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Solvable rational extensions of the Morse and Kepler-Coulomb potentials

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We show that it is possible to generate an infinite set of solvable rational extensions from every exceptional first category translationally shape invariant potential. This is made by using Darboux-

Bäcklund transformations based on unphysical regular Riccati-Schrödinger functions which are obtained from specific symmetries associated to the considered family of potentials.

PACS numbers:

I.

II. INTRODUCTION

In the recent years, several notable progresses have been made in the research and characterization of new closed-form exactly solvable systems in quantum mechanics [4–21]. The obtained systems are regular rational extensions of some shape-invariant potentials [1–3] and are associated to families of exceptional orthogonal polynomials (EOP) built from the Laguerre or Jacobi classical orthogonal polynomials. In all the considered cases, the initial potentials belong to the second category (as defined in [22]) of primary translationally shape-invariant potentials (TSIP): the extended potentials of the $J1$ and $J2$ series (associated to the Jacobi EOP) are obtained from the generic second category potentials (Darboux-Pöschl-Teller or Scarf of the hyperbolic or trigonometric types), as for the extended potentials of the $L1$, $L2$ and $L3$ series, they are obtained from the unique exceptional second category potential which is the isotonic one.

If we except the specific case of the harmonic oscillator which has been extensively treated [4, 20, 23–27], the solvable extensions of first category potentials have been much less studied. Referring to the classification established in [22], the exceptional first category primary TSIP are the one-dimensional harmonic oscillator (HO), the Morse potential and the effective radial Kepler-Coulomb (ERKC) system, whereas the generic first category primary TSIP include the trigonometric and hyperbolic Rosen-Morse potentials as well as the Eckardt potential. A general study of the possible extensions of a large number of exactly solvable potentials from the point of view of conditionally solvable potentials has been made by Junker and Roy [35]. The case of the Morse potential has been also considered by Gomez-Ullate et al [4] who have determined the algebraic deformations of this system which are solvable by polynomials.

In [21] we have developed a new approach which allows to generate an infinite set of regular exactly solvable extensions starting from every TSIP in a very direct and systematic way without taking recourse to any ansatz. This approach is based on a generalization of the usual SUSY partnership built from excited states. The corresponding Darboux-Bäcklund Transformations (DBT), which are covariance transformations for the class of Riccati-Schrödinger (RS) equations [22], are based on regularized RS functions which are obtained by using discrete symmetries acting on the parameters of the considered family of potentials. Considering the isotonic oscillator, we have obtained the three infinite sets $L1$, $L2$ and $L3$ of regular rationally solvable extensions of this potential and have given a simple and transparent proof of the shape-invariance of the potentials belonging to the $L1$ and $L2$ series. In the present paper we show that the same approach can be applied to generate infinite towers of solvable rational extensions from every exceptional first category potential. As shown in [22], the first category primary TSIP can be reduced into a harmonic one by a change of the variable which satisfies a constant coefficient Riccati equation. The exceptional cases correspond to the cases where this equation degenerates into a linear equation or a Riccati equation with a double root in the right-hand member, namely the HO, the Morse and ERKC potentials. In this cases the bound states are expressible in terms of generalized Laguerre Polynomials (GLP) [1, 2].

The paper is organized as follows. After recalling briefly the basic elements of our approach, we test its efficiency on the simple and exhaustively studied case of one-dimensional HO, retrieving very simply the results already obtained in [4, 20, 23–27]. In the second and third parts, we treat successively the Morse and ERKC systems, building the associated towers of solvable regular extensions and characterizing their eigenstates. For the Morse potential we recover the algebraic deformations described by Gomez-Ullate et al [4], the extensions being not strictly isospectral to the primary potential. For the ERKC potential we obtain two distinct regimes with respect to the value of the "angular momentum" parameter. In the first regime the extensions are strictly isospectral to the primary potential whereas in the second regime they are not.

Contrarily to the case of the second category potentials the extensions of the exceptional first category potentials do not inherit of the shape invariance properties of the primary potential.

III. DARBOUX-BÄCKLUND TRANSFORMATIONS (DBT) AND REGULAR EXTENSIONS

A. General scheme

Consider a family of one-dimensional hamiltonians indexed by a multiparameter a

$$\hat{H}(a) = -d^2/dx^2 + V(x; a), \quad a \in \mathbb{R}^m, \quad x \in I \subset \mathbb{R}$$

If $\psi_\lambda(x; a)$ is an eigenstate of $\hat{H}(a)$ associated to the eigenvalue $E_\lambda(a)$, then its logarithmic derivative $w_\lambda(x; a) = -\psi'_\lambda(x; a)/\psi_\lambda(x; a)$, that we will call a Riccati-Schrödinger (RS) function, satisfies a particular Riccati equation of the following form

$$-w'_\lambda(x; a) + w_\lambda^2(x; a) = V(x; a) - E_\lambda(a). \quad (1)$$

Eq(1) is called the Riccati-Schrödinger (RS) equation [22] for the level $E_\lambda(a)$. The RS function $w_\lambda(x; a)$ presents a simple pole at each node of the eigenstates $\psi_\lambda(x; a)$.

