $c < \frac{q(q-1)}{4}$, then for every fixed $\mu = 1, \dots, M$

$$\mathbb{P}(T(\xi^\mu) = \xi^\mu) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

- (2) If $c < \frac{q(q-1)}{8}$, then it holds that

$$\mathbb{P}(\forall \mu = 1, \dots, M : T(\xi^\mu) = \xi^\mu) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

- (3) If $c < \frac{q(q-1)}{12}$, then

$$\mathbb{P}(\liminf_{N \rightarrow \infty} \bigcap_{\mu=1}^M T(\xi^\mu) = \xi^\mu) = 1.$$

Remark 2.2. Note that for $q = 2$ we obtain the known bounds for the Hopfield model (see [11],[14], [12]). Also note that the capacity increases with q as $q(q-1)$. This agrees with the findings in [9] in a similar context.

The next theorem, that is a generalization of Theorems by Loukianova [10] and Bovier [2] for the Hopfield model, shows that the bound in part (1) of Theorem 2.1 is optimal.

Theorem 2.3. Write $M = c \frac{N}{\log N}$. Then, if $c > \frac{q(q-1)}{4}$

$$\mathbb{P}(T(\xi^\mu) = \xi^\mu) \rightarrow 0 \quad \text{as } N \rightarrow \infty \text{ for all } \mu = 1, \dots, M.$$

The next two theorems deal with the reconstruction of corrupted patterns.

Theorem 2.4. Let $M = c \frac{N}{\log N}$. Then if $\varrho < \frac{q-1}{q}$ and $c < \frac{q^2(1-\frac{1}{q}-\varrho)^2}{4(1-\frac{1}{q})}$, for any fixed μ and any y^μ taken at random from $S_{\varrho N}(\xi^\mu) := \{x \in S^N, d(x, \xi^\mu) = \varrho N\}$ (where $d(\cdot, \cdot)$ denotes the Hamming distance and we assume that ϱN is an integer) it holds that

$$\mathbb{P}(T(y^\mu) = \xi^\mu) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

Remark 2.5. Analogous results to parts (2) and (3) of Theorem 2.1 can also be proven. The corresponding constants are $c < \frac{q^2(1-\frac{1}{q}-\varrho)^2}{8(1-\frac{1}{q})}$ and $c < \frac{q^2(1-\frac{1}{q}-\varrho)^2}{12(1-\frac{1}{q})}$, respectively.

Theorem 2.4 immediately raises the questions, whether for larger M , maybe even for $M = c \frac{N}{\log N}$ with $c < q(q-1)/4$, corrupted patterns can be reconstructed by several steps of the dynamics. This is answered in the affirmative by the following theorem, which basically completes the picture of exact retrieval in q -state Potts networks.

Theorem 2.6. Let $M = c \frac{N}{\log N}$. Then, if $c < q(q-1)/4$ for any fixed $\mu = 1, \dots, M$, any $0 \leq \varrho < \frac{q-1}{q}$, and any y^μ taken at random from $S_{\varrho N}(\xi^\mu)$ there exists a k (depending on ϱ) such that

$$\mathbb{P}(T^k(y^\mu) = \xi^\mu) \rightarrow 1$$

as $N \rightarrow \infty$. Here T^k is defined as the k 'th iterate of the map T .

- Remarks 2.7.** (1) *Analogous results for all patterns (as in part (2) of Theorem 2.1) or for sequential dynamics (i.e. if we apply the T_i one after another instead of simultaneously) can also be proven.*
- (2) *Theorem 2.6 generalizes a result by Burshtein [3] to the case of q -state Potts networks. However, our proof is different.*

3. PROOFS

In this section we give the proofs of Theorems 2.1, 2.3, 2.4, and 2.6. The proofs of Theorems 2.1 and 2.4 are rather elementary and similar to the proofs of the corresponding results in the Hopfield model. We will keep them rather short, sketch the proof of 2.1 and leave the proof of Theorem 2.4 to the reader.

