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# ON STABILITY OF STANDING WAVES OF NONLINEAR DIRAC EQUATIONS

NABILE BOUSSAID AND SCIPIO CUCCAGNA

ABSTRACT. We consider the stability problem for standing waves of nonlinear Dirac models. Under a suitable definition of linear stability, and under some restriction on the spectrum, we prove at the same time orbital and asymptotic stability. We are not able to get the full result proved in [Cuc10] for the nonlinear Schrödinger equation, because of the strong indefiniteness of the energy.

## 1. INTRODUCTION

In this paper we study the stability of standing waves of a class of nonlinear Dirac equations (NLDE). We assume a number of hypotheses on these standing waves, about their smoothness and exponential decay to 0 at infinity. We also assume that they form families smoothly dependent on a parameter. We then partially characterize, under a number of further technical hypotheses, their stability and their instability. We succeed partially in transposing to NLDE results proved for the nonlinear Schrödinger equations (NLS) in [Cuc10] and in previous references. We recall that [CL82, Sh83, SS85, Wei85, Wei86, GSS87, GSS90] contain a quite satisfactory characterization of the orbital stability of standing waves of the NLS. They do not apply to the Dirac equation, due to the strong indefiniteness of the energy. In this paper we initiate a theory of stability in the case of the NLDE, using ideas coming from the theory of asymptotic stability which are less sensitive to indefiniteness of the energy. This idea is explored also in [PS10] in a very special situation.

**1.1. The nonlinear Dirac equation.** We consider for  $m > 0$  a NLDE

$$(1.1) \quad \begin{cases} iu_t - D_m u + g(u\bar{u})\beta u = 0 \\ u(0, x) = u_0(x) \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3$$

where  $D_m = -i \sum_{j=1}^3 \alpha_j \partial_{x_j} + m\beta$ , with for  $j = 1, 2, 3$

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The unknown  $u$  is a  $C^4$ -valued function and given two vectors of  $\mathbb{C}^4$ ,  $uv = u \cdot v$  is the inner product in  $\mathbb{C}^4$ ,  $v^*$  is the complex conjugate,  $u \cdot v^*$  is the hermitian product in  $\mathbb{C}^4$ , which we write as  $uv^* = u \cdot v^*$  and denote

$$\bar{u} = \beta u^*$$

so that  $u\bar{u} = u \cdot \beta u^*$ .

Note that

$$(1.2) \quad \alpha_j \alpha_\ell + \alpha_\ell \alpha_j = 2\delta_{j\ell} I_{\mathbb{C}^4}, \quad \alpha_j \beta + \beta \alpha_j = 0, \quad \beta^2 = I_{\mathbb{C}^4}$$

thus the operator  $D_m$  is self-adjoint on  $L^2(\mathbb{R}^3, \mathbb{C}^4)$ , with domain  $H^1(\mathbb{R}^3, \mathbb{C}^4)$  and we have  $D_m^2 = -\Delta + m^2$ . The spectrum is  $\sigma(D_m) = (-\infty, -m] \cup [m, +\infty)$ , see [Tha92, Theorem 1.1].

**1.2. State of the art.** The equation in §1.1 arises in Dirac models used to model either extended particles with self-interaction or particles in space-time with geometrical structure. In the latter case, physicists have shown that a relativistic theory sometimes imposes a fourth order nonlinear potential (i.e., a cubic nonlinearity) such as the square of a quadratic form on  $\mathbb{C}^4$ ; see Rañada [Ran] and the references therein. The associated stationary equation is called the Soler model, [Sol70], as it was proposed by Soler to model the elementary fermions.

In our study, we assume the existence of stationary solutions as well as a number of properties like the smooth dependence on a parameter, the smoothness and the fact that they are rapidly decaying. These are not well established properties. Stationary solutions were actively studied in the last thirty years. The following authors used a dynamical systems approach: Cazenave and Vázquez [CV86], Merle [Mer88], Balabane, Cazenave, Douady and Merle [BCDM88] and [BCV90]. It is also possible to exploit the variational structure of the stationary equation, see Esteban and Séré [ES95]. A perturbation approach

yielding stationary solutions of the NLDE from solutions of the NLS is considered in Ounaies [Oun00] and Guan [Gua08].

Turning to the stability of stationary solutions, [SV86] frames the problem of stability of the Soler model within the framework of [SS85], without attempting a proof.

Some partial results involving small standing waves obtained by bifurcation from linear ones, when  $D_m$  replaced by  $H := D_m + V$  with  $V$  a nice potential, are obtained in [Bou06, Bou08b]. [Bou08b] shows that if a resonance condition holds, the space splits into a stable manifold outside which any initial condition leads to instability. If the resonance condition is not fulfilled, the stability problem is left open. The results we present here answer this question and can be used to clarify [Bou06].

Komech and Komech [KK10] prove the existence of global attractors in model involving a Dirac equation coupled to an harmonic oscillator.

The stability problem for the 1 dimensional NLDE is discussed under very restrictive hypotheses by Pelinovsky and Stefanov [PS10], who reproduce for the 1 D NLDE an analogue of the result in [SW92]. Notice that our theory can be adapted to extend these results.

**1.3. Hypotheses.** We assume the following hypotheses **(H:1)**–**(H:12)**.

**(H:1)**  $g(0) = 0$ ,  $g \in C^\infty(\mathbb{R}, \mathbb{R})$ ;  $g$  even,  $g(-r) = g(r)$ .

**(H:2)** There exists an open interval  $\mathcal{O} \subseteq (m/3, m)$  such that  $D_m u - \omega u - g(u\bar{u})\beta u = 0$  admits a  $C^\infty$  family of solutions  $\omega \in \mathcal{O} \rightarrow \phi_\omega \in H^{k,\tau}(\mathbb{R}^3)$  for any  $(k, \tau)$ , see (1.3) for a definition. In spherical coordinates  $x = \rho \cos(\vartheta) \sin(\varphi)$ ,  $y = \rho \sin(\vartheta) \sin(\varphi)$ ,  $z = \rho \cos(\varphi)$ , the standing waves are of the form

$$\phi_\omega(x) = \begin{bmatrix} a(\rho) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ ib(\rho) \begin{bmatrix} \cos \varphi \\ e^{i\vartheta} \sin \varphi \end{bmatrix} \end{bmatrix}$$

with  $a(\rho)$  and  $b(\rho)$  real valued and satisfying the following properties:

$$a, b \in C^\infty([0, \infty), \mathbb{R}),$$

$$\forall r \geq 0, a^2(\rho) - b^2(\rho) \geq 0,$$

$$a^{(j)}, b^{(j)} \text{ decay exponentially at infinity for all } j.$$

Moreover, notice that  $\phi_\omega(-x) = \beta \phi_\omega(x)$ .

**(H:3)** Let  $q(\omega) = \|\phi_\omega\|_{L^2}^2$ . We assume  $q'(\omega) \neq 0$  for all  $\omega \in \mathcal{O}$ .

**(H:4)** For any  $x \in \mathbb{R}^3$  we consider in (1.1) initial data s.t.  $u_0(-x) = \beta u_0(x)$ .

**(H:5)** Let  $\mathcal{H}_\omega$  be the linearized operator around  $e^{it\omega} \phi_\omega$ , see Section 2. We assume that  $\mathcal{H}_\omega$  satisfies the definition of linear stability in Definition 2.7.

**(H:6)**  $\mathcal{H}_\omega$  has  $2n$  nonzero eigenvalues, counted with multiplicity, all contained in  $(\omega - m, m - \omega)$ . The positive eigenvalues can be listed as

$$0 < \lambda_1(\omega) \leq \dots \leq \lambda_n(\omega) < m - \omega$$

where we repeat each eigenvalue according to the multiplicity. For each  $\lambda_j(\omega)$ , also  $-\lambda_j(\omega)$  is an eigenvalue (this symmetry follows from (2.14)). There are no other eigenvalues except for 0.

**(H:7)** The points and  $\pm(m - \omega)$  and  $\pm(m + \omega)$  are not resonances for  $\mathcal{H}_\omega$ , see Definition A.1.

**(H:8)** There are no resonances for  $\mathcal{H}_\omega$  in the essential spectrum  $\sigma_{ess}(\mathcal{H}_\omega)$ .

**(H:9)** There are natural numbers  $N_j$  defined by the property  $0 < N_j \lambda_j(\omega) < m - \omega < (N_j + 1) \lambda_j(\omega)$ .

**(H:10)** There is no multi index  $\mu \in \mathbb{Z}^k$  with  $|\mu| := |\mu_1| + \dots + |\mu_k| \leq 2N_1 + 3$  such that  $\mu \cdot \lambda = m \pm \omega$ .

**(H:11)** If  $\lambda_{j_1} < \dots < \lambda_{j_k}$  are  $k$  distinct  $\lambda$ 's, and  $\mu \in \mathbb{Z}^k$  satisfies  $|\mu| \leq 2N_1 + 3$ , then we have

$$\mu_1 \lambda_{j_1} + \dots + \mu_k \lambda_{j_k} = 0 \iff \mu = 0.$$

**(H:12)** The nonlinear Fermi golden rule (4.57) is true.

**1.4. Main results.** The main result in this article is the following one.

**Theorem 1.1.** *Suppose that  $\mathcal{O} \subset (m/3, m)$  and fix  $k_0 > 3$ . Pick  $\omega_0 \in \mathcal{O}$  and let  $\phi_{\omega_1}(x)$  be a standing wave of (1.1). Let  $u(t, x)$  be a solution to (1.1). Assume **(H:1)**–**(H:12)**. Then, there exist an  $\epsilon_0 > 0$  and a  $C > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  and for any  $u_0$  with  $\|u_0 - e^{i\gamma_0} \phi_{\omega_1}\|_{H^{k_0}} < \epsilon$ , there exist  $\omega_+ \in \mathcal{O}$ ,  $\theta \in C^1(\mathbb{R}; \mathbb{R})$  and  $h_+ \in H^{k_0}$  with  $\|h_+\|_{H^{k_0}} + |\omega_+ - \omega_1| \leq C\epsilon$  such that*

$$\lim_{t \rightarrow +\infty} \|u(t, \cdot) - e^{i\theta(t)} \phi_{\omega_+} - e^{-itD_m} h_+\|_{H^{k_0}} = 0.$$

*Remark 1.2.* The constraint  $3\omega > m$  allows to exploit the Fermi golden rule in the same way of what done in the case of the NLS in [Cuc10]. The constraint  $3\omega > m$  can be eased to  $(2N_j + 1)\omega > m$  for all  $j = 1, \dots, n$ , see Remark 4.10. These constraints are a consequence of the strong indefiniteness of the Dirac system. We expect that these constraints can be eased.

*Remark 1.3.* It is possible to construct examples of “small solitons” by bifurcation from linear standing waves like in [Bou06, Bou08b, PS10, SW89, SW92], to which our result will apply if we assume an analogue of **(H:12)** and after some relatively minor points of linear theory have been worked out.

*Remark 1.4.* Except for the smoothness with respect to the parameter  $\omega$ , for some non-linearities **(H:2)** is a consequence of [ES95]. Notice that [Gua08] proves continuous dependence on  $\omega$  for some examples.

*Remark 1.5.* The regularity and the exponential decay of the solution can be proved by the Combes-Thomas method, see [His00].

*Remark 1.6.* The hypothesis that there are no eigenvalues  $\lambda \in (m - \omega, m + \omega)$  is not obvious. There is such an eigenvalue in 1D, see [BC09].

*Remark 1.7.* Assumption **(H:7)** is just part of **(H:8)**. In the case of the NLS it is proved that a resonance in the interior (in  $\mathbb{R}$ ) of the continuous spectrum of  $\mathcal{H}_\omega$  is necessarily an eigenvalue, see [CPV05]. In this case **(H:8)** is a consequence of **(H:7)** and **(H:6)**. Unfortunately, in the case of the Dirac system we are not able to prove an analogous result, except for resonances contained in  $(-\omega + m, \omega - m)$  or for large energies. This is yet a consequence of the strong indefiniteness of the energy of the Dirac system. We expect that **(H:8)** can be eliminated whenever Assumption **(H:6)** holds.

Consider  $\xi \in \ker(\mathcal{H}_\omega - \lambda_j(\omega))$ . One of the requirements for linear stability in Definition 2.7 is that if  $\xi \neq 0$  then  $\langle \xi, \Sigma_3 \xi^* \rangle > 0$ . As it might seem artificial, we prove what follows.

**Theorem 1.8.** *Suppose that  $\mathcal{O} \subset (m/3, m)$ . Pick  $\omega \in \mathcal{O}$  and let  $\phi_\omega(x)$  be a standing wave of (1.1). Replace **(H:5)** with the following assumption:*

**(H:5')** *We assume that  $\mathcal{H}_\omega$  satisfies all the conditions of Definition 2.7 except for condition (4) which we restate as follows. That is, we assume that for any eigenvalue  $\lambda > 0$  the quadratic form  $\xi \rightarrow \langle \xi, \Sigma_3 \xi^* \rangle$  is non degenerate in  $\ker(\mathcal{H}_\omega - \lambda)$ . We assume that there exists at least one eigenvalue  $\lambda > 0$  such that the quadratic form is non positive in  $\ker(\mathcal{H}_\omega - \lambda)$ .*

*Assume **(H:1)**–**(H:4)**, **(H:5')** and **(H:6)**–**(H:12)**. Then  $\phi_\omega(x)$  is orbitally unstable.*

In this article we follow the argument developed in [Cuc10] for the NLS. The Dirac equation is harder than the NLS also because less is known about the existence of families of solutions in  $C^1(\mathcal{O}, H^1(\mathbb{R}^3))$  of standing waves. It is well known that the classical methods to prove orbital stability, see [CL82, Wei86, GSS87, GSS90], which are based on the positivity of certain functionals, do not apply because of the strong indefiniteness of the energy. As already mentioned, some initial results for the Dirac equation are in [Bou06, Bou08b, PS10]. Like in these articles, we exploit the dispersive properties of the linearizations, adapting the methods used to prove asymptotic stability for the Schrödinger equation, which were initiated in [SW89, SW92, BP92, BP95] and developed by a substantial number of authors, see the references in [Cuc10]. One of the difficult issues for the NLS, is to prove that the energy of the discrete modes associated to the eigenvalues in **(H:6)** leaks either in the radiation part or in the standing wave. The solution to this problem was initiated in [BP95], where the eigenvalues are close to the continuous spectrum, and solved in quite general form in [Cuc10], see also [BC09, Cuc11]. We refer to [Cuc10] for a discussion of the fact that it is essential to exploit the hamiltonian structure of the equation. In this article we follow the same framework of [Cuc10] obtaining similar results. We need to develop some of the linear theory of dispersion, which in the case of the NLS had been developed in the course of a decade, see [Cuc01, CPV05]. Key to dispersion theory is the proof of smoothing estimates for Schrödinger operators with magnetic potentials in [EGS09]. There are two points in the article where the strong indefiniteness of the energy interferes with our method. We expect these difficulties to be technical and solvable. The main difficulty occurs in the proof of the positive semidefiniteness of the key coefficients in the Fermi golden rule (FGR). Another difficulty occurs with resonances, requiring the explicit assumption of their absence inside the continuous spectrum, see Remark 1.7. Notice that in the case of small solitons considered in [Bou06, Bou08b, PS10] the absence of resonances comes for free, while these references do not have to address the FGR because of their restrictive hypotheses. To prove the positive semidefiniteness and overcome the FGR difficulty, we use the hypothesis  $3\omega > m$ . This hypothesis is unnecessary if there are no eigenvalues  $\lambda$ 's, or can be weakened if they are close to 0.

Indeed, in the case all  $N_j$  in **(H:9)** are large, then the hypothesis  $3\omega > m$  can be considerably relaxed, see Remark 4.10. This is somewhat odd, considering that the case with all  $N_j$  large was the hardest in [Cuc10].

The instability result in Theorem 1.8 arises from our desire to justify hypothesis (4) in our definition of linear stability, see Definition 2.7. The proof of Theorem 1.8 is similar to [Cuc09]. That is, we show that orbital stability implies asymptotic stability, and we then show that this is incompatible with **(H:5')**. All the proofs are conditional on **(H:12)**, that is that a certain non negative quantity is actually positive. Presumably this is true generically.

**1.5. Notation and preliminaries.** We consider spaces

$$(1.3) \quad H^{k,s}(\mathbb{R}^3, \mathbb{C}^4) = \{f \in \mathcal{S}'(\mathbb{R}^3), \|\langle x \rangle^s \langle \nabla \rangle^k f\|_2 < \infty\}$$

for  $s, k \in \mathbb{R}$  with norm  $\|f\|_{H^{k,s}} = \|\langle x \rangle^s \langle \nabla \rangle^k f\|_2$ . Sometimes we will write  $H_x^{k,s}$  to emphasize the independent variable  $x$ . If  $k = 0$ , we write  $L^{2,s}$  instead of  $H^{0,s}$ .

For  $k \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , the Besov space  $B_{p,q}^k(\mathbb{R}^3, \mathbb{C}^d)$  is the space of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^d)$  such that

$$\|f\|_{B_{p,q}^k} = \left( \sum_{j \in \mathbb{N}} 2^{jkq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

with  $\widehat{\varphi} \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$  such that  $\sum_{j \in \mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$  for all  $\xi \in \mathbb{R}^3 \setminus \{0\}$ ,  $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$  for all  $j \in \mathbb{N}^*$  and for all  $\xi \in \mathbb{R}^3$ , and  $\widehat{\varphi}_0 = 1 - \sum_{j \in \mathbb{N}^*} \widehat{\varphi}_j$ . It is endowed with the norm  $\|f\|_{B_{p,q}^k}$ .

Given a vector  $u$ , by  $u^*$  we will denote the vector whose coordinates are the complex conjugates of those of  $u$ .

Given two vector of  $\mathbb{C}^4$  or  $\mathbb{C}^8$ ,  $uv = u \cdot v$  is their inner product,  $u \cdot v^*$  is their hermitian product, which we write as  $uv^* = u \cdot v^*$ . We denote by  $\beta$  either the 4-dimensional hermitian matrix defined above or (by an abuse of notation and depending on the context) the 8-dimensional hermitian matrix

$$(1.4) \quad \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}.$$

For  $A$  a closed operator on a Hilbert space  $X$ , we will denote by  $R_A(z)$  the resolvent of  $A$  at any  $z$  in the resolvent set of  $A$ . That is the inverse of  $A - z$  whenever it is invertible with bounded inverse from the domain of  $A$  to  $X$ .

**1.6. Structure of the article.** The paper is organized as follows.

In Section 2, we study of the linearization of (1.1) around a stationary solution. In particular in a neighborhood of a stationary state we introduce an appropriate coordinate system related to the spectral decomposition of the linearized operator. Estimates on such operators are discussed in part here and in part in the appendix.

In Section 3 we discuss the Hamiltonian structure of the system, and in particular we look for canonical coordinates. We then apply the method of Birkhoff normal forms, referring for proofs to [Cuc10].

In Section 4 we discuss scattering of the continuous modes and dissipation of discrete modes, proving the semipositivity of in the Fermi golden rule for  $\omega > m/3$ , or more generally if the elements of  $\sigma_d(\mathcal{H}_\omega)$  are all close to 0.

## 2. SET UP AND LINEAR ESTIMATES

**2.1. Linearization, modulation and set up.** Since our ambient space is  $H^{k_0}(\mathbb{R}^3, \mathbb{C}^4)$  with  $k_0 > 3$  and so in particular  $k_0 > 3/2$ , under **(H:1)** the functional  $u \rightarrow g(u\bar{u})\beta u$  is locally Lipschitz and (1.1) is locally well posed, see pp. 293–294 volume III [Tay96]. Consider the solution  $u(t, x)$  of (1.1). Then by **(H:4)** we have  $u(t, -x) = \beta u(t, x)$ . We write the ansatz

$$(2.1) \quad u(t, x) = e^{i\vartheta(t)}(\phi_{\omega(t)}(x) + r(t, x)).$$

Inserting (2.1) in (1.1) we get from the definition of  $\phi_\omega$

$$(2.2) \quad \begin{aligned} ir_t &= D_m r - \omega(t)r - g(\phi_{\omega(t)}\bar{\phi}_{\omega(t)})\beta r - g'(\phi_{\omega(t)}\bar{\phi}_{\omega(t)})(r\bar{\phi}_{\omega(t)})\beta\phi_{\omega(t)} \\ &\quad - g'(\phi_{\omega(t)}\bar{\phi}_{\omega(t)})(\phi_{\omega(t)}\bar{r})\beta\phi_{\omega(t)} + (i\dot{\vartheta}(t) + \omega(t))(\phi_{\omega(t)} + r) - i\dot{\omega}(t)\partial_\omega\phi_{\omega(t)} + n(r), \end{aligned}$$

where  $n(r) = O(r^2)$  is defined by

$$n(r) := g((\phi_{\omega(t)} + r)\overline{\phi_{\omega(t)} + r})\beta(\phi_{\omega(t)} + r) - g(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})\beta\phi_{\omega(t)} \\ - g'(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})(r\overline{\phi_{\omega(t)}})\beta\phi_{\omega(t)} - g'(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})(\phi_{\omega(t)}\overline{r})\beta\phi_{\omega(t)}.$$

We denote by  $C : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  the charge conjugation operator:

$$(2.3) \quad u^c := Cu := i\beta\alpha_2 u^*.$$

it satisfies the following properties (see [Tha92, Section 1.4.6]) :

$$\forall j \in \{1, 2, 3\}, \alpha_j C = C\alpha_j \text{ and } \beta C = -C\beta,$$

and since it is anti-linear for any  $u \in \mathbb{C}^4$ ,  $C(u^*) = (Cu)^*$ .

*Remark 2.1.* For more details we refer to [Tha92, Section 1.4.6]. This choice for the charge conjugation is due the choice we made for the coefficient of the Dirac operator, which is the standard one. If we had chosen the Majorana representation for the coefficient then the charge conjugation would have been simply the complex conjugation. These two representations of Dirac operators are unitary equivalent.

We thus obtain the following lemma.

**Lemma 2.2.** *For any vector  $v \in \mathbb{C}^4$  we have  $C^2 v = v$ . Moreover if **(H:1)** holds, then we have:*

$$(2.4) \quad C(iv) = -iv^c, \quad v\overline{v} = -Cv\overline{Cv}, \quad g(v\overline{v}) = g(v^c\overline{v^c}), \quad g'(v\overline{v}) = -g'(v^c\overline{v^c}), \quad C(\beta v) = -\beta v^c.$$

*For any function  $w \in C^1(\mathbb{R}^3, \mathbb{C}^4)$  we have  $C(D_m w) = -D_m w^c$ . Finally, for  $u_0$  satisfying **(H:4)** we have  $u_0^c(-x) = -\beta u_0^c(x)$ .*

Applying  $-C$  to (2.2) or (1.1) we obtain

$$(2.5) \quad i\dot{r}_t^c = D_m r^c + \omega(t)r^c - g(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})\beta r^c + g'(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})(r^c\overline{\phi_{\omega(t)}})\beta\phi_{\omega(t)}^c \\ + g'(\phi_{\omega(t)}\overline{\phi_{\omega(t)}})(\phi_{\omega(t)}^c\overline{r^c})\beta\phi_{\omega(t)}^c - (\dot{\vartheta}(t) + \omega(t))(\phi_{\omega(t)}^c + r^c) - i\dot{\omega}(t)\partial_\omega\phi_{\omega(t)}^c - Cn(r).$$

We set

$$(2.6) \quad U = \begin{pmatrix} u \\ u^c \end{pmatrix}, \quad R = \begin{pmatrix} r \\ r^c \end{pmatrix}, \quad \Phi_\omega = \begin{pmatrix} \phi_\omega \\ \phi_\omega^c \end{pmatrix}, \quad N(R) = \begin{pmatrix} n(r) \\ -Cn(r) \end{pmatrix} = \begin{pmatrix} n(r) \\ -n(r^c) \end{pmatrix} \\ \mathcal{H}_\omega = \mathcal{H}_{\omega,0} + V_\omega, \quad \mathcal{H}_{\omega,0} = \begin{pmatrix} D_m - \omega & 0 \\ 0 & D_m + \omega \end{pmatrix}, \\ V_\omega = g(\phi_\omega\overline{\phi_\omega})\beta + g'(\phi_\omega\overline{\phi_\omega}) \begin{pmatrix} -(\beta\phi_\omega^*) & \beta\phi_\omega & (\beta(\phi_\omega^c)^*) & \beta\phi_\omega^c \\ -(\beta\phi_\omega^*) & \beta\phi_\omega^c & (\beta(\phi_\omega^c)^*) & \beta\phi_\omega^c \end{pmatrix}.$$

where in this context  $\beta$  is meant in the sense of (1.4) or (2.8) below and where the parenthesis  $(\phi \quad )$  stands for the map  $r \mapsto \phi r$ .