It is a well-known fact that the set of general Riccati equations is invariant under the group \mathcal{G} of smooth $SL(2, \mathbb{R})$ -valued curves $Map(\mathbb{R}, SL(2, \mathbb{R}))$ [28, 29]. The particular of Riccati-Schrödinger equations is, as for it, preserved by a specific subset of \mathcal{G} . These transformations, called Darboux-Bäcklund Transformations (DBT), are build from any solution $w_\nu(x; a)$ of the initial RS equation Eq(1) as [22, 28, 29]

$$w_\lambda(x; a) \xrightarrow{A(w_\nu)} w_\lambda^{(\nu)}(x; a) = -w_\nu(x; a) + \frac{E_\lambda(a) - E_\nu(a)}{w_\nu(x; a) - w_\lambda(x; a)}, \quad (2)$$

where $E_\lambda(a) > E_\nu(a)$. Then $w_\lambda^{(\nu)}$ is a solution of the RS equation:

$$-w_\lambda^{(\nu)'}(x; a) + \left(w_\lambda^{(\nu)}(x; a)\right)^2 = V^{(\nu)}(x; a) - E_\lambda(a), \quad (3)$$

with the same energy $E_\lambda(a)$ as in Eq(1) but with a modified potential

$$V^{(\nu)}(x; a) = V(x; a) + 2w'_\nu(x; a). \quad (4)$$

The corresponding eigenstate of $\hat{H}^{(\nu)}(a) = -d^2/dx^2 + V^{(\nu)}(x; a)$ can be written

$$\psi_\lambda^{(\nu)}(x; a) = \exp\left(-\int dx w_\lambda^{(\nu)}(x; a)\right) \sim \frac{1}{\sqrt{E_\lambda(a) - E_\nu(a)}} \hat{A}(w_\nu) \psi_\lambda(x; a), \quad (5)$$

where $\hat{A}(a)$ is a first order operator given by

$$\hat{A}(w_\nu) = d/dx + w_\nu(x; a). \quad (6)$$

From V , the DBT generates a new potential $V^{(\nu)}$ (quasi) isospectral to the original one and its eigenstates are directly obtained from those of V via Eq(5). If the initial system is exactly solvable, which is the case of the translationally shape invariant potentials (TSIP), this scheme allows to build new exactly solvable potentials.

Nevertheless, in general, $w_\nu(x; a)$ and then the transformed potential $V^{(\nu)}(x; a)$ are singular at the nodes of $\psi_\nu(x; a)$. For instance, if $\psi_n(x; a)$ ($\nu = n$) is a bound state of $\hat{H}(a)$, $V^{(n)}$ is regular only when $n = 0$, that is when $\psi_{n=0}$ is the ground state of \hat{H} , and we recover the usual SUSY partnership in quantum mechanics. Starting from an excited state, that is for $n \geq 1$, the transformed potential presents exactly n second order poles and a priori we cannot use

$A(w_n)$ to build a regular potential. We can however envisage to use any other regular solution of Eq(1) as long as it has no zero on the considered real interval I , even if it does not correspond to a physical state. As shown in the case of the isotonic oscillator, we can obtain such solutions by using specific discrete symmetries Γ_a which are covariance transformations for the considered family of potentials. Γ_a acts on the parameters of the potential and transforms the RS function of a physical eigenstate w_n into a non singular but unphysical RS function $v_n(x; a) = \Gamma_a(w_n(x; a))$ associated to the eigenvalue $\tilde{E}_n(a) = \Gamma_a(E_n(a))$. For a solvable TSIP, w_n and v_n are known in closed form and the regular extended potential (see Eq(4) and Eq(5))

$$\tilde{V}^{(n)}(x; a) = V(x; a) + 2v'_n(x; a) \quad (7)$$

is then (quasi) isospectral to $V(x; a)$ with eigenstates given by (see Eq(2))

$$\left\{ \begin{array}{l} w_\lambda^{(n)}(x; a) = -v_n(x; a) + \frac{E_k(a) - \tilde{E}_n(a)}{v_n(x; a) - w_k(x; a)} \\ \psi_k^{(n)}(x; a) = \exp\left(-\int dx w_k^{(n)}(x; a)\right) \sim \frac{1}{\sqrt{E_k(a) - \tilde{E}_n(a)}} \hat{A}(v_n) \psi_k(x; a) \end{array} \right. , \quad (8)$$

for the energy $E_k(a)$.

Interestingly, such combinations of Darboux-Bäcklund transformations and discrete symmetries appears as natural covariance groups for Painlevé equations [30]. Very recently another type of discrete symmetries have been also considered by Plyushchay et al [31, 32] in a different context.

B. One dimensional harmonic oscillator

To illustrate this general scheme we consider the well studied example [4, 20, 23–27] of the 1D HO which is simplest exceptional first category TSIP. The corresponding potential with zero ground level ($E_0(\omega) = 0$) is given by

$$V(x, \omega) = \frac{\omega^2}{4} x^2 - \frac{\omega}{2}. \quad (9)$$

Its spectrum is well known

$$E_n(\omega) = n\omega; \quad \psi_n(x) \sim H_n\left(\sqrt{\omega/2}x\right) \exp(-\omega x^2/4) \quad (10)$$

and the corresponding RS functions $w_n(x)$ can be written as terminating continued fractions [22] as

$$w_n(x, \omega) = w_0(x, \omega) + R_n(x, \omega), \quad (11)$$

where

$$\left\{ \begin{array}{l} w_0(x, \omega) = \frac{\omega}{2} x \\ R_n(x, \omega) = -\frac{n\omega}{\omega x -} \uparrow \dots \uparrow \frac{(n-j+1)\omega}{\omega x -} \uparrow \dots \uparrow \frac{1}{x} = -\left(\log H_n\left(\sqrt{\omega/2}x\right)\right)' \end{array} \right. \quad (12)$$

