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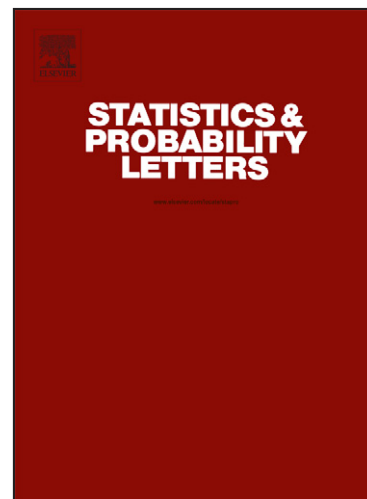
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Limit laws for multidimensional extremes.

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Abstract

A definition of the ordering of a multivariate sample - based on the isobar surfaces - is used in order to obtain limit laws for the extreme values of a multidimensional sample.

Key-words : isobars , order statistics, extreme values, stability, limit laws, concomitant.

A.M.S. Subject classification : 62 G 30, 60 D 05

1 Introduction

Today many papers on extreme values deal with multivariate extreme values, see [Nadarajah, Anderson, Tawn 98], [Schlatler, Tawn 03], [Fougères 04], [Heffernan, Tawn 04], [Falk, Reiss 05]; see also for recent developments about max-stable processes and multivariate extremal index, [Smith, Weissman 96], [Zhang, Smith 04]. Unfortunately, ordering multivariate data can be done in various ways and many definitions have been proposed e.g. [Barnett 76], [Maller 90]. Some papers of [Abdous, Theodorescu 92], [Einmahl, Mason 92], [Massé, Theodorescu 94], [De Haan, Huang 95], and more recently, of [Berlinet, Gannoun, Matzner-Lober 01], [Serfling 02] develop the notion of multivariate quantiles. In the classical scheme (cartesian coordinates) the multivariate variables are ordered coordinate by coordinate - see for example [Galambos 87] and the references therein. And in this way the maximum value thus obtained is not a sample point.

In this paper we will use the definition of the maximum value of a multidimensional sample, given in [Delcroix, Jacob 91]. This definition is based on surfaces we have called isobars. Our approach is more geometric and as recalled just below, we use the level-surfaces of the conditional distribution function of the radius R given the angle Θ . So the maximum value is a sample point. The first motivation, [Barme 93], was to find the asymptotic location of a multidimensional sample without using the convex hull of the sample as it is done classically in [Geffroy 61], [Geffroy 58]. By a unidimensional approach, stability, strong behaviour and outlier-proneness properties have been explored for this new notion of extreme value in [Barme, Brito 01] and [Barme, Gather 02]. In this paper, the goal is to find limit laws for such mul-

tidimensional extremes. Through the use of the isobar surfaces we will see that the classical results -[Gnedenko 43] - about limit laws for unidimensional extremes are still efficient in this multidimensional context.

2 Preliminaries

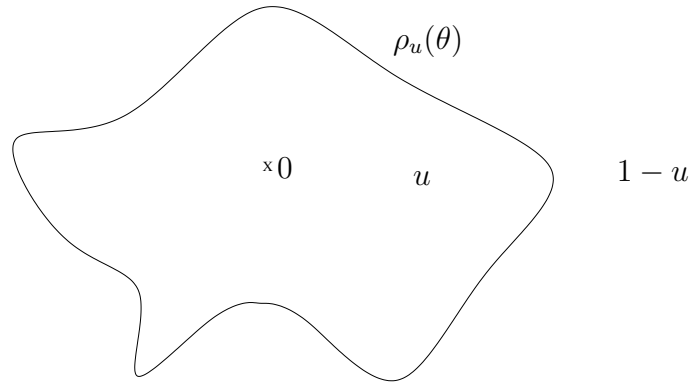
We consider random variables defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and with values in the Euclidean space \mathbf{R}^k .