Therefore we have:

$$(2.7) \quad i\dot{R} = \mathcal{H}_\omega R + (\dot{\vartheta}(t) + \omega(t))(\Sigma_3\Phi_\omega + \Sigma_3R) - i\dot{\omega}\partial_\omega\Phi_\omega + N(R),$$

where

$$\Sigma_1 = \begin{pmatrix} 0 & I_{\mathbb{C}^4} \\ I_{\mathbb{C}^4} & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & iI_{\mathbb{C}^4} \\ -iI_{\mathbb{C}^4} & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} I_{\mathbb{C}^4} & 0 \\ 0 & -I_{\mathbb{C}^4} \end{pmatrix}.$$

Notice that by **(H:4)** and Lemma 2.2 we have for  $\Upsilon(x) \in \{\Phi_\omega(x), R(t, x)\}$

$$(2.8) \quad \Upsilon(-x) = \beta\Sigma_3\Upsilon(x) \text{ where } \beta = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}.$$

**2.2. Energy, charge and symmetries.** The following result is an elementary but crucial remark in our study. It expresses the energy and the charge as a symmetric bilinear (block anti-diagonal) forms on  $L^2(\mathbb{R}^3, \mathbb{C}^8)$ . In the following lemma, we denote by  $\langle \cdot, \cdot \rangle$  the inner product of  $L^2(\mathbb{R}^3, \mathbb{C}^8)$ .

**Lemma 2.3.** *Let  $U^T = (u, Cu)$ . Set for  $G(0) = 0$  and  $G'(s) = g(s)$*

$$(2.9) \quad E(U) = E_K(U) + E_P(U), \quad E_K(U) = \int_{\mathbb{R}^3} (D_m u)u^* dx, \quad E_P(U) = - \int_{\mathbb{R}^3} G(u\overline{u})dx, \\ Q(U) = \int_{\mathbb{R}^3} uu^* dx.$$

Then  $E(U)$  and  $Q(U)$  are invariants of motion for (1.1) and we have

$$(2.10) \quad \begin{aligned} E(U) &= \frac{1}{2} \langle i\beta\alpha_2 \Sigma_3 \Sigma_1 D_m U, U \rangle - \int_{\mathbb{R}^3} G \left( \frac{1}{2} U \cdot i\alpha_2 \Sigma_3 \Sigma_1 U \right) dx \\ Q(U) &= \frac{1}{2} \langle U, i\beta\alpha_2 \Sigma_1 U \rangle \end{aligned}$$

and  $U$  satisfies system

$$(2.11) \quad i\dot{U} = i\beta\alpha_2 \Sigma_3 \Sigma_1 \nabla E(U).$$

*Proof.* For any symmetric operator  $A$  acting on  $L^2(\mathbb{R}^3, \mathbb{C}^4)$  with the domain invariant by  $C$  and anticommuting to  $C$  and any  $u \in D(A)$ ,

$$\begin{aligned} u \cdot (Au)^* &= \frac{u \cdot (Au)^* + u^* \cdot (Au)}{2} = \frac{u \cdot i\beta\alpha_2 C A u + i\beta\alpha_2 C u \cdot Au}{2} \\ &= \frac{-u \cdot i\beta\alpha_2 A u^c + i\beta\alpha_2 u^c \cdot Au}{2} = \frac{-u \cdot i\beta\alpha_2 A u^c + u^c i\beta\alpha_2 \cdot Au}{2} \\ &= \frac{i}{2} U \cdot \beta\alpha_2 \Sigma_3 \Sigma_1 A U \end{aligned}$$

where we write  $A$  for

$$A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

If  $A$  is commuting to  $C$  then a similar calculation shows

$$\langle u, (Au)^* \rangle = \frac{i}{2} \langle U, \beta\alpha_2 \Sigma_1 A U \rangle$$

This identities for  $A = D_m$ ,  $A = \beta$  or  $A = I$  proves the lemma.  $\square$

We consider now the bilinear map

$$(2.12) \quad \left\langle \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \begin{pmatrix} s_1^* \\ s_2^* \end{pmatrix} \right\rangle = \int_{\mathbb{R}^3} (r_1 \cdot s_1^* + r_2 \cdot s_2^*) dx.$$

By  $\mathcal{H}_\omega^*$  we denote the adjoint of  $\mathcal{H}_\omega$  w.r.t. (2.12). We have:

**Lemma 2.4.** *We have*

$$(2.13) \quad \mathcal{H}_\omega^* = \Sigma_3 \mathcal{H}_\omega \Sigma_3$$

$$(2.14) \quad \mathcal{H}_\omega = -C \Sigma_1 \mathcal{H}_\omega C \Sigma_1 \text{ where } C = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$$

$$(2.15) \quad V_\omega(-x) = \beta \Sigma_3 V_\omega(x) \beta \Sigma_3 \text{ with } \beta \text{ in the sense of (2.8).}$$

*Proof.* First of all, (2.13)–(2.14) hold with  $\mathcal{H}_\omega$  replaced by  $\mathcal{H}_{\omega,0}$ . It remains to check them with  $\mathcal{H}_\omega$  replaced by  $V_\omega$ . We have  $V_\omega^* = \Sigma_3 V_\omega \Sigma_3$  by

$$(2.16) \quad \Sigma_3 \begin{pmatrix} -(\beta\phi_\omega^*) \beta\phi_\omega & (\beta(\phi_\omega^c)^*) \beta\phi_\omega \\ -(\beta\phi_\omega^*) \beta\phi_\omega^c & (\beta(\phi_\omega^c)^*) \beta\phi_\omega^c \end{pmatrix} \Sigma_3 = \begin{pmatrix} -(\beta\phi_\omega^*) \beta\phi_\omega & -(\beta(\phi_\omega^c)^*) \beta\phi_\omega \\ (\beta\phi_\omega^*) \beta\phi_\omega^c & (\beta(\phi_\omega^c)^*) \beta\phi_\omega^c \end{pmatrix}$$

and from the fact that the matrix in rhs(2.16) is the adjoint of the matrix in lhs(2.16). (2.14) holds with  $\mathcal{H}_\omega$  replaced by  $\mathcal{H}_{\omega,0}$  by Lemma 2.2. We have

$$(2.17) \quad \begin{aligned} C \Sigma_1 \begin{pmatrix} -(\beta\phi_\omega^*) \beta\phi_\omega & (\beta(\phi_\omega^c)^*) \beta\phi_\omega \\ -(\beta\phi_\omega^*) \beta\phi_\omega^c & (\beta(\phi_\omega^c)^*) \beta\phi_\omega^c \end{pmatrix} = \\ -C \begin{pmatrix} -(\beta(\phi_\omega^c)^*) \beta\phi_\omega^c & (\beta\phi_\omega^*) \beta\phi_\omega^c \\ -(\beta(\phi_\omega^c)^*) \beta\phi_\omega & (\beta\phi_\omega^*) \beta\phi_\omega \end{pmatrix} \Sigma_1 = \\ - \begin{pmatrix} (\beta(\phi_\omega^c)^*) \beta\phi_\omega & -(\beta\phi_\omega^*) \beta\phi_\omega \\ (\beta(\phi_\omega^c)^*) \beta\phi_\omega^c & -(\beta\phi_\omega^*) \beta\phi_\omega^c \end{pmatrix} \Sigma_1. \end{aligned}$$

We have for  $v \in \mathbb{C}^4$

$$\begin{aligned} (\beta(\phi_\omega^c)^* v)^* &= \beta(i\beta\alpha_2 \phi_\omega^*) v^* = -\beta\phi_\omega^* C(v), \\ (\beta\phi_\omega^* v)^* &= \beta\phi_\omega v^* = -\beta(i\beta\alpha_2 \phi_\omega)(i\beta\alpha_2 v^*) = -\beta(\phi_\omega^c)^* C(v). \end{aligned}$$



Then

$$(2.18) \quad \text{rhs}(2.17) = - \begin{pmatrix} -(\beta\phi_\omega^*) & \beta\phi_\omega & (\beta(\phi_\omega^c)^*) & \beta\phi_\omega \\ -(\beta\phi_\omega^*) & \beta\phi_\omega^c & (\beta(\phi_\omega^c)^*) & \beta\phi_\omega^c \end{pmatrix} C\Sigma_1.$$

This yields (2.14). The proof of (2.15) goes as follows. Using  $\phi(-x) = \beta\phi(x)$  and  $\phi^c(-x) = -\beta\phi^c(x)$ , where we omit the subindex  $\omega$ , we have

$$(2.19) \quad \begin{aligned} V(-x)\beta\Sigma_3 &= g(\phi(x)\bar{\phi}(x))\Sigma_3 + \begin{pmatrix} -(\phi^*(x)) & \phi(x) & -((\phi^c(x))^*) & \phi(x) \\ (\phi^*(x)) & \phi^c(x) & ((\phi^c(x))^*) & \phi^c(x) \end{pmatrix} \beta\Sigma_3 \\ &= g(\phi(x)\bar{\phi}(x))\Sigma_3 + \begin{pmatrix} -(\beta\phi^*(x)) & \phi(x) & (\beta(\phi^c(x))^*) & \phi(x) \\ (\beta\phi^*(x)) & \phi^c(x) & -(\beta(\phi^c(x))^*) & \phi^c(x) \end{pmatrix}. \end{aligned}$$

Similarly

$$(2.20) \quad \begin{aligned} \beta\Sigma_3 V(x) &= g(\phi(x)\bar{\phi}(x))\Sigma_3 + \beta\Sigma_3 \begin{pmatrix} -(\beta\phi^*(x)) & \beta\phi(x) & (\beta(\phi^c(x))^*) & \beta\phi(x) \\ -(\beta\phi^*(x)) & \beta\phi^c(x) & (\beta(\phi^c(x))^*) & \beta\phi^c(x) \end{pmatrix} \\ &= \text{second line of (2.19)}. \end{aligned}$$

The last two formulas yield (2.15).  $\square$

**2.3. Spectrum, coordinates, linear stability.** Identity (2.15) and the definition of  $\mathcal{H}_{\omega,0}$  imply that  $\mathcal{H}_\omega$  leaves invariant the space of functions for which the symmetry  $\Upsilon(-x) = \beta\Sigma_3\Upsilon(x)$  holds for all  $x \in \mathbb{R}^3$ . From now on we focus on the space of vector valued functions satisfying this symmetry.

**2.3.1. Spectrum.** We consider the spectrum

$$\sigma(\mathcal{H}_\omega) = \{\lambda \in \mathbb{C}, \mathcal{H}_\omega - \lambda Id : H^1(\mathbb{R}^3, \mathbb{C}^4) \mapsto L^2(\mathbb{R}^3, \mathbb{C}^4) \text{ is not invertible}\}$$

A point  $\lambda \in \sigma(\mathcal{H}_\omega)$  is in the essential spectrum  $\sigma_{\text{ess}}(\mathcal{H}_\omega)$  if  $\mathcal{H}_\omega - \lambda Id$  is not a Fredholm operator of index 0. This set is stable under relatively compact perturbation (such as a change on the nonlinearity). The following lemma summarizes what we know about the spectrum.

- Lemma 2.5.** (1) For the essential spectrum we have,  $\sigma_{\text{ess}}(\mathcal{H}_\omega) = (-\infty, \omega - m] \cup [m - \omega, +\infty)$ .  
(2) For each  $z \in \sigma_p(\mathcal{H}_\omega)$  the corresponding generalized eigenspace  $N_g(\mathcal{H}_\omega - z)$  has finite dimension.  
(3) If  $z \in \sigma(\mathcal{H}_\omega)$  then also  $-z \in \sigma(\mathcal{H}_\omega)$ .  
(4) For the generalized kernel we have  $N_g(\mathcal{H}_\omega^*) \supseteq \{\Phi_\omega, \Sigma_3 \partial_\omega \Phi_\omega\}$ .  
(5)  $\partial_\omega \|\phi_\omega\|_2^2 \neq 0$  implies that there are no  $v$  such that  $\mathcal{H}_\omega v = \partial_\omega \Phi_\omega$ .

*Proof.* We have that (1) and (2) are consequences of the above discussion. If  $z \in \sigma_{\text{ess}}(\mathcal{H}_\omega)$  then (3) is a consequence of (1). If  $z$  is an eigenvalue, then (3) is a consequence of (2.14). (4) is a consequence of  $N_g(\mathcal{H}_\omega) \supseteq \{\Sigma_3 \Phi_\omega, \partial_\omega \Phi_\omega\}$  which can be seen as follows. By the gauge invariance of the nonlinearity,  $G((e^{i\theta} u)(\overline{e^{i\theta} u})) = G(u\bar{u})$ , where  $G$  is a primitive of  $g$ , we have

$$\mathcal{H}_\omega \begin{pmatrix} i\phi_\omega \\ Ci\phi_\omega \end{pmatrix} = 0 \text{ or } \mathcal{H}_\omega \Sigma_3 \Phi_\omega = 0.$$

Then differentiating (1.1) and its image by  $C$  with respect to  $\omega$ , we obtain

$$(2.21) \quad \mathcal{H}_\omega \partial_\omega \Phi_\omega = -\Sigma_3 \Phi_\omega.$$

(5) follows by the following argument, if we assume existence of  $v$  s.t.  $\mathcal{H}_\omega v = \partial_\omega \Phi_\omega$ ,

$$(2.22) \quad \begin{aligned} 0 &= \langle v, (\mathcal{H}_\omega^* \Phi_\omega)^* \rangle = \langle \partial_\omega \Phi_\omega, \Phi_\omega^* \rangle = \langle \partial_\omega \phi_\omega, \phi_\omega^* \rangle + \langle \partial_\omega i\beta\alpha_2 \phi_\omega^*, i\beta\alpha_2 \phi_\omega \rangle \\ &= \langle \partial_\omega \phi_\omega, \phi_\omega^* \rangle + \langle \partial_\omega \phi_\omega^*, \phi_\omega \rangle = \partial_\omega \|\phi_\omega\|_2^2 \neq 0. \end{aligned}$$

$\square$

*Remark 2.6.* Notice that from (2.13), if  $z \in \sigma(\mathcal{H}_\omega)$  then  $\bar{z} \in \sigma(\mathcal{H}_\omega)$ . Hence if  $z \in \sigma(\mathcal{H}_\omega)$  then  $\{z, -z, \bar{z}, -\bar{z}\} \subseteq \sigma(\mathcal{H}_\omega)$ .

We have the beginning of  $\mathcal{H}_\omega$  invariant Jordan block decomposition

$$(2.23) \quad L^2 = N_g(\mathcal{H}_\omega) \oplus N_g^\perp(\mathcal{H}_\omega^*).$$

Linear stability means to us what follows, see [Cuc09].

**Definition 2.7** (Linear Stability). We will say that a standing wave  $e^{it\omega} \phi_\omega$  is linearly stable when the following conditions hold.

- (1)  $\sigma(\mathcal{H}_\omega) \subset \mathbb{R}$ .



- (2)  $N_g(\mathcal{H}) = \{\Sigma_3 \Phi_\omega, \partial_\omega \Phi_\omega\}$ .
- (3) For any eigenvalue  $z \neq 0$  of  $\mathcal{H}_\omega$  we have  $N_g(\mathcal{H}_\omega - z) = \ker(\mathcal{H}_\omega - z)$ .
- (4) For any positive eigenvalue  $\lambda > 0$  and for any  $\xi \in \ker(\mathcal{H}_\omega - \lambda)$ , we have  $\langle \xi, \Sigma_3 \xi^* \rangle > 0$ .

As a consequence of **(H:5)**, the Jordan decomposition can be continued as follows:

$$(2.24) \quad L^2 = N_g(\mathcal{H}_\omega) \oplus \left( \bigoplus_{j,\pm} \ker(\mathcal{H}_\omega \mp \lambda_j(\omega)) \right) \oplus L_c^2(\mathcal{H}_\omega) \text{ with } L_c^2(\mathcal{H}_\omega) = \{L_d^2(\mathcal{H}_\omega^*)\}^\perp,$$

where for  $K = \mathcal{H}_\omega^*, \mathcal{H}_\omega$  we set  $L_d^2(K_\omega) := N_g(K) \oplus \bigoplus_{j,\pm} \ker(K \mp \lambda_j(\omega))$ .

Let  $(\xi_j(\omega, x))_j$  be a basis of  $\bigoplus_{j=1}^n \ker(\mathcal{H}_\omega - \lambda_j(\omega))$  so that each vector is smooth in both variables, with  $|\partial_{\omega x}^\alpha \xi_j(\omega, x)| < c_\alpha e^{-a_\alpha |x|}$  for some  $c_\alpha > 0$  and  $a_\alpha > 0$ . Notice that this is just exponential decay of eigenvectors with the smoothness assumption in **(H:2)**.

We normalize  $\xi_j(\omega, x)$  so that  $\varepsilon_j = \langle \xi_j, \Sigma_3 \xi_j^* \rangle \in \{1, -1\}$  and  $\langle \xi_j, \Sigma_3 \xi_i^* \rangle = 0$  for  $j \neq i$ . Notice that in Theorem 1.1 for all  $j$  we have  $\varepsilon_j = 1$  while for Theorem 1.8 we have  $\varepsilon_j = -1$  for at least one  $j$ .

From the calculations of this section, we have built a dual basis. Hence, given any vector  $X$ , we have

$$(2.25) \quad X = \frac{\langle X, (e^{i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi)^* \rangle}{q'(\omega)} e^{i\Sigma_3 \vartheta} \Sigma_3 \Phi + \frac{\langle X, (e^{i\Sigma_3 \vartheta} \Phi)^* \rangle}{q'(\omega)} e^{i\Sigma_3 \vartheta} \partial_\omega \Phi +$$

$$\sum_{j=1}^n \varepsilon_j \langle X, (e^{i\Sigma_3 \vartheta} \Sigma_3 \xi_j)^* \rangle e^{i\Sigma_3 \vartheta} \xi_j + \sum_{j=1}^n \varepsilon_j \langle X, (e^{i\Sigma_3 \vartheta} \Sigma_1 \Sigma_3 C \xi_j)^* \rangle e^{i\Sigma_3 \vartheta} \Sigma_1 C \xi_j + e^{i\Sigma_3 \vartheta} P_c(\mathcal{H}_\omega) e^{-i\Sigma_3 \vartheta} X,$$

with  $P_c(\mathcal{H}_\omega)$  the projector onto  $L_c^2(\mathcal{H}_\omega)$  with respect to decomposition (2.24). By duality, we have the following lemma.

**Lemma 2.8.** *Suppose that for a given  $\omega$  the conditions of Definition 2.7 are satisfied. Then*

$$(2.26) \quad L^2 = N_g(\mathcal{H}_\omega^*) \oplus \left( \bigoplus_{j,\pm} \ker(\mathcal{H}_\omega^* \mp \lambda_j(\omega)) \right) \oplus L_c^2(\mathcal{H}_\omega^*) \text{ with } L_c^2(\mathcal{H}_\omega^*) := \{L_d^2(\mathcal{H}_\omega)\}^\perp.$$

Any 1 form  $\alpha = \langle \alpha^\sharp, \cdot \rangle$  can be decomposed as follows:

$$(2.27) \quad \alpha^\sharp = \frac{\langle \alpha^\sharp, e^{i\Sigma_3 \vartheta} \partial_\omega \Phi \rangle}{q'(\omega)} (e^{i\Sigma_3 \vartheta} \Phi)^* + \frac{\langle \alpha^\sharp, e^{i\Sigma_3 \vartheta} \Sigma_3 \Phi \rangle}{q'(\omega)} (e^{i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi)^*$$

$$+ \sum_{j=1}^n \varepsilon_j \langle \alpha^\sharp, e^{i\Sigma_3 \vartheta} \xi_j \rangle (e^{i\Sigma_3 \vartheta} \Sigma_3 \xi_j)^* - \sum_{j=1}^n \varepsilon_j \langle \alpha^\sharp, e^{i\Sigma_3 \vartheta} \Sigma_1 C \xi_j \rangle (e^{i\Sigma_3 \vartheta} \Sigma_3 \Sigma_1 C \xi_j)^*$$

$$+ e^{-i\Sigma_3 \vartheta} (P_c(\mathcal{H}_\omega^*) e^{-i\Sigma_3 \vartheta} (\alpha^\sharp)^*)^*.$$

**2.3.2. Modulation.** Consider the  $U$  in (2.6). Then, in the notation of (2.6), (2.1) can be written as

$$(2.28) \quad U = e^{i\Sigma_3 \vartheta} (\Phi_\omega + R).$$

Consider the following two functions

$$\mathcal{F}(U, \omega, \vartheta) := \langle e^{-i\Sigma_3 \vartheta} U - \Phi_\omega, \Phi_\omega^* \rangle, \quad \mathcal{G}(U, \omega, \vartheta) := \langle e^{-i\Sigma_3 \vartheta} U, \Sigma_3 \partial_\omega \Phi_\omega^* \rangle.$$

Notice that  $R \in N_g^\perp(\mathcal{H}_\omega^*)$  if and only if  $\mathcal{F}(U, \omega, \vartheta) = \mathcal{G}(U, \omega, \vartheta) = 0$ . By **(H:2)** the map  $\omega \in \mathcal{O} \rightarrow \phi_\omega \in H^1(\mathbb{R}^3)$  is  $C^\infty$ . Then  $\mathcal{F}$  and  $\mathcal{G}$  are  $C^\infty$  functions with

$$(2.29) \quad \begin{aligned} \mathcal{F}_\vartheta(U, \omega, \vartheta) &= -i \langle \Sigma_3 e^{-i\Sigma_3 \vartheta} U, \Phi_\omega^* \rangle \\ \mathcal{F}_\omega(U, \omega, \vartheta) &= -2q'(\omega) + \langle e^{-i\Sigma_3 \vartheta} U, \partial_\omega \Phi_\omega^* \rangle, \\ \nabla_U \mathcal{F}(U, \omega, \vartheta) &= e^{-i\Sigma_3 \vartheta} \Phi_\omega^*, \quad \nabla_U \mathcal{G}(U, \omega, \vartheta) = e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi_\omega^* \\ \mathcal{G}_\vartheta(U, \omega, \vartheta) &= -i \langle e^{-i\Sigma_3 \vartheta} U, \partial_\omega \Phi_\omega^* \rangle \\ \mathcal{G}_\omega(U, \omega, \vartheta) &= \langle e^{-i\Sigma_3 \vartheta} U, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle. \end{aligned}$$

We have  $\mathcal{F}(e^{i\Sigma_3 \vartheta} \Phi_\omega, \omega, \vartheta) = \mathcal{G}(e^{i\Sigma_3 \vartheta} \Phi_\omega, \omega, \vartheta) = 0$ . For  $U = e^{i\Sigma_3 \vartheta} \Phi_\omega$  in (2.29) we get

$$(2.30) \quad \begin{aligned} \mathcal{F}_\vartheta(e^{i\Sigma_3 \vartheta} \Phi_\omega, \omega, \vartheta) &= 0 \\ \mathcal{F}_\omega(U, \omega, \vartheta) &= -q'(\omega), \\ \mathcal{G}_\vartheta(e^{i\Sigma_3 \vartheta} \Phi_\omega, \omega, \vartheta) &= -iq'(\omega) \\ \mathcal{G}_\omega(e^{i\Sigma_3 \vartheta} \Phi_\omega, \omega, \vartheta) &= 0. \end{aligned}$$

Then by the implicit function theorem and **(H:3)** there is a unique choice of functions  $\theta = \theta(U)$ ,  $\omega = \omega(U)$  which are  $C^\infty$  and yield to the following Lemma.