The unique parameter transformation which preserves the functional form $V(x, \omega)$ is the ω inversion

$$\omega \xrightarrow{\Gamma_\omega} (-\omega), \quad \left\{ \begin{array}{l} V(x; \omega) \xrightarrow{\Gamma_\omega} V(x; \omega) + \omega \\ w_n(x; \omega) \xrightarrow{\Gamma_\omega} v_n(x; \omega) = w_n(x; -\omega), \end{array} \right. \quad (13)$$

$v_n(x; \omega)$ satisfying

$$-v'_n(x; \omega) + v_n^2(x; \omega) = V(x; \omega) - E_{-(n+1)}(\omega), \quad (14)$$

that is, $E_n(\omega) \xrightarrow{\Gamma_\omega} E_{-(n+1)}(\omega)$. From Eq.(11) and Eq.(12) we deduce

$$v_n(x; \omega) = v_0(x; \omega) + Q_n(x; \omega), \quad (15)$$

with

$$v_0(x; \omega) = -\frac{\omega}{2}x \quad (16)$$

and

$$\begin{aligned} Q_n(x; \omega) &= -\frac{n\omega}{\omega x +} \uparrow \dots \uparrow \frac{(n-j+1)\omega}{\omega x +} \uparrow \dots \uparrow \frac{1}{x} \\ &= -\left(\log H_n\left(i\sqrt{\omega/2x}\right)\right)'. \end{aligned} \quad (17)$$

Clearly, $Q_n(x; \omega)$ does not present any singularity on the real line, except possibly one at the origin. Indeed the terminating continued fraction has only positive terms which implies that there is no positive singularity and then, since the potential has a even parity, any singularity on the whole real axis, except one at the origin when the number n of denominators is odd. This can be recovered more directly from the expression of Q_n in terms of Hermite polynomials of imaginary argument since the Hermite polynomials have all their zeros on the real line, with a zero at the origin for odd n . Using the correspondence between Hermite and Laguerre polynomials given by

$$H_n\left(i\sqrt{\omega/2x}\right) = \begin{cases} (-1)^m 2^{2m} m! L_m^{-1/2}(-\omega x^2/2), & n = 2m \\ (-1)^m 2^{2m+1} m! i \sqrt{\omega/2x} L_m^{1/2}(-\omega x^2/2), & n = 2m+1, \end{cases} \quad (18)$$

the regularity properties of are direct consequences of the Kienast-Lawton-Hahn theorem [33, 34] which establishes

Kienast-Lawton-Hahn's Theorem

Suppose that $\alpha \notin \{-n, \dots, -1\}$. Then $L_n^{(\alpha)}(z)$ admits

- 1) n positive zeros if $\alpha > -1$
- 2) $n + [\alpha] + 1$ positive zeros if $-n < \alpha < -1$ ($[\alpha]$ means the integer part of α)
- 3) No positive zero if $\alpha < -n$

The number of negative zeros is always 0 or 1.

- 1) 0 if $\alpha > -1$
- 2) 0 if $-2k-1 < \alpha < -2k$ and 1 if $-2k < \alpha < -2k+1$, with $-n < \alpha < -1$
- 3) 0 if n is even and 1 if n is odd, with $\alpha < -n$

Only when $\alpha \in \{-n, \dots, -1\}$, we have a zero of $L_n^{(\alpha)}(z)$ at the origin with multiplicity $|\alpha|$. If α decreases through an odd value in $\{-n, \dots, -1\}$, a negative zero is gained and a positive one is lost. If the crossed value is even, simultaneously two zeros, one negative and one positive, disappear.

Applying the DBT $A(v_n)$ to w_k (see Eq(2)), we obtain

$$w_k(x; \omega) \xrightarrow{A(v_n)} w_k^{(n)}(x; \omega) = -v_n(x; \omega) + \frac{E_{n+1+k}(\omega)}{v_n(x; \omega) - w_k(x; \omega)}, \quad (19)$$

where $w_k^{(n)}(x; \omega)$ is an RS function at energy $E_k(\omega)$ for the extended potential

$$V^{(n)}(x; \omega) = V(x; \omega) + 2v_n'(x; \omega) = V(x; \omega) - \omega + 2Q_n'(x; \omega). \quad (20)$$

We recover here the results obtained in [4, 20, 23–27]. In particular, for $n = 1$ $\tilde{V}^{(1)}(x)$ is the $l = 1$ isotonic potential

$$V^{(1)}(x; \omega) = V(x; \omega) - \omega + \frac{2}{x^2} = \frac{\omega^2}{4}x^2 + \frac{2}{x^2} - \frac{3\omega}{2} \quad (21)$$

and for $n = 2$, $V^{(2)}(x; \omega)$ is the CPRS [27] potential

$$V^{(2)}(x; \omega) = \frac{\omega^2}{4}x^2 + 4\omega \frac{\omega^2 x^2 - 1}{(\omega^2 x^2 + 1)^2} - \frac{3}{2}\omega. \quad (22)$$

For every $n \geq 0$, $V^{(n)}(x; \omega)$ is (quasi)isospectral to $V(x; \omega)$

$$V^{(n)}(x; \omega) \underset{iso}{=} V(x; \omega). \quad (23)$$

and regular on the real line \mathbb{R} if n is even and on the positive half real line \mathbb{R}^{+*} if n is odd. To keep the same definition domain for the initial and extended potentials, we then must consider only even values of $n = 2m$.