Proof of Theorem 2.1. Assume without loss of generality that $\mu = 1$ and that ξ^1 is the vector consisting of N coordinates that are all equal to $v_1 = (1 - \frac{1}{q}, -\frac{1}{q}, \dots, -\frac{1}{q})^t$. Then, e.g. $T_1(\xi^1) \neq \xi^1$ if and only if the projection of $\sum_{\mu=1}^M \sum_{j=1}^N \xi_1^\mu (\xi_j^\mu)^t \xi_j^1$ is not v_1 and this is the case if and only if

$$\langle v_k, \sum_{\mu=1}^M \sum_{j=1}^N \xi_1^\mu (\xi_j^\mu)^t \xi_j^1 \rangle \geq \langle v_1, \sum_{\mu=1}^M \sum_{j=1}^N \xi_1^\mu (\xi_j^\mu)^t \xi_j^1 \rangle \quad (2)$$

for some $k = 2, \dots, q$ (here $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^q). For $k = 2$, formula (2) is true if and only if

$$\sum_{j=1}^N \sum_{\mu=2}^M x_1^\mu \eta_j^\mu \leq -N(1 - \frac{1}{q}). \quad (3)$$

Here we have set $x_i^\mu := \xi_{i,1}^\mu - \xi_{i,2}^\mu$, $\eta_j^\mu := (\xi_j^\mu)^t \xi_j^1$ and $\xi_{i,1}^\mu$ and $\xi_{i,2}^\mu$ are the first and second coordinate of ξ_i^μ , respectively. The x_i^μ as well as the η_j^μ are identically distributed with

$$\mathbb{P}(x_i^\mu = x) = \begin{cases} \frac{1}{q} & x = 1 \\ \frac{1}{q} & x = -1 \\ 1 - \frac{2}{q} & x = 0 \end{cases} \quad \text{and} \quad \mathbb{P}(\eta_j^\mu = y) = \begin{cases} \frac{1}{q} & y = 1 - \frac{1}{q} \\ 1 - \frac{1}{q} & y = -\frac{1}{q} \end{cases}$$

The probability of (3) can be bounded by an exponential Chebyshev inequality: for $t > 0$,

$$\mathbb{P}\left(\sum_{j=1}^N \sum_{\mu=2}^M x_1^\mu \eta_j^\mu \leq -N(1 - \frac{1}{q})\right) \leq e^{-tN(1 - \frac{1}{q})} \mathbb{E}\left(\exp -t \sum_{j=1}^N \sum_{\mu=2}^M x_1^\mu \eta_j^\mu\right).$$

Using the independence of $x_1^\mu \eta_j^\mu$ in μ and the pairwise independence of these variables in j one computes that

$$\mathbb{E}\left(\exp -t \sum_{j=1}^N \sum_{\mu=2}^M x_1^\mu \eta_j^\mu\right) \leq \text{const.} \cdot e^{t^2 NM \frac{1 - \frac{1}{q}}{q^2}}$$

The optimal choice of $t = \frac{q^2}{2M}$ and the ansatz $M = cN/(\log N)$ give the result. \square

Proof of Theorem 2.3. Now, for an upper bound of

$$\mathbb{P}(\forall i = 1, \dots, N, \sum_{j=1}^N \sum_{\mu=2}^M x_i^\mu \eta_j^\mu \geq -N(1 - \frac{1}{q}))$$

one cannot simply work with the union bound as in the previous proof. An extra argument is provided by using negative association. The idea of negative association in this context was introduced by Loukianova [10] and was later also used by Bovier [2] to obtain a result similar to ours in the context of the Hopfield model.

Assume that $\mu = 1$ and ξ^1 consists of N coordinates that are all equal to $v_1 = (1 - \frac{1}{q}, -\frac{1}{q}, \dots, -\frac{1}{q})^t$. Let $y = (y^\mu, \mu = 1, \dots, M)$, $z = (z^\mu, \mu = 1, \dots, M)$ and $\mathbb{P}_{y,z}$ the probability measure \mathbb{P} conditioned on $Y^\mu : \sum_j \eta_j^\mu = y^\mu$ and $Z^\mu : \sum_i \delta_{x_i^\mu = -1} = z^\mu$ for each μ . We write

$$\begin{aligned} \mathbb{P}(\forall i, \sum_{j=1}^N \sum_{\mu=2}^M x_i^\mu \eta_j^\mu \geq -N(1 - \frac{1}{q})) &= \sum_{y,z} \mathbb{P}_{y,z}(\forall i, \sum_{\mu=2}^M x_i^\mu y^\mu \geq -N(1 - \frac{1}{q})) \\ &\quad \times \mathbb{P}(\forall \mu, \sum_j \eta_j^\mu = y^\mu, \sum_i \delta_{x_i^\mu = -1} = z^\mu). \end{aligned}$$