**Lemma 2.9** (Modulation). *For any  $\omega_1 \in \mathcal{O}$  there exist  $\varepsilon > 0$  and  $C > 0$  such that for any  $u \in H^1(\mathbb{R}^3)$  with  $\|u - e^{i\vartheta_1} \phi_{\omega_1}\| < \varepsilon < \varepsilon$ , there exists a unique choice of  $(\vartheta, \omega, r)$  such that  $|\omega - \omega_1| + |\vartheta - \vartheta_1| < C\varepsilon$  for a fixed  $C$  and  $R \in N_g^\perp(\mathcal{H}_\omega^*)$ .*

Consider now the two  $C^\infty$  functions  $\vartheta, \omega : U \in B_{H^1}(e^{i\Sigma_3\vartheta_0} \Phi_{\omega_1}, \varepsilon) \rightarrow \mathbb{R}$ . Inserting (2.28) in (2.29) we get

$$\begin{aligned} \mathcal{F}_\vartheta &= -i\langle \Sigma_3 R, \Phi_\omega^* \rangle; & \mathcal{F}_\omega &= -q'(\omega) + \langle R, \partial_\omega \Phi_\omega^* \rangle; \\ \nabla_U \mathcal{F} &= e^{-i\Sigma_3\vartheta} \Phi_\omega^*; & \nabla_U \mathcal{G} &= e^{-i\Sigma_3\vartheta} \Sigma_3 \partial_\omega \Phi_\omega^*; \\ \mathcal{G}_\vartheta &= -i(q'(\omega) + \langle R, \partial_\omega \Phi_\omega^* \rangle); & \mathcal{G}_\omega &= \langle R, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle. \end{aligned}$$

Then, if we set

$$(2.31) \quad \mathcal{A} := \begin{pmatrix} -q'(\omega) + \langle R, \partial_\omega \Phi_\omega^* \rangle & -i\langle \Sigma_3 R, \Phi_\omega^* \rangle \\ \langle R, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle & -i(q'(\omega) + \langle R, \partial_\omega \Phi_\omega^* \rangle) \end{pmatrix}$$

we have the following equality

$$(2.32) \quad \mathcal{A} \begin{pmatrix} \nabla \omega \\ \nabla \vartheta \end{pmatrix} = \begin{pmatrix} -e^{-i\Sigma_3\vartheta} \Phi_\omega^* \\ -e^{-i\Sigma_3\vartheta} \Sigma_3 \partial_\omega \Phi_\omega^* \end{pmatrix}.$$

This yields the following lemma.

**Lemma 2.10.** *We have the following formulas:*

$$(2.33) \quad \begin{aligned} \nabla \omega &= \frac{(q'(\omega) + \langle R, \partial_\omega \Phi_\omega^* \rangle) (e^{i\Sigma_3\vartheta} \Phi)^* - \langle \Sigma_3 R, \Phi_\omega^* \rangle (e^{i\Sigma_3\vartheta} \Sigma_3 \partial_\omega \Phi)^*}{(q'(\omega))^2 - \langle R, \partial_\omega \Phi_\omega^* \rangle^2 + \langle \Sigma_3 R, \Phi_\omega^* \rangle \langle R, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle} \\ \nabla \vartheta &= \frac{\langle R, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle (e^{i\Sigma_3\vartheta} \Phi)^* + (q'(\omega) - \langle R, \partial_\omega \Phi_\omega^* \rangle) (e^{i\Sigma_3\vartheta} \Sigma_3 \partial_\omega \Phi)^*}{i[q'(\omega))^2 - \langle R, \partial_\omega \Phi_\omega^* \rangle^2 + \langle \Sigma_3 R, \Phi_\omega^* \rangle \langle R, \Sigma_3 \partial_\omega^2 \Phi_\omega^* \rangle}. \end{aligned}$$

**2.3.3. Coordinates.** For  $\omega \in \mathcal{O}$  we consider decomposition (2.24). By  $P_c(\mathcal{H}_\omega)$  (resp.  $P_d(\mathcal{H}_\omega)$ ), or simply by  $P_c(\omega)$  (resp.  $P_d(\omega)$ ), we denote the projection on  $L_c^2(\mathcal{H}_\omega)$  (resp.  $L_d^2(\mathcal{H}_\omega)$ ). The space  $L_c^2(\mathcal{H}_\omega)$  "depends continuously" on  $\omega$ , as  $P_c(\omega) = 1 - P_d(\omega)$  depends smoothly on  $\omega$ .

By Lemma 2.9 we specify the ansatz imposing that

$$(2.34) \quad U = e^{i\Sigma_3\vartheta} (\Phi_\omega + R) \text{ with } \omega \in \mathcal{O}, \vartheta \in \mathbb{R} \text{ and } R \in N_g^\perp(\mathcal{H}_\omega^*).$$

Fix  $\omega_0$ , where  $q(\omega_0) = \|u_0\|_{L^2}^2$ . For  $\omega$  close to  $\omega_0$  the map  $P_c(\mathcal{H}_\omega)$  is an isomorphism from  $L_c^2(\mathcal{H}_{\omega_0})$  to  $L_c^2(\mathcal{H}_\omega)$ . In particular we write

$$(2.35) \quad N_g^\perp(\mathcal{H}_\omega^*) \ni R = \sum_{j=1}^n z_j \xi_j(\omega) + \sum_{j=1}^n \bar{z}_j \Sigma_1 C \xi_j(\omega) + P_c(\mathcal{H}_\omega) f, \quad f \in L_c^2(\mathcal{H}_{\omega_0}).$$

Setting  $z \cdot \xi = \sum_{j=1}^n z_j \xi_j$  and  $\bar{z} \cdot \Sigma_1 C \xi = \sum_{j=1}^n \bar{z}_j \Sigma_1 C \xi_j$ , we write

$$(2.36) \quad U = e^{i\Sigma_3\vartheta} (\Phi_\omega + z \cdot \xi(\omega) + \bar{z} \cdot \Sigma_1 C \xi(\omega) + P_c(\mathcal{H}_\omega) f)$$

$\omega \in \mathcal{O}$  close to  $\omega_0$ ,  $(z, f) \in \mathbb{C}^n \times L_c^2(\mathcal{H}_{\omega_0})$  close to 0, are our coordinates. In the sequel, we set

$$(2.37) \quad \partial_\omega R := \sum_{j=1}^n z_j \partial_\omega \xi_j(\omega) + \sum_{j=1}^n \bar{z}_j \Sigma_1 C \partial_\omega \xi_j(\omega) + \partial_\omega P_c(\mathcal{H}_\omega) f.$$

Then we have the vector fields

$$(2.38) \quad \begin{aligned} \frac{\partial}{\partial \omega} &= e^{i\Sigma_3\vartheta} \partial_\omega (\Phi + R), & \frac{\partial}{\partial \vartheta} &= i e^{i\Sigma_3\vartheta} \Sigma_3 (\Phi + R), \\ \frac{\partial}{\partial z_j} &= e^{i\Sigma_3\vartheta} \xi_j, & \frac{\partial}{\partial \bar{z}_j} &= e^{i\Sigma_3\vartheta} \Sigma_1 C \xi_j. \end{aligned}$$

In particular, given a scalar function  $F$ , we have

$$(2.39) \quad \begin{aligned} \partial_\omega F &= \langle \nabla F, e^{i\Sigma_3\vartheta} \partial_\omega (\Phi + R) \rangle, & \partial_\vartheta F &= i \langle \nabla F, e^{i\Sigma_3\vartheta} \Sigma_3 (\Phi + R) \rangle, \\ \partial_{z_j} F &= \langle \nabla F, e^{i\Sigma_3\vartheta} \xi_j \rangle, & \partial_{\bar{z}_j} F &= \langle \nabla F, e^{i\Sigma_3\vartheta} \Sigma_1 C \xi_j \rangle, \end{aligned}$$

where by definition, given a vector field  $X$ , we have  $XF = \langle \nabla F, X \rangle = dF(X)$ , with  $dF$  the exterior differential and  $\nabla F$  the gradient.

**Lemma 2.11.** *We have the following formulas:*

$$(2.40) \quad \begin{aligned} \varepsilon_j \nabla z_j &= -\langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle \nabla \omega - i \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle \nabla \vartheta + e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^* \\ \varepsilon_j \nabla \bar{z}_j &= -\langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \partial_\omega R \rangle \nabla \omega - i \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \Sigma_3 R \rangle \nabla \vartheta + e^{-i\Sigma_3 \vartheta} \Sigma_1 \Sigma_3 (C \xi_j)^* \\ f'(U) &= (P_c(\omega) P_c(\omega_0))^{-1} P_c(\omega) [-\partial_\omega R d\omega - i \Sigma_3 R d\vartheta + e^{-i\Sigma_3 \vartheta} \mathbb{1}], \end{aligned}$$

with  $(P_c(\omega) P_c(\omega_0))^{-1} : L_c^2(\mathcal{H}_\omega) \rightarrow L_c^2(\mathcal{H}_{\omega_0})$  the inverse of  $P_c(\omega) P_c(\omega_0) : L_c^2(\mathcal{H}_{\omega_0}) \rightarrow L_c^2(\mathcal{H}_\omega)$  and  $\varepsilon_j = \langle \xi_j, \Sigma_3 \xi_j \rangle$ .

*Proof.* The proof is similar to the proof of [Cuc10, Lemmas 4.1–4.2]. Let us see for example the proof of the first formula. Equalities  $\frac{\partial z_j}{\partial z_\ell} = \delta_{j\ell}$ ,  $\frac{\partial z_j}{\partial \bar{z}_\ell} = \frac{\partial z_j}{\partial \omega} = \frac{\partial z_j}{\partial \vartheta} = 0$  and  $\nabla_f z_j = 0$  are equivalent to

$$(2.41) \quad \begin{aligned} \langle \nabla z_j, e^{i\Sigma_3 \vartheta} \xi_\ell \rangle &= \delta_{j\ell}, \langle \nabla z_j, e^{i\Sigma_3 \vartheta} \Sigma_1 C \xi_\ell \rangle \equiv 0 = \langle \nabla z_j, e^{i\Sigma_3 \vartheta} \Sigma_3 (\Phi + R) \rangle \\ \langle \nabla z_j, e^{i\Sigma_3 \vartheta} \partial_\omega (\Phi + R) \rangle &= 0 \equiv \langle \nabla z_j, e^{i\Sigma_3 \vartheta} P_c(\omega) P_c(\omega_0) g \rangle \forall g \in L_c^2(\mathcal{H}_{\omega_0}). \end{aligned}$$

Notice that the last identity implies  $P_c(\mathcal{H}_{\omega_0}^*) P_c(\mathcal{H}_\omega^*) e^{i\Sigma_3 \vartheta} \nabla z_j = 0$  which in turn implies  $P_c(\mathcal{H}_\omega^*) e^{i\Sigma_3 \vartheta} \nabla z_j = 0$ . Then, applying (2.25) and using the product row column, we get for some pair of numbers  $(a, b)$

$$\begin{aligned} \nabla z_j &= a e^{-i\Sigma_3 \vartheta} \Phi^* + b e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi^* + \varepsilon_j e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^* \\ &= (a, b) \begin{pmatrix} e^{-i\Sigma_3 \vartheta} \Phi^* \\ e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi^* \end{pmatrix} + \varepsilon_j e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^* = -(a, b) \mathcal{A} \begin{pmatrix} \nabla \omega \\ \nabla \vartheta \end{pmatrix} + \varepsilon_j e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*, \end{aligned}$$

where in the last line we used (2.32). Equating the two extreme sides and applying to the formula  $\langle \cdot, \frac{\partial}{\partial \omega} \rangle$  and  $\langle \cdot, \frac{\partial}{\partial \vartheta} \rangle$ , by  $\langle \nabla z_j, \frac{\partial}{\partial \omega} \rangle = \langle \nabla z_j, \frac{\partial}{\partial \vartheta} \rangle = \langle \nabla \vartheta, \frac{\partial}{\partial \omega} \rangle = \langle \nabla \omega, \frac{\partial}{\partial \vartheta} \rangle = 0$ , by  $\langle \nabla \vartheta, \frac{\partial}{\partial \vartheta} \rangle = \langle \nabla \omega, \frac{\partial}{\partial \vartheta} \rangle = 1$  and by (2.38) and (2.41), we get

$$\mathcal{A}^* \begin{pmatrix} a \\ b \end{pmatrix} = \varepsilon_j \begin{pmatrix} \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle \\ i \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle \end{pmatrix}.$$

This implies

$$\nabla z_j = -\varepsilon_j (\langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle, i \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle) \begin{pmatrix} \nabla \omega \\ \nabla \vartheta \end{pmatrix} + \varepsilon_j e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*.$$

□

**2.4. Smoothing and dispersive estimates.** In this subsection we collect the statements on linear theory needed later to prove the nonlinear estimates.

**Lemma 2.12.** *The following facts are true.*

(i) *For any  $\tau \geq 1$  there exists  $C$  independent of  $\omega$  s.t.*

$$(2.42) \quad \|R_{D_m}(z)\psi\|_{L^{2,-\tau}} \leq C \|\psi\|_{L^{2,\tau}} \text{ for all } z \notin \mathbb{R}$$

$$(2.43) \quad \|R_{\mathcal{H}_{\omega,0}}(z)\psi\|_{L^{2,-\tau}} \leq C \|\psi\|_{L^{2,\tau}} \text{ for all } z \notin \mathbb{R}.$$

(ii) *For any  $\tau > 1$  the following limits*

$$(2.44) \quad R_{D_m}^+(\lambda) = \lim_{\varepsilon \searrow 0} R_{D_m}(\lambda \pm i\varepsilon) \text{ and } R_{\mathcal{H}_{\omega,0}}^+(\lambda) = \lim_{\varepsilon \searrow 0} R_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon)$$

*exist in  $B(H_x^{1,\tau}, L_x^{2,-\tau})$  and the convergence is uniform for  $\lambda$  in compact sets.*

*Proof.* Estimate (2.42) implies (2.43). Then (i) is the content of [IM99, Theorem 2.1] while (ii) is contained in [GM01, Theorem 1.6]. □

**Lemma 2.13.** *We have  $R_{\mathcal{H}_{\omega,0}}(x, y, \lambda) = R_{\mathcal{H}_{\omega,0}}(x - y, \lambda) = \begin{pmatrix} R_{D_m}(x - y, \lambda + \omega) & 0 \\ 0 & R_{D_m}(x - y, \lambda - \omega) \end{pmatrix}$*

*for  $\lambda \notin \sigma(\mathcal{H}_{\omega,0})$  with*

$$(2.45) \quad R_{D_m}(x, \Lambda) = \begin{pmatrix} (\Lambda + m)I_2 & i\sqrt{m^2 - \Lambda^2} \sigma \cdot \hat{x} \\ i\sqrt{m^2 - \Lambda^2} \sigma \cdot \hat{x} & (\Lambda - m)I_2 \end{pmatrix} \frac{e^{-\sqrt{m^2 - \Lambda^2}|x|}}{4\pi|x|} + i \frac{\alpha \cdot \hat{x}}{4\pi|x|^2} e^{-\sqrt{m^2 - \Lambda^2}|x|}$$

*where  $\hat{x} = x/|x|$  and where for  $\zeta = e^{i\vartheta} r$  with  $r \geq 0$  and  $\vartheta \in (-\pi, \pi)$  we set  $\sqrt{\zeta} = e^{i\vartheta/2} \sqrt{r}$ .*

*Proof.* This is [Tha92, Identity (1.263) section 1.E]. □

*Remark 2.14.*  $R_{D_m}^+(x, \Lambda)$  for  $\Lambda > m$  (resp.  $\Lambda < -m$ ) is obtained substituting  $\sqrt{m^2 - \Lambda^2}$  in (2.45) with  $-i\sqrt{\Lambda^2 - m^2} = \lim_{\varepsilon \searrow 0} \sqrt{m^2 - (\Lambda + i\varepsilon)^2}$  (resp.  $i\sqrt{\Lambda^2 - m^2} = \lim_{\varepsilon \searrow 0} \sqrt{m^2 - (\Lambda + i\varepsilon)^2}$ ).

**Theorem 2.15.** *For any  $\tau > 1$  and  $k \in \mathbb{R} \exists C$  s.t.*

$$(2.46) \quad \begin{aligned} & \|e^{-itD_m}\psi\|_{L_t^2(\mathbb{R}, H^{k, -\tau})} \leq C\|\psi\|_{H^k}, \\ & \left\| \int_{\mathbb{R}} e^{itD_m} F(t) dt \right\|_{H^k} \leq C\|F\|_{L_t^2(\mathbb{R}, H^{k, \tau})}, \\ & \left\| \int_{t' < t} e^{-i(t-t')D_m} F(t') dt' \right\|_{L_t^2(\mathbb{R}, H^{k, -\tau})} \leq C\|F\|_{L_t^2(\mathbb{R}, H^{k, \tau})}. \end{aligned}$$

The same estimates with the same constants hold when we replace  $D_m$  with  $\mathcal{H}_{\omega, 0}$ .

*Proof.* This is [Bou08b, Theorem 1.1] in the free case. But can be easily deduced from Lemma 2.12 using tools in [RS78, Section XIII.7].  $\square$

The following theorem is a special case of Theorem 1.1 [Bou06].

**Theorem 2.16.** *For any  $\tau > 5/2$  and  $k \in \mathbb{R} \exists C$  s.t.*

$$(2.47) \quad \|e^{-itD_m}\psi\|_{H^{k, -\tau}(\mathbb{R}^3)} \leq C\langle t \rangle^{-\frac{3}{2}}\|\psi\|_{H^{k, \tau}},$$

The same estimates with the same constants hold when we replace  $D_m$  with  $\mathcal{H}_{\omega, 0}$ .

**Theorem 2.17.** *For any  $2 \leq p, q \leq \infty$ ,  $\theta \in [0, 1]$ , with  $(1 - \frac{\theta}{2})(1 \pm \frac{\theta}{2}) = \frac{2}{p}$  and  $(p, \theta) \neq (2, 0)$ , and for any reals  $k, k'$  with  $k' - k \geq \alpha(q)$ , where  $\alpha(q) = (1 + \frac{\theta}{2})(1 - \frac{\theta}{2})$ , there exists a positive constant  $C$  such that*

$$(2.48) \quad \begin{aligned} & \|e^{-itD_m}\psi\|_{L_t^p(\mathbb{R}, B_{q, 2}^k(\mathbb{R}^3, \mathbb{C}^4))} \leq C\|\psi\|_{H^{k'}(\mathbb{R}^3, \mathbb{C}^4)}, \\ & \left\| \int_{\mathbb{R}} e^{itD_m} F(t) dt \right\|_{H^k} \leq C\|F\|_{L_t^{p'}(\mathbb{R}, B_{q', 2}^{k'}(\mathbb{R}^3, \mathbb{C}^4))}, \\ & \left\| \int_{t' < t} e^{-i(t-t')D_m} F(t') dt' \right\|_{L_t^p(\mathbb{R}, B_{q, 2}^k(\mathbb{R}^3, \mathbb{C}^4))} \leq C\|F\|_{L_t^{a'}(\mathbb{R}, B_{b', 2}^k(\mathbb{R}^3, \mathbb{C}^4))}, \end{aligned}$$

for any  $(a, b)$  chosen like  $(p, q)$ , and  $h - k \geq \alpha(q) + \alpha(b)$ . Exactly the same estimates hold with  $D_m$  replaced by  $\mathcal{H}_{\omega, 0}$ .

*Proof.* For  $D_m$  see [Bou08b], see also [Bre84] for the Klein-Gordon case. For  $\mathcal{H}_{\omega, 0}$  the statement is an immediate consequence of the case  $D_m$ .  $\square$

**Lemma 2.18.** *Consider pairs  $(p, q)$  as in Theorem 2.17 with  $p > 2$ ,  $k \in \mathbb{R}$  arbitrary and  $k' - k \geq \alpha(q)$ . Then for any  $\tau > 1$  there is a constant  $C_0 = C_0(\tau, k, p, q)$  such that*

$$(2.49) \quad \left\| \int_0^t e^{iD_m(t-t')} F(t') dt' \right\|_{L_t^p B_{q, 2}^k} \leq C_0 \|F\|_{L_t^2 H^{k', \tau}}.$$

The same estimates hold with  $D_m$  replaced by  $\mathcal{H}_{\omega, 0}$

*Proof.* For  $F(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$  set

$$TF(t) = \int_0^{+\infty} e^{i(t'-t)D_m} F(t') dt', \quad f = \int_0^{+\infty} e^{it'D_m} F(t') dt'.$$

Theorem 2.17 implies  $\|TF\|_{L_t^p B_{q, 2}^k} \leq \|f\|_{H^{k'}}$  for  $k' - k = \alpha(q)$ . By Theorem 2.15 we have  $\|f\|_{H^{k'}} \leq C\|F\|_{L_t^2 H^{k', \tau}}$ . Since  $p > 2$ , by a well known lemma due to Christ and Kiselev [CK01], see Lemma 3.1 [SS00], the statement of Lemma 2.18 follows.  $\square$

**Lemma 2.19.** *Let  $\tau_1 > 1$  and  $\mathcal{K}$  be a compact subset of  $\mathcal{O}$  and let  $I$  be a compact subset of  $\sigma_\varepsilon(\mathcal{H}_\omega) \setminus \{\pm(m \pm \omega)\}$ . Assume that  $\omega \rightarrow V_\omega$  is continuous with values in the Schwartz space  $\mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$ . Assume furthermore that for any  $\omega \in \mathcal{O}$  there are no eigenvalues of  $\mathcal{H}_\omega$  in the continuous spectrum and the points  $\pm\omega$  are not resonances. Then there exists a  $C > 0$ , such that*

$$(2.50) \quad \|e^{-it\mathcal{H}_{\omega, 0}} R_{\mathcal{H}_\omega}^+(\lambda) P_c(\omega) \psi_0\|_{L^{2, -\tau_1}(\mathbb{R}^3)} \leq C\langle t \rangle^{-\frac{3}{2}} \|\psi_0\|_{L^{2, \tau_1+1}(\mathbb{R}^3)}$$

for every  $t \geq 0$ ,  $\lambda \in I$ ,  $\omega \in \mathcal{K}$  and  $\psi_0 \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^2)$ .

*Proof.* We expand  $R_{\mathcal{H}_\omega}^+(\lambda) = R_{\mathcal{H}_{\omega,0}}^+(\lambda) - R_{\mathcal{H}_{\omega,0}}^+(\lambda)V_\omega R_{\mathcal{H}_\omega}^+(\lambda)$ . We have from [BG87, Theorem 2 ]

$$(2.51) \quad \|e^{-it\mathcal{H}_{\omega,0}}R_{\mathcal{H}_{\omega,0}}^+(\lambda)\psi_0\|_{L^{2,-\tau_1}(\mathbb{R}^3)} \leq C\langle t \rangle^{-\frac{3}{2}}\|R_{\mathcal{H}_{\omega,0}}^+(\lambda)\psi_0\|_{L^{2,\tau_1}(\mathbb{R}^3)} \leq C_1\langle t \rangle^{-\frac{3}{2}}\|\psi_0\|_{L^{2,\tau_1+1}(\mathbb{R}^3)},$$

with  $C_1$  locally bounded in  $\lambda$  and  $\tau_1$ . Hence, by exponential decay of  $\phi_\omega$ ,

$$\begin{aligned} & \|e^{-it\mathcal{H}_{\omega,0}}R_{\mathcal{H}_{\omega,0}}^+(\lambda)V_\omega R_{\mathcal{H}_\omega}^+(\lambda)P_c(\omega)\psi_0\|_{L^{2,\tau_1}} \\ & \leq C_1\langle t \rangle^{-\frac{3}{2}}\|V_\omega\|_{L^{2,-\tau_1},L^{2,\tau_1+1}}\|R_{\mathcal{H}_\omega}^+(\lambda)P_c(\omega)\|_{L^{2,\tau_1},L^{2,-\tau_1}}\|\psi_0\|_{L^{2,\tau_1}}. \end{aligned}$$

The key fact that  $\|R_{\mathcal{H}_\omega}^+(\lambda)\|_{L^{2,\tau_1},L^{2,-\tau_1}} \leq C$  for a fixed  $C > 0$  and for all  $\lambda \in I$  is a consequence of (A.5), of Lemma A.3 and of Lemma 2.12.  $\square$

### 3. HAMILTONIAN STRUCTURE

The discussion in this section is essentially adapted from [Cuc10], rewritten in the context of the Dirac systems.

