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## APPLICATION OF A BERNSTEIN TYPE INEQUALITY TO RATIONAL INTERPOLATION IN THE DIRICHLET SPACE

RACHID ZAROUF

ABSTRACT. We prove a Bernstein-type inequality involving the Bergman and the Hardy norms, for rational functions in the unit disc  $\mathbb{D}$  having at most  $n$  poles all outside of  $\frac{1}{r}\mathbb{D}$ ,  $0 < r < 1$ . The asymptotic sharpness of this inequality is shown as  $n \rightarrow \infty$  and  $r \rightarrow 1^-$ . We apply our Bernstein-type inequality to an effective Nevanlinna-Pick interpolation problem in the standard Dirichlet space, constrained by the  $H^2$ - norm.

### INTRODUCTION

#### a. Statement of the problems.

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disc of the complex plane and let  $\text{Hol}(\mathbb{D})$  be the space of holomorphic functions on  $\mathbb{D}$ . Let also  $X$  and  $Y$  be two Banach spaces of holomorphic functions on the unit disc  $\mathbb{D}$ ,  $X, Y \subset \text{Hol}(\mathbb{D})$ . Here and later on,  $H^\infty$  stands for the space (algebra) of bounded holomorphic functions in the unit disc  $\mathbb{D}$  endowed with the norm  $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$ . We suppose that  $n \geq 1$  is an integer,  $r \in [0, 1)$  and we consider the two following problems.

**Problem 1.** Let  $\mathcal{P}_n$  be the complex space of analytic polynomials of degree less or equal than  $n$ , and

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q} : q \in \mathcal{P}_n, d^\circ p < d^\circ q, q(\zeta) = 0 \implies |\zeta| \geq \frac{1}{r} \right\},$$

(where  $d^\circ p$  means the degree of any  $p \in \mathcal{P}_n$ ) be the set of all rational functions in  $\mathbb{D}$  of degree less or equal than  $n \geq 1$ , having at most  $n$  poles all outside of  $\frac{1}{r}\mathbb{D}$ . Notice that for  $r = 0$ , we get  $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$ . Our first problem is to search for the “best possible” constant  $\mathcal{C}_{n,r}(X, Y)$  such that

$$\|f'\|_X \leq \mathcal{C}_{n,r}(X, Y) \|f\|_Y$$

for all  $f \in \mathcal{R}_{n,r}$ .

**Problem 2.** Let  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a finite subset of  $\mathbb{D}$ . What is the best possible interpolation by functions of the space  $Y$  for the traces  $f|_\sigma$  of functions of the space  $X$ , in the worst case? The case  $X \subset Y$  is of no interest, and so one can suppose that either  $Y \subset X$  or  $X$  and  $Y$  are incomparable. More precisely, our second problem is to compute or estimate the following interpolation constant

$$I(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \}.$$

We also define

$$\mathcal{I}_{n,r}(X, Y) = \sup \{I(\sigma, X, Y) : \text{card } \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma\}.$$

## b. Motivations.

**Problem 1.** Bernstein-type inequalities for rational functions are applied

**1.1.** in matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [LeTr, Sp] or [Z1, Z4] for resolvent estimates of power bounded matrices),

**1.2.** to “inverse theorems of rational approximation” using the *classical Bernstein decomposition* (see [Da, Pel, Pek]),

**1.3.** to effective  $H^\infty$  interpolation problems (see [Z3] and our Theorem B below in Subsection d), and more generally to our Problem 1.

**Problem 2.** We can give three main motivations for Problem 2.

**2.1.** It is explained in [Z3] (the case  $Y = H^\infty$ ) why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [N2] p.231 for these two problems), on the one hand and Carleson’s free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

**2.2.** It is also explained in [Z3] why this constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

**2.3.** It has already been proved in [Z3] that for  $X = H^2$  (see Subsection c. for the definition of  $H^2$ ) and  $Y = H^\infty$ ,

$$(1) \quad \frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq \mathcal{I}_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}.$$

The above estimate (1) answers a question of L. Baratchart (private communication), which is part of a more complicated question arising in an applied situation in [BL1] and [BL2]: given a set  $\sigma \subset \mathbb{D}$ , how to estimate  $I(\sigma, H^2, H^\infty)$  in terms of  $n = \text{card}(\sigma)$  and  $\max_{\lambda \in \sigma} |\lambda| = r$  only?