For every x in $\mathbf{R}^k \setminus \{0\}$ we define a pair $(\|x\|, \frac{x}{\|x\|}) = (r, \theta)$ in $\mathbf{R}^{+*} \times \mathbf{S}^{k-1}$, where $\|\cdot\|$ is the Euclidean norm and \mathbf{R}^{+*} is the set of the positive real numbers. The unit sphere \mathbf{S}^{k-1} in \mathbf{R}^k is endowed with the induced topology of \mathbf{R}^k .

For each random variable $X = (R, \Theta) = (\|X\|, \frac{X}{\|X\|}) \in \mathbf{R}^k$, we assume that the distribution of Θ , and for all θ , the distribution of R given $\Theta = \theta$, have a continuous density. F_θ denotes the continuous conditional distribution function of R , given $\Theta = \theta$ and F_θ^{-1} the generalized inverse of F_θ .

For each $0 < u < 1$, we call the mapping $\theta \rightsquigarrow F_\theta^{-1}(u)$ a u -level **isobar** from the distribution of R given $\Theta = \theta$. We suppose that this mapping is continuous and strictly positive; the surface given by $\rho = F_\theta^{-1}(u)$ is also called an isobar, see Figure 1.

Let $E_n = (X_1, \dots, X_n)$ be a sample of independent random variables in \mathbf{R}^k each with the same distribution as X . Clearly for each $1 \leq k \leq n$ there is almost surely a unique u_k -level isobar from the distribution of R given $\Theta = \theta$ which contains $(R_k, \Theta_k) = (\|X_k\|, \frac{X_k}{\|X_k\|})$. We define **the maximum value** in E_n as the point of the sample $X_n^* = (R_n^*, \Theta_n^*)$ which belongs to the upper level isobar, i.e. the isobar with level $\max_{1 \leq k \leq n} u_k$. See Figure 2. The

Figure 1: u -level isobar

multivariate sample is then ordered according to the increasing levels of the corresponding isobar surfaces, and the order statistics of the n -sample are denoted by

$$X_{n,1}, \dots, X_{n,i}, \dots, X_{n,n-1}, X_{n,n} = X_n^*$$

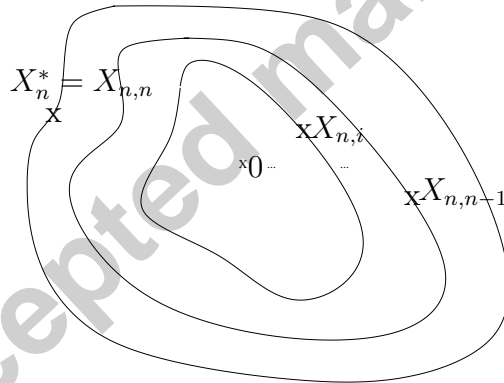


Figure 2: isobar-maximum

Remark 1 *We could imagine a more general way to order the sample. For example, by considering an increasing sequence of borelians, according to a criterion to define, and not necessarily related to the Euclidean norm. But it is not the purpose of this paper.*

It has been shown in [Delcroix, Jacob 91] that the conditional distribution of R_n^* given Θ_n^* is F_θ^n . More exactly we have the following theorem.

Theorem 1 a) *The variables Θ_n^* and Θ have the same distribution.*

b) *The conditional distribution function of R_n^* given $\Theta_n^* = \theta$ is given by $P(R_n^* \leq r / \Theta_n^* = \theta) = (F_\theta(r))^n$.*

Proof : see [Delcroix, Jacob 91]. ■

The second assertion of the previous theorem shows that this multidimensional problem can be reduced to a unidimensional one . That will be done in the next section to find limit laws for the pair (R_n^*, Θ_n^*) . In a third section, we will focus on limit laws for the variables R_n^* .

From the previous theorem, we know that the distributions of (R_n^*, Θ_n^*) and (R, Θ) possess the same set of isobars. So it has been possible in [Delcroix, Jacob 91] to give the following definition.