**3.1. Symplectic structure.** We recall that in view of Theorem 1.8 we set  $\varepsilon_j = \langle \xi_j, \Sigma_3 \xi_j^* \rangle$  where  $\varepsilon_j \in \{1, -1\}$ . Notice that in Theorem 1.1 and in [Cuc10], we have  $\varepsilon_j \equiv 1$ .

Our ambient space is  $H^1(\mathbb{R}^3, \mathbb{C}^4) \times H^1(\mathbb{R}^3, \mathbb{C}^4)$ . We focus only on the subspace formed by the points which satisfy  $\Sigma_1 U = CU$ . In view of (2.11), the natural symplectic structure for our problem is

$$(3.1) \quad \Omega(X, Y) = \langle X, i\beta\alpha_2\Sigma_1\Sigma_3Y \rangle.$$

The Hamiltonian vector field  $X_G$  of a scalar function  $G$  is defined by the equation  $\Omega(X_G, Y) = -i\langle \nabla G, Y \rangle$  for any vector  $Y$  and is  $X_G = \beta\alpha_2\Sigma_3\Sigma_1\nabla G$ .

We call Poisson bracket of a pair of scalar valued functions  $F$  and  $G$  the scalar valued function

$$(3.2) \quad \{F, G\} = \langle \nabla F, X_G \rangle = i\Omega(X_F, X_G) = i\Omega(\nabla F, \nabla G).$$

This can be extended to vector valued function using 1-forms or equivalently defining the extension the following way.

**Definition 3.1.** Given a function  $\mathcal{G}(U)$  with values in  $L_c^2(\mathcal{H}_{\omega_0})$ , a symplectic form  $\Omega$  and a scalar function  $F(U)$ , we define  $\{\mathcal{G}, F\} = \mathcal{G}'(U)X_F(U)$ , with  $X_F$  the Hamiltonian vector field associated to  $F$ . We set  $\{F, \mathcal{G}\} := -\{\mathcal{G}, F\}$ .

**Lemma 3.2.** *Let  $Q$  be the function defined by (2.10) and let  $X_Q$  its Hamiltonian vectorfield of  $Q$ . Then*

$$(3.3) \quad X_Q = -\frac{\partial}{\partial \vartheta}.$$

*We have the following formulas :*

$$(3.4) \quad \{Q, \omega\} = 0, \quad \{Q, \vartheta\} = 1, \quad \{Q, z_j\} = \{Q, \bar{z}_j\} = 0, \quad \{Q, f\} = 0.$$

*Proof.* (3.4) follows from (3.3). The latter follows from (2.38):

$$(3.5) \quad X_Q := \beta\alpha_2\Sigma_3\Sigma_1\nabla Q = \beta\alpha_2\Sigma_3\Sigma_1i\beta\alpha_2\Sigma_1U = -i\Sigma_3U = -\frac{\partial}{\partial \vartheta}.$$

$\square$

**3.2. Hamiltonian reformulation of the system.** For any scalar function  $F$ , the time derivative of  $F(U(t))$  is  $\langle \nabla F(U), \dot{U} \rangle$  and thus if  $U$  satisfies (2.11) it is  $\{F, E\}$ . A similar identity holds for vector valued function and thus as in [Cuc10] we write our system as

$$(3.6) \quad \dot{\omega} = \{\omega, E\}, \quad \dot{f} = \{f, E\}, \quad \dot{z}_j = \{z_j, E\}, \quad \dot{\vartheta} = \{\vartheta, E\}.$$

For  $u_0$  the initial datum in (1.1), we introduce a new Hamiltonian for which the stationary solution  $\Phi_{\omega_0}$ , with  $q(\omega_0) = \|u_0\|_{L_x^2}^2$ , is a critical point :

$$(3.7) \quad K(U) = E(U) + \omega(U)Q(U) - \omega(U)\|u_0\|_{L_x^2}^2.$$

By Lemma 3.2 and since  $Q(U)$  is an invariant of the motion, see Lemma 2.3, the solution of the initial value problem in (1.1) solves also

$$(3.8) \quad \dot{\omega} = \{\omega, K\}, \quad \dot{f} = \{f, K\}, \quad \dot{z}_j = \{z_j, K\}, \quad \dot{\vartheta} - \omega = \{\vartheta, K\}.$$

By  $\frac{\partial}{\partial \vartheta} K = 0$  and (3.4) the right hand sides in the equations (3.8) do not depend on  $\vartheta$ . Hence, if we look at the new system

$$(3.9) \quad \dot{\omega} = \{\omega, K\}, \quad \dot{f} = \{f, K\}, \quad \dot{z}_j = \{z_j, K\}, \quad \dot{\vartheta} = \{\vartheta, K\},$$

the evolution of the crucial variables  $(\omega, z, f)$  in (3.6) and (3.9) is the same. Therefore, to prove Theorem 1.1 it is sufficient to consider system (3.9).

**3.3. Application of the Darboux Theorem.** In the sequel we will show that a resonances phenomena is responsible of energy leaking from discrete to continuous spectrum. The main ingredient will be the use of Birkhoff normal forms. Since the coordinates (2.36) are not canonical coordinates for the natural symplectic structure  $\Omega$  in (3.1), it is natural to apply Darboux theorem, moving to a different set of coordinates. It is natural and elementary to reduce as in Theorem 6.35 p. 412 [Ol93] the number of coordinates, using the invariance of  $Q$ . However one key issue is that we want our nonlinear Dirac equation to remain semilinear. Hence we follow the argument of [Cuc10, Section 7], which takes care of this, and to which we refer for more details.

*Strategy of the proof.* For  $q = q(\omega) = \|\phi_\omega\|_{L^2}^2$ , we introduce the 2-form

$$(3.10) \quad \Omega_0 = id\vartheta \wedge dq + \varepsilon_j dz_j \wedge d\bar{z}_j + \langle f'(U) \cdot, i\beta\alpha_2 \Sigma_3 \Sigma_1 f'(U) \cdot \rangle,$$

summing on repeated indexes, with  $f(U)$  the function in Lemma 2.11,  $f'(U)$  its Frechét derivative and the last term in (3.10) acting on pairs  $(X, Y)$  like  $\langle f'(U)X, i\beta\alpha_2 \Sigma_3 \Sigma_1 f'(U)Y \rangle$ .

The proof of the Darboux Theorem goes as follows. First consider

$$(3.11) \quad \Omega_\tau = (1 - \tau)\Omega_0 + \tau\Omega = \Omega_0 + \tau\tilde{\Omega} \text{ with } \tilde{\Omega} := \Omega - \Omega_0.$$

In Lemma 3.3, we check that  $\Omega_0(U) = \Omega(U)$  at  $U = e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$ . Then near  $e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$   $\Omega_\tau$  is non degenerate. One considers a differential form  $\gamma(\tau, U)$  such that  $d\gamma(\tau, U) = \tilde{\Omega}$  with  $\gamma(U) = 0$  at  $U = e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$  (external differentiation will always be on the  $U$  variable only) and the dual vector field  $\mathcal{Y}^\tau$  such that  $i_{\mathcal{Y}^\tau} \Omega_\tau = -\gamma$ . The flow  $\mathfrak{F}_\tau$  generated by  $\mathcal{Y}^\tau$ , close the points  $e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$  is defined up to time 1, and is such that  $\mathfrak{F}_1^* \Omega = \Omega_0$  by

$$(3.12) \quad \begin{aligned} \frac{d}{d\tau} (\mathfrak{F}_\tau^* \Omega_\tau) &= \mathfrak{F}_\tau^* (L_{\mathcal{Y}^\tau} \Omega_\tau) + \mathfrak{F}_\tau^* \frac{d}{d\tau} \Omega_\tau = \\ &= \mathfrak{F}_\tau^* d(i_{\mathcal{Y}^\tau} \Omega_\tau) + \mathfrak{F}_\tau^* \tilde{\Omega} = \mathfrak{F}_\tau^* (-d\gamma + \tilde{\Omega}) = 0. \end{aligned}$$

This procedure can be carried out abstractly. But here we need to be careful, choosing  $\gamma$  appropriately, because we want the new Hamiltonian  $\tilde{K} = K \circ \mathfrak{F}_1$  to be  $\vartheta$  invariant and yield a semilinear Dirac equation. It is interesting that the two issues are solved simultaneously.

In the sequel of this section all the work is finalized to the correct choice of  $\gamma$ . In Lemma 3.4 we compute explicitly a differential form  $\alpha$  and we make the preliminary choice  $\gamma = \alpha$ . This is not yet the right choice. By the computations in Lemma 3.5 we find the obstruction to the fact that  $\tilde{K}$  is of the desired type. Lemmas 3.7–3.10 are necessary to find an appropriate solution  $F$  of a differential equation in Lemma 3.11. Then  $\gamma = \alpha + idF$  is the right choice of  $\gamma$ . In Lemma 3.13 we collect a number of useful estimates for  $\mathfrak{F}_1$ . Lemma 3.14 is valid independently of the precise  $\gamma$  chosen and contains information necessary for the reformulation of our system in (3.70)–(3.71).

*Preliminary remarks.* Note that for  $U$  in a sufficiently small neighborhood of  $\Phi_\omega$ , that is  $R$  small, from (2.25) the vector fields defined in (2.38) can be completed into a basis of  $T_U L^2$  (tangent space at  $U$ ). For any vector  $Y \in T_U L^2$ , we have

$$(3.13) \quad Y = Y_\vartheta \frac{\partial}{\partial \vartheta} + Y_\omega \frac{\partial}{\partial \omega} + \sum Y_j \frac{\partial}{\partial z_j} + \sum Y_{\bar{j}} \frac{\partial}{\partial \bar{z}_j} + e^{i\Sigma_3 \vartheta} P_c(\omega) Y_f$$

and defining the dual basis we set

$$(3.14) \quad \begin{aligned} Y_\vartheta &= d\vartheta(Y), & Y_\omega &= d\omega(Y), & Y_j &= dz_j(Y) \\ Y_{\bar{j}} &= d\bar{z}_j(Y), & Y_f &= f'(U)Y. \end{aligned}$$

So similarly, a differential 1-form  $\gamma$  decomposes as

$$(3.15) \quad \gamma = \gamma^\vartheta d\vartheta + \gamma^\omega d\omega + \sum \gamma^j dz_j + \sum \gamma^{\bar{j}} d\bar{z}_j + \langle \gamma^f, f' \cdot \rangle,$$

where  $\langle \gamma^f, f' \cdot \rangle$  acts on a vector  $Y$  as  $\langle \gamma^f, f'Y \rangle$ , with here  $\gamma^f \in L_c^2(\mathcal{H}_{\omega_0}^*)$ ;  $\gamma^\vartheta, \gamma^\omega, \gamma^j$  and  $\gamma^{\bar{j}}$  are in  $\mathbb{C}$ .

Notice that we are reversing the standard notation on super and subscripts for forms and vector fields.

In the sequel, given a differential 1-form  $\gamma$  and a point  $U$ , we will denote by  $\gamma_U$  the value of  $\gamma$  at  $U$ .

Given a function  $\chi$ , denote its hamiltonian vector field with respect to  $\Omega_\tau$  by  $X_\chi^\tau : i_{X_\chi^\tau} \Omega_\tau = -i d\chi$ . By (3.10) we have :

$$(3.16) \quad X_{q(\omega)}^0 = -\frac{\partial}{\partial \vartheta}.$$

*The proof.* We have the following preliminary observation ensuring that  $\Omega_\tau$  is a non degenerate 2-form in a neighborhood of  $e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$ .

**Lemma 3.3.** *At  $U = e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$ , for any  $\vartheta$ , we have  $\Omega_0(U) = \Omega(U)$ .*

*Proof.* See also [Cuc10, Lemma 7.1]. Using (2.25) we get, summing on repeated indexes,

$$(3.17) \quad \begin{aligned} \Omega(X, Y) &= \langle X, i\beta\alpha_2\Sigma_3\Sigma_1 Y \rangle = \\ &= \frac{1}{q'} \langle \cdot, e^{-i\Sigma_3\vartheta}\Sigma_3\partial_\omega\Phi^* \rangle \wedge \langle \cdot, e^{-i\Sigma_3\vartheta}\Phi^* \rangle (X, Y) + \\ &+ \varepsilon_j \langle \cdot, e^{-i\Sigma_3\vartheta}\Sigma_3\xi_j^* \rangle \wedge \langle \cdot, e^{-i\Sigma_3\vartheta}\Sigma_1\Sigma_3(C\xi_j)^* \rangle (X, Y) \\ &+ \langle P_c(\mathcal{H}_\omega)e^{-i\Sigma_3\vartheta}X, i\beta\alpha_2\Sigma_3\Sigma_1P_c(\mathcal{H}_\omega)e^{-i\Sigma_3\vartheta}Y \rangle. \end{aligned}$$

Consider

$$(3.18) \quad a_1 = a_1(\omega, z, f) := \Omega(\partial_\omega\Phi_\omega, \Sigma_3R) + \Omega(R, \Sigma_3\Phi_\omega) + \Omega(R, \Sigma_3R)$$

where  $R = z \cdot \xi(\omega) + \bar{z} \cdot C\xi(\omega) + P_c(\omega)f$ . Notice that  $\Omega(R, \Sigma_3R) = -\langle \Sigma_3R, (CR)^* \rangle$  and as we focus on the kernel of  $1 - C\Sigma_1$  this is  $-\langle \Sigma_3R, R^* \rangle$  and thus 0. Hence the function  $a_1$  is smooth in the arguments  $\omega \in \mathcal{O}$ ,  $z \in \mathbb{C}^n$  and  $f \in H^{-K', -S'}$  (see (1.3) for the definition) for any pair  $(K', S')$  with, for  $(z, f)$  near 0,

$$(3.19) \quad |a_1| \leq C(K', S')(|z| + \|f\|_{H^{-K', -S'}})^2,$$

we get by Lemmas 2.10 and 2.11, summing on repeated indexes,

$$(3.20) \quad \begin{aligned} \Omega &= (iq' + a_1)d\vartheta \wedge d\omega + \varepsilon_j dz_j \wedge d\bar{z}_j \\ &+ dz_j \wedge (\langle \Sigma_1\Sigma_3(C\xi_j)^*, \partial_\omega R \rangle d\omega + i\langle \Sigma_1\Sigma_3(C\xi_j)^*, \Sigma_3R \rangle d\vartheta) \\ &- d\bar{z}_j \wedge (\langle \Sigma_3\xi_j^*, \partial_\omega R \rangle d\omega + i\langle \Sigma_3\xi_j^*, \Sigma_3R \rangle d\vartheta) + \\ &+ \langle P_c(\omega)P_c(\omega_0)f', i\beta\alpha_2\Sigma_3\Sigma_1P_c(\omega)P_c(\omega_0)f' \rangle + \\ &+ \langle P_c(\omega)P_c(\omega_0)f', i\beta\alpha_2\Sigma_3\Sigma_1P_c(\omega)\partial_\omega R \rangle \wedge d\omega + \\ &+ i\langle P_c(\omega)P_c(\omega_0)f', i\beta\alpha_2\Sigma_3\Sigma_1P_c(\omega)\Sigma_3R \rangle \wedge d\vartheta. \end{aligned}$$

At points  $U = e^{i\Sigma_3\vartheta}\Phi_\omega$ , that is for  $R = 0$ , we have

$$(3.21) \quad \Omega = id\vartheta \wedge dq + \varepsilon_j dz_j \wedge d\bar{z}_j + \langle P_c(\omega)P_c(\omega_0)f', i\beta\alpha_2\Sigma_3\Sigma_1P_c(\omega)P_c(\omega_0)f' \rangle.$$

which at  $\omega = \omega_0$  gives  $\Omega = \Omega_0$ . □

Since  $\Omega_\tau = \Omega_0 + \tau(\Omega - \Omega_0)$  with  $\tau \in [0, 1]$  and  $\Omega = \Omega_0$  at  $e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$ , and since  $\Omega_0$  is a non degenerate 2-form,  $\Omega_\tau$  is also non degenerate in a neighborhood of  $e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$ . Thus the map  $X \mapsto i_X\Omega_\tau$  from vector fields to 1-forms is bijective at any point in the neighborhood of  $e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$ . Notice that Lemma 3.3 is claimed at  $\omega_0$  and not at different standing waves, and that the  $e^{i\Sigma_3\vartheta}\Phi_{\omega_0}$  are the only stationary solutions preserved by our changes of coordinates.

The next lemma states as candidate for the 1 form  $\gamma$  the choice  $\gamma = -\alpha$ , for  $\alpha$  see below. This is not yet the final choice of  $\gamma$ .

**Lemma 3.4.** *Consider the forms, summing on repeated indexes,*

$$(3.22) \quad \begin{aligned} \varpi(U)Y &:= \frac{1}{2} \langle i\beta\alpha_2\Sigma_3\Sigma_1U, Y \rangle \\ \varpi_0(U) &:= -iqd\vartheta - \varepsilon_j \frac{\bar{z}_j dz_j - z_j d\bar{z}_j}{2} + \frac{1}{2} \langle f(U), i\beta\alpha_2\Sigma_3\Sigma_1f'(U) \rangle. \end{aligned}$$

Then

$$(3.23) \quad d\varpi_0 = \Omega_0, \quad d\varpi = \Omega.$$

Set

$$(3.24) \quad \alpha(U) := \varpi(U) - \varpi_0(U) + d\psi(U) \text{ where } \psi(U) := \frac{1}{2} \langle \Sigma_3\Phi^*, R \rangle.$$



We have  $\alpha = \alpha^\vartheta d\vartheta + \alpha^\omega d\omega + \langle \alpha^f, f' \rangle$  with

$$(3.25) \quad \begin{aligned} \alpha^\vartheta + \frac{1}{2} \|f\|_2^2 &= -\frac{1}{2} \|z \cdot \xi + \bar{z} \cdot \Sigma_1 C \xi\|_2^2 - i \Re(z \cdot \xi + \bar{z} \cdot \Sigma_1 C \xi, (P_c(\omega)f)^*) \\ &\quad - i \Re((P_c(\omega) - P_c(\omega_0))f, (P_c(\omega)f)^*), \\ \alpha^\omega &= -\frac{1}{2} \langle R^*, \Sigma_3 \partial_\omega R \rangle, \\ \alpha^f &= \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 P_c(\mathcal{H}_{\omega_0}) (P_c(\mathcal{H}_\omega) - P_c(\mathcal{H}_{\omega_0})) f. \end{aligned}$$

*Proof.* Here the proof is almost the same of [Cuc10, Lemma 7.2]. We focus on (3.25), which is the only nontrivial statement. We will sum over repeated indexes. We have

$$(3.26) \quad \begin{aligned} \varpi &= \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \cdot \rangle + \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 P_c(\omega) f, \cdot \rangle \\ &\quad + \frac{1}{2} z_j \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 \xi_j, \cdot \rangle - \frac{1}{2} \bar{z}_j \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_3 C \xi_j, \cdot \rangle. \end{aligned}$$

By Lemma 2.8 and summing on repeated indexes we obtain

$$(3.27) \quad \begin{aligned} \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \cdot \rangle &= \frac{\langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \partial_\omega \Phi \rangle}{q'(\omega)} \langle e^{-i\Sigma_3 \vartheta} \Phi^*, \cdot \rangle \\ &\quad + \frac{\langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \Sigma_3 \Phi \rangle}{q'(\omega)} \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi^*, \cdot \rangle + \varepsilon_j \langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*, \cdot \rangle \\ &\quad - \varepsilon_j \langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \Sigma_1 C \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \Sigma_1 (C \xi_j)^*, \cdot \rangle + \langle e^{-i\Sigma_3 \vartheta} (P_c(\mathcal{H}_\omega^*) \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi^*)^*, \cdot \rangle. \end{aligned}$$

By  $i \beta \alpha_2 \Sigma_1 \Phi = i \beta \alpha_2 C \Phi = (i \beta \alpha_2)^2 \Phi^* = \Phi^*$  we have

$$(3.28) \quad \langle i \beta \alpha_2 \Sigma_3 \Sigma_1 \Phi, \partial_\omega \Phi \rangle = \langle \phi^*, \partial_\omega \phi \rangle - \langle \phi, \partial_\omega \phi^* \rangle = 0,$$

by  $\langle \phi, \partial_\omega \phi^* \rangle = \int_{\mathbb{R}^3} (a \partial_\omega a + b \partial_\omega b) dx = \langle \phi^*, \partial_\omega \phi \rangle$ , see **(H:2)**. Then

$$(3.29) \quad \begin{aligned} \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \cdot \rangle &= \\ &\quad - \frac{q}{q'} \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi^*, \cdot \rangle + \varepsilon_j \langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*, \cdot \rangle \\ &\quad - \varepsilon_j \langle \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi, \Sigma_1 C \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \Sigma_1 (C \xi_j)^*, \cdot \rangle + \langle e^{-i\Sigma_3 \vartheta} (P_c(\mathcal{H}_\omega^*) \frac{1}{2} i \beta \alpha_2 \Sigma_1 \Sigma_3 \Phi^*)^*, \cdot \rangle. \end{aligned}$$

with by (2.32)

$$(3.30) \quad -\frac{q}{q'} \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \partial_\omega \Phi^*, \cdot \rangle = \frac{q}{q'} \langle R, \Sigma_3 \partial_\omega^2 \Phi^* \rangle d\omega - i \frac{q}{q'} (q' + \langle R, \partial_\omega \Phi^* \rangle) d\vartheta.$$

Applying Lemma 2.11, we get (by  $i \beta \alpha_2 \Sigma_1 f = f^*$  which follows from  $\Sigma_1 U = CU$ )

$$(3.31) \quad \begin{aligned} \varpi_0 &= -iq d\vartheta - \varepsilon_j \frac{\bar{z}_j dz_j - z_j d\bar{z}_j}{2} + \frac{1}{2} \langle f(U), i \beta \alpha_2 \Sigma_3 \Sigma_1 f'(U) \cdot \rangle \\ &= i \left( -q + \frac{1}{2} \|R\|_{L^2}^2 \right) d\vartheta + \frac{1}{2} \langle \Sigma_3 R^*, \partial_\omega R \rangle d\omega + \frac{1}{2} \langle i \beta \alpha_2 \Sigma_1 \Sigma_3 (1 - P_c(\omega_0)) P_c(\omega) f, f' \rangle + \\ &\quad + \frac{1}{2} z_j \langle e^{-i\Sigma_3 \vartheta} \Sigma_1 \Sigma_3 (C \xi_j)^*, \cdot \rangle - \frac{1}{2} \bar{z}_j \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*, \cdot \rangle + \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} i \beta \alpha_2 \Sigma_1 \Sigma_3 P_c(\omega) f, \cdot \rangle. \end{aligned}$$

By (2.36) we have

$$(3.32) \quad d\psi = \frac{1}{2} \langle \Sigma_3 \Phi^*, \partial_\omega R \rangle d\omega + \frac{1}{2} \langle \Sigma_3 \Phi^*, \xi_j \rangle dz_j + \frac{1}{2} \langle \Sigma_3 \Phi^*, \Sigma_1 C \xi_j \rangle d\bar{z}_j + \frac{1}{2} \langle \Sigma_3 \Phi^*, P_c(\omega) f' \cdot \rangle.$$

Applying to (3.32) Lemma 2.11 and the identities (3.34) below, we get  $d\psi =$

$$\begin{aligned}
(3.33) \quad d\psi &= \frac{1}{2} \langle \Sigma_3 \Phi^*, \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_3 \xi_j^*, \cdot \rangle + \frac{1}{2} \langle \Sigma_3 \Phi^*, \Sigma_1 C \xi_j \rangle \langle e^{-i\Sigma_3 \vartheta} \Sigma_1 \Sigma_3 (C \xi_j)^*, \cdot \rangle \\
&+ \frac{1}{2} \langle e^{-i\Sigma_3 \vartheta} (P_c(\mathcal{H}_\omega^*) \Sigma_3 \Phi)^*, \cdot \rangle \\
&+ \frac{q}{q'} \langle \Sigma_3 \partial_\omega \Phi^*, \partial_\omega R \rangle d\omega \\
&- \frac{i}{2} \left\langle \underbrace{\langle \Sigma_3 \Phi^*, \xi_j \rangle \Sigma_3 \xi_j^* + \langle \Sigma_3 \Phi^*, \Sigma_1 C \xi_j \rangle \Sigma_1 \Sigma_3 (C \xi_j)^* + (P_c(\mathcal{H}_\omega^*) \Sigma_3 \Phi)^*}_{P_{N_g^\perp(\mathcal{H}_\omega)} \Sigma_3 \Phi^*}, \Sigma_3 R \right\rangle d\vartheta.
\end{aligned}$$