The isospectrality established above is not strict. Indeed, we have clearly

$$v'_n(x; \omega) + v_n^2(x; \omega) = V^{(n)}(x; \omega) - E_{-(n+1)}(\omega), \quad (24)$$

that is, $-v_n(x; \omega)$ is a regular RS function for the extended potential $V^{(n)}(x; \omega)$, associated to the eigenvalue $E_{-(n+1)}(\omega) < 0$. Then

$$\psi_-^{(2m)}(x; \omega) \sim \exp\left(+ \int v_{2m}(x; \omega) dx\right) = \frac{\exp\left(-\frac{\omega x^2}{4}\right)}{H_{2m}\left(i\sqrt{\omega/2}x\right)} \sim \frac{\exp\left(-\frac{\omega x^2}{4}\right)}{L_m^{-1/2}(-\omega x^2/2)} \quad (25)$$

is a physical eigenstate of $\hat{H}^{(2m)}$ and more precisely its fundamental state. Consequently the superpartner of the extended potential $V^{(2m)}(x; \omega)$ is

$$\tilde{V}^{(2m)}(x; \omega) = V^{(2m)}(x; \omega) - 2v'_{2m}(x; \omega) = V(x; \omega), \quad m \geq 1 \quad (26)$$

and we recover the fact that the DBT $A(v_n)$ is a backward SUSY partnership.

The eigenfunctions of $\hat{H}^{(2m)}(\omega) = -d^2/dx^2 + V^{(2m)}(x; \omega)$ corresponding to the energies $E_k(\omega)$ are given by

$$\psi_k^{(2m)}(x; \omega) = \exp\left(- \int dx w_k^{(2m)}(x; a)\right) \sim \frac{1}{\sqrt{E_{2m+1+k}(\omega)}} \hat{A}(v_{2m}) \psi_k(x; \omega), \quad (27)$$

that is, using Eq(18)

$$\psi_k^{(2m)}(x; \omega) \sim P_{(m,k)}(x) \frac{\exp(-\omega x^2/4)}{L_m^{-1/2}(-\omega x^2/2)}, \quad (28)$$

where the polynomials

$$P_{(m,k)}(x) = \frac{1}{2} L_m^{-1/2}(-\omega x^2/2) H_{k+1}\left(\sqrt{\omega/2}x\right) + \sqrt{\omega/2}x L_{m-1}^{1/2}(-\omega x^2/2) H_k\left(\sqrt{\omega/2}x\right) \quad (29)$$

of respective degrees $2m + k + 1$ constitute, with the constant 1, an orthogonal family on the real line with respect to the weight

$$w_m(x) = \frac{\exp(-\omega x^2/2)}{\left(L_m^{-1/2}(-\omega x^2/2)\right)^2}. \quad (30)$$

Note that for $k = 2l$, $l \in \mathbb{N}$, we have

$$\begin{aligned} P_{(m,2l)}(x) &\sim \sqrt{\omega/2}x \left(L_m^{-1/2}(-\omega x^2/2) L_l^{1/2}(\omega x^2/2) + L_{m-1}^{1/2}(-\omega x^2/2) L_l^{-1/2}(\omega x^2/2) \right) \\ &= \sqrt{\omega/2}x L_l^{1/2}(\omega x^2/2) \end{aligned} \quad (31)$$

where $L_l^{1/2}(z)$ is an EOP of the $L1$ series [7]. This is coherent with the fact that the 1D HO on is obtained as the singular limit at $a \rightarrow 0$ of the isotonic potential

$$V(x; \omega, a) = \frac{\omega^2}{4}x^2 + \frac{a(a-1)}{x^2} - \omega \left(a + \frac{1}{2} \right). \quad (32)$$

For $a > 0$ the presence of the centrifugal barrier restricts the definition domain to the positive half line $x > 0$ and the energy spectrum include only the level $E_k(\omega) = k\omega$ associated to an even quantum number $k = 2l$. The $L1$ series of rational extensions of $V(x; \omega, a)$ is then built using DBT based on RS functions actually regularized via the same ω inversion Γ_ω that we used above for the HO [21].

Finally, for the odd values of the quantum number $k = 2l + 1$ we can write

$$P_{(m, 2l+1)}(x) \sim (l+1) L_m^{-1/2}(-\omega x^2/2) L_{l+1}^{-1/2}(\omega x^2/2) - \sqrt{\omega/2} x L_{m-1}^{1/2}(-\omega x^2/2) L_l^{1/2}(\omega x^2/2). \quad (33)$$

IV. MORSE POTENTIAL

The Morse potential with zero ground level ($E_0(a, b) = 0$) is the second exceptional primary TSIP of the first category [22]. It is given by [1, 2, 36]

$$V(y; a, b) = b^2 y^2 - 2 \left(a + \frac{\alpha}{2} \right) b y + a^2, \quad a, b > 0 \quad (34)$$

where $y = \exp(-\alpha x) > 0$, $x \in \mathbb{R}$. It possesses exactly $[a]$ bound states ($[a]$ being the integer part of a) which are given by

$$\psi_n(x; a, b) \sim y^{a/\alpha - n} e^{-by/\alpha} L_n^{2(a/\alpha - n)}(2by/\alpha), \quad n \in \{0, \dots, [a] - 1\}, \quad (35)$$

with the corresponding energies $E_n(a) = a^2 - a_n^2$, where $a_k = a - k\alpha$.