Definition 1 $(X_n^*)_n$ is called stable in probability (resp. a.s.) if and only if there is a sequence $(g_n)_n$ of isobars satisfying

$$(1) \quad R_n^* - g_n(\Theta_n^*) \xrightarrow[(a.s.)]{P} 0 .$$

We are well aware that the above definition of multivariate maxima depends on the underlying distribution. This may be crucial in practical situations when given data are to be ordered¹. To find out more about the limit distribution of this maximum however it is not a severe additional obstacle. Note, that even to check the type of extreme value distribution in one dimension affords at least the knowledge of the tail of the underlying distribution.

¹Another work is in progress on the estimation of isobars.

3 Limit laws for some functionals of (R_n^*, Θ_n^*)

We know the three classical types of limit distributions for unidimensional extreme values and their domains of attraction, [Gnedenko 43], [Leadbetter 83], [Galambos 87]. Recall the three corresponding distribution functions with the Galambos notations :

i) $G_{3,0}(x) = \exp(-e^{-x})$ for all x .

ii) $G_{1,\alpha}(x) = \exp(-x^{-\alpha})$ for $x > 0$, with $\alpha > 0$.

iii) $G_{2,\alpha}(x) = \exp(-(-x)^\alpha) \mathbf{1}_{(x \leq 0)} + \mathbf{1}_{(x > 0)}$, with $\alpha > 0$.

Similarly, in this section, the goal is to find a nondegenerate distribution function \tilde{H} and sequences $a_n(\cdot) > 0$ and $b_n(\cdot)$ such that for all x

$$P\{a_n(\Theta_n^*)(R_n^* - b_n(\Theta_n^*)) \leq x\} \xrightarrow{w} \tilde{H}(x)$$

where \xrightarrow{w} denotes convergence at continuity points of \tilde{H} .

Let $g_n(\theta)$ denote the $(1 - \frac{1}{n})$ -level isobar; that is $g_n(\theta) = F_\theta^{-1}(1 - \frac{1}{n})$. Under given conditions, [Delcroix, Jacob 91], we have proved that these isobars stabilize the sequence (R_n^*, Θ_n^*) , according to the definition given in the first section.

Theorem 2 *Suppose that for all θ , $\sup_x \{F_\theta(x) < 1\} = +\infty$. If for all θ the distribution function F_θ satisfies*

$$\lim_{t \rightarrow \infty} \frac{1 - F_\theta(tx)}{1 - F_\theta(t)} = x^{-\alpha(\theta)} \text{ for all } x > 0,$$

where α is a strictly positive function, then for all $x > 0$

$$P\{R_n^* \leq x g_n(\Theta_n^*)\} \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

Proof : From Theorem 1.6.2 of [Leadbetter 83], for all θ there exist sequences a_n and b_n depending on θ such that

$$F_\theta^n \left(\frac{x}{a_n(\theta)} + b_n(\theta) \right) \rightarrow \exp(-x^{-\alpha(\theta)}).$$

Then from the Dominated Convergence Theorem we have

$$\int_{S^{k-1}} F_\theta^n \left(\frac{x}{a_n(\theta)} + b_n(\theta) \right) P_\Theta(d\theta) \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

But, we know that F_θ^n is the conditional distribution function of R_n^* given $\Theta_n^* = \theta$, so

$$\int_{S^{k-1}} P(R_n^* \leq \frac{x}{a_n(\theta)} + b_n(\theta) / \Theta_n^* = \theta) P_\Theta(d\theta) \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

And since Θ_n^* and Θ have the same distribution,

$$P(R_n^* \leq \frac{x}{a_n(\Theta_n^*)} + b_n(\Theta_n^*)) \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

Hence

$$P(a_n(\Theta_n^*)(R_n^* - b_n(\Theta_n^*)) \leq x) \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

Moreover from Corollary 1.6.3. of [Leadbetter 83] we know that for all θ $b_n = 0$ and $(a_n(\theta))^{-1} = F_\theta^{-1}(1 - \frac{1}{n}) = g_n(\theta)$. ■