From general results on negative association (see [10]), one derives that, conditionally on $Y^\mu = y^\mu$ and $Z^\mu = z^\mu$ for each μ , the random variables $(x_i^\mu)_{i=1, \dots, N}^{\mu=1, \dots, M}$, are negatively associated. This implies that

$$\mathbb{P}_{y,z}(\forall i, \sum_{\mu=2}^M x_i^\mu y^\mu \geq -N(1 - \frac{1}{q})) \leq \prod_{i=1}^N \mathbb{P}_{y,z}(\sum_{\mu=2}^M x_i^\mu y^\mu \geq -N(1 - \frac{1}{q})) \quad (4)$$

All terms in the product on the right hand side in (4) are equal and can now be estimated by using the technique of tilted measures which is known from large deviation theory : for $t \leq 0$, let $\mathbb{P}_{y,z}^t$ be the probability measure

$$\mathbb{P}_{y,z}^t(\cdot) := \frac{\mathbb{E}_{y,z}[\mathbf{1}_{\{\cdot\}} \exp(\frac{t}{N} \sum_{\mu \geq 2} x_1^\mu y^\mu)]}{\mathbb{E}_{y,z}[\exp(\frac{t}{N} \sum_{\mu \geq 2} x_1^\mu y^\mu)]}.$$

Now, $\mathbb{P}_{y,z}(\sum_{\mu=2}^M x_1^\mu y^\mu \geq -N(1 - \frac{1}{q})) = 1 - \mathbb{P}_{y,z}(\sum_{\mu=2}^M x_1^\mu y^\mu < -N(1 - \frac{1}{q}))$ and for any $\delta > 0$,

$$\begin{aligned} \mathbb{P}_{y,z}(\sum_{\mu=2}^M x_1^\mu y^\mu < -N(1 - \frac{1}{q})) &\geq \mathbb{P}_{y,z}(-N(1 - \frac{1}{q} + \delta) < \sum_{\mu=2}^M x_1^\mu y^\mu < -N(1 - \frac{1}{q})) \\ &\geq e^{-t(-1 + \frac{1}{q} - \delta)} \mathbb{E}_{y,z}[e^{\frac{t}{N} \sum_{\mu=2}^M x_1^\mu y^\mu}] \mathbb{P}_{y,z}^t(-N(1 - \frac{1}{q} + \delta) < \sum_{\mu=2}^M x_1^\mu y^\mu < -N(1 - \frac{1}{q})) \end{aligned}$$

Under $\mathbb{P}_{y,z}$, the random variables (x_1^μ) are still independent and their law is given by $\mathbb{P}_{y,z}(x_1^\mu = 1) = \frac{w^\mu}{N}$, $\mathbb{P}_{y,z}(x_1^\mu = -1) = \frac{z^\mu}{N}$ and $\mathbb{P}_{y,z}(x_1^\mu = 0) = 1 - \frac{w^\mu}{N} - \frac{z^\mu}{N}$, where we set $w^\mu := y^\mu + \frac{N}{q}$. Computing $\mathbb{E}_{y,z}[e^{\frac{t}{N} \sum_{\mu=2}^M x_1^\mu y^\mu}]$ and $\mathbb{E}_{y,z}[x_1^\mu e^{\frac{t}{N} \sum_{\mu=2}^M x_1^\mu y^\mu}]$, we obtain

$$\mathbb{E}_{y,z}^t[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu] = \frac{1}{N} \sum_{\mu} y^\mu \frac{w^\mu - z^\mu}{N} + (y^\mu)^2 (\frac{w^\mu + z^\mu}{N} - (\frac{w^\mu - z^\mu}{N})^2) \frac{t}{N} + \mathcal{O}((y^\mu)^3 \frac{t^2}{N^2}).$$

This quantity can be evaluated for typical values of w^μ, z^μ and y^μ . More precisely, let

$$h(Y^\mu, Z^\mu) := Y^\mu \frac{W^\mu - Z^\mu}{N} + (Y^\mu)^2 (\frac{W^\mu + Z^\mu}{N} - (\frac{W^\mu - Z^\mu}{N})^2) \frac{t}{N},$$

where $W^\mu := Y^\mu + \frac{N}{q}$. Then there exist constants $c_1, c_2 > 0$ such that the set

$$A(c_1, c_2) := \{y, z : |\frac{1}{M} \sum_{\mu=2}^M h(y^\mu, z^\mu) - \mathbb{E}(h(Y^1, Z^1))| \leq c_1 \sqrt{\frac{\log M}{M}}; \sup_{\mu=2}^M |y^\mu| \leq c_2 \sqrt{\log M}\}$$

satisfies $\mathbb{P}((Y^\mu, Z^\mu)_{\mu=1, \dots, M} \in A(c_1, c_2)) \rightarrow 1$ as $N \rightarrow \infty$. We can then restrict to $y, z \in A(c_1, c_2)$, for which we have

$$\mathbb{E}_{y,z}^t[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu] \simeq a(\frac{1}{q} + t \frac{2}{q}(1 - \frac{1}{q})),$$

where $a := \frac{M}{N}$, and the rest is negligible. Choosing $t = t^* := -\frac{1}{a}(\frac{q^2}{2} + \varepsilon_N)$ for some $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$, we obtain

$$\mathbb{E}_{y,z}^{t^*} \left[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu \right] = -1 + \frac{1}{q} - \frac{\delta}{2}.$$