In terms of the y variable, the associated RS equation is

$$\alpha y w'_n(y; a, b) + w_n^2(y; a, b) = V(y; a, b) - E_n(a) \quad (36)$$

and its solutions associated to the physical eigenstates Eq(35) are

$$w_n(y; a, b) = w_0(y; a, b) + R_n(y; a, b), \quad (37)$$

where

$$w_0(y; a, b) = -by + a \quad (38)$$

and

$$\begin{aligned} R_n(y; a, b) &= -\frac{E_n(a)}{a + a_1 - 2by} \uparrow \dots \uparrow \frac{E_n(a) - E_{j-1}(a)}{a_{j-1} + a_j - 2by} \uparrow \dots \uparrow \frac{E_n(a) - E_{n-1}(a)}{a_{n-1} + a_n - 2by} \\ &= -n\alpha + \alpha y \left(\log L_n^{2(a/\alpha - n)}(2by/\alpha) \right)'. \end{aligned} \quad (39)$$

The only parameters transformation under which the Morse potential Eq(34) is covariant, is

$$(a, b) \xrightarrow{\Gamma_{a,b}} \left(\underbrace{-a-1}_{-a_{-1}}, -b \right), \left\{ \begin{array}{l} V(x; a, b) \xrightarrow{\Gamma_{a,b}} V(x; a, b) - E_{-1}(a) \\ w_n(x; a, b) \xrightarrow{\Gamma_{a,b}} v_n(x; a, b) = w_n(x; -a_{-1}, -b), \end{array} \right. \quad (40)$$

where

$$\alpha y v'_n(y; a, b) + v_n^2(y; a, b) = V(y; a, b) - E_{-(n+1)}(a), \quad (41)$$

since $a_k \xrightarrow{\Gamma_{a,b}} a_{-(k+1)}$ and $E_n(a) \xrightarrow{\Gamma_{a,b}} a_{-1}^2 - a_{-(n+1)}^2 = E_{-(n+1)}(a) - E_{-1}(a)$.

From Eq.(38) and Eq.(39), we deduce

$$v_n(x; a, b) = v_0(x; a, b) + Q_n(x; a, b), \quad (42)$$

where

$$v_0(y, a, b) = by - a_{-1} \quad (43)$$

and

$$\begin{aligned} Q_n(y, a, b) &= -\frac{E_{-(n+1)}(a) - E_{-1}(a)}{-(a_{-1} + a_{-2}) + 2by} \overset{\Gamma}{\rightarrow} \dots \overset{\Gamma}{\rightarrow} \frac{E_{-(n+1)}(a) - E_{-j}(a)}{-(a_{-j} + a_{-j-1}) + 2by} \overset{\Gamma}{\rightarrow} \dots \overset{\Gamma}{\rightarrow} \frac{E_{-(n+1)}(a) - E_{-n}(a)}{-(a_{-n} + a_{-n-1}) + 2by} \\ &= -n\alpha + \alpha y \left(\log L_n^{-2(a/\alpha+1+n)}(-2by/\alpha) \right)'. \end{aligned} \quad (44)$$

The Kienast-Lawton-Hahn's theorem ensures that for even values of n , $Q_n(y, a, b)$ and then $v_n(x; a, b)$ are regular for every $y > 0$, that is, every $x \in \mathbb{R}$. Applying the DBT $A(v_n)$ (see Eq(2)) to w_k gives

$$w_k(x; a, b) \xrightarrow{A(v_n)} w_k^{(n)}(x; a, b) = -v_n(x; a, b) + \frac{E_k(a) - E_{-(n+1)}(a)}{v_n(x; a, b) - w_k(x; a, b)}, \quad (45)$$

where $w_k^{(n)}(x; \omega)$ satisfies

$$-w_k^{(n)'}(x; a, b) + \left(w_k^{(n)}(x; a, b) \right)^2 = V^{(n)}(x; a, b) - E_k(a), \quad (46)$$

with

$$V^{(n)}(x; a, b) = V(x; a, b) + 2v'_n(x; a, b) = V(y; a_{-1}, b) + E_{-1}(a) - 2\alpha y Q'_n(y; a, b). \quad (47)$$

In the following we consider the case where n takes even values $n = 2m$. $V^{(2m)}(x; a, b)$ is then regular on the positive half line and isospectral to $V(x; a, b)$

$$V^{(2m)}(x; a, b) \underset{iso}{=} V(x; a, b) \quad (48)$$

Again, as in the preceding case, the isospectrality is not strict since

$$v'_n(x; a, b) + v_n^2(x; a, b) = V^{(2m)}(x; a, b) - E_{-(n+1)}(a), \quad (49)$$

that is, $-v_n(x; a, b)$ is a regular RS function for the extended potential $V^{(2m)}(x; a, b)$, associated to the eigenvalue $E_{-(n+1)}(a) < 0$. The asymptotic behaviour of the corresponding eigenstate is