Example 1 If $F_\theta(x) = (1 - K(\theta)x^{-\alpha(\theta)})\mathbf{1}_{\left(x > (K(\theta))^{\frac{1}{\alpha(\theta)}}\right)}$ where α and K are strictly and continuous functions, then for all $x > 0$

$$P\{R_n^* \leq x(nK(\Theta_{n,n}))^{1/\alpha(\Theta_n^*)}\} \rightarrow \int_{S^{k-1}} \exp(-x^{-\alpha(\theta)}) P_\Theta(d\theta).$$

For the two others domains of attraction it is possible to give similar limit laws according to the results of [Leadbetter 83]. For all θ we use the notation $x_{F_\theta} = \sup\{x : F_\theta(x) < 1\}$.

Theorem 3 *If for all θ $x_{F_\theta} < +\infty$ and if for all $x > 0$ and for all θ there exists a strictly positive function α such that*

$$\lim_{h \rightarrow 0} \frac{1 - F_\theta(x_{F_\theta} - xh)}{1 - F_\theta(x_{F_\theta} - h)} = x^{\alpha(\theta)}$$

then

$$P\left\{\left(x_{F_{\Theta_n^*}} - g_n(\Theta_n^*)\right)^{-1} (R_n^* - x_{F_{\Theta_n^*}}) \leq x\right\} \xrightarrow{w} \int_{S^{k-1}} \left(\exp(-(-x)^{\alpha(\theta)}) \mathbf{1}_{(x \leq 0)} + \mathbf{1}_{(x > 0)}\right) P_\Theta(d\theta).$$

Theorem 4 *If for all θ there exists a strictly positive function l_θ such that for all θ and for all x*

$$\lim_{t \rightarrow x_{F_\theta}} \frac{1 - F_\theta(t + xl_\theta(t))}{1 - F_\theta(t)} = e^{-x}$$

then for all x

$$P\left\{\left(l_{\Theta_n^*}(g_n(\Theta_n^*))\right)^{-1} (R_n^* - g_n(\Theta_n^*)) \leq x\right\} \rightarrow \int_{S^{k-1}} \exp(-e^{-x}) P_\Theta(d\theta) = \exp(-e^{-x}).$$

Example 2 *If $F_\theta(x) = (1 - e^{-\alpha(\theta)x}) \mathbf{1}_{(x > 0)}$ where α is a strictly positive continuous function, then for all x*

$$P\left\{\alpha(\Theta_n^*) \left(R_n^* - \frac{\text{Log}(n)}{\alpha(\Theta_n^*)}\right) \leq x\right\} \rightarrow \exp(-e^{-x}).$$

Example 3 *For a bivariate Gaussian sample with covariance matrix $\begin{pmatrix} \sigma^2 & 0 \\ 0 & \tau^2 \end{pmatrix}$,*

the density over \mathbf{R}^2 is $f(x, y) = \frac{1}{2\pi\sigma\tau} e^{-\frac{x^2}{2\sigma^2} - \frac{y^2}{2\tau^2}}$; the conditional distribution of R given $\Theta = \theta$ is then given by $F_\theta(x) = (1 - e^{-x^2\Phi(\theta)}) \mathbf{1}_{(x > 0)}$, where $\Phi(\theta) = \frac{\cos^2(\theta)}{2\sigma^2} + \frac{\sin^2(\theta)}{2\tau^2}$. Note that in this case, the isobars are also the level curves of the bivariate density. The condition of the previous theorem is satisfied and for all x ,

$$P\left\{2\sqrt{\Phi(\Theta_n^*)\text{Log}(n)} \left(R_n^* - \sqrt{\frac{\text{Log}(n)}{\Phi(\Theta_n^*)}}\right) \leq x\right\} \rightarrow \exp(-e^{-x}).$$

4 Limit law of R_n^*

Now, we consider the pairs (R_k, U_k) . In this part we use the following remark. By definition, R_n^* is the concomitant of $U_{n,n} = \text{Max}(U_1, \dots, U_n)$ where for all $1 \leq k \leq n$, $U_k = F_{\Theta_k}(R_k)$; see e.g. [Galambos 87], [David 81]. The two properties of the next theorem will be used in this section.