## c. The spaces $X$ and $Y$ considered here.

Now let us define some Banach spaces  $X$  and  $Y$  of holomorphic functions in  $\mathbb{D}$  which we will consider throughout this paper. From now on, if  $f \in \text{Hol}(\mathbb{D})$  and  $k \in \mathbb{N}$ ,  $\hat{f}(k)$  stands for the  $k^{\text{th}}$  Taylor coefficient of  $f$ .

1. The standard Hardy space  $H^2 = H^2(\mathbb{D})$ ,

$$H^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^2}^2 = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

where  $m$  stands for the normalized Lebesgue measure on  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ . An equivalent description of the space  $H^2$  is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

2. The standard Bergman space  $L_a^2 = L_a^2(\mathbb{D})$ ,

$$L_a^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{L_a^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\},$$

where  $A$  is the standard area measure, also defined by

$$L_a^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{L_a^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$

3. The analytic Besov space  $B_{2,2}^{\frac{1}{2}}$  (also known as the standard Dirichlet space) defined by

$$B_{2,2}^{\frac{1}{2}} = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{B_{2,2}^{\frac{1}{2}}} = \left( \sum_{k \geq 0} (k+1) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Then if  $f \in B_{2,2}^{\frac{1}{2}}$ , we have the following equality

$$(2) \quad \|f\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|f'\|_{L_a^2}^2 + \|f\|_{H^2}^2,$$

which establishes a link between the spaces  $B_{2,2}^{\frac{1}{2}}$  and  $L_a^2$ .

**d. The results.** Here and later on, the letter  $c$  denotes a positive constant that may change from one step to the next. For two positive functions  $a$  and  $b$ , we say that  $a$  is dominated by  $b$ , denoted by  $a = O(b)$ , if there is a constant  $c > 0$  such that  $a \leq cb$ ; and we say that  $a$  and  $b$  are comparable, denoted by  $a \asymp b$ , if both  $a = O(b)$  and  $b = O(a)$  hold.

**Problem 1.** Our first result (Theorem A, below) is a partial case ( $p = q = 2$ ,  $s = \frac{1}{2}$ ) of the following K. Dyakonov's result [Dy]: if  $p \in [1, \infty)$ ,  $s \in (0, +\infty)$ ,  $q \in [1, +\infty]$ , then there exists a constant  $c_{p,s} > 0$  such that

$$(3) \quad \mathcal{C}_{n,r} (B_{p,p}^{s-1}, H^q) \leq c_{p,s} \sup \|B'\|_{H^\gamma}^s,$$

where  $\gamma$  is such that  $\frac{s}{\gamma} + \frac{1}{q} = \frac{1}{p}$ , and the supremum is taken over all finite Blaschke products  $B$  of order  $n$  with  $n$  zeros outside of  $\frac{1}{r}\mathbb{D}$ . Here  $B_{p,p}^s$  stands for the Hardy-Besov space which consists of

analytic functions  $f$  on  $\mathbb{D}$  satisfying

$$\|f\|_{B_{p,p}^s} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} (1-|w|)^{(n-s)p-1} |f^{(n)}(w)|^p dA(w) < \infty.$$

For the (tiny) partial case considered here, our proof is different and the constant  $c_{2, \frac{1}{2}}$  is asymptotically sharp as  $r$  tends to  $1^-$  and  $n$  tends to  $+\infty$ .

**Theorem A.** *Let  $n \geq 1$  and  $r \in [0, 1)$ . We have*

(i)

$$(4) \quad \tilde{a}(n, r) \sqrt{\frac{n}{1-r}} \leq \mathcal{C}_{n,r}(L_a^2, H^2) \leq \tilde{A}(n, r) \sqrt{\frac{n}{1-r}},$$

where

$$\tilde{a}(n, r) \geq \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{A}(n, r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

(ii) Moreover, the sequence

$$\left(\frac{\mathcal{C}_{n,r}(L_a^2, H^2)}{\sqrt{n}}\right)_{n \geq 1}$$

is convergent and there exists a limit

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(L_a^2, H^2)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}.$$

for all  $r \in [0, 1)$ .

Notice that it has already been proved in [Z2] that there exists a limit

$$(6) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(H^2, H^2)}{n} = \frac{1+r}{1-r},$$

for every  $r$ ,  $0 \leq r < 1$ .