$$\psi_-^{(2m)}(x; a, b) = \exp \left(+ \int v_{2m}(x; a, b) dx \right) \underset{x \rightarrow \pm\infty}{\sim} e^{-(a+(2m+1)\alpha)x} \exp \left(-\frac{b}{\alpha} e^{-\alpha x} \right) \quad (50)$$

from which we deduce that $\psi_-^{(2m)}$ is the fundamental state for $H^{(2m)}$. The superpartner of the extended potential $V^{(2m)}(x; a, b) = V(x; a, b) + 2v'_{2m}(x; a, b)$ is then defined as

$$\tilde{V}^{(2m)}(x; a, b) = V^{(2m)}(x; a, b) + 2(-v'_{2m}(x; a, b)) = V(x; a, b), \quad n \geq 1 \quad (51)$$

and the DBT $A(v_{2m})$ is a backward SUSY partnership. We recover here the results obtained by Gómez-Ullate, Kamran and Milson [4] in a different way.

The excited physical eigenstate of $\hat{H}^{(2m)}(a, b) = -d^2/dx^2 + V^{(2m)}(x; a, b)$ at the energy $E_k(a)$, $k \geq 0$, is given by (see Eq(8))

$$\psi_k^{(2m)}(x; a, b) = \exp\left(\frac{1}{\alpha} \int dy \frac{w_k^{(2m)}(y; a, b)}{y}\right) \sim \frac{1}{\sqrt{E_k(a) - E_{-(2m+1)}(a)}} \hat{A}(v_{2m}) \psi_k(x; a, b). \quad (52)$$

Inserting Eq(43), Eq(44) and Eq(35) into Eq(52) and using the following identities for GLP

$$\begin{cases} L_n^{(\beta)}(z) + L_{n-1}^{(\beta+1)}(z) = L_n^{(\beta+1)}(z) \\ z L_{n-1}^{(\beta+1)}(z) = (n + \beta) L_{n-1}^{(\beta)}(z) - n L_n^{(\beta)}(z), \end{cases} \quad (53)$$

we obtain (in order to simplify the expressions we fix the x scale such that $\alpha = 1$)

$$\psi_k^{(2m)}(x; a, b) \sim M_{a,k}^{(2m)}(z) \frac{z^{a-k} e^{-z/2}}{L_{2m}^{-2(a+1+2m)}(-z)}, \quad \psi_{-}^{(2m)}(x; a, b) \sim \frac{z^{a+1+2m} e^{-z/2}}{L_{2m}^{-2(a+1+2m)}(-z)} \quad (54)$$

where $z = 2by$ and

$$M_{a,k}^{(2m)}(z) = 2(m + a + 1) L_{2m-1}^{-2(a+2m+1)}(-z) L_k^{2(a-k)}(z) - (k + 1) L_{k+1}^{2(a-k)}(z) L_{2m}^{-2(a+2m+1)}(-z) \quad (55)$$

which is a polynomial of degree $2m + k + 1$ with

$$M_{a,k}^{(2m)}(0) = -\frac{(2a + 2m + 2)_{2m} (2a - 2k + 1)_k}{(2m)! k!}, \quad (56)$$

$(a)_n$ being the usual Pochhammer function $(a)_n = a(a+1)\dots(a+n-1)$ [34].

From the orthonormality conditions $\langle \psi_k^{(2m)}(x; a, b) | \psi_{k'}^{(2m)}(x; a, b) \rangle = \delta_{k,k'}$ we deduce that the polynomials

$$\begin{cases} B_{-}^{(2m)}(z, a) = 1 \\ B_k^{(2m)}(z, a) = z^{k+2m+1} M_{a,k}^{(2m)}\left(\frac{1}{z}\right), \quad k \in \{0, \dots, [a] - 1\}, \end{cases} \quad (57)$$

are orthogonal on the positive half line with respect to the weight

$$w^{(2m)}(z, a) = \frac{e^{-1/z}}{z^{2(a+2m)+3} \left(L_{2m}^{-2(a+1+2m)}(-1/z)\right)^2}. \quad (58)$$

V. RADIAL EFFECTIVE KEPLER-COULOMB

The effective radial Kepler-Coulomb (ERKC) potential with zero ground level ($E_0(a) = 0$) is the third and last exceptional primary TSIP of the first category [22]. It is defined on the positive half line as

$$V(x; a) = \frac{a(a-1)}{x^2} - \frac{\gamma}{x} + V_0(a), \quad \gamma > 0, \quad a > 1 \quad (59)$$

where $x > 0$ and $V_0(a) = \gamma^2/4a^2$.

Its bound states are given by

$$\psi_n(x; a) = \exp\left(-\int dx w_n(x; a)\right) \sim x^a e^{-\gamma x/2a_n} L_n^{(2a-1)}(\gamma x/a_n), \quad n \geq 0, \quad (60)$$

with the corresponding energies $E_n(a) = V_0(a) - V_0(a_n)$, where $a_k = a + k$.