Theorem 5 a) $(U_1 = F_{\Theta_1}(R_1), \dots, U_n = F_{\Theta_n}(R_n))$ is a sample from the uniform distribution on $[0, 1]$.

b) U_k is independant of Θ_k for all $1 \leq k \leq n$.

Proof : see [Delcroix, Jacob 91]. ■

Note that the uniform distribution belongs to the type III -according to the notation of [Leadbetter 83]- of extreme values with $\alpha = 1$. It will be useful for the following. From Galambos results on concomitants it is possible to state this limit law for R_n^* .

Theorem 6 If there exist $A_n, B_n > 0$ such that $\lim_{n \rightarrow \infty} P\{R \leq A_n + B_n u/U = 1 + z/n\} = T(u, z)$ (for $-n \leq z \leq 0$ and $u > 0$) is a non degenerate distribution function, then $\lim_{n \rightarrow \infty} P\{R_n^* < A_n + B_n u\} = T(u)$ where

$$T(u) = \int_{-\infty}^0 T(u, z)e^z dz.$$

Proof : We have

$$P(R_n^* \leq r) = \int_{[0,1]} P(R_n^* \leq r/U_{n,n} = u)P_{U_{n,n}}(du).$$

But $(R_n^*, U_{n,n})$ is a point of the sample $((R_1, U_1), \dots, (R_n, U_n))$ and the (R_k, U_k) are independent, so

$$P(R_n^* \leq r) = \int_{[0,1]} P(R \leq r/U = u)nF_1^{n-1}(u)f_1(u)du$$

where F_1 and f_1 denote the uniform distribution function over $[0, 1]$ and the uniform density function respectively.

Now, put $u = 1 + \frac{z}{n}$. We obtain

$$P(R_n^* \leq r) = \int_{[-n,0]} P\left(R \leq r/U = 1 + \frac{z}{n}\right) \left(1 + \frac{z}{n}\right)^{n-1} dz.$$

And finally put $r = A_n + B_n u$, hence

$$P(R_n^* \leq r) = \int_{[-n,0]} P\left(R \leq A_n + B_n u/U = 1 + \frac{z}{n}\right) \left(1 + \frac{z}{n}\right)^{n-1} dz.$$

We conclude with the Dominated Convergence Theorem. ■

Remark 2 *The condition of the previous theorem may be written more easily with the conditional law of R given $U = u$ if we suppose F_θ one-to-one. Since $U = F_\Theta(R)$, we have*

$$P(R \leq r/U = u) = P(U \leq F_\Theta(r)/U = u).$$

And since the variables U and Θ are independent, we obtain

$$P(R \leq r/U = u) = P(u \leq F_\Theta(r)).$$

So the condition of the theorem is

$$P\left(1 + \frac{z}{n} \leq F_\Theta(A_n + B_n u)\right) \rightarrow T(u, z).$$

Of course this condition depends heavily on the form of the conditional distribution of R given $\Theta = \theta$.

Example 4 *If $F_\theta(x) = (1 - e^{\alpha(\theta)x})\mathbf{1}_{(x>0)}$ and $\alpha(\Theta)$ is uniformly distributed over $[0, 1]$ then the previous theorem is true with $A_n = \text{Log } n$, $B_n = \text{Log } n$, the limit law is Pareto.*

5 Concluding remarks

It would be possible to consider the pairs (U_k, Θ_k) as done in the last section for the variables U and R , but we obtain only the asymptotic independence between Θ and U and the fact that asymptotically the variables Θ_n^* and Θ have the same distribution, which is weaker than the results stated in Theorems 1 and 5.

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