**Problem 2.** Looking at motivation 2.3, we replace the algebra  $H^\infty$  by the Dirichlet space  $B_{2,2}^{\frac{1}{2}}$ . We show that the ‘‘gap’’ between  $X = H^2$  and  $Y = H^\infty$  (see (1)) is asymptotically the same as the one which exists between  $X = H^2$  and  $Y = B_{2,2}^{\frac{1}{2}}$ . In other words,

$$(7) \quad \mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \asymp \mathcal{I}_{n,r}(H^2, H^\infty) \asymp \sqrt{\frac{n}{1-r}}.$$

More precisely, we prove the following Theorem B, in which the right-hand side inequality of (10) is a consequence of the right-hand side inequality of (4) in the above Theorem A.

**Theorem B.** *Let  $n \geq 1$ , and  $r \in [0, 1)$ . Then,*

$$(8) \quad \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \left[ (\mathcal{C}_{n,r} (L_a^2, H^2))^2 + 1 \right]^{\frac{1}{2}}.$$

Let  $\lambda \in \mathbb{D}$  and the corresponding one-point interpolation set  $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$ . We have,

$$(9) \quad I \left( \sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}} \right) \geq \sqrt{\frac{n}{1-|\lambda|}} \left[ \frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)} \right]^{\frac{1}{2}}.$$

In particular,

$$(10) \quad \left[ \frac{1+r}{2} \left( 1 - \frac{1}{n} \right) \right]^{\frac{1}{2}} \sqrt{\frac{n}{1-r}} \leq \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \left( 1+r + \frac{1}{\sqrt{n}} + \frac{1-r}{n} \right)^{\frac{1}{2}} \sqrt{\frac{n}{1-r}},$$

$$(11) \quad \sqrt{\frac{\frac{1+r}{2}}{1-r}} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right)}{\sqrt{n}} \leq \limsup_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right)}{\sqrt{n}} \leq \sqrt{\frac{1+r}{1-r}},$$

and

$$(12) \quad \frac{\sqrt{2}}{2} \leq \liminf_{r \rightarrow 1^-} \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \leq \sqrt{2}.$$

In the next Section, we first give some definitions introducing the main tools used in the proofs of Theorem A and Theorem B. After that, we prove these theorems.

#### PROOFS OF THEOREMS A AND B

From now on, if  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$  is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^n b_{\lambda_j}$$

is the corresponding finite Blaschke product where  $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ ,  $\lambda \in \mathbb{D}$ . In Definitions 1, 2, 3 and in Remark 4 below,  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  is a sequence in the unit disc  $\mathbb{D}$  and  $B_\sigma$  is the corresponding Blaschke product.

**Definition 1.** *Malmquist family.* For  $k \in [1, n]$ , we set  $f_k = \frac{1}{1-\lambda_k z}$ , and define the family  $(e_k)_{1 \leq k \leq n}$ , (which is known as Malmquist basis, see [N1, p.117]), by

$$(13) \quad e_1 = \frac{f_1}{\|f_1\|_2} \quad \text{and} \quad e_k = \left( \prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},$$

for  $k \in [2, n]$ ; we have  $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$ .

**Definition 2.** *The model space  $K_{B_\sigma}$ .* We define  $K_{B_\sigma}$  to be the  $n$ -dimensional space:

$$(14) \quad K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

**Definition 3.** *The orthogonal projection  $P_{B_\sigma}$  on  $K_{B_\sigma}$ .* We define  $P_{B_\sigma}$  to be the orthogonal projection of  $H^2$  on its  $n$ -dimensional subspace  $K_{B_\sigma}$ .

**Remark 4.** The Malmquist family  $(e_k)_{1 \leq k \leq n}$  corresponding to  $\sigma$  is an orthonormal basis of  $K_{B_\sigma}$ . In particular,

$$(15) \quad P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where  $(\cdot, \cdot)_{H^2}$  means the scalar product on  $H^2$ .