The associated RS equation is

$$-w'_n(x; a) + w_n^2(x; a) = V(x; a) - E_n(a) \quad (61)$$

The solutions of eq(61) corresponding to the physical eigenstates are given by

$$w_n(x; a) = w_0(x; a) + R_n(x; a), \quad (62)$$

where

$$w_0(x; a) = -\frac{a}{x} + \gamma/2a \quad (63)$$

and

$$\begin{aligned} R_n(y; a) &= -\frac{E_n(a)}{w_0(x; a) + w_0(y; a_1)} \uparrow \dots \uparrow \frac{E_n(a) - E_{j-1}(a)}{w_0(x; a_{j-1}) + w_0(x; a_j)} \uparrow \dots \uparrow \frac{E_n(a) - E_{n-1}(a)}{w_0(x; a_{n-1}) + w_0(x; a_n)} \\ &= \frac{\gamma}{2a_n} - \frac{\gamma}{2a} - \left(\log\left(L_n^{(2a-1)}(\gamma x/a_n)\right)\right)'. \end{aligned} \quad (64)$$

The only covariance transformation for the ERKC potentials is given by

$$a \xrightarrow{\Gamma_3} \underbrace{1-a}_{-a_{-1}}, \quad \begin{cases} V(x; a) \xrightarrow{\Gamma_3} V(x; a) - E_{-1}(a) \\ w_n(x; a) \xrightarrow{\Gamma_3} v_n(x; a) = w_n(x; -a_{-1}), \end{cases} \quad (65)$$

with

$$a_k \xrightarrow{\Gamma_3} 1 - a + k = -a_{-(k+1)}, \quad E_n(a) \xrightarrow{\Gamma_3} \gamma^2/4 \left(\frac{1}{a_{-1}^2} - \frac{1}{a_{-(n+1)}^2} \right) = E_{-(n+1)}(a) - E_{-1}(a). \quad (66)$$

We then have

$$-v'_n(x; a) + v_n^2(x; a) = V(x; a) - E_{-(n+1)}(a) \quad (67)$$

and from Eq.(63) and Eq.(64), we deduce

$$v_n(x; a) = v_0(x; a) + Q_n(x; a), \quad (68)$$

where

$$\begin{cases} v_0(x, a) = \frac{a-1}{x} - \frac{\gamma}{2a_{-1}} \\ Q_n(x, a) = -\frac{\gamma}{2a_{-(n+1)}} + \frac{\gamma}{2a_{-1}} - \left(\log\left(L_n^{(1-2a)}(-\gamma x/a_{-(n+1)})\right)\right)'. \end{cases} \quad (69)$$

If the argument of the GLP is positive, that is, if $a < n+1$ the Kienast-Lawton-Hahn theorem ensures that $Q_n(x, a)$ is regular for $x > 0$ if $1 - 2a < -n$, that is, if

$$\frac{n+1}{2} < a < n+1 \quad (70)$$

Another possibility to ensure the regularity of $Q_n(x, a)$ is to consider values of a such that $a > n + 1$, where the argument of the GLP is now negative. From the Kienast-Lawton-Hahn theorem, we then deduce that in this case for each even value of $n = 2m$, $Q_{2m}(x, a)$ is regular.

The DBT $A(v_n)$ applied to w_k gives

$$w_k(x; a) \xrightarrow{A(v_n)} w_k^{(n)}(x; a) = -v_n(x; a) + \frac{E_k(a) - E_{-(n+1)}(a)}{v_n(x; a) - w_k(x; a)}, \quad (71)$$

where $w_k^{(n)}(x; a)$ satisfies

$$-w_k^{(n)'}(x; a) + \left(w_k^{(n)}(x; a)\right)^2 = V^{(n)}(x; a) - E_k(a), \quad (72)$$

with

$$V^{(n)}(x; a) = V(x; a) + 2v_n'(x; a) = V(x; a_{-1}) + E_{-1}(a) + 2Q_n'(x; a). \quad (73)$$

In the cases where

$$\begin{cases} \frac{n+1}{2} < a < n+1 & (i) \\ n = 2m, \quad a > n+1, & (ii) \end{cases} \quad (74)$$

$V^{(n)}(x; a)$ is regular on the positive half line and isospectral to $V(x; a)$

$$V^{(n)}(x; a) \underset{iso}{=} V(x; a). \quad (75)$$

We have also

$$v_n'(x; a) + v_n^2(x; a) = V^{(n)}(x; a) - E_{-(n+1)}(a), \quad (76)$$

that is, $-v_n(x; a)$ is a regular RS function for the extended potential $V^{(n)}(x; a)$, associated to the eigenvalue $E_{-(n+1)}(a) < 0$, when $\frac{n+1}{2} < a$. Moreover

$$\psi_-^{(n)}(x; a) = \exp\left(+ \int v_n(x; a) dx\right) \sim \frac{x^{a-1} \exp\left(-\frac{\gamma}{2a_{-(n+1)}}x\right)}{L_n^{(1-2a)}(-\gamma x/a_{-(n+1)})}. \quad (77)$$

In the case (ii) (see Eq.(74)), $a_{-(2m+1)} > 0$ and $\psi_-^{(2m)}$ is a physical eigenstate for $\hat{H}^{(2m)}$ with the lowest eigenvalue. In other words, $\psi_-^{(2m)}$ is the fundamental state for $\hat{H}^{(2m)}$ and, as for the two preceding exceptional primary TSIP of the first category, the isospectrality is not strict. On the other hand, in the case (i) ($a_{-(n+1)} < 0$), $\psi_-^{(n)}$ is not in the physical spectrum and in this regime the isospectrality between $\hat{H}^{(n)}$ and \hat{H} becomes strict.