To get the third line of (3.33) we have used:

$$\begin{aligned}
(3.34) \quad &\frac{1}{2} \langle \Sigma_3 \Phi^*, \partial_\omega R \rangle - \frac{1}{2} \langle \Sigma_3 \Phi^*, \xi_j \rangle \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle - \\
&\frac{1}{2} \langle \Sigma_3 \Phi^*, \Sigma_1 C \xi_j \rangle \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \partial_\omega R \rangle - \frac{1}{2} \langle (P_c(\mathcal{H}_\omega^*) \Sigma_3 \Phi)^*, \partial_\omega R \rangle = \frac{1}{2} \langle \Sigma_3 \Phi^*, \partial_\omega R \rangle; \\
&- \frac{1}{2} \left[ \langle \Sigma_3 \Phi^*, \partial_\omega R \rangle - \frac{1}{q'} \langle \Sigma_3 \Phi^*, \Sigma_3 \Phi \rangle \langle \Sigma_3 \partial_\omega \Phi^*, \partial_\omega R \rangle \right] = \frac{2q}{2q'} \langle \Sigma_3 \partial_\omega \Phi^*, \partial_\omega R \rangle.
\end{aligned}$$

Let us consider the sum (3.24). There are various cancelations. The first and second (resp. the first term of the third) line of (3.33) cancel with the second and third lines of (3.29) (resp. the first term of the rhs of (3.30)). The last three terms in rhs(3.26) cancel with the last two lines of (3.31). The  $-iqd\vartheta$  term in the rhs of (3.31) cancels with the  $-iqd\vartheta$  term in (3.30). Adding the fourth line of (3.33) with the last term of rhs(3.30) we get the product of  $i$  times the following quantities:

$$\begin{aligned}
(3.35) \quad &- \frac{1}{2} \langle P_{N_g^\perp(\mathcal{H}_\omega)} \Sigma_3 \Phi^*, \Sigma_3 R \rangle - \frac{q}{q'} \langle R, \partial_\omega \Phi^* \rangle = -\frac{1}{2} \langle \Phi^*, R \rangle \\
&+ \frac{1}{2} \langle P_{N_g(\mathcal{H}_\omega^*)} \Sigma_3 \Phi^*, \Sigma_3 R \rangle - \frac{q}{q'} \langle R, \partial_\omega \Phi^* \rangle \\
&= -\frac{1}{2} \langle \Phi^*, R \rangle + \frac{1}{2q'} \langle \Phi^*, \Sigma_3 R \rangle \langle \partial_\omega \Phi, \Sigma_3 \Phi^* \rangle \\
&+ \frac{1}{2q'} \langle \Sigma_3 \partial_\omega \Phi^*, \Sigma_3 R \rangle \langle \Sigma_3 \Phi^*, \Sigma_3 \Phi \rangle - \frac{q}{q'} \langle R, \partial_\omega \Phi^* \rangle = 0,
\end{aligned}$$

where for the second equality we have used

$$P_{N_g(\mathcal{H}_\omega^*)} = \frac{1}{q'} \Phi^* \langle \partial_\omega \Phi, \cdot \rangle + \frac{1}{q'} \Sigma_3 \partial_\omega \Phi^* \langle \Sigma_3 \Phi, \cdot \rangle.$$

The last equality in (3.35) can be seen as follows. The two terms in the third line in (3.35) are both equal to 0. Indeed,  $\langle \Sigma_3 \Phi^*, \partial_\omega \Phi \rangle = 0$  by (3.28) and, by  $R \in N_g^\perp(\mathcal{H}_\omega^*)$  and  $\Phi^* \in N_g(\mathcal{H}_\omega)$ ,  $\langle R, \Phi^* \rangle = 0$ . The two terms in the fourth line in (3.35) cancel each other. Then we get formulas for  $\alpha^\omega$  and  $\alpha^f$ . We get  $\alpha^\vartheta$  also by  $\|P_c(\omega)f\|_2^2 = \|f\|_2^2 + 2\Re(\langle P_c(\omega) - P_c(\omega_0) \rangle f, (P_c(\omega)f)^*)$ .  $\square$

**Lemma 3.5.** *We have, summing over repeated indexes (also on  $j$  and  $\bar{j}$ ):*

$$(3.36) \quad i_Y \Omega_0 = iq' Y_\vartheta d\omega - iq' Y_\omega d\vartheta + \varepsilon_j (Y_j d\bar{z}_j - Y_{\bar{j}} dz_j) + \langle i\beta \alpha_2 \Sigma_1 \Sigma_3 Y_f, f' \cdot \rangle.$$

For the  $a_1$  in (3.18), and for  $\Gamma = i_Y \tilde{\Omega}$ , we have

$$\begin{aligned}
(3.37) \quad \Gamma_\omega &= a_1 Y_\vartheta + Y_j \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \partial_\omega R \rangle - Y_{\bar{j}} \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle + \langle Y_f, i\beta \alpha_2 \Sigma_3 \Sigma_1 P_c \partial_\omega R \rangle; \\
-\Gamma_\vartheta &= a_1 Y_\omega - i Y_j \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \Sigma_3 R \rangle + i Y_{\bar{j}} \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle - i \langle Y_f, i\beta \alpha_2 \Sigma_3 \Sigma_1 P_c \Sigma_3 R \rangle; \\
-\Gamma_j &= \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \partial_\omega R \rangle Y_\omega + i \langle \Sigma_1 \Sigma_3 (C \xi_j)^*, \Sigma_3 R \rangle Y_\vartheta; \\
\Gamma_{\bar{j}} &= \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle Y_\omega + i \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle Y_\vartheta; \\
i\beta \alpha_2 \Sigma_3 \Sigma_1 \Gamma_f &= (P_c(\omega_0) P_c(\omega) - 1) Y_f + Y_\omega P_c(\omega_0) P_c(\omega) \partial_\omega R + i Y_\vartheta P_c(\omega_0) P_c(\omega) \Sigma_3 R.
\end{aligned}$$

In particular, for  $\gamma = i_{Y^\tau} \Omega_\tau = i_{Y^\tau} \Omega_0 + \tau i_{Y^\tau} \tilde{\Omega}$  we have

$$\begin{aligned}
(3.38) \quad & \gamma_\omega = (iq' + \tau a_1) Y_\vartheta^\tau + \tau Y_j^\tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle - \tau Y_j^\tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle \\
& \quad + \tau \langle Y_f^\tau, i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c \partial_\omega R \rangle; \\
& -\gamma_\vartheta = (iq' + \tau a_1) Y_\omega^\tau - \tau i Y_j^\tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \Sigma_3 R \rangle + \tau i Y_j^\tau \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle \\
& \quad - i\tau \langle Y_f^\tau, i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c \Sigma_3 R \rangle; \\
& -\gamma_j = \varepsilon_j (Y^\tau)_{\bar{j}} + \tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle Y_\omega^\tau + i\tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \Sigma_3 R \rangle Y_\vartheta^\tau; \\
& \quad \gamma_{\bar{j}} = \varepsilon_j (Y^\tau)_j + \tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle Y_\omega^\tau + i\tau \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle Y_\vartheta; \\
& i\beta\alpha_2 \Sigma_3 \Sigma_1 \gamma_f = (Y^\tau)_f + \tau (P_c(\omega_0) P_c(\omega) - 1) Y_f^\tau + \\
& \quad \tau Y_\omega^\tau P_c(\omega_0) P_c(\omega) \partial_\omega R + i\tau Y_\vartheta^\tau P_c(\omega_0) P_c(\omega) \Sigma_3 R.
\end{aligned}$$

*Proof.* Identity (3.36) is straightforward. Identity (3.38) follows immediately from (3.36)–(3.37). Finally, (3.37) is elementary linear algebra, and basically the same of [Cuc10, Lemma 7.3].  $\square$

*Remark 3.6.* If we choose  $\gamma = -i\alpha$  in Lemma 3.5 with the  $\alpha$  of (3.24), and if  $\mathcal{F}_\tau$  is the flow of  $Y^\tau$ , then the component  $(Y^\tau)_\vartheta$  is an obstruction to the fact that, for  $0 < \tau \leq 1$ ,  $K \circ \mathcal{F}_\tau$  is a  $\vartheta$  invariant Hamiltonian and that the hamiltonian  $K \circ \mathcal{F}_1$  yields a semilinear Dirac equation. We want flows defined from fields with  $(Y^\tau)_\vartheta = 0$  or  $d\vartheta(Y^\tau) = i\Omega_\tau(X_\vartheta^\tau, Y^\tau) = 0$ , with  $X_\vartheta^\tau$  the Hamiltonian fields of  $\vartheta$ . To this effect we add a correction to  $\alpha$  and define  $Y^\tau$  from  $\alpha + dF$  where  $(\alpha + dF)(X_\vartheta^\tau) = 0$ .

**Lemma 3.7.** *Consider the vector field  $X_\vartheta^\tau$  (resp.  $X_\omega^\tau$ ) defined by  $i_{X_\vartheta^\tau} \Omega_\tau = -id\vartheta$  (resp.  $i_{X_\omega^\tau} \Omega_\tau = -id\omega$ ). Then we have (here  $P_c = P_c(\mathcal{H}_\omega)$  and  $P_c^0 = P_c(\mathcal{H}_{\omega_0})$ ):*

$$\begin{aligned}
(3.39) \quad & X_\vartheta^\tau = (X_\vartheta^\tau)_\omega \left[ \frac{\partial}{\partial \omega} - \tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle \frac{\partial}{\partial z_j} - \tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle \frac{\partial}{\partial \bar{z}_j} \right. \\
& \quad \left. - \tau P_c^0 (1 + \tau P_c - \tau P_c^0)^{-1} P_c^0 P_c \partial_\omega R \right], \\
& X_\omega^\tau = (X_\omega^\tau)_\vartheta \left[ \frac{\partial}{\partial \vartheta} - i\tau \langle \xi_j^*, R \rangle \frac{\partial}{\partial z_j} + i\tau \langle \Sigma_1 (C\xi_j)^*, R \rangle \frac{\partial}{\partial \bar{z}_j} \right. \\
& \quad \left. - i\tau P_c^0 (1 + \tau P_c - \tau P_c^0)^{-1} P_c^0 P_c \Sigma_3 R \right],
\end{aligned}$$

where, for the  $a_1$  in (3.18), we have

$$(3.40) \quad (X_\vartheta^\tau)_\omega = \frac{i}{iq' + \tau a_1 + \tau a_2} = -(X_\omega^\tau)_\vartheta$$

$$(3.41) \quad a_2 := i\tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \Sigma_3 R \rangle - i\tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle \langle \xi_j^*, R \rangle + \\ + i\tau \langle P_c^0 (1 + \tau P_c - \tau P_c^0)^{-1} P_c^0 P_c \partial_\omega R, i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c \Sigma_3 R \rangle.$$

*Proof.* The proof is almost the same of [Cuc10, Lemma 7.5]. By (3.38) for  $\gamma = -id\vartheta$ ,  $X_\vartheta^\tau$  satisfies

$$\begin{aligned}
(3.42) \quad & (X_\vartheta^\tau)_\vartheta = 0; \\
& i = (iq' + \tau a_1) (X_\vartheta^\tau)_\omega - i\tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \Sigma_3 R \rangle (X_\vartheta^\tau)_j + \\
& \quad + i\tau \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle (X_\vartheta^\tau)_{\bar{j}} - i\tau \langle (X_\vartheta^\tau)_f, i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c \Sigma_3 R \rangle; \\
& (X_\vartheta^\tau)_f = \tau (1 - P_c^0 P_c) (X_\vartheta^\tau)_f - \tau (X_\vartheta^\tau)_\omega P_c^0 P_c \partial_\omega R; \\
& (X_\vartheta^\tau)_{\bar{j}} = -\tau (X_\vartheta^\tau)_\omega \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle; \quad (X_\vartheta^\tau)_j = -\tau (X_\vartheta^\tau)_\omega \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle.
\end{aligned}$$

This yields (3.39) for  $X_\vartheta^\tau$  and the first equality in (3.40). By (3.38) for  $\gamma = -id\omega$ ,  $X_\omega^\tau$  satisfies

$$\begin{aligned}
(3.43) \quad & (X_\omega^\tau)_\omega = 0; \\
& -i - iq' (X_\omega^\tau)_\vartheta = \tau a_1 (X_\omega^\tau)_\vartheta + \tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle (X_\omega^\tau)_j - \\
& \quad - \tau \langle \Sigma_1 \Sigma_3 \xi_j^*, \partial_\omega R \rangle (X_\omega^\tau)_{\bar{j}} + \tau \langle (X_\omega^\tau)_f, i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c \partial_\omega R \rangle; \\
& (X_\omega^\tau)_f = \tau (1 - P_c^0 P_c) (X_\omega^\tau)_f - i\tau (X_\omega^\tau)_\vartheta P_c^0 P_c \Sigma_3 R; \\
& (X_\omega^\tau)_{\bar{j}} = -i\tau (X_\omega^\tau)_\vartheta \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \Sigma_3 R \rangle; \quad (X_\omega^\tau)_j = -i\tau (X_\omega^\tau)_\vartheta \langle \Sigma_3 \xi_j^*, \Sigma_3 R \rangle.
\end{aligned}$$

This yields the rest of (3.39)–(3.40).  $\square$

The following lemma is an immediate consequence of the formulas in Lemma 3.7 and of (3.19).

**Lemma 3.8.** For any  $(K', S', K, S)$  we have

$$(3.44) \quad \begin{aligned} |1 - (X_\vartheta^\tau)_\omega q'| &\lesssim \|R\|_{H^{-K', -S'}}^2 \\ |(X_\vartheta^\tau)_j| + |(X_\vartheta^\tau)_{\bar{j}}| + \|(X_\vartheta^\tau)_f\|_{H^{K, S}} &\lesssim \|R\|_{H^{-K', -S'}}. \end{aligned}$$

and

$$(3.45) \quad \begin{aligned} |1 + (X_\omega^\tau)_\vartheta q'| &\lesssim \|R\|_{H^{-K', -S'}}^2, \\ |(X_\omega^\tau)_j| + |(X_\omega^\tau)_{\bar{j}}| + \|(X_\omega^\tau)_f\|_{H^{-K', -S'}} &\lesssim \|R\|_{H^{-K', -S'}}. \end{aligned}$$

**Definition 3.9.** Set  $H_c^{K, S}(\omega) = P_c(\omega)H^{K, S}$  and denote

$$(3.46) \quad \tilde{\mathcal{P}}^{K, S} = \mathbb{C}^n \times H_c^{K, S}(\omega_0), \quad \mathcal{P}^{K, S} = \mathbb{R}^2 \times \tilde{\mathcal{P}}^{K, S}$$

with elements  $(\vartheta, \omega, z, f) \in \mathcal{P}^{K, S}$  and  $(z, f) \in \tilde{\mathcal{P}}^{K, S}$ .

**Lemma 3.10.** We consider  $\forall \tau \in [0, 1]$  the hamiltonian field  $X_\vartheta^\tau$  and the flow

$$(3.47) \quad \frac{d}{ds}\Phi_s(\tau, U) = X_\vartheta^\tau(\Phi_s(\tau, U)), \quad \Phi_0(\tau, U) = U.$$

(1) For any  $(K', S')$  there is a  $s_0 > 0$  and a neighborhood  $\mathcal{U}$  of  $\mathbb{R} \times \{(\omega_0, 0, 0)\}$  in  $\mathcal{P}^{-K', -S'}$  such that the map  $(s, \tau, U) \rightarrow \Phi_s(\tau, U)$  is smooth

$$(3.48) \quad (-s_0, s_0) \times [0, 1] \times (\mathcal{U} \cap \{\omega = \omega_0\}) \rightarrow \mathcal{P}^{-K', -S'}.$$

(2)  $\mathcal{U}$  can be chosen so that for any  $\tau \in [0, 1]$  there is another neighborhood  $\mathcal{V}_\tau$  of  $\mathbb{R} \times \{(\omega_0, 0, 0)\}$  in  $\mathcal{P}^{-K', -S'}$  s.t. the above map establishes a diffeomorphism

$$(3.49) \quad (-s_0, s_0) \times (\mathcal{U} \cap \{\omega = \omega_0\}) \rightarrow \mathcal{V}_\tau.$$

(3)  $f(\Phi_s(\tau, U)) - f(U) = G(t, s, z, f)$  is a smooth map for all  $(K, S)$

$$\begin{aligned} &(-s_0, s_0) \times [0, 1] \times (\mathcal{U} \cap \{\omega = \omega_0\}) \rightarrow H^{K, S} \\ &\text{with } \|G(t, s, z, f)\|_{H^{K, S}} \leq C|s|(|z| + \|f\|_{H^{-K', -S'}}). \end{aligned}$$

*Proof.* The proof is exactly the same of Lemma 7.7 [Cuc10]. We only remark, that the field  $X_\vartheta^\tau$ , the flow  $\Phi_s(\tau, U)$  and the function  $F(\tau, U)$  in Lemma 3.11 are defined intrinsically, and so are periodic in  $\vartheta$ . This is because  $X_\vartheta^\tau$  satisfies these properties, since  $i_{X_\vartheta^\tau}\Omega_\tau = -id\vartheta$  with both  $\Omega_\tau$  and  $d\vartheta$  intrinsically defined and periodic in  $\vartheta$ .  $\square$

**Lemma 3.11.** We consider a scalar function  $F(\tau, U)$  defined as follows:

$$(3.50) \quad F(\tau, \Phi_s(\tau, U)) = i \int_0^s \alpha_{\Phi_{s'}(t, U)}(X_\vartheta^\tau(\Phi_{s'}(t, U))) ds', \quad \text{where } \omega(U) = \omega_0.$$

We have  $F \in C^\infty([0, 1] \times \mathcal{U}, \mathbb{R})$  for a neighborhood  $\mathcal{U}$  of  $\mathbb{R} \times \{(\omega_0, 0, 0)\}$  in  $\mathcal{P}^{-K', -S'}$ . We have

$$(3.51) \quad |F(t, U)| \leq C(K', S')|\omega - \omega_0| (|z| + \|f\|_{H^{-K', -S'}})^2$$

We have (exterior differentiation only in  $U$ )

$$(3.52) \quad (\alpha + i dF)(X_\vartheta^\tau) = 0.$$

*Proof.* The proof is elementary and is exactly the same of Lemma 7.8 [Cuc10].  $\square$

We now have the desired correction for  $\alpha$  and below we introduce the vector field whose flow yields the wanted change of coordinates.

**Lemma 3.12.** Denote by  $\mathcal{X}^\tau$  the vector field which solves

$$(3.53) \quad i_{\mathcal{X}^\tau}\Omega_\tau = -\alpha - i dF(\tau).$$

Then the following properties hold.

- (1) There is a neighborhood  $\mathcal{U}$  of  $\mathbb{R} \times \{(\omega_0, 0, 0)\}$  in  $\mathcal{P}^{1, 0}$  such that  $\mathcal{X}^\tau(U) \in C^\infty([0, 1] \times \mathcal{U}, \mathcal{P}^{1, 0})$ .
- (2) We have  $(\mathcal{X}^\tau)_\vartheta \equiv 0$ .

(3) For constants  $C(K, S, K', S')$

$$(3.54) \quad \begin{aligned} & \left| (\mathcal{X}^\tau)_\omega + \frac{\|f\|_2^2}{2q'(\omega)} \right| \lesssim (|z| + \|f\|_{H^{-K', -S'}})^2; \\ & |(\mathcal{X}^\tau)_j| + |(\mathcal{X}^\tau)_{\bar{j}}| + \|(\mathcal{X}^\tau)_f\|_{H^{K, S}} \lesssim (|z| + \|f\|_{H^{-K', -S'}}) \times \\ & \quad \times (|\omega - \omega_0| + |z| + \|f\|_{H^{-K', -S'}} + \|f\|_{L^2}^2). \end{aligned}$$

(4) We have

$$(3.55) \quad L_{\mathcal{X}^\tau} \frac{\partial}{\partial \vartheta} := \left[ \mathcal{X}^\tau, \frac{\partial}{\partial \vartheta} \right] = 0.$$

*Proof.* The proof is almost the same of [Cuc10, Lemma 7.9]. Claim (1) follows from the regularity properties of  $\alpha$ ,  $F$  and  $\Omega_\tau$  and from equations (3.56) and (3.58) below. (3.52) implies (2) by

$$i(\mathcal{X}^\tau)_\vartheta = id\vartheta(\mathcal{X}^\tau) = -i_{X_\vartheta} \Omega_\tau(\mathcal{X}^\tau) = i_{\mathcal{X}^\tau} \Omega_\tau(X_\vartheta^\tau) = -(\alpha + idF)(X_\vartheta^\tau) = 0.$$

We have  $i(\mathcal{X}^\tau)_\omega = id\omega(\mathcal{X}^\tau) = -i_{X_\omega} \Omega_\tau(\mathcal{X}^\tau)$ , so

$$(3.56) \quad \begin{aligned} i(\mathcal{X}^\tau)_\omega &= i_{\mathcal{X}^\tau} \Omega_\tau(X_\omega^\tau) = -(X_\omega^\tau)_\vartheta [\alpha^\vartheta + \tau \partial_j F \langle \xi_j^*, R \rangle - \tau \partial_{\bar{j}} F \langle \Sigma_1(C\xi_j)^*, R \rangle \\ & \quad + \tau \langle \nabla_f F + i\alpha^f, P_c^0(1 + \tau P_c - \tau P_c^0)^{-1} P_c^0 P_c \Sigma_3 R \rangle]. \end{aligned}$$

Then by (3.25), (3.40) and (3.41), we get the first inequality in (3.54):

$$(3.57) \quad \left| (\mathcal{X}^\tau)_\omega + \frac{\|f\|_2^2}{2q'(\omega)} \right| \leq C(|z| + \|f\|_{H^{-K', -S'}})^2.$$

By (3.38) we have the following equations

$$(3.58) \quad \begin{aligned} i\partial_j F &= \varepsilon_j(\mathcal{X}^\tau)_{\bar{j}} + \tau \langle \Sigma_1 \Sigma_3(C\xi_j)^*, \partial_\omega R \rangle (\mathcal{X}^\tau)_\omega \\ -i\partial_{\bar{j}} F &= \varepsilon_j(\mathcal{X}^\tau)_j + \tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle (\mathcal{X}^\tau)_\omega \\ i\beta \alpha_2 \Sigma_3 \Sigma_1(\alpha^f + i\nabla_f F) &= -(\mathcal{X}^\tau)_f - \tau(P_c^0 P_c - 1)(\mathcal{X}^\tau)_f - \tau(\mathcal{X}^\tau)_\omega P_c^0 P_c \partial_\omega R. \end{aligned}$$

Formulas (3.58) imply

$$\begin{aligned} |(\mathcal{X}^\tau)_{\bar{j}}| &\leq |\partial_j F| + C(|z| + \|f\|_{H^{-K', -S'}}) |(\mathcal{X}^\tau)_\omega| \\ |(\mathcal{X}^\tau)_j| &\leq |\partial_{\bar{j}} F| + C(|z| + \|f\|_{H^{-K', -S'}}) |(\mathcal{X}^\tau)_\omega| \\ \|(\mathcal{X}^\tau)_f\|_{H^{K, S}} &\leq \|\alpha^f\|_{H^{K, S}} + \|\nabla_f F\|_{H^{K, S}} + C(|z| + \|f\|_{H^{-K', -S'}}) |(\mathcal{X}^\tau)_\omega| \end{aligned}$$

which with (3.57), (3.25) and Lemma (3.51) imply (3.54). (3.55) follows by  $L_{\frac{\partial}{\partial \vartheta}}(\alpha + idF) = 0$  and by the product rule for the Lie derivative,

$$L_{\frac{\partial}{\partial \vartheta}}(i_{\mathcal{X}^\tau} \Omega_\tau) = i_{[\frac{\partial}{\partial \vartheta}, \mathcal{X}^\tau]} \Omega_\tau + i_{\mathcal{X}^\tau} L_{\frac{\partial}{\partial \vartheta}} \Omega_\tau = i_{[\frac{\partial}{\partial \vartheta}, \mathcal{X}^\tau]} \Omega_\tau.$$

□

The following lemma gathers some properties of the change of coordinates.