**Proof of Theorem A.**

*Proof of (i).* 1) We first prove the the right-hand side inequality of (4). Using both Cauchy-Schwarz inequality and the fact that  $\widehat{f}'(k) = (k+1)\widehat{f}(k+1)$  for all  $k \geq 0$ , we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \sum_{k \geq 0} \frac{|\widehat{f}'(k)|^2}{k+1} = \sum_{k \geq 0} \frac{(k+1)^2 |\widehat{f}(k+1)|^2}{k+1} = \\ &= \sum_{k \geq 1} k |\widehat{f}(k)|^2 \leq \left( \sum_{k \geq 1} k^2 |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} = \\ &= \|f'\|_{H^2} \|f\|_{H^2} \leq \mathcal{C}_{n,r}(H^2, H^2) \|f\|_{H^2}^2, \end{aligned}$$

and hence,

$$\|f'\|_{L_a^2} \leq \sqrt{\mathcal{C}_{n,r}(H^2, H^2)} \|f\|_{H^2},$$

which means

$$\mathcal{C}_{n,r}(L_a^2, H^2) \leq \sqrt{\mathcal{C}_{n,r}(H^2, H^2)}.$$

Then it remains to use [Z2, p.2]:

$$\mathcal{C}_{n,r}(H^2, H^2) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right) \frac{n}{1-r},$$

for all  $n \geq 1$  and  $r \in [0, 1)$ .

2) The proof of the left-hand side inequality of (4) repeats the one of [Z2, (i)] (for the left-hand side inequality) excepted that this time, we replace the Hardy norm  $\|\cdot\|_{H^2}$  by the Bergman one

$\|\cdot\|_{L_a^2}$ . Indeed, we use the same test function  $e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}$  (the  $n^{\text{th}}$  vector of the Malmquist family associated with the one-point set  $\sigma_{n,r} = \underbrace{\{r, r, \dots, r\}}_n$  see Definition 1) and prove by the

same changing of variable  $ob_r$  (in the integral on the unit disc  $\mathbb{D}$  which defines the  $L_a^2$ -norm) that

$$\|e'_n\|_{L_a^2}^2 = \frac{n}{1-r} \left(1 - \frac{1-r}{n}\right),$$

which gives

$$\mathcal{C}_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{n}{1-r}} \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Here are the details of the proof. We have  $e_n \in K_{b_r^n}$  and  $\|e_n\|_{H^2} = 1$ , (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e'_n &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2} = \\ &= -\frac{r}{(1-r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2}, \end{aligned}$$

since  $b'_r = \frac{r^2-1}{(1-rz)^2}$ . Then,

$$e'_n = b'_r \left[ -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |b'_r(w)|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) =$$



$$= \frac{1}{2\pi} \int_{\mathbb{D}} |b'_r(w)|^2 |(b_r(w))^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right| dm(w),$$

which gives, using the variables  $u = b_r(w)$ ,

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But  $1 - rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$  and  $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$ . This implies

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) = \\ &= \frac{1}{(1-r^2)} \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 |(-ru + (n-1)(1-ru))|^2 dm(u), \end{aligned}$$

which gives

$$\|e'_n\|_{L_a^2} = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_2,$$

where  $\varphi_n = z^{n-2}(-rz + (n-1)(1-rz))$ . Expanding, we get

$$\begin{aligned} \varphi_n &= z^{n-2}(-rz + n - 1 + rz - nrz) = \\ &= z^{n-2}(-nrz + n - 1) = (n-1)z^{n-2} - nrz^{n-1}, \end{aligned}$$

and

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{(1-r^2)} \left( \frac{(n-1)^2}{n-1} + \frac{n^2}{n} r^2 \right) = \frac{1}{(1-r^2)} (n(1+r) - 1) \\ &= \frac{n}{(1-r)(1+r)} \left( (1+r) - \frac{1}{n} \right) = \frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right), \end{aligned}$$

which gives

$$\mathcal{C}_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{n}{1-r}} \left( 1 - \frac{1-r}{n} \right)^{\frac{1}{2}}.$$

*Proof of (ii).* This is again the same proof as [Z2, (ii)] (the three steps). More precisely in Step 2, we use the same test function

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k},$$

(where  $s = (s_n)$  is defined in [Z2, p.8]), and the same changing of variable  $\circ b_r$  in the integral on  $\mathbb{D}$ . Here are the details of the proof.

**Step 1.** We first prove the right-hand-side inequality:

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) \leq \sqrt{\frac{1+r}{1-r}},$$

which becomes obvious since

$$\frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) \leq \frac{1}{\sqrt{n}} \sqrt{\mathcal{C}_{n,r}(H^2, H^2)},$$

and

$$\frac{1}{\sqrt{n}} \sqrt{\mathcal{C}_{n,r}(H^2, H^2)} \rightarrow \sqrt{\frac{1+r}{1-r}},$$

as  $n$  tends to infinity, see [Z1] p. 2.