Consider first the case (ii). The superpartner of the extended potential $V^{(2m)}(x; a) = V(x; a) + 2v_{2m}'(x; a)$ is given by

$$\tilde{V}^{(2m)}(x; a) = V^{(2m)}(x; a) + 2(-v_{2m}'(x; a)) = V(x; a) \quad (78)$$

and the DBT $A(v_{2m})$ corresponds to a backward SUSY partnership.

The fundamental eigenstate of $\hat{H}^{(2m)}(a) = -d^2/dx^2 + V^{(2m)}(x; a)$ at the energy $E_{-(2m+1)}(a)$ is

$$\psi_-^{(2m)}(x; a) \sim \frac{x^{a-1} \exp\left(-\frac{\gamma x}{2|a_{-(2m+1)}|}\right)}{L_{2m}^{(1-2a)}(-\gamma x/|a_{-(2m+1)}|)} \quad (79)$$

and the excited eigenstates at energy $E_k(a)$, $k \geq 0$ are (see Eq.(5))

$$\psi_k^{(2m)}(x; a) \sim \frac{1}{\sqrt{E_k(a) - E_{-(2m+1)}(a)}} \hat{A}(v_{2m}) \psi_k(x; a), \quad (80)$$

that is,

$$\psi_k^{(2m)}(x; a) \sim x^{a-1} e^{-\gamma x/2a_k} \frac{N_{a,k}^{(2m)}(x)}{L_{2m}^{(1-2a)}(-\gamma x/|a_{-(2m+1)}|)}, \quad (81)$$

where

$$\begin{aligned} N_{a,k}^{(n)}(x) = & (1-2a) L_k^{(2a-1)}(\gamma x/a_k) L_n^{(1-2a)}(-\gamma x/a_{-(n+1)}) \\ & + \left(a - \frac{n+1}{2}\right) L_k^{(2a-1)}(\gamma x/a_k) L_n^{(-2a)}(-\gamma x/a_{-(n+1)}) + \left(a + \frac{k-1}{2}\right) L_k^{(2a-2)}(\gamma x/a_k) L_n^{(1-2a)}(-\gamma x/a_{-(n+1)}) \\ & + \frac{k+1}{2} L_{k+1}^{(2a-1)}(\gamma x/a_k) L_n^{(1-2a)}(-\gamma x/a_{-(n+1)}) - \frac{n+1}{2} L_k^{(2a-1)}(\gamma x/a_k) L_{n+1}^{(-2a)}(-\gamma x/a_{-(n+1)}), \end{aligned} \quad (82)$$

is a polynomial of degree $n+k+1$. From the orthonormality condition of the eigenstates of $\hat{H}^{(2m)}(a)$ we obtain that the functions $C_-^{(2m)}(x, a) = 1$ and

$$C_k^{(2m)}(x, a) = e^{-\gamma x/2a_k} N_{a,k}^{(2m)}(x), \quad k \geq 0, \quad (83)$$

constitute an orthogonal family on the positive half line with respect to the weight

$$w^{(2m)}(x, a) = \frac{x^{2(a-1)}}{\left(L_{2m}^{(1-2a)}(-\gamma x/|a_{-(2m+1)}|)\right)^2}. \quad (84)$$

In the case (i), the situation is quite different since the ground state of $V^{(n)}$ is associated to the RS function $w_0^{(n)}(x; a)$ and the superpartner of the extended potential $V^{(n)}(x; a)$ is now given by

$$\tilde{V}^{(n)}(x; a) = V^{(n)}(x; a) + 2w_0^{(n)'}(x; a), \quad n \geq 0, \quad (85)$$

as for the $L1$ and $L2$ extensions of the isotonic oscillator [21] but in the ERKC case $V^{(n)}$ does not inherit of the shape invariance properties of the initial TSIP.

The physical eigenstates for the energies $E_k(a)$, $k \geq 0$ of $\hat{H}^{(n)}(a)$ are given by

$$\psi_k^{(n)}(x; a, \gamma) \sim x^{a-1} e^{-\gamma x/2a_k} \frac{N_{a,k}^{(n)}(x)}{L_n^{(1-2a)}(\gamma x/|a_{-(n+1)}|)}, \quad (86)$$

and the functions

$$C_k^{(n)}(x, a) = e^{-\gamma x/2a_k} N_{a,k}^{(n)}(x), \quad k \geq 0, \quad (87)$$

are orthogonal on the positive half line with respect to the weight

$$w^{(n)}(x, a) = \frac{x^{2(a-1)}}{\left(L_n^{(1-2a)}(\gamma x/|a_{-(n+1)}|)\right)^2}. \quad (88)$$

VI. CONCLUSION

In this paper we have shown that the method previously developed for the isotonic potential [21], can be used to generate in a direct and systematic way the solvable regular rational extensions for all the exceptional first category TSIP. This approach is based on DBT transformations built from excited states RS functions regularized via the use of discrete symmetries of the initial potential.

The results are quite different from those obtained for the isotonic oscillator (which is the unique exceptional second category TSIP). Each exceptional first category TSIP admits only one series of regular rational extensions. Generally, as for the $L3$ series of rational extensions of the isotonic potential, it can be obtained only from regularized excited states associated to even quantum numbers and the DBT can be viewed as a backward SUSY partnership. The isospectrality is not strict and the spectrum of the extended potential presents a supplementary lower level. The ERKC potential constitutes an exception since extended potentials can be also obtained from regularized excited states RS functions associated to odd quantum numbers for some range of values of the "angular momentum" parameter a . They are in this case strictly isospectral to the original potential.

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