**Lemma 3.13.** *Consider the vectorfield  $\mathcal{X}^\tau$  in Lemma 3.11 and denote by  $\mathcal{F}_\tau(U)$  the corresponding flow. Then the flow  $\mathcal{F}_\tau(U)$  for  $U$  near  $e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$  is defined for all  $\tau \in [0, 1]$ . We have  $\vartheta \circ \mathcal{F}_1 = \vartheta$ . We have*

$$(3.59) \quad \begin{aligned} q(\omega(\mathcal{F}_1(U))) &= q(\omega(U)) - \frac{\|f\|_2^2}{2} + \mathcal{E}_\omega(U) \\ z_j(\mathcal{F}_1(U)) &= z_j(U) + \mathcal{E}_j(U) \\ f(\mathcal{F}_1(U)) &= f(U) + \mathcal{E}_f(U) \end{aligned}$$

with

$$(3.60) \quad |\mathcal{E}_\omega(U)| \lesssim (|\omega - \omega_0| + |z| + \|f\|_{H^{-K', -S'}})^2,$$

$$(3.61) \quad \begin{aligned} |\mathcal{E}_j(U)| + \|\mathcal{E}_f(U)\|_{H^{K, S}} &\lesssim (|\omega - \omega_0| + |z| + \|f\|_{H^{-K', -S'}} + \|f\|_{L^2}^2) \\ &\quad \times (|\omega - \omega_0| + |z| + \|f\|_{H^{-K', -S'}}). \end{aligned}$$

For each  $\zeta = \omega, z_j, f$  we have

$$(3.62) \quad \mathcal{E}_\zeta(U) = \mathcal{E}_\zeta(\|f\|_{L^2}^2, \omega, z, f)$$

with, for a neighborhood  $\mathcal{U}^{-K', -S'}$  of  $\{(\omega_0, 0, 0)\}$  in  $\mathcal{P}^{-K', -S'} \cap \{\vartheta = 0\}$  and for some fixed  $a_0 > 0$

$$(3.63) \quad \mathcal{E}_\zeta(\varrho, \omega, z, f) \in C^\infty((-a_0, a_0) \times \mathcal{U}^{-K', -S'}, \mathbb{C})$$

for  $\zeta = \omega, z_j$  and with

$$(3.64) \quad \mathcal{E}_f(\varrho, \omega, z, f) \in C^\infty((-a_0, a_0) \times \mathcal{U}^{-K', -S'}, H^{K, S}).$$

*Proof.* The argument is the same of Lemma 7.10 [Cuc10], but we review it for the sake of the reader. We add a new variable  $\varrho$ . We define a new field by

$$(3.65) \quad \begin{aligned} \mathfrak{i}(Y^\tau)_\omega &= -(X_\omega^\tau)_\vartheta [\alpha^\vartheta + \mathfrak{i} \frac{\|f\|_2^2 - \rho}{2} + \tau \partial_j F \langle \xi_j^*, R \rangle - \tau \partial_{\bar{j}} F \langle \Sigma_1(C\xi_j)^*, R \rangle \\ &\quad + \tau \langle \nabla_f F + \mathfrak{i}\alpha^f, P_c^0(1 + \tau P_c - \tau P_c^0)^{-1} P_c^0 P_c \Sigma_3 R \rangle], \end{aligned}$$

by

$$(3.66) \quad \begin{aligned} \mathfrak{i} \partial_j F &= \varepsilon_j (Y^\tau)_{\bar{j}} + \tau \langle \Sigma_1 \Sigma_3 (C\xi_j)^*, \partial_\omega R \rangle (Y^\tau)_\omega \\ -\mathfrak{i} \partial_{\bar{j}} F &= \varepsilon_j (Y^\tau)_j + \tau \langle \Sigma_3 \xi_j^*, \partial_\omega R \rangle (Y^\tau)_\omega \\ \mathfrak{i} \beta \alpha_2 \Sigma_3 \Sigma_1 (\alpha^f + \mathfrak{i} \nabla_f F) &= -(Y^\tau)_f - \tau (P_c^0 P_c - 1) (Y^\tau)_f - \tau (Y^\tau)_\omega P_c^0 P_c \partial_\omega R. \end{aligned}$$

and by  $Y_\rho^\tau = 2 \langle (Y^\tau)_f, \mathfrak{i} \beta \alpha_2 \Sigma_1 f \rangle$ . Then  $Y^\tau = Y^\tau(\omega, \rho, z, f)$  defines a new flow  $\mathcal{G}_\tau(\rho, U)$ , which reduces to  $\mathcal{F}_\tau(U)$  in the invariant manifold defined by  $\rho = \|f\|_2^2$ . Notice that by  $\rho(t) = \rho(0) + \int_0^t Y_\rho^s ds$  it is easy to conclude  $\rho(\mathcal{G}_1(\rho, U)) = \rho(U) + O(\text{rhs}(3.60))$ . Using (3.45), (3.25) and (3.65) it is then easy to get

$$q(\omega(t)) = q(\omega(0)) + \int_0^t q'(\omega(s)) Y_\omega^s ds = q(\omega(0)) - \int_0^t \frac{\rho(s)}{2} ds + O(\text{rhs}(3.60)).$$

By standard arguments, see for example the proof of Lemma 4.3 [BC09], we get

$$(3.67) \quad \begin{aligned} q(\omega(\mathcal{G}_1(\rho, U))) &= q(\omega(U)) - \frac{\rho}{2} + \mathcal{E}_\omega(\rho, U) \\ z_\ell(\mathcal{G}_1(\rho, U)) &= z_\ell(U) + \mathcal{E}_\ell(\rho, U) \\ f(\mathcal{G}_1(\rho, U)) &= f(U) + \mathcal{E}_f(\rho, U) \end{aligned}$$

with  $\mathcal{E}_\zeta(\rho, U)$  satisfying (3.63) for  $\zeta = \omega, z_\ell$  and (3.64) for  $\zeta = f$ . We have  $\mathcal{E}_\zeta(\cdot, U) = \mathcal{E}_\zeta(\|f\|_2, U)$  satisfying (3.60) for  $\zeta = \omega$  and (3.61) for  $\zeta = z_\ell, f$ .  $\square$

Eventually we have the desired Darboux type result:

**Lemma 3.14. (Darboux Theorem)** *Consider the flow  $\mathcal{F}_\tau$  of Lemma 3.13. Then we have*

$$(3.68) \quad \mathcal{F}_\tau^* \Omega_\tau = \Omega_0.$$

We have

$$(3.69) \quad Q \circ \mathcal{F}_1 = q.$$

If  $\chi$  is a function with  $\partial_\vartheta \chi \equiv 0$ , then  $\partial_\vartheta(\chi \circ \mathcal{F}_t) \equiv 0$ .

*Proof.* Identity (3.68) is Darboux Theorem and the proof of the lemma is the same of Lemma 7.11 [Cuc10].  $\square$

**3.4. Reformulation of (3.9) in the new coordinates.** We set

$$(3.70) \quad H = K \circ \mathcal{F}_1.$$

In the new coordinates (3.9) becomes

$$(3.71) \quad q' \dot{\omega} = \frac{\partial H}{\partial \vartheta} \equiv 0, \quad q' \dot{\vartheta} = -\frac{\partial H}{\partial \omega}$$

and

$$(3.72) \quad \mathfrak{i} \dot{z}_j = \varepsilon_j \frac{\partial H}{\partial \bar{z}_j}, \quad \mathfrak{i} \dot{f} = \mathfrak{i} \beta \alpha_2 \Sigma_3 \Sigma_1 \nabla_f H.$$

Recall that we are solving the initial value problem (1.1) and that we have chosen  $\omega_0$  with  $q(\omega_0) = \|u_0\|_{L_x^2}^2$ . Correspondingly it is enough to focus on (3.72) with  $\omega = \omega_0$ . Consider the notation of Theorem 1.1. Let us focus for the moment on the case  $\varepsilon_j \equiv 1$  in system (3.72). Then we prove :

**Theorem 3.15.** *Assume (H:1)–(H:12). Then for any  $k_0 > 3$  there exist  $\epsilon_0 > 0$  and  $C > 0$  such that for  $|z(0)| + \|f(0)\|_{H^{k_0}} \leq \epsilon < \epsilon_0$  the corresponding solution of (3.72) is globally defined and there are  $f_{\pm} \in H^{k_0}$  with  $\|f_{\pm}\|_{H^{k_0}} \leq C\epsilon$  such that*

$$(3.73) \quad \lim_{t \rightarrow \pm\infty} \|e^{i\vartheta(t)\Sigma_3} f(t) - e^{-itD_m} e^{i\vartheta(0)\Sigma_3} f_{\pm}\|_{H^{k_0}} = 0$$

and

$$(3.74) \quad \lim_{t \rightarrow \infty} z(t) = 0.$$

Fix  $p_0 > 2$  and  $\tau_0 > 1$ . Let  $\frac{1}{p} = \frac{1}{2} - \frac{1}{q}$  and  $\alpha(q) = \frac{3}{p}$ . Then, we can choose  $\epsilon_0$  small enough such that

$$f(t, x) = A(t, x) + \tilde{f}(t, x)$$

with

$$\forall n \in \mathbb{N}, C_n(t) := \sup_{x \in \mathbb{R}^3, \alpha \in \mathbb{N}^3} \langle x \rangle^n |\partial_x^\alpha A(t, x)| \rightarrow 0 \text{ as } t \rightarrow \infty$$

and for some fixed  $C$

$$(3.75) \quad \|\tilde{f}\|_{L_t^p([0, \infty), B_{q,2}^{k_0 - \frac{3}{p}}) \cap L_t^2([0, \infty), H_x^{k_0 - \tau_0}) \cap L_t^2([0, \infty), L_x^\infty)} \leq C\epsilon.$$

There exist  $\omega_+$  such that  $|\omega_+ - \omega_0| = O(\|f_+\|_2^2)$  such that

$$\lim_{t \rightarrow +\infty} \omega(t) = \omega_+.$$

*Proof that Theorem 3.15 implies Theorem 1.1.* . If we denote  $(\omega, z', f')$  the initial coordinates, and  $(\omega_0, z, f)$  the coordinates in (3.72), we have from Lemma 3.13 :

$$|z' - z| = O(|z| + \|f\|_{L_x^{2,-2}}) \text{ and } \|f' - f\|_{H^{\kappa,S}} = O(|z| + \|f\|_{L_x^{2,-2}})$$

for any  $(K, S) \in (\mathbb{R}^+)^2$ . The two error terms  $O$  converge to 0 as  $t \rightarrow \infty$ . Hence the asymptotic behavior of  $(z', f')$  and of  $(z, f)$  is the same. We also have, from Lemma 3.13,  $q(\omega(t)) = q(\omega_0) - \frac{\|f(t)\|_2^2}{2} + O(|z(t)| + \|f(t)\|_{L_x^{2,-2}})$  which implies, say at  $+\infty$

$$\lim_{t \rightarrow +\infty} q(\omega(t)) = \lim_{t \rightarrow +\infty} \left( q(\omega_0) - \frac{\|e^{-it\mathcal{H}_{\omega_0,0}} f_+\|_2^2}{2} \right) = q(\omega_0) - \frac{\|f_+\|_2^2}{2} = q(\omega_+)$$

for  $\omega_+$  the unique element near  $\omega_0$  for which the last inequality holds. So  $\lim_{t \rightarrow +\infty} \omega(t) = \omega_+$ .  $\square$

In the case  $\varepsilon_j \in \{1, -1\}$  with  $\varepsilon_j \neq 1$ , using the same argument of Theorem 3.15, we prove that solutions which remain close to the standing wave, actually have remainder which scatters. We state this in terms of the system (3.72) and the coordinates after Darboux, but of course it can be stated also in terms of the original coordinates, as in Theorems 1.1 and 1.8.

**Theorem 3.16.** *Assume (H:1)–(H:4), (H:5') and (H:6)–(H:12). Then there exist  $\epsilon_0 > 0$  with the following property. Suppose that  $(z(t), f(t))$  is a solution of (3.72) such that  $|z(t)| + \|f(0)\|_{H^{k_0}} \leq \epsilon < \epsilon_0$  for all  $t \geq 0$ . Suppose furthermore that there exists a fixed  $C > 0$  such that  $\|f(t)\|_{H^{k_0}} \leq C\epsilon$  for all  $t \geq 0$ . Then there exists  $f_+ \in H^{k_0}$  with*

$$(3.76) \quad \lim_{t \rightarrow \pm\infty} \|e^{i\vartheta(t)\Sigma_3} f(t) - e^{-itD_m} e^{i\vartheta(0)\Sigma_3} f_{\pm}\|_{H^{k_0}} = 0$$

and we have

$$(3.77) \quad \lim_{t \rightarrow +\infty} z(t) = 0.$$

Furthermore, we can write  $f(t, x) = A(t, x) + \tilde{f}(t, x)$  as in Theorem 3.15 in such a way that the same conclusions of Theorem 3.15 regarding  $A(t, x)$  and  $\tilde{f}$  hold.

*Remark 3.17.* Theorem 3.16 is analogous to an observation in [MM08] regarding the fact that solutions remaining for all times close to a standing wave, stable or unstable, converge to it. Among other references see also [Bec08, NS10].

Finally, Theorem 1.8, that is orbital instability, is a consequence of the following theorem.

**Theorem 3.18.** *Assume (H:1)–(H:4), (H:5') and (H:6)–(H:12). Then we prove that there is a  $\epsilon_1 > 0$  such that for any  $\delta > 0$  there is a solution  $(z(t), f(t))$  of (3.72) such that  $|z(0)| + \|f(0)\|_{H^{k_0}} \leq \delta$  but there exists  $t \geq 0$  such that  $|z(t)| \geq \epsilon_1$ .*



3.4.1. *Taylor expansions.* We recall that  $\varepsilon_j = \langle \xi_j, \Sigma_3 \xi_j \rangle \in \{1, -1\}$  is the Krein signature of the eigenvalues of  $\mathcal{H}_\omega$ . We set

$$d(\omega) = E(\Phi_\omega) + \omega Q(\Phi_\omega).$$

We recall that  $\omega_0$  is the unique element such that  $q(\omega_0) = \|u_0\|_2^2$  and  $G$  is the primitive of the non-linearity  $g$  vanishing at 0.

**Lemma 3.19.** *The following statements hold.*

$$(3.78) \quad \begin{aligned} K &= d(\omega) - \omega \|u_0\|_2^2 + K_2 + K_P; \\ K_2 &= \sum_j \varepsilon_j \lambda_j(\omega) |z_j|^2 + \frac{1}{2} \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_\omega f, f \rangle \\ K_P &= \langle G_6(\omega, f(x)), 1 \rangle + \sum_{|\mu+\nu|=3} \langle k_{\mu\nu}(\omega, z), 1 \rangle z^\mu \bar{z}^\nu + \sum_{|\mu+\nu|=2} z^\mu \bar{z}^\nu \langle K_{\mu\nu}(\omega, z), i\beta\alpha_2 \Sigma_3 \Sigma_1 P_c(\omega) f \rangle \\ &\quad + \sum_{d=2}^4 \langle G_d(\omega, z), (P_c(\omega) f)^{\otimes d} \rangle + \int_{\mathbb{R}^3} \langle G_5(x, \omega, z, f(x)), f^{\otimes 5}(x) \rangle dx, \end{aligned}$$

where for a small neighborhood  $\mathcal{U}$  of  $(\omega_0, 0)$  in  $\mathcal{O} \times \mathbb{C}^n$ , we have what follows for  $\mathcal{U}$  like in (1), possibly smaller.

- (1)  $G_6(x, \omega, f) = G\left(\frac{1}{2}(P_c(\omega) f(x)) \cdot i\alpha_2 \Sigma_3 \Sigma_1 (P_c(\omega) f(x))\right)$ ,
- (2)  $k_{\mu\nu}(\cdot, \omega, z) \in C^\infty(\mathcal{U}, H_x^{K,S}(\mathbb{R}^3, \mathbb{C}^8))$ ,
- (3)  $K_{\mu\nu}(\cdot, \omega, z) \in C^\infty(\mathcal{U}, H_x^{K,S}(\mathbb{R}^3, \mathbb{C}^8))$ ,
- (4)  $G_d(\cdot, \omega, z) \in C^\infty(\mathcal{U}, H_x^{K,S}(\mathbb{R}^3, B((\mathbb{C}^8)^{\otimes d}, \mathbb{C})))$ , for  $2 \leq d \leq 4$  and  $G_2(\cdot, \omega, 0) \equiv 0$ .
- (5) Let  ${}^t\eta = (\zeta, C\zeta)$  for  $\zeta \in \mathbb{C}^4$ . Then for  $G_5(\cdot, \omega, z, \eta)$  we have

$$\forall l \in \mathbb{N} \cup \{0\}, \|\nabla_{\omega, z, \bar{z}, \zeta, C\zeta}^l G_5(\omega, z, \eta)\|_{H_x^{K,S}(\mathbb{R}^3, B((\mathbb{C}^8)^{\otimes 5}, \mathbb{C}))} \leq C_l.$$

- (6) We have  $k_{\mu\nu} = k_{\nu\mu}^*$ ,  $K_{\mu\nu} = -C\Sigma_1 K_{\nu\mu}$ .

*Proof.* Consider  $U = e^{i\Sigma_3 \vartheta}(\Phi_\omega + R)$  as in (2.28). Decompose  $R$  as in (2.35). Set  $U = \varphi(\omega, z) + P_c(\omega) f$ . Let  $K_P(U) = \int h(U(x)) dx$ , see Lemma 2.3, then after first a Taylor integral expansion around  $f$  at first order and a Taylor integral expansion around  $\phi$  at fourth order, we have

$$(3.79) \quad \begin{aligned} h(U) &= h(P_c(\omega) f) + \int_0^1 dh(t\varphi + P_c(\omega) f) \varphi dt \\ &= h(P_c(\omega) f) + \int_0^1 \sum_{i \leq 4} \frac{1}{i!} d^{i+1} h(t\varphi)(P_c(\omega) f)^i \varphi dt + \\ &\quad + 5 \int_{[0,1]^2} (1-s)^4 \frac{1}{5!} d^6 h(t\varphi + sP_c(\omega) f)(P_c(\omega) f)^4 \varphi dt ds \end{aligned}$$

Notice that  $\Phi_\omega$  is a critical point of  $K$  as it is in the kernel of  $\mathcal{H}_\omega \Sigma_3$ . So in the Taylor expansion of  $K$  around  $\Phi_\omega$  there is no first order term. Notice that the second derivative of  $K$  is the bilinear form  $\frac{1}{2} \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_\omega \cdot, \cdot \rangle$ . This gives  $K_2$ .

The term  $K_P$  contains all terms of order higher than 2 in  $f$  and  $z$ . Thus coincides with the term of order higher than 2 in  $f$  and  $z$  in the above expansion after integration in  $x$ .

The Hamiltonian  $K$  is a real quantity and considering its conjugate will exchange  $\bar{z}$  and  $z$  and lead by a straightforward calculation to the last assertion.  $\square$

The following lemma is a reformulation with some rearrangements of the above one in the canonical coordinates provided by Lemma 3.13. We set  $\delta_j$  be for  $j \in \{1, \dots, n\}$  the multi index  $\delta_j = (\delta_{1j}, \dots, \delta_{nj})$ . Let  $\lambda_j^0 = \lambda_j(\omega_0)$  and  $\lambda^0 = (\lambda_1^0, \dots, \lambda_n^0)$ .

**Lemma 3.20.** *Let  $H = K \circ \mathcal{F}_1$ . Then, around  $e^{i\Sigma_3 \vartheta} \Phi_{\omega_0}$  we have the expansion*

$$(3.80) \quad H = d(\omega_0) - \omega_0 \|u_0\|_2^2 + \psi(\|f\|_2^2) + H_2^{(1)} + \mathcal{R}^{(1)}$$

where

$$(3.81) \quad H_2^{(1)} = \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0 \cdot (\mu-\nu)=0}} k_{\mu\nu}^{(1)}(\|f\|_2^2) z^\mu \bar{z}^\nu + \frac{1}{2} \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_{\omega_0} f, f \rangle.$$

and  $\mathcal{R}^{(1)} = \widetilde{\mathcal{R}}^{(1)} + \widetilde{\mathcal{R}}^{(2)}$ , with

$$(3.82) \quad \begin{aligned} \widetilde{\mathcal{R}}^{(1)} &= \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0 \cdot (\mu-\nu) \neq 0}} k_{\mu\nu}^{(1)} (\|f\|_2^2) z^\mu \bar{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \bar{z}^\nu \langle H_{\nu\mu} (\|f\|_2^2), i\beta\alpha_2 \Sigma_3 \Sigma_1 f \rangle, \\ \widetilde{\mathcal{R}}^{(2)} &= \int_{\mathbb{R}^3} G \left( \frac{1}{2} (P_c(\omega_0) f(x)) \cdot i\alpha_2 \Sigma_3 \Sigma_1 (P_c(\omega_0) f(x)) \right) dx + \sum_{|\mu+\nu|=3} z^\mu \bar{z}^\nu \int_{\mathbb{R}^3} k_{\mu\nu}(x, z, f, f(x), \|f\|_2^2) dx \\ &+ \sum_{|\mu+\nu|=2} z^\mu \bar{z}^\nu \int_{\mathbb{R}^3} [i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{\nu\mu}(x, z, f, f(x), \|f\|_2^2)]^T f(x) dx \\ &+ \sum_{j=2}^5 \mathcal{R}_j^{(1)} + \widehat{\mathcal{R}}_2^{(1)}(z, f, \|f\|_2^2) \\ \text{and } \mathcal{R}_j^{(1)} &= \int_{\mathbb{R}^3} F_j(x, z, f, f(x), \|f\|_2^2) f^{\otimes j}(x) dx. \end{aligned}$$

and where the following holds.

(1) We have  $\psi(s)$  is smooth with  $\psi(0) = \psi'(0) = 0$ .

(2) At  $\|f\|_2 = 0$  we have:

$$(3.83) \quad \begin{aligned} k_{\mu\nu}^{(1)}(0) &= 0 \text{ for } |\mu + \nu| = 2 \text{ with } (\mu, \nu) \neq (\delta_j, \delta_j) \text{ for all } j; \\ k_{\delta_j \delta_j}^{(1)}(0) &= \varepsilon_j \lambda_j(\omega_0), \text{ where } \delta_j = (\delta_{1j}, \dots, \delta_{mj}) \text{ and here we are not summing in } j, \\ H_{\nu\mu}(0) &= 0 \text{ for } |\mu + \nu| = 1 \end{aligned}$$

These  $k_{\mu\nu}^{(1)}(\varrho)$  and  $H_{\nu\mu}(x, \varrho)$  are smooth in all variables with  $H_{\nu\mu}(\cdot, \cdot) \in C^\infty(\mathbb{R}_\varrho, H_x^{K,S}(\mathbb{R}^3, \mathbb{C}^8))$  for all  $(K, S)$ .

(3) We have for all indexes

$$(3.84) \quad k_{\mu\nu}^{(1)} = (k_{\nu\mu}^{(1)})^*, \quad k_{\mu\nu} = k_{\nu\mu}^*, \quad H_{\nu\mu} = -C \Sigma_1 H_{\mu\nu}.$$

(4) We have  $F_2(x, 0, 0, 0, 0) = 0$ .