**Step 2.** We now prove the left-hand-side inequality:

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{1+r}{1-r}}.$$

More precisely, we show that

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|D\|_{(K_{b_r^n}, \|\cdot\|_{L_a^2}) \rightarrow H^2} \geq \sqrt{\frac{1+r}{1-r}}.$$

Let  $f \in K_{b_r^n}$ . Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b_r'}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k = \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k = \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} = \\ &= -b_r' \left[ \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right]. \end{aligned}$$

Now using the change of variables  $v = b_r(u)$ , we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} |b_r'(u)|^2 \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du = \\ &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

Now,  $b_r - r = \frac{r-z-r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz}$ , which gives

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv = \\ &= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-2} \right|^2 dv = \\ &= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right|^2 dv. \end{aligned}$$

Thus,

$$\begin{aligned} (16) \quad & \frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[ \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} + \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right] \geq \\ & \geq \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \\ & \geq \frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[ \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} - \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right]. \end{aligned}$$

Now,

$$\begin{aligned} & (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k = \\ & = \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^{k+1} = \\ & = \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k - r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^k = \\ & = (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\ & \quad -r [(f, e_2)_{H^2} v + (n-1) (f, e_n)_{H^2} v^{n-1}] = \\ & = (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r (f, e_2)_{H^2}] v + \sum_{k=2}^{n-2} [(k+1) (f, e_{k+2})_{H^2} - rk (f, e_{k+1})_{H^2}] v^k + \\ & \quad -r(n-1) (f, e_n)_{H^2} v^{n-1}, \end{aligned}$$

which gives

$$\begin{aligned}
(17) \quad & \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\
& = |(f, e_2)_{H^2}|^2 + \frac{1}{2} |(f, e_3)_{H^2} - r (f, e_2)_{H^2}|^2 + \\
& + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \sum_{k=2}^{n-2} \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2.
\end{aligned}$$

On the other hand,

$$(18) \quad \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \leq r \left( \sum_{k=0}^{n-1} \frac{1}{k+1} |(f, e_{k+1})_{H^2}|^2 \right)^{1/2} \leq r \|f\|_{H^2},$$

Now, let  $s = (s_n)$  be a sequence of even integers such that

$$\lim_{n \rightarrow \infty} s_n = \infty \text{ and } s_n = o(n) \text{ as } n \rightarrow \infty.$$

Then we consider the following function  $f$  in  $K_{b_r}$ :

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k}.$$

Applying (17) with such an  $f$ , we get

$$\begin{aligned}
& \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = \\
& = r^4 \frac{(n-1)^2}{n} + \\
& + \sum_{l=2}^{n-2} (n-l+1) \left| (f, e_{n-l+2})_{H^2} - \frac{r(n-l)}{n-l+1} (f, e_{n-l+1})_{H^2} \right|^2,
\end{aligned}$$

setting the change of index  $l = n - k$  in the last sum. This finally gives

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 =$$

$$\begin{aligned}
&= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left| 1 + \frac{r(n-l)}{n-l+1} \right|^2 = \\
&= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[ 1 + r \left( 1 - \frac{1}{n-l+1} \right) \right]^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq \\
&\geq r^4 \frac{(n-1)^2}{n} + (s+1-2+1)(n-(s+1)+1) \left[ 1 + r \left( 1 - \frac{1}{n-(s+1)+1} \right) \right]^2 = \\
&= r^4 \frac{(n-1)^2}{n} + s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2.
\end{aligned}$$

In particular,

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2.$$

Now, since  $\|f\|_{H^2}^2 = s_n + 3$ , we get

$$\begin{aligned}
&\liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq \\
&\geq \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \|f\|_{H^2}^2 (n - \|f\|_{H^2}^2) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = \\
&= \lim_{n \rightarrow \infty} \left( 1 - \frac{s_n}{n} \right) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = (1+r)^2.
\end{aligned}$$

On the other hand, applying (18) with this  $f$ , we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \|f\|_{H^2}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0.$$

Thus, we can conclude passing after to the limit as  $n$  tends to  $+\infty$  in (16), that