A similar computation shows that the variance satisfies

$$\mathbb{V}_{y,z}^{t^*} \left[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu \right] = \mathbb{E}_{y,z}^{t^*} \left[\left(\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu - \mathbb{E}_{y,z}^{t^*} \left[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu \right] \right)^2 \right] \simeq c_3 a(1 + \varepsilon_N),$$

for a constant $c_3 > 0$ (depending only on q) and some $\varepsilon_N \rightarrow 0$ as $N \rightarrow +\infty$. Employing Chebychev's inequality, we arrive at

$$\mathbb{P}_{y,z}^{t^*} \left(-N \left(1 - \frac{1}{q} + \delta \right) < \sum_{\mu=2}^M x_1^\mu y^\mu < -N \left(1 - \frac{1}{q} \right) \right) \geq 1 - \frac{4}{\delta^2} \mathbb{V}_{y,z}^{t^*} \left[\frac{1}{N} \sum_{\mu=2}^M x_1^\mu y^\mu \right] \simeq 1 - \frac{4}{\delta^2} c_3 a,$$

so that choosing $\delta := 4\sqrt{c_3 a}$, we get $\mathbb{P}_{y,z}^{t^*} \left(-N \left(1 - \frac{1}{q} + \delta \right) < \sum_{\mu=2}^M x_1^\mu y^\mu < -N \left(1 - \frac{1}{q} \right) \right) \geq \frac{1}{2}$ for N large enough. Similarly we obtain

$$e^{-t^* \left(-1 + \frac{1}{q} - \delta \right)} \mathbb{E}_{y,z}^{t^*} \left[e^{\frac{t^*}{N} \sum_{\mu=2}^M x_1^\mu y^\mu} \right] \geq e^{-\frac{q(q-1)}{4a} (1 + \varepsilon_N)}$$

for some $\varepsilon_N \rightarrow 0$ as $N \rightarrow +\infty$. We finally arrive at

$$\begin{aligned} \mathbb{P}_{y,z}(\forall i, \sum_{\mu=2}^M x_i^\mu y^\mu \geq -N \left(1 - \frac{1}{q} \right)) &\leq \left(1 - \frac{1}{2} \exp \left(-\frac{q(q-1)}{4a} (1 + \varepsilon_N) \right) \right)^N \\ &\leq \exp \left(-\frac{N}{2} \exp \left(-\frac{q(q-1)}{4a} (1 + \varepsilon_N) \right) \right), \end{aligned}$$

which goes to zero for $a = c/\log N$, if $c > \frac{q(q-1)}{4}$. Since this is true for all $y, z \in A(c_1, c_2)$, this proves the theorem. \square

Eventually the proof of Theorem 2.6 is split into different lemmas. We only need to consider the case $M = (1 - 1/q - \kappa)^2 \frac{q^2 N}{4(1-1/q)\log N}$ for some positive $\kappa \leq \varrho$, since the case $\kappa \in]\varrho, 1 - 1/q[$ is identical to Theorem 2.4, with $k = 1$.

Lemma 3.1. *Let $c > 0$, $M = c \frac{N}{\log N}$ and y^μ be randomly chosen from $S_{\varrho N}(\xi^\mu)$, $0 \leq \varrho < (q-1)/q$. Then for $\alpha = 1 - \frac{(1-1/q-\varrho)^2 q^2}{4(1-1/q)c} + \varepsilon$ for some $\varepsilon > 0$ it holds that*

$$\mathbb{P}(d(T(y^\mu), \xi^\mu) \geq N^\alpha) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Proof. Assume that $\mu = 1$ and that ξ^1 consists of N coordinates that are all equal to $v_1 = (1 - \frac{1}{q}, -\frac{1}{q}, \dots, -\frac{1}{q})^t$. For y^1 and α defined as in the theorem let $A := \{d(T(y^1), \xi^1) \geq N^\alpha\}$. In other words

$$A = \left\{ \exists I \mid |I| = N^\alpha, \forall i \in I \exists k_i \in \{2, \dots, q\} : \langle v_{k_i}, \sum_{\mu=1}^M \sum_{j=1}^N \xi_i^\mu (\xi_j^\mu)^t y_j^1 \rangle \geq \langle v_1, \sum_{\mu=1}^M \sum_{j=1}^N \xi_i^\mu (\xi_j^\mu)^t y_j^1 \rangle \right\}.$$