(5) For all  $(K, S, K', S')$  positives there is a neighborhood  $\mathcal{U}^{-K', -S'}$  of  $\{(0, 0)\}$  in  $\widetilde{\mathcal{P}}^{-K', -S'}$ , see (3.46), such that

(a) for  ${}^t\eta = (\zeta, C\zeta)$  where  $\zeta \in \mathbb{C}^4$ . we have, for  $k_{\mu\nu}(x, z, f, \eta, \varrho)$  with  $(z, f, \zeta, \varrho) \in \mathcal{U}^{-K', -S'} \times \mathbb{C}^4 \times \mathbb{R}$

$$(3.85) \quad \forall l \in \mathbb{N}^6, \quad \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l k_{\mu\nu}\|_{H_x^{K,S}(\mathbb{R}^3, \mathbb{C})} \leq C_l;$$

(b) for  $H_{\nu\mu}(x, z, f, g, \varrho)$ ,

$$(3.86) \quad \forall l \in \mathbb{N}^6, \quad \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l H_{\nu\mu}\|_{H_x^{K,S}(\mathbb{R}^3, \mathbb{C}^2)} \leq C_l;$$

(c) for  $F_j(x, z, f, g, \varrho)$ ,

$$\forall l \in \mathbb{N}^6, \quad \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l F_j\|_{H_x^{K,S}(\mathbb{R}^3, B((\mathbb{C}^2)^{\otimes j}, \mathbb{C}))} \leq C_l;$$

(d) we have  $\widehat{\mathcal{R}}_2^{(1)}(z, f, \varrho) \in C^\infty(\mathcal{U}^{-K', -S'} \times \mathbb{R}, \mathbb{R})$  with

$$|\widehat{\mathcal{R}}_2^{(1)}(z, f, \varrho)| \leq C(|z| + |\varrho| + \|f\|_{H^{-K', -S'}}) \|f\|_{H^{-K', -S'}}^2.$$

*Proof.* The following proof is a continuation of proof of Lemma 3.13. We thus consider  $H = K \circ \mathcal{G}_1$  as a function of  $(\varrho, U)$ . By  $\mathcal{G}_1(0, \Phi_{\omega_0}) = \mathcal{F}_1(\Phi_{\omega_0}) = \Phi_{\omega_0}$ ,  $K'(\Phi_{\omega_0}) = 0$  and  $\|\mathcal{F}_1(U) - U\|_{\mathcal{P}K, S} \lesssim \|R\|_{L^2}^2$  we conclude  $H'(\Phi_{\omega_0}) = 0$  and  $H''(\Phi_{\omega_0}) = K''(\Phi_{\omega_0})$ . In particular, this yields the formula for  $H_2^{(1)} + \widetilde{\mathcal{R}}^{(1)}$  for  $\varrho = \|f\|_2^2 = 0$ .

The other terms are obtained by substituting in  $K_P$  of (3.78) the formula (3.59). The term  $\psi(\varrho)$  arises from  $d(\omega \circ \mathcal{G}_1) - \omega \circ \mathcal{G}_1 \|u_0\|_2^2$ . There are no monomials  $\|f\|_2^j z^\mu \bar{z}^\nu \langle H, f \rangle^i$  with  $|\mu + \nu| + i = 1$ , due to (3.60) (applied for  $\omega = \omega_0$ ). By  $\langle i\beta\alpha_2 \Sigma_1 \Sigma_3 f, f \rangle = \|f\|_2^2$ , we have  $\langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_{\omega_0 + \delta\omega} f, f \rangle = \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_{\omega_0} f, f \rangle + \frac{\|f\|_2^2 \varrho}{2} + \widetilde{F}_2$  where  $\widetilde{F}_2$  can be absorbed in  $j = 2$  in  $\widetilde{\mathcal{R}}^{(2)}$  and  $\frac{\|f\|_2^2 \varrho}{2}$  can be absorbed in  $\psi$  when restricted to  $\varrho = \|f\|_2^2$ .

Notice that  $\widehat{\mathcal{R}}_2^{(1)}$  is a remainder term obtained from terms in  $\mathcal{E}$  of Lemme 3.13.  $\square$

3.4.2. *Normal form.* Here again and in the following sections, we use the notation  $\lambda_j^0 = \lambda_j(\omega_0)$ .

Let

$$(3.87) \quad \mathcal{H} = \mathcal{H}_{\omega_0} P_c(\mathcal{H}_{\omega_0}) \quad \text{and} \quad \mathcal{H}_0 = \mathcal{H}_{\omega_0, 0}.$$

**Definition 3.21.** A function  $Z(z, f)$  is in normal form if it is of the form

$$(3.88) \quad Z = Z_0 + Z_1$$

where we have finite sums of the following types:

$$(3.89) \quad Z_1 = \sum_{|\lambda^0 \cdot (\nu - \mu)| > m - \omega_0} z^\mu \bar{z}^\nu \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 G_{\mu\nu}(\|f\|_2^2), f \rangle$$

with  $H_{\mu\nu}(x, \varrho) \in C^\infty(\mathbb{R}_\varrho, H_x^{K,S})$  for all  $K, S$ ;

$$(3.90) \quad Z_0 = \sum_{\lambda^0 \cdot (\mu - \nu) = 0} a_{\mu,\nu}(\|f\|_2^2) z^\mu \bar{z}^\nu$$

and  $a_{\mu,\nu}(\varrho) \in C^\infty(\mathbb{R}_\varrho, \mathbb{C})$ . We will always assume the symmetries (3.84).  $\square$

We consider the coefficients of the type of (3.81) (below it will be those of the  $H_2^{(r)}$  in Theorem 3.25) and thus let, for  $\delta_j = (\delta_{1j}, \dots, \delta_{nj})$ ,

$$(3.91) \quad \lambda_j = \lambda_j(\|f\|_2^2) = \lambda_j^0 + k_{\delta_j \delta_j}(\|f\|_2^2), \quad \lambda = (\lambda_1, \dots, \lambda_m).$$

Let

$$(3.92) \quad D_2 = \sum_{j=1}^n \varepsilon_j \lambda_j(\|f\|_2^2) |z^j| + \frac{1}{2} \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}_{\omega_0} f, f \rangle.$$

We have  $(\lambda'_j(\varrho))$  is the derivative in  $\varrho$  for  $F$  a scalar valued function that, summing on repeated indexes,

$$(3.93) \quad \begin{aligned} \{D_2, F\} &:= dD_2(X_F) = \partial_j D_2(X_F)_j + \partial_{\bar{j}} D_2(X_F)_{\bar{j}} + \langle \nabla_f D_2, (X_F)_f \rangle \\ &= -i\partial_j D_2 \partial_{\bar{j}} F + i\partial_{\bar{j}} D_2 \partial_j F - \langle \nabla_f D_2, \beta\alpha_2 \Sigma_3 \Sigma_1 \nabla_f F \rangle = \\ &= i\lambda_j z_j \partial_j F - i\lambda_{\bar{j}} \bar{z}_{\bar{j}} \partial_{\bar{j}} F + i\langle \mathcal{H}f, \nabla_f F \rangle + 2i\lambda'_j(\|f\|_2^2) |z_j|^2 \langle f, \Sigma_3 \nabla_f F \rangle. \end{aligned}$$

In particular, we have, for  $G = G(x)$ , (we use  $\Sigma_1 i \Sigma_2 = \Sigma_3$ )

$$(3.94) \quad \begin{aligned} \{D_2, z^\mu \bar{z}^\nu\} &= i\lambda \cdot (\mu - \nu) z^\mu \bar{z}^\nu, \\ \{D_2, \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 G, f \rangle\} &= i\langle \mathcal{H}f, i\beta\alpha_2 \Sigma_1 \Sigma_3 G \rangle - 2i \sum_{j=1}^n \lambda'_j |z_j|^2 \langle i\beta\alpha_2 \Sigma_1 f, G \rangle \\ &= -i\langle f, i\beta\alpha_2 \Sigma_1 \Sigma_3 \mathcal{H}G \rangle - 2i \sum_{j=1}^n \lambda'_j |z_j|^2 \langle i\beta\alpha_2 \Sigma_1 f, G \rangle, \\ \{D_2, \frac{1}{2}\|f\|_2^2\} &= \{D_2, \frac{1}{2}\langle f, i\beta\alpha_2 \Sigma_1 f \rangle\} = -i\langle \mathcal{H}f, i\beta\alpha_2 \Sigma_1 f \rangle = -i\langle V_{\omega_0} f, i\beta\alpha_2 \Sigma_1 f \rangle. \end{aligned}$$

In the sequel we will prove that  $\|f\|_2$  is small.

*Remark 3.22.* We will consider only

$$|\mu + \nu| \leq 2N + 3.$$

Then,  $\lambda^0 \cdot (\mu - \nu) \neq 0$  implies  $|\lambda^0 \cdot (\mu - \nu)| \geq c > 0$  for some fixed  $c$ , and so we can assume also  $|\lambda \cdot (\mu - \nu)| \geq c/2$ . Similarly  $|\lambda^0 \cdot (\mu - \nu)| < m - \omega_0$  (resp.  $|\lambda^0 \cdot (\mu - \nu)| > m - \omega_0$ ) will be assumed equivalent to  $|\lambda \cdot (\mu - \nu)| < m - \omega_0$  (resp.  $|\lambda \cdot (\mu - \nu)| > m - \omega_0$ ).

**Lemma 3.23** (Homological equation). *Consider*

$$(3.95) \quad K = \sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\|f\|_2^2) z^\mu \bar{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \bar{z}^\nu \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 K_{\mu\nu}(\|f\|_2^2), f \rangle.$$

Suppose that all the terms in (3.95) are not in normal form and that the symmetries (3.84) hold. Consider

$$(3.96) \quad \begin{aligned} \chi = & \sum_{|\mu+\nu|=M_0+1} \frac{k_{\mu\nu}(\|f\|_2^2)}{i\lambda \cdot (\mu - \nu)} z^\mu \bar{z}^\nu \\ & + \sum_{|\mu+\nu|=M_0} z^\mu \bar{z}^\nu \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 \frac{1}{i(\lambda \cdot (\mu - \nu) - \mathcal{H})} K_{\mu\nu}(\|f\|_2^2), f \rangle. \end{aligned}$$

Then we have

$$(3.97) \quad \{D_2, \chi\} = K + L$$

with, summing on repeated indexes,

$$(3.98) \quad \begin{aligned} L = & -2 \frac{k'_{\mu\nu}}{(\mu - \nu) \cdot \lambda} z^\mu \bar{z}^\nu \langle V_{\omega_0} f, i\beta\alpha_2 \Sigma_1 f \rangle \\ & - 2\lambda'_j z^\mu \bar{z}^\nu |z_j|^2 \left\langle i\beta\alpha_2 \Sigma_1 f, \frac{1}{(\mu - \nu) \cdot \lambda - \mathcal{H}} K_{\mu\nu} \right\rangle \\ & + 2\lambda' \cdot (\mu - \nu) z^\mu \bar{z}^\nu |z_j|^2 \left\langle f, i\beta\alpha_2 \frac{1}{((\mu - \nu) \cdot \lambda - \mathcal{H})^2} K_{\mu\nu} \right\rangle \langle V_{\omega_0} f, i\beta\alpha_2 \Sigma_1 f \rangle \\ & - 2z^\mu \bar{z}^\nu \left\langle f, \Sigma_3 \Sigma_1 \frac{1}{(\mu - \nu) \cdot \lambda - \mathcal{H}} K'_{\mu\nu} \right\rangle \langle V_{\omega_0} f, i\beta\alpha_2 \Sigma_1 f \rangle. \end{aligned}$$

The coefficients in (3.96) satisfy (3.84).

*Proof.* The proof follows by the tables (3.94), by the product rule for the derivative and by the symmetry properties of  $\mathcal{H}$ .  $\square$

3.4.3. *Canonical transformations.* First we consider functions

$$(3.99) \quad \chi = \sum_{|\mu+\nu|=M_0+1} b_{\mu\nu}(\|f\|_2^2) z^\mu \bar{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \bar{z}^\nu \langle i\beta\alpha_2 \Sigma_1 \Sigma_3 B_{\mu\nu}(\|f\|_2^2), f \rangle$$

where  $b_{\mu\nu}(\varrho) \in C^\infty(\mathbb{R}_\varrho, \mathbb{C})$  and  $B_{\mu\nu}(x, \varrho) \in C^\infty(\mathbb{R}, P_c(\omega_0) H_x^{k,s}(\mathbb{R}^3, \mathbb{C}^8))$  for all  $k$  and  $s$ . Assume

$$(3.100) \quad b_{\mu\nu} = (b_{\nu\mu})^* \text{ and } i\beta\alpha_2 \Sigma_1 B_{\mu\nu} = -(B_{\nu\mu})^* \text{ for all indexes.}$$

The canonical transformations used in the proof of Theorem 3.25 are compositions of the Lie transforms :

$$\phi = \phi^\tau \Big|_{\tau=1},$$

with  $\phi^\tau$  the flow of the Hamiltonian vector field  $X_\chi$  (with respect to  $\Omega_0$  and only in  $(z, f)$ ). Let for  $K > 0$  and  $S > 0$  fixed and large

$$(3.101) \quad \|\chi\| = \sum |b_{\mu\nu}(\|f\|_2^2)| + \sum \|B_{\mu\nu}(\|f\|_2^2)\|_{H^{K,S}}.$$

Then, the following lemma can be proved like Lemma 9.2 [Cuc10].

**Lemma 3.24.** *Consider the  $\chi$  in (3.99) and its Lie transform  $\phi$ . Set  $(z', f') = \phi(z, f)$ . Then there are  $\mathcal{G}(z, f, \varrho)$ ,  $\Gamma(z, f, \varrho)$ ,  $\Gamma_0(z, f, \rho)$  and  $\Gamma_1(z, f, \rho)$  with the following properties.*

- (1)  $\Gamma \in C^\infty(\mathcal{U}^{-K', -S'}, \mathbb{C}^n)$ ,  $\Gamma_0, \Gamma_1 \in C^\infty(\mathcal{U}^{-K', -S'}, \mathbb{R})$ , with  $\mathcal{U}^{-K', -S'} \subset \mathbb{C}^n \times H_c^{-K', -S'}(\omega_0) \times \mathbb{R}$  an appropriately small neighborhood of the origin.
- (2)  $\mathcal{G} \in C^\infty(\mathcal{U}^{-K', -S'}, H_c^{K,S}(\omega_0))$  for any  $K, S$ .
- (3) The transformation  $\phi$  is of the following form:

$$(3.102) \quad z' = z + \Gamma(z, f, \|f\|_2^2),$$

$$(3.103) \quad f' = e^{i\Gamma_0(z, f, \|f\|_2^2) P_c(\omega_0) \Sigma_3} f + \mathcal{G}(z, f, \|f\|_2^2).$$

- (4) There are constants  $c_{K', S'}$  and  $c_{K, S, K', S'}$  such that

$$(3.104) \quad |\Gamma(z, f, \|f\|_2^2)| \leq c_{K', S'} (\|\chi\| + (3.108)) |z|^{M_0-1} (|z| + \|f\|_{H^{-K', -S'}}),$$

$$(3.105) \quad \|\mathcal{G}(z, f, \|f\|_2^2)\|_{H^{K,S}} \leq c_{K, S, K', S'} (\|\chi\| + (3.108)) |z|^{M_0},$$

$$(3.106) \quad |\Gamma_0(z, f, \|f\|_2^2)| \leq c_{K', S'} |z|^{M_0-1} (|z| + \|f\|_{H^{-K', -S'}})^2.$$

(5) We have

$$(3.107) \quad \|f'\|_2^2 = \|f\|_2^2 + \Gamma_1(z, f, \|f\|_2^2),$$

$$(3.108) \quad \begin{aligned} & |\Gamma_1(z, f, \|f\|_2^2)| \leq \\ & C|z|^{M_0-1}(|z|^{M_0+2} + |z|^2\|f\|_{H^{-K'}, -S'} + \|f\|_{H^{-K'}, -S'}^3). \end{aligned}$$

(6) We have

$$(3.109) \quad e^{i\Gamma_0 P_c(\omega_0)\Sigma_3} = e^{i\Gamma_0\Sigma_3} + T(\Gamma_0),$$

where  $T(r) \in C^\infty(\mathbb{R}, B(H^{-K'}, -S', H^{K, S}))$  for all  $(K, S, K', S')$ , with norm

$$\|T(r)\|_{B(H^{-K'}, -S', H^{K, S})} \leq C(K, S, K', S')|r|.$$

More specifically, the range of  $T(r)$  is a subspace of  $L_d^2(\mathcal{H}) + L_d^2(\mathcal{H}^*)$ .

The crux of this section is the following result.

**Theorem 3.25.** *For any integer  $r \geq 2$  there are a neighborhood  $\mathcal{U}^{1,0}$  of  $\{(0,0)\}$  in  $\tilde{\mathcal{P}}^{1,0}$ , see (3.46), and a smooth canonical transformation  $\mathcal{T}_r : \mathcal{U}^{1,0} \rightarrow \tilde{\mathcal{P}}^{1,0}$  s.t.*

$$(3.110) \quad H^{(r)} := H \circ \mathcal{T}_r = d(\omega_0) - \omega_0 \|u_0\|_2^2 + \psi(\|f\|_2^2) + H_2^{(r)} + Z^{(r)} + \mathcal{R}^{(r)}.$$

where:

- (i)  $H_2^{(r)} = H_2^{(2)}$  for  $r \geq 2$ , is of the form (3.81) where  $k_{\mu\nu}^{(r)}(\|f\|_2)$  satisfy (3.83)–(3.84);
  - (ii)  $Z^{(r)}$  is in normal form, in the sense of Definition 3.21 below, with monomials of degree  $\leq r$  whose coefficients satisfy (3.84);
  - (iii) the transformation  $\mathcal{T}_r$  is of the form (3.102)–(3.103) and satisfies (3.104)–(3.106) for  $M_0 = 1$ ;
  - (iv) we have  $\mathcal{R}^{(r)} = \sum_{d=0}^6 \mathcal{R}_d^{(r)}$  and for all  $(K, S, K', S')$  positives there is a neighbourhood  $\mathcal{U}^{-K', -S'}$  of  $\{(0,0)\}$  in  $\tilde{\mathcal{P}}^{-K', -S'}$  such that
- (iv.0)

$$\mathcal{R}_0^{(r)} = \sum_{|\mu+\nu|=r+1} z^\mu \bar{z}^\nu \int_{\mathbb{R}^3} k_{\mu\nu}^{(r)}(x, z, f, f(x), \|f\|_2^2) dx$$

and for  $k_{\mu\nu}^{(r)}(z, f, \eta, \varrho)$  with  ${}^t\eta = (\zeta, C\zeta)$ ,  $\zeta \in \mathbb{C}^4$  we have for  $(z, f) \in \mathcal{U}^{-K', -S'}$  and  $|\varrho| \leq 1$

$$(3.111) \quad \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l k_{\mu\nu}^{(r)}(\cdot, z, f, \eta, \varrho)\|_{H^{K, S}(\mathbb{R}^3, \mathbb{C})} \leq C_l \text{ for all } l;$$

(iv.1)

$$\mathcal{R}_1^{(r)} = \sum_{|\mu+\nu|=r} z^\mu \bar{z}^\nu \int_{\mathbb{R}^3} \left[ i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{\mu\nu}^{(r)}(x, z, f, f(x), \|f\|_2^2) \right]^T f(x) dx$$

$$(3.112) \quad \text{with } \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l H_{\nu\mu}^{(r)}(\cdot, z, f, \eta, \varrho)\|_{H^{K, S}(\mathbb{R}^3, \mathbb{C}^8)} \leq C_l \text{ for all } l;$$

(iv.2–5) for  $2 \leq d \leq 5$ ,

$$\mathcal{R}_d^{(r)} = \int_{\mathbb{R}^3} F_d^{(r)}(x, z, f, f(x), \|f\|_2^2) f^{\otimes d}(x) dx + \widehat{\mathcal{R}}_d^{(r)},$$

with for any  $l$

$$(3.113) \quad \|\nabla_{z, \bar{z}, \zeta, C\zeta, f, \varrho}^l F_d^{(r)}(\cdot, z, f, \eta, \varrho)\|_{H^{K, S}(\mathbb{R}^3, B((\mathbb{C}^8)^{\otimes d}, \mathbb{C}))} \leq C_l,$$

with  $F_2^{(r)}(x, 0, 0, 0, 0) = 0$  and with  $\widehat{\mathcal{R}}_d^{(r)}(z, f, \|f\|_2^2)$  s.t.

$$(3.114) \quad \begin{aligned} & \widehat{\mathcal{R}}_d^{(r)}(z, f, \varrho) \in C^\infty(\mathcal{U}^{-K', -S'} \times \mathbb{R}, \mathbb{R}), \\ & |\widehat{\mathcal{R}}_d^{(r)}(z, f, \varrho)| \leq C \|f\|_{H^{-K'}, -S'}^d, \\ & |\widehat{\mathcal{R}}_2^{(r)}(z, f, \varrho)| \leq C(|z| + |\varrho| + \|f\|_{H^{-K'}, -S'}) \|f\|_{H^{-K'}, -S'}^2; \end{aligned}$$

$$(iv.6) \quad \mathcal{R}_6^{(r)} = \int_{\mathbb{R}^3} G\left(\frac{1}{2}(P_c(\omega)f(x)) \cdot i\alpha_2 \Sigma_3 \Sigma_1(P_c(\omega)f(x))\right) dx.$$

The proof of Theorem 3.25 is the same of Theorem 9.1 in [Cuc10] and we skip it. The ingredients needed in the proof (in particular the notion of normal form) are described above.

## 4. NON LINEAR DYNAMICS

**4.1. Dispersion.**  $\lambda$  We apply Theorem 3.25 for  $r = 2N_1 + 1$  (recall  $N_j \lambda_j < m - \omega_0 < (N_j + 1)\lambda_j$ ). In the rest of the article we work with the Hamiltonian  $H^{(r)}$ . We will drop the upper index. So we will set  $H = H^{(r)}$ ,  $H_2 = H_2^{(r)}$ ,  $\lambda_j = \lambda_j^{(r)}$ ,  $\lambda = \lambda^{(r)}$ ,  $Z_a = Z_a^{(r)}$  for  $a = 0, 1$  and  $\mathcal{R} = \mathcal{R}^{(r)}$ . In particular we will denote by  $H_{\mu\nu}$  the coefficients  $G_{\mu\nu}^{(r)}$  of  $Z_1^{(r)}$ . We will show:

**Theorem 4.1.** *Fix  $p_0 > 2$  and  $\tau_0 > 1$ . Let  $\frac{2}{p} = \frac{3}{2}(1 - \frac{2}{q})$  and  $\alpha(q) = \frac{2}{p}$ , i.e.  $(1 + \frac{\theta}{2})(1 - \frac{2}{q}) = \frac{2}{p}$  with  $\theta = 1$  in Theorem 2.17. Consider  $k_0 > 3$  (as in Theorem 1.1). There is a fixed  $C > 0$  such that for  $\varepsilon_0 > 0$  sufficiently small, for  $\epsilon \in (0, \varepsilon_0)$  and for  $p \geq p_0$  we have the following inequalities:*

$$(4.1) \quad \|f\|_{L_t^p([0, \infty), B_{q,2}^{k_0 - \frac{2}{p}})} \leq C\epsilon;$$

$$(4.2) \quad \|f\|_{L_t^2([0, \infty), H_x^{k_0, -\tau_0})} \leq C\epsilon$$

$$(4.3) \quad \|f\|_{L_t^2([0, \infty), L_x^\infty)} \leq C\epsilon$$

$$(4.4) \quad \|z^\mu\|_{L_t^2([0, \infty))} \leq C\epsilon \text{ for all multi indexes } \mu \text{ with } \lambda \cdot \mu > m - \omega_0$$

$$(4.5) \quad \|z_j\|_{W_t^{1, \infty}([0, \infty))} \leq C\epsilon \text{ for all } j \in \{1, \dots, n\}.$$

Notice that, due to time reversibility, it is easy to conclude that (4.1)–(4.5) are true over the whole real line.

The proof of Theorem 4.1 involves a standard continuation argument following [Sog95, End of proof of Theorem II.2.1]. We assume

$$(4.6) \quad \|f\|_{L_t^p([0, T], B_{q,2}^{k_0 - \frac{2}{p}})} + \|f\|_{L_t^2([0, T], H_x^{k_0, -\tau_0})} + \|f\|_{L_t^2([0, T], L_x^\infty)} \leq C_1\epsilon$$

$$(4.7) \quad \|z^\mu\|_{L_t^2([0, T])} \leq C_2\epsilon \text{ for all multi indexes } \mu \text{ with } \omega \cdot \mu > m - \omega_0$$

$$(4.8) \quad \|z_j\|_{W_t^{1, \infty}([0, T])} \leq C_3\epsilon \text{ for all } j \in \{1, \dots, n\}$$

for fixed sufficiently large constants  $C_1$ – $C_3$ . Notice that there is an  $\varepsilon_1 > 0$  such that this assumption is true for all  $|z(0)| + \|f(0)\|_{H^{k_0}} < \varepsilon_1$  if say  $T \in (0, 1]$ . We then prove that there exists a fixed  $\varepsilon_0 \in (0, \varepsilon_1)$ , with  $\varepsilon_0 = \varepsilon_0(C_1, C_2, C_3)$ , such that for  $\epsilon \in (0, \varepsilon_0)$ , (4.6)–(4.8) imply the same estimate but with  $C_1$ – $C_3$  replaced by  $C_1/2$ – $C_3/2$ . This implies that the set of  $T$  such that (4.6)–(4.8) is open in  $\mathbb{R}^+$ . Since it is also closed, it is all  $\mathbb{R}^+$ . Then (4.6)–(4.8) hold with  $[0, T]$  replaced by  $[0, \infty)$  for all  $|z(0)| + \|f(0)\|_{H^{k_0}} < \epsilon < \varepsilon_0$ .

The proof of Theorem 4.1 consists in three main steps.

- (i) Estimate  $f$  in terms of  $z$ .
- (ii) Substitute the variable  $f$  with a new "smaller" variable  $g$  and find smoothing estimates for  $g$ .
- (iii) Reduce the system for  $z$  to a closed system involving only the  $z$  variables, by insulating the part of  $f$  which interacts with  $z$ , and by decoupling the rest (this remainder is  $g$ ). Then clarify the nonlinear Fermi golden rule.

Step (i). Using the Proposition 4.2 below, we will choose  $C_1 > 2K_1(C_2)$ . This tells us that if we get upper bounds on  $C_2$  and  $C_3$ , and this is done in Subsection 4.2, then we will have proved Theorem 4.1.

**Proposition 4.2.** *Assume (4.6)–(4.8). Then there exist constants  $C = C(C_1, C_2, C_3), K_1(C_2)$ , such that, if  $C(C_1, C_2, C_3)\epsilon$  is sufficiently small, then we have*

$$(4.9) \quad \|f\|_{L_t^p([0, T], B_{q,2}^{k_0 - \frac{2}{p}})} + \|f\|_{L_t^2([0, T], H_x^{k_0, -\tau_0})} + \|f\|_{L_t^2([0, T], L_x^\infty)} \leq K_1(C_2)\epsilon.$$

*Proof.* Consider  $Z_1$  of the form (3.89). Set:

$$(4.10) \quad H_{\mu\nu}^0 = H_{\mu\nu}(\|f\|_2^2) \text{ for } \|f\|_2^2 = 0; \lambda_j^0 = \lambda_j(\omega_0).$$

Then we have (with finite sums)

$$(4.11) \quad \begin{aligned} i\dot{f} - \mathcal{H}f - 2(\partial_{\|f\|_2^2} H)P_c(\omega_0)\Sigma_3 f &= \sum_{\substack{|\lambda^0 \cdot (\nu - \mu)| > m - \omega_0, \\ |\mu + \nu| \leq 2N_1 + 1}} z^\mu \bar{z}^\nu H_{\mu\nu}^0 \\ &+ \sum_{\substack{|\lambda^0 \cdot (\nu - \mu)| > m - \omega_0, \\ |\mu + \nu| \leq 2N_1 + 1}} z^\mu \bar{z}^\nu (H_{\mu\nu} - H_{\mu\nu}^0) + i\beta\alpha_2 \Sigma_3 \Sigma_1 \nabla_f \mathcal{R} - 2(\partial_{\|f\|_2^2} \mathcal{R})P_c(\omega_0)\Sigma_3 f. \end{aligned}$$

In order to obtain bounds on  $f$ , we need bounds on the right hand term of the equation especially the last two terms. They are provided by the following lemma.

**Lemma 4.3.** *Assume (4.6)–(4.8) and consider a fixed  $\tau_0 > 1$ . Then there is a constant  $C = C(C_1, C_2, C_3)$  independent of  $\epsilon$  such that the following is true: we have  $\beta\alpha_2\Sigma_3\Sigma_1\nabla_f\mathcal{R} - 2(\partial_{\|f\|_2^2}\mathcal{R})P_c(\omega_0)\Sigma_3f = R_1 + R_2$  with*

$$(4.12) \quad \begin{aligned} \|R_1\|_{H_x^{k_0}} &\leq C(C_1, C_2, C_3)(|z|^{2N_1+2} + \|f\|_{L^\infty}^2\|f\|_{H_x^{k_0}}) \\ \|R_2\|_{H_x^{k_0, \tau_0}} &\leq C(C_1, C_2, C_3)(|z| + \|f\|_{L_x^2}^2 + \|f\|_{H_x^{k_0, -\tau_0}})\|f\|_{H_x^{k_0, -\tau_0}}. \end{aligned}$$

In particular we have for some other fixed constant  $C = C(C_1, C_2, C_3)$ ,

$$(4.13) \quad \|R_1\|_{L_x^1([0, T], H_x^{k_0})} + \|R_2\|_{L_x^2([0, T], H_x^{k_0, \tau_0})} \leq C(C_1, C_2, C_3)\epsilon^2.$$

*Proof.* (4.13) is a consequence of (4.12) and (4.6)–(4.8). We focus on (4.12). For  $d \leq 1$  and arbitrary fixed  $(S, K)$  we have  $\nabla_f\mathcal{R}_d \in H^{S, K}$ . By (iv0–iv1) Theorem 3.25

$$(4.14) \quad \|\nabla_f\mathcal{R}_0\|_{H^{S, K}} + \|\nabla_f\mathcal{R}_1\|_{H^{S, K}} \leq C|z|^{2N_1+2}.$$

These terms can be absorbed in  $R_1$ . For  $2 \leq d \leq 5$  we have

$$(4.15) \quad \Sigma_3\Sigma_1\nabla_f\widehat{\mathcal{R}}_d - 2(\partial_{\|f\|_2^2}\widehat{\mathcal{R}}_d)P_c(\omega_0)\Sigma_3f = \Sigma_3\Sigma_1\nabla_f\widehat{\mathcal{R}}_d(z, f, \rho),$$

computed at  $\rho = \|f\|_2^2$ . By (3.114) we obtain

$$(4.16) \quad \begin{aligned} \|\nabla_f\widehat{\mathcal{R}}_d(z, f, \rho)\|_{H^{K', S'}} &\leq C\|f\|_{H^{-K', -S'}}^{d-1} \text{ for } 3 \leq d \leq 5 \text{ and} \\ \|\nabla_f\widehat{\mathcal{R}}_2(z, f, \rho)\|_{H^{K', S'}} &\leq C\|f\|_{H^{-K', -S'}}^2 + C|z|\|f\|_{H^{-K', -S'}}. \end{aligned}$$

Since  $K'$  and  $S'$  are arbitrarily large, we have  $\|f\|_{H^{-K', -S'}} \leq \|f\|_{H^{k_0, -\tau_0}}$ . So these terms can be absorbed in  $R_2$ . Other terms are treated as in [BC09, Lemma 7.5]: For  $d = 2, 3, 4, 5$  we have schematically

$$(4.17) \quad \begin{aligned} &F_d(x, z, f, f(t, \cdot), \rho)f^{\otimes(d-1)}(t, \cdot) + \partial_w F_d(x, z, f, w, \rho)_{w=f(t, \cdot)}f^{\otimes d}(t, \cdot) \\ &+ \nabla_g \left( \int_{\mathbb{R}^3} F_d(x, z, g, f(t, x), \|f(t)\|_{L_x^2}^2)[f(t, x)]^{\otimes d} dx \right)_{g=f}. \end{aligned}$$

The first line of (4.17) has  $H_x^{k_0, \tau_0}$  norm bounded, for some fixed sufficiently large  $\mathbf{N}$ , by

$$(4.18) \quad \begin{aligned} &\widetilde{C}\|\langle x \rangle^{\mathbf{N}} F_d(x, z, f, f(t, x), \rho)\|_{W_x^{k_0, \infty}}\|f\|_{H_x^{k_0, -\tau_0}}^{d-1} \\ &+ \widetilde{C}\|\langle x \rangle^{\mathbf{N}} \partial_w F_d(x, z, f, w, \rho)_{w=f(t, x)}\|_{W_x^{k_0, \infty}}\|f\|_{H_x^{k_0, -\tau_0}}^d \leq C\|f\|_{H_x^{k_0, -\tau_0}}^{d-1} + C\|f\|_{H_x^{k_0, -\tau_0}}^d. \end{aligned}$$

When these terms are bounded by  $\|f\|_{H_x^{k_0, -\tau_0}}^{d_1}$  for  $d_1 \geq 2$ , we can absorb them in  $R_2$ . Cases  $d_1 = 1$  come from terms in the first line of (4.18) with  $d = 2$ . By  $F_2(x, 0, 0, 0, 0) = 0$  these are less than

$$(4.19) \quad (|z| + \|f\|_{H_x^{-K', -S'}} + \|f\|_{L_x^2}^2)\|f\|_{H_x^{k_0, -\tau_0}}$$

and can be absorbed in  $R_2$ . Looking at the second line of (4.17) and for  $\mathbf{N}$  sufficiently large, we have

$$(4.20) \quad \begin{aligned} &\|\nabla_g \left( \int_{\mathbb{R}^3} F_d(x, z, g, f(t, x), \|f(t)\|_{L_x^2}^2)[f(t, x)]^{\otimes d} dx \right)_{g=f}\|_{H_x^{k_0}} = \\ &\left| \sup_{\|\psi\|_{H_x^{-k_0}}=1} \int_{\mathbb{R}^3} D_g F_d(x, z, g, f(t, x), \|f(t)\|_{L_x^2}^2)_{g=f}[\psi][f(t, x)]^{\otimes d} dx \right| \\ &\leq C \sup_{\|\psi\|_{H_x^{-k_0}}=1} \|D_g F_d(x, z, g, f(t, x), \|f(t)\|_{L_x^2}^2)_{g=f}[\psi]\|_{L_x^{\infty, \mathbf{N}}}\|f\|_{H_x^{k_0, -\tau_0}}^d \leq C\|f\|_{H_x^{k_0, -\tau_0}}^d. \end{aligned}$$

So the second line of (4.17) can be absorbed in  $R_2$ . Finally we consider  $\nabla_f\mathcal{R}_6 = \Sigma_1g(|f(t, x)|^2/2)f(t, x)$ . Then for a fixed  $C$  we have

$$(4.21) \quad \|\nabla_f\mathcal{R}_6\|_{H_x^{k_0}} \leq C\|f\|_{L_x^\infty}^2\|f\|_{H_x^{k_0}}.$$

□

Denote by  $F$  the rhs of (4.11) and set  $\varphi = 2\partial_{\|f\|_2^2}H$ .



**Lemma 4.4.** *Consider  $i\dot{\psi} - \mathcal{H}\psi - \varphi(t)\Sigma_3 P_c \psi = F$  where  $P_c = P_c(\omega_0)$  and  $\psi = P_c \psi$ . Let  $k \in \mathbb{R}$  and  $\tau_0 > 1$ . Then there exist  $c_0 > 0$  and  $C > 0$  such that if  $\|\varphi\|_{L_t^\infty[0,T]} < c_0$  then for  $p \geq p_0 > 2$  and for  $(p, q)$  as in Theorem 4.1 we have*

$$(4.22) \quad \|\psi\|_{L_t^p([0,T], B_{q,2}^{k-\frac{2}{p}}) \cap L_t^2([0,T], H_x^{k,-\tau_0})} \leq C\|\psi(0)\|_{H^k} + C\|F\|_{L_t^1([0,T], H_x^k) + L_t^2([0,T], H_x^{k,\tau_0})}$$

*Proof.* We apply the argument for the NLS in Lemma B.2 [NS10], see also Theorem 1.5 [Bec08]. A more precise statement than Lemma B.2 [NS10] is in [BP95, Cuc08], but the proof does not seem easy to reproduce for Dirac. We fix any  $\delta > 0$ . Let  $P_d = P_d(\omega_0)$  and  $\mathcal{H}_0 = \mathcal{H}_{\omega_0,0}$ . Consider

$$(4.23) \quad i\dot{Z} - \mathcal{H}P_c Z + i\delta P_d Z - \varphi\Sigma_3 P_c Z = F.$$

Then notice that for  $Z(0) = \psi(0)$  the solution of (4.23) satisfies  $Z(t) \equiv \psi(t)$ . We rewrite (4.23) as

$$(4.24) \quad i\dot{Z} - \mathcal{H}_0 Z - \varphi\Sigma_3 Z = F + (V - \mathcal{H}P_d - i\delta P_d)Z - \varphi\Sigma_3 P_d Z.$$

Let  $(V - \mathcal{H}P_d - i\delta P_d) = V_1 V_2$  with  $V_2(x)$  a smooth exponentially decaying and invertible matrix, and with  $V_1$  bounded from  $H^{k,s'} \rightarrow H^{k,s}$  for all  $k, s$  and  $s'$ . For  $\mathcal{U}(t) = e^{-i\Sigma_3 \int_0^t \varphi(t') dt'}$  we have

$$(4.25) \quad Z(t) = \mathcal{U}(t)e^{-i\mathcal{H}_0 t} Z(0) - i \int_0^t e^{i\mathcal{H}_0(t-t')} \mathcal{U}(t) \mathcal{U}^{-1}(t') [F(t') + V_1 V_2 Z(t') - \varphi(t') \Sigma_3 P_d Z(t')] dt'.$$

$c_0 P_d V_2^{-1}$  maps  $H^{-K',-S'} \rightarrow H^{K,S}$  for arbitrarily fixed pairs  $(K, S)$  and  $(K', S')$ . By picking  $c_0$  small enough, we can assume that the related operator norms are small. By Theorems 2.15 and 2.17

$$(4.26) \quad \begin{aligned} \|Z\|_{L_t^p B_{q,2}^{k-\frac{2}{p}} \cap L_t^2 H_x^{k,-\tau_0}} &\leq C\|Z(0)\|_{H^k} + C\|F\|_{L_t^1 H_x^k + L_t^2 H_x^{k,\tau_0}} \\ &+ \|V_1 - \varphi(t)\Sigma_3 P_d V_2^{-1}\|_{L_t^\infty B(H_x^k, H_x^{k,\tau_0})} \|V_2 Z(t)\|_{L_t^2 H_x^k}. \end{aligned}$$

For  $\tilde{T}_0 f(t) = V_2 \int_0^t e^{i\mathcal{H}_0(t-t')} \mathcal{U}(t) \mathcal{U}^{-1}(t') V_1 f(t') dt'$ , by (4.25) we obtain

$$(I + i\tilde{T}_0) V_2 Z(t) = V_2 \mathcal{U}(t) e^{-i\mathcal{H}_0 t} Z(0) - i V_2 \int_0^t e^{i\mathcal{H}_0(t-t')} \mathcal{U}(t) \mathcal{U}^{-1}(t') [F(t') - \varphi(t') \Sigma_3 P_d Z(t')] dt'$$

We then obtain (4.22) if we can show that

$$(4.27) \quad \|(I + i\tilde{T}_0)^{-1} : L_t^2([0, T], H^k(\mathbb{R}^3)) \rightarrow L_t^2([0, T], H^k(\mathbb{R}^3))\| < C_1,$$

for  $c_0 C_1$  smaller than a fixed number. It is enough to prove (4.27) with  $\tilde{T}_0$  replaced by

$$T_0 f(t) = V_2 \int_0^t e^{i\mathcal{H}_0(t-t')} V_1 f(t') dt'.$$

Indeed by Theorem 2.16 we have

$$\begin{aligned} \|(\tilde{T}_0 - T_0) f\|_{L_t^2 H_x^k} &\leq \left\| \int_0^t \|V_2 e^{i\mathcal{H}_0(t-t')} (e^{i\Sigma_3 \int_t^{t'} \varphi(t'') dt''} - 1) V_1 f(t')\|_{H_x^k} dt' \right\|_{L_t^2} \\ &\leq \tilde{C} c_0^{\frac{1}{4}} \left\| \int_0^t \langle t' - t \rangle^{-\frac{5}{4}} \|f(t')\|_{H_x^k} dt' \right\|_{L_t^2} \leq C c_0^{\frac{1}{4}} \|f(t')\|_{L_t^2 H_x^k}. \end{aligned}$$

Set

$$T_1 f(t) = V_2 \int_0^t e^{(i\mathcal{H}P_c + \delta P_d)(t-t')} V_1 f(t') dt' = V_2 \int_0^t (e^{i\mathcal{H}(t-t')} P_c + e^{-\delta|t-t|} P_d) V_1 f(t') dt'.$$

We have  $\|T_1 : L_t^2([0, T], H^k(\mathbb{R}^3)) \rightarrow L_t^2([0, T], H^k(\mathbb{R}^3))\| < C_2$  for a fixed  $C_2$ . For exactly the same reasons of [NS10] we have

$$(I + iT_0)(I - iT_1) = (I - iT_1)(I + iT_0) = I.$$

This yields (4.27) with  $\tilde{T}_0$  replaced by  $T_0$  and with  $C_1 = 1 + C_2$ .  $\square$

**Lemma 4.5.** *Using the notation of Lemma 4.4, but this time picking  $\tau_0 > 3/2$ , we have*

$$(4.28) \quad \|\psi\|_{L_t^2([0,T], L^\infty)} \leq C\|\psi(0)\|_{H^{k_0}} + C\|F\|_{L_t^1([0,T], H_x^{k_0}) + L_t^2([0,T], H_x^{k_0, \tau_0})}$$

*Proof.* We proceed as above until (4.25). We claim we have

$$(4.29) \quad \begin{aligned} \|Z\|_{L_t^2 L_x^\infty} &\leq C \|Z(0)\|_{H^{k_0}} + C \|F\|_{L_t^1 H_x^{k_0} + L_t^2 H_x^{k_0, \tau_0}} \\ &+ \|V_1 - \varphi(t) \Sigma_3 P_d V_2^{-1}\|_{L_t^\infty B(H_x^{k_0}, H_x^{k_0, \tau_0})} \|V_2 Z(t)\|_{L_t^2 H_x^{k_0}}. \end{aligned}$$

(4.29) will yield (4.28) by the argument in Lemma 4.4. So now we prove (4.29). We have for  $k > 1/2$

$$\|e^{-i\mathcal{H}_0 t} Z(0)\|_{L_t^2 L_x^\infty} \leq C \|e^{-i\mathcal{H}_0 t} Z(0)\|_{L_t^2 B_{6,2}^k} \leq C' \|Z(0)\|_{H^{k+1}} \leq C' \|Z(0)\|_{H^{k_0}}$$

by Theorem 2.17. Similarly, splitting  $F = F_1 + F_2$ , we have

$$\begin{aligned} \left\| \int_0^t e^{i\mathcal{H}_0(t'-t)} \mathcal{U}^{-1}(t') F_1(t') dt' \right\|_{L_t^2 L_x^\infty} &\leq C \left\| \int_0^t e^{i\mathcal{H}_0(t'-t)} \mathcal{U}^{-1}(t') F_1(t') dt' \right\|_{L_t^2 B_{6,2}^k} \\ &\leq C' \|F_1\|_{L_t^1 H^{k+1}} \leq C' \|F_1\|_{L_t^1 H^{k_0}}. \end{aligned}$$

Using  $B_{\infty,2}^k \subset L^\infty$  for  $k > 0$ , by Theorem 3.1 [Bou06] we have for  $k_0 > 3$

$$\begin{aligned} \left\| \int_0^t e^{i\mathcal{H}_0(t'-t)} \mathcal{U}^{-1}(t') F_2(t') dt' \right\|_{L_t^2 L_x^\infty} &\leq C \left\| \int_0^t \min\{|t-t'|^{-\frac{1}{2}}, |t-t'|^{-\frac{3}{2}}\} \|F_2(t')\|_{B_{1,2}^{k_0}} dt' \right\|_{L_t^2} \\ &\leq C' \|F_2\|_{L_t^2 B_{1,2}^{k_0}} \leq C'' \|\langle x \rangle^{\tau_0} F_2\|_{L_t^2 B_{2,2}^{k_0}} = C'' \|F_2\|_{L_t^2 H^{k_0, \tau_0}}, \end{aligned}$$

where we have used  $\|\varphi_j * F_2\|_{L_x^1} \leq \|\langle x \rangle^{-\tau_0}\|_{L_x^2} \|\langle x \rangle^{\tau_0} \varphi_j * F_2\|_{L_x^2} \leq C''' \|\varphi_j * (\langle \cdot \rangle^{\tau_0} F_2)\|_{L_x^2}$  for fixed  $C''' > 0$  and fixed  $\tau_0 > 3/2$ . With  $F_2$  replaced by  $(V_1 V_2 - \varphi \Sigma_3 P_d) Z$  we get a similar estimate. This yields inequality (4.29).  $\square$

*Continuation of the proof of Proposition 4.1.* By (4.11) we can apply to  $f$  Lemmas 4.4 and 4.5 by taking  $\varphi(t) = 2(\partial_{\|f\|_2^2} H)$  and  $F = \text{rhs}(4.11) - \varphi(t) [\Sigma_3, P_d] f$ . Then

$$\|f\|_{L_t^p([0,T], B_{q,2}^{k_0 - \frac{2}{p}}) \cap L_t^2([0,T], H_x^{k_0, -\tau_0}) \cap L_t^2([0,T], L_x^\infty)} \leq C \|f(0)\|_{H^{k_0}} + C \|F\|_{L_t^1([0,T], H_x^{k_0}) \cap L_t^2([0,T], H_x^{k_0, s})}.$$

We have

$$(4.30) \quad \|F\|_{L_t^1 H_x^{k_0} + L_t^2 H_x^{k_0, \tau_0}} \lesssim \sum_{\lambda \cdot \mu > m - \omega_0} \|z^\mu\|_{L_t^2}^2 + \|R_1\|_{L_t^1 H_x^{k_0}} + \|R_2\|_{L_t^2 H_x^{k_0, \tau_0}} + \epsilon \|f\|_{L_t^2 H_x^{-k_0, -\tau_0}}.$$

For  $\epsilon$  small this yields Proposition 4.1 by Lemma 4.4 and by (4.7).  $\square$

**Lemma 4.6.** *Assume the conclusions of Theorem 4.1. Then there exists a fixed  $C > 0$  and  $f'_+ \in H^{k_0}$  with  $\|f'_+\|_{H^{k_0}} < C\epsilon$  such that for  $\vartheta(t)$  the phase in the ansatz (2.1) we have*

$$(4.31) \quad \lim_{t \rightarrow +\infty} \left\| e^{i\vartheta(t) \Sigma_3} f(t) - e^{-itD_m} e^{i\vartheta(0) \Sigma_3} f'_+ \right\|_{H^{k_0}} = 0.$$

*Proof.* For  $\psi(t) = f(t)$ , for  $F = \text{rhs}(4.11) - \varphi(t) [\Sigma_3, P_d] f$  and for  $t_1 < t_2$ , we have

$$\begin{aligned} &\|\mathcal{U}^{-1}(t_2) e^{i\mathcal{H}_0 t_2} f(t_2) - \mathcal{U}^{-1}(t_1) e^{i\mathcal{H}_0 t_1} f(t_1)\|_{H^{k_0}} \\ &\leq \left\| \int_{t_1}^{t_2} e^{i\mathcal{H}_0 t'} \mathcal{U}^{-1}(t') [F(t') + V f(t') - \varphi(t') \mathcal{U}^{-1} \Sigma_3 P_d f(t')] dt' \right\|_{H^{k_0}} \leq \\ &C \left( \sum_{|\lambda^0 \cdot \mu| > m - \omega_0} \|z^\mu\|_{L^2(t_1, t_2)} + \|R_1\|_{L_t^1([t_1, t_2], H_x^{k_0})} + \|R_2\|_{L_t^2([t_1, t_2], H_x^{k_0, s})} + \|f\|_{L_t^2([t_1, t_2], H_x^{k_0, -\tau_0})} \right). \end{aligned}$$

Since the latter has limit 0 as  $t_1 \rightarrow +\infty$ , there exists  $f'_+ \in H^{k_0}$  such that

$$\lim_{t \rightarrow +\infty} \|\mathcal{U}^{-1}(t) f(t) - e^{-i\mathcal{H}_0 t} f'_+\|_{H^{k_0}} = 0.$$

Hence, from  $\mathcal{H}_0 = D_m - \omega_0 \Sigma_3$  and  $\mathcal{U}^{-1}(t) = e^{i\Sigma_3 \int_0^t \varphi(t') dt'}$  we have  $\theta(t) = -t\omega_0 + \int_0^t \varphi(t') dt'$

$$(4.32) \quad \lim_{t \rightarrow +\infty} \left\| e^{i\theta(t) \Sigma_3} f(t) - e^{