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} = \frac{1}{\sqrt{1+r}} \liminf_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq$$

$$\geq \frac{1+r}{\sqrt{1+r}} = \sqrt{1+r},$$

and

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \|D\|_{K_{b_n^2} \rightarrow H^2} \geq \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \sqrt{1+r}.$$

**Step 3. Conclusion.** Using both **Step 1** and **Step 2**, we get

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = 1+r,$$

which means that the sequence  $\left(\frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2)\right)_{n \geq 1}$  is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,r}(L_a^2, H^2) = \sqrt{\frac{1+r}{1-r}}.$$

□

### Proof of Theorem B.

*Proofs of inequality (8) and of the right-hand side inequality of (10).* Let  $\sigma$  be a sequence in  $\mathbb{D}$ , and  $B = B_\sigma$  the finite Blaschke product corresponding to  $\sigma$ . If  $f \in H^2$ , we use the same function  $g$  as in [Z3] which satisfies  $g|_\sigma = f|_\sigma$ . More precisely, let  $g = P_B f \in K_B$  (see Definitions 2, 3 and Remark 4 above for the definitions of  $K_B$  and  $P_B$ ). Then  $g - f \in BH^2$  and using the definition of  $\mathcal{C}_{n,r}(L_a^2, H^2)$ ,

$$\|g'\|_{L_a^2}^2 \leq (\mathcal{C}_{n,r}(L_a^2, H^2))^2 \|g\|_{H^2}^2.$$

Now applying the identity (2) to  $g$  we get

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \leq \left[ (\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1 \right] \|g\|_{H^2}^2.$$

Using the fact that  $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$ , we finally get

$$\|g\|_{B_{2,2}^{\frac{1}{2}}} \leq \left[ (\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1 \right]^{\frac{1}{2}} \|f\|_{H^2},$$

and as a result,

$$I\left(\sigma, H^2, B_{2,2}^{\frac{1}{2}}\right) \leq \left[ (\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1 \right]^{\frac{1}{2}}.$$

It remains to apply the right-hand side inequality of (4) in Theorem A to prove the right-hand side one of (10).

*Proof of inequality (9).* 1) We use the same test function

$$f = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{\frac{1}{2}} b_\lambda^k (1 - \bar{\lambda}z)^{-1},$$

as the one used in the proof of [Z3, Theorem B] (the lower bound, page 11 of [Z3]).  $f$  being the sum of  $n$  elements of  $H^2$  which are an orthonormal family known as Malmquist's basis (associated with  $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$ , see Remark 4 above or [N1, p.117]), we have  $\|f\|_{H^2}^2 = n$ .

2) Since the spaces  $H^2$  and  $B_{2,2}^{\frac{1}{2}}$  are rotation invariant, we have  $I(\sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}}) = I(\sigma_{n,\mu}, H^2, B_{2,2}^{\frac{1}{2}})$  for every  $\lambda, \mu$  with  $|\lambda| = |\mu| = r$ . Let  $\lambda = -r$ . To get a lower estimate for  $\|f\|_{B_{2,2}^{\frac{1}{2}}/b_\lambda B_{2,2}^{\frac{1}{2}}}$  consider  $g$  such that  $f - g \in b_\lambda^n \text{Hol}(\mathbb{D})$ , i.e. such that  $f \circ b_\lambda - g \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$ .

3) First, we notice that

$$\begin{aligned} \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 &= \left\| (g \circ b_\lambda)' \right\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|b_\lambda \cdot (g' \circ b_\lambda)\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \\ &= \int_{\mathbb{D}} |b_\lambda(u)|^2 |g'(b_\lambda(u))|^2 du + \|g \circ b_\lambda\|_{H^2}^2 = \int_{\mathbb{D}} |g'(w)|^2 dw + \|g \circ b_\lambda\|_{H^2}^2, \end{aligned}$$

using the changing of variable  $w = b_\lambda(u)$ . We get

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g'\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 + \|g \circ b_\lambda\|_{H^2}^2 - \|g\|_{H^2}^2,$$

and

$$\begin{aligned} \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 &= \|g\|_{H^2}^2 + \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 = \\ &\geq \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2. \end{aligned}$$

Now, we notice that

$$\begin{aligned} f \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-\frac{1}{2}} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right). \end{aligned}$$