Therefore with C defined as

$$C := \left\{ \exists I : |I| = N^\alpha, \forall i \in I \exists k_i \in \{2, \dots, q\} : \sum_{i \in I} \langle v_1 - v_{k_i}, \sum_{\mu=1}^M \sum_{j=1}^N \xi_i^\mu (\xi_j^\mu)^t y_j^1 \rangle \leq 0 \right\}$$

we have $A \subseteq C$. Now define x_i^μ and η_j^μ by $x_i^\mu := \xi_{i,1}^\mu - \xi_{i,2}^\mu$, and $\eta_j^\mu := (\xi_j^\mu)^t y_j^1$. Their laws are given by

$$\mathbb{P}[x_i^\mu = 1] = \mathbb{P}[x_i^\mu = -1] = \frac{1}{q}, \quad \mathbb{P}[x_i^\mu = 0] = 1 - \frac{2}{q}; \quad \mathbb{P}[\eta_j^\mu = 1 - \frac{1}{q}] = \frac{1}{q}, \quad \mathbb{P}[\eta_j^\mu = -\frac{1}{q}] = 1 - \frac{1}{q}.$$

We can bound the probability of A by

$$\mathbb{P}(A) \leq P(C) \leq \binom{N}{N^\alpha} (q-1)^{N^\alpha} \mathbb{P}\left(\sum_{i=1}^{N^\alpha} \sum_{j=1}^N \sum_{\mu=1}^M x_i^\mu \eta_j^\mu \leq 0\right).$$

By Stirling's formula $\binom{N}{N^\alpha} \sim e^{(1-\alpha)N^\alpha \log N}$ which also shows that $(q-1)^{N^\alpha}$ (even though being huge) is negligible with respect to $\binom{N}{N^\alpha}$. The probability is again estimated by an exponential Chebyshev inequality. Observe that the summand for $\mu = 1$ is a number:

$$\sum_{i=1}^{N^\alpha} \sum_{j=1}^N x_i^1 \eta_j^1 = N^{1+\alpha} \left(\left(1 - \frac{1}{q}\right)(1 - \varrho) - \frac{1}{q} \varrho \right) = N^{1+\alpha} \left(1 - \frac{1}{q} - \varrho\right).$$

Therefore

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{N^\alpha} \sum_{j=1}^N \sum_{\mu=1}^M x_i^\mu \eta_j^\mu \leq 0\right) &\leq e^{-tN^{1+\alpha}(1-\frac{1}{q}-\varrho)} \mathbb{E} \exp\left(-t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} \sum_{j=1}^N x_i^\mu \eta_j^\mu\right) \\ &\leq e^{-tN^{1+\alpha}(1-\frac{1}{q}-\varrho)} \left(\mathbb{E} e^{-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=N^\alpha+1}^N \eta_j^\mu}\right)^{1/2} \left(\mathbb{E} e^{-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=1}^{N^\alpha} \eta_j^\mu}\right)^{1/2}. \end{aligned}$$

To compute the expectation of the first term, we use the fact that $\sum_{i=1}^{N^\alpha} x_i^\mu$ and $\sum_{j=N^\alpha+1}^N \eta_j^\mu$ are independent, and the quantities are also independent for different μ , which implies

$$\mathbb{E} \exp\left(-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=N^\alpha+1}^N \eta_j^\mu\right) = \prod_{\mu \geq 2} \mathbb{E}_{(\xi_i^\mu)_{i \leq N^\alpha}} \prod_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} e^{-2tX^\mu \eta_j^\mu}, \quad (5)$$

where $\mathbb{E}_{\xi_j^\mu}$ denotes expectation with respect to the random variable ξ_j^μ and $X^\mu := \sum_{i=1}^{N^\alpha} x_i^\mu$. To compute the second factor on the right, observe that since the η_j^μ are centered

$$\begin{aligned} \prod_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} e^{-2tX^\mu \eta_j^\mu} &= \exp\left(\sum_{j > N^\alpha} \log \mathbb{E}_{\xi_j^\mu} e^{-2tX^\mu \eta_j^\mu}\right) \leq \exp\left(\sum_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} [e^{-2tX^\mu \eta_j^\mu} - 1]\right) \\ &= \exp\left(\sum_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} [e^{-2tX^\mu \eta_j^\mu} + 2tX^\mu \eta_j^\mu - 1]\right), \end{aligned}$$

Now, $-2X^\mu \eta_j^\mu \leq K := 2(1 - \frac{1}{q})N^\alpha$ and $x \mapsto x^{-2}(e^x - x - 1)$ is increasing, yielding that

$$\begin{aligned} \mathbb{E}_{\xi_j^\mu} [e^{-2tX^\mu \eta_j^\mu} - 2tX^\mu \eta_j^\mu - 1] &\leq 4(X^\mu)^2 \mathbb{E}_{\xi_j^\mu} [(\eta_j^\mu)^2] \frac{e^{tK} - tK - 1}{K^2} \\ &\leq 4(X^\mu)^2 \mathbb{E}_{\xi_j^\mu} [(\eta_j^\mu)^2] \frac{t^2}{2(1 - Kt/3)} \end{aligned}$$

Now $\mathbb{E}_{\xi_j^\mu} [(\eta_j^\mu)^2] = \frac{1}{q}(1 - \frac{1}{q})$. Moreover,

$$\prod_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} e^{-2tX^\mu \eta_j^\mu} \leq e^{2(1-\frac{1}{q})\frac{1}{q}t^2(N-N^\alpha)(X^\mu)^2(1+\mathcal{O}(tN^\alpha))} = \mathbb{E}_U \exp(\gamma tX^\mu U),$$

for a standard normal random variable U and $\gamma := \sqrt{2(1-\frac{1}{q})\frac{1}{q}(N-N^\alpha)(1+\mathcal{O}(tN^\alpha))}$.

Now, integrating with respect to $\xi_i^\mu, i \leq N^\alpha$, and using $\cosh(x) \leq \exp(\frac{x^2}{2})$ for all x , we get

$$\begin{aligned} \prod_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} e^{-2tX^\mu \eta_j^\mu} &\leq \mathbb{E}_U \left(\frac{2}{q} \cosh(\gamma tU) + \left(1 - \frac{2}{q}\right)\right)^{N^\alpha} \\ &\leq \sum_{k=0}^{N^\alpha} \binom{N^\alpha}{k} \left(\frac{2}{q}\right)^k (1 - \gamma^2 t^2 k)^{-1/2} \left(1 - \frac{2}{q}\right)^{N^\alpha - k} \end{aligned}$$

where we anticipated the choice $t = \mathcal{O}(1/M)$ and $M = c \frac{N}{\log N}$, hence $\gamma^2 t^2 N^\alpha < 1$ and

$$(1 - \gamma^2 t^2 k)^{-1/2} = e^{-\frac{1}{2} \log(1 - \gamma^2 t^2 k)} \leq e^{\gamma^2 \frac{t^2}{2} k + \gamma^4 \frac{t^2}{2} N^{2\alpha}}.$$

This implies

$$\prod_{j > N^\alpha} \mathbb{E}_{\xi_j^\mu} e^{-2t X^\mu \eta_j^\mu} \leq e^{\gamma^4 \frac{t^2}{2} N^{2\alpha}} \left(\frac{2}{q} e^{\gamma^2 \frac{t^2}{2}} + 1 - \frac{2}{q} \right)^{N^\alpha}$$

Hence

$$\left(\mathbb{E} \exp \left(-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=N^\alpha+1}^N \eta_j^\mu \right) \right)^{1/2} \leq \exp(MN^{1+\alpha} (1 - \frac{1}{q}) \frac{1}{q^2} t^2 (1 + \varepsilon_N)),$$

with $\varepsilon_N \rightarrow 0$ as $N \rightarrow +\infty$. The first factor on the right in (5) is negligible. More precisely we have

$$\mathbb{E} \exp \left(-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=1}^{N^\alpha} \eta_j^\mu \right) = \prod_{\mu \geq 2} \mathbb{E} \exp \left(-2t \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=1}^{N^\alpha} \eta_j^\mu \right) \leq \exp(CMtN^\alpha),$$

for some $C > 0$ and N large enough. Indeed, let $X^\mu := \sum_{i \leq N^\alpha} x_i^\mu$ and $Y^\mu := \sum_{i \leq N^\alpha} \eta_i^\mu$. We use the trivial inequality $-2X^\mu Y^\mu \leq (X^\mu)^2 + (Y^\mu)^2$ to get

$$\begin{aligned} \mathbb{E} \exp \left(-2t \sum_{\mu \geq 2} \sum_{i=1}^{N^\alpha} x_i^\mu \sum_{j=1}^{N^\alpha} \eta_j^\mu \right) &\leq \prod_{\mu \geq 2} \mathbb{E} \exp(t(X^\mu)^2 + t(Y^\mu)^2) \\ &\leq (\mathbb{E} \exp 2t(X^\mu)^2)^{1/2} (\mathbb{E} \exp 2t(Y^\mu)^2)^{1/2} \\ &\leq \exp(CMtN^\alpha), \end{aligned}$$

again by introducing a normal random variable to linearize the exponent in the expectation.

Keeping only the dominating term, choose $t = \frac{(1 - \frac{1}{q} - \varrho)q^2}{2(1 - \frac{1}{q})M}$. This yields

$$\mathbb{P} \left(\sum_{i=1}^{N^\alpha} \sum_{j=1}^N \sum_{\mu=1}^M x_i^\mu \eta_j^\mu \leq 0 \right) \leq e^{-\frac{N^{1+\alpha} (1 - \frac{1}{q} - \varrho)^2 q^2}{M \cdot 4(1 - \frac{1}{q})}}.$$

The choice $M = c \frac{N}{\log N}$ together with our previous calculations implies

$$\begin{aligned} P(C) &\leq \binom{N}{N^\alpha} (q-1)^{N^\alpha} \exp \left(-\frac{(1 - \frac{1}{q} - \varrho)^2 q^2}{4(1 - \frac{1}{q})c} N^\alpha \log N \right) \\ &\approx (q-1)^{N^\alpha} \exp \left(\left(-\frac{(1 - \frac{1}{q} - \varrho)^2 q^2}{4(1 - \frac{1}{q})c} + (1 - \alpha) \right) N^\alpha \log N \right). \end{aligned}$$

This term clearly converges to zero, if $\alpha = 1 - \frac{(1 - \frac{1}{q} - \varrho)^2 q^2}{4(1 - \frac{1}{q})c} + \varepsilon$ for some $\varepsilon > 0$. \square

Lemma 3.2. *Let $c < q(q-1)/4$, $M = c \frac{N}{\log N}$ and $\alpha \in [1/2, 1[$. Then for every fixed μ , $B_{N^\alpha}(\xi^\mu) := \{x : d(x, \xi^\mu) \leq N^\alpha\}$, and $\beta = 2\alpha - 1 + \varepsilon$ for some $\varepsilon > 0$ it holds*

$$\mathbb{P}(\exists x \in B_{N^\alpha}(\xi^\mu) : d(T(x), \xi^\mu) \geq N^\beta) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Proof. This again requires some exponential estimate. Let

$$A := \{\exists y \in B_{N^\alpha}(\xi^\mu) : d(T(y), \xi^\mu) \geq N^\beta\}.$$

As before we may assume that $\mu = 1$ and ξ^1 is the vector consisting of N coordinates that are all equal to v_1 . We have again that $A \subseteq C$, where $C := \cup_{y \in B_{N^\alpha}(\xi^1)} C(y)$, and

$$C(y) := \{ \exists I : |I| = N^\beta, \forall i \in I, \exists k_i \in \{2, \dots, q\} : \sum_{i \in I} < v_1 - v_{k_i}, \sum_{\mu=1}^M \sum_{j=1}^N \xi_i^\mu (\xi_j^\mu)^t y_j > \leq 0 \}.$$

Following the lines of the proof of Lemma 3.1 one shows that the probability of the event

$$C(\xi^1, \delta) := \{ \exists I : |I| = N^\beta, \forall i \in I, \exists k_i : \sum_{i \in I} < v_1 - v_{k_i}, \sum_{\mu=2}^M \sum_{j=1}^N \xi_i^\mu (\xi_j^\mu)^t \xi_j^1 > \leq -N^{1+\beta} (1 - \frac{1}{q})(1-\delta) \}$$

converges to zero for $\delta \in]0, 1[$, $\beta = 1 - \frac{(1-\delta)^2 q^2 (1-\frac{1}{q})}{4c} + \varepsilon$ and $\varepsilon > 0$. Write

$$\mathbb{P}[A] \leq \mathbb{P}[C(\xi^1, \delta)] + \sum_{y \in B_{N^\alpha}(\xi^1)} \mathbb{P}[C(y) \cap \overline{C(\xi^1, \delta)}]$$

(where the bar denotes the complement of an event). The first term converges to 0 and the second term satisfies

$$\mathbb{P}[C(y) \cap \overline{C(\xi^1, \delta)}] \leq \sum_{I: |I|=N^\beta} \mathbb{P}[\sum_{i \in I} \sum_{\mu=2}^M \sum_{j \in J} x_i^\mu (\xi_j^\mu)^t (y_j - \xi_j^1) \leq -N^{1+\beta} (1 - \frac{1}{q})\delta],$$

where y_j are the coordinates of the vector $y \in B_{N^\alpha}(\xi^1)$, $J \subseteq \{1, \dots, N\}$ is the set of size at most N^α such that y differs from ξ^1 exactly in the coordinates J and we again write $x_i^\mu := \xi_{i,1}^\mu - \xi_{i,2}^\mu$. For every fixed $y \in B_{N^\alpha}(\xi^1)$, $\mu \geq 2$ and $j \in J$, $\delta_j^\mu(y) := (\xi_j^\mu)^t (y_j - \xi_j^1)$ has the same law as x_i^μ , that is

$$\mathbb{P}[\delta_j^\mu(y) = 1] = \mathbb{P}[\delta_j^\mu(y) = -1] = \frac{1}{q}, \quad \mathbb{P}[\delta_j^\mu(y) = 0] = 1 - \frac{2}{q}.$$

Now the probability of this event doesn't change much, if we just concentrate on sets J with $|J| = N^\alpha$. As in the proof of Lemma 3.1 one shows that for each fixed J with $|J| = N^\alpha$

$$\mathbb{P}[\sum_{i \in I} \sum_{\mu=2}^M \sum_{j \in J} x_i^\mu (\xi_j^\mu)^t (y_j - \xi_j^1) \leq -N^{1+\beta} (1 - \frac{1}{q})\delta] \leq C_1 \exp(-C_2 N^{1+\beta-\alpha} \log N)$$

for some positive constants C_1 and C_2 depending only on q and δ . On the other hand, there are $\binom{N}{N^\alpha} \sim e^{(1-\alpha)N^\alpha \log N}$ choices for J and $\binom{N}{N^\beta} \sim e^{(1-\beta)N^\beta \log N}$ choices for I . Therefore,

$$\sum_{y \in B_{N^\alpha}(\xi^1)} \mathbb{P}[C(y) \cap \overline{C(\xi^1, \delta)}] \leq C_1 e^{(1-\alpha)N^\alpha \log N} e^{(1-\beta)N^\beta \log N} \exp(-C_2 N^{1+\beta-\alpha} \log N) \quad (6)$$

with $\beta = 1 - \frac{(1-\delta)^2}{\lambda} + \varepsilon$, $\lambda := \frac{4c}{q(q-1)} \in]0, 1[$ and $\alpha \in [1/2, 1[$. Now choose

$$\delta = 1 - \sqrt{2\lambda(1-\alpha)} \in]0, 1[$$

such that $\beta = 2\alpha - 1 + \varepsilon$ for $\varepsilon > 0$ small enough. In particular $\beta < \alpha$ and we obtain

$$\sum_{y \in B_{N^\alpha}(\xi^1)} \mathbb{P}[C(y) \cap \overline{C(\xi^1, \delta)}] \leq C_1 e^{(1-\alpha)N^\alpha \log N} e^{(1-\beta)N^\beta \log N} \exp(-C_2 N^{\alpha+\varepsilon} \log N)$$

which goes to 0 as $N \rightarrow \infty$. This proves the lemma. \square

Lemma 3.3. *Let $c < q(q-1)/4$, $M = c \frac{N}{\log N}$ and $\beta \in]0, 1/2[$. Then for every fixed μ it holds*

$$\mathbb{P}(\exists x \in B_{N^\beta}(\xi^\mu) : T(x) \neq \xi^\mu) \rightarrow 0$$

as $N \rightarrow \infty$.

Proof. As the proof of Lemma 3.3 is very similar to the proof of Lemma 3.2 we will refrain from proving it here. \square

These three lemmas now immediately imply Theorem 2.6.

Proof of Theorem 2.6. The proof of Theorem 2.6 consists of one application of Lemma 3.1 that states that $T(y^\mu)$ is in distance N^α to ξ^μ for some $\alpha < 1$, several applications of Lemma 3.2 to reduce the distance of $T^{k-1}(y^\mu)$ to N^β for some $\beta < 1/2$, and a final application of Lemma 3.3. \square

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