4) Next,

$$\begin{aligned} \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 &= \sum_{k \geq 1} k \left| \widehat{g \circ b_\lambda}(k) \right|^2 \geq \\ &\geq \sum_{k=1}^{n-1} k \left| \widehat{g \circ b_\lambda}(k) \right|^2 = \sum_{k=1}^{n-1} k \left| \widehat{f \circ b_\lambda}(k) \right|^2, \end{aligned}$$

since  $\widehat{g \circ b_\lambda}(k) = \widehat{f \circ b_\lambda}(k)$ ,  $\forall k \in [0, n-1]$ . This gives

$$\begin{aligned} \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 &\geq \frac{1}{1-r^2} \left( (1+r)^2 \sum_{k=1}^{n-1} k \right) = \\ &= \frac{(1+r)^2 n(n-1)}{1-r^2} \frac{1}{2} = \frac{1+r}{1-r} \frac{n(n-1)}{2} = \frac{1+r}{1-r} \frac{(n-1)}{2} \|f\|_{H^2}^2, \end{aligned}$$

for all  $n \geq 2$  since  $\|f\|_{H^2}^2 = n$ . Finally,

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \geq \frac{n}{1-r} \frac{1+r}{2} \left(1 - \frac{1}{n}\right) \|f\|_{H^2}^2.$$

In particular,

$$\mathcal{I}_{n,r} \left( H^2, B_{2,2}^{\frac{1}{2}} \right) \geq \sqrt{\frac{n}{1-r}} \left[ \frac{1+r}{2} \left(1 - \frac{1}{n}\right) \right]^{\frac{1}{2}}.$$

□

### Some comments.

**a. Extension of Theorem A to spaces  $B_{2,2}^s$ ,  $s \geq 0$ .** Using the techniques developed in the proof of our Theorem A (combined with complex interpolation (between Banach spaces) and a reasoning by induction), it is possible both to precise the sharp numerical constant  $c_{2,s}$  in K. Dyakonov's result (3) (mentioned above in paragraph d. of the Introduction) and to prove the asymptotic sharpness (at least for  $s \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$ ) of the right-hand side inequality of (3). In the same spirit, we would obtain that there exists a limit:

$$(19) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r} \left( B_{2,2}^{s-1}, H^2 \right)}{n^s} = \left( \frac{1+r}{1-r} \right)^s.$$

Our Theorem A corresponds to the case  $s = \frac{1}{2}$ .

**b. Extension of Theorem B to spaces  $B_{2,2}^s$ ,  $s \geq 0$ .** The proof of the upper bound in our Theorem B can be extended so as to give an upper (asymptotic) estimate of the interpolation constant  $\mathcal{I}_{n,r} \left( H^2, B_{2,2}^s \right)$ ,  $s \geq 0$ . More precisely, applying K. Dyakonov's result (3) (mentioned above in paragraph d. of the Introduction) we get

$$(20) \quad \mathcal{I}_{n,r} \left( H^2, B_{2,2}^s \right) \leq \tilde{c}_s \left( \frac{n}{1-r} \right)^s, \text{ with } \tilde{c}_s \asymp c_{2,s},$$

where  $c_{2,s}$  is defined in (3) and precised in (19). Looking at the above comment 1,  $\tilde{c}_s \asymp (1+r)^s$  for sufficiently large values of  $n$ . Our Theorem B corresponds again to the case  $s = \frac{1}{2}$ . In this Theorem B, we prove the sharpness of the right-hand side inequality in (20) for  $s = \frac{1}{2}$ . However, for the general case  $s \geq 0$ , the asymptotic sharpness of  $\left(\frac{n}{1-r}\right)^s$  as  $r \rightarrow 1^-$  and  $n \rightarrow \infty$  is less obvious. Indeed, the key of the proof (for the sharpness) is based on the property that the Dirichlet norm



(the one of  $B_{2,2}^{1/2}$ ) is “nearly” invariant composing by an elementary Blaschke factor  $b_\lambda$ , as this is the case for the  $H^\infty$  norm. A conjecture given by N. K. Nikolski is the following:

$$(21) \quad \mathcal{I}_{n,r}(H^2, B_{2,2}^s) \asymp \begin{cases} \frac{n^s}{\sqrt{1-r}} & \text{if } s \geq \frac{1}{2} \\ \left(\frac{n}{1-r}\right)^s & \text{if } 0 \leq s \leq \frac{1}{2} \end{cases},$$

and is due to the position of the spaces  $B_{2,2}^s$ ,  $s \geq 0$  with respect to the algebra  $H^\infty$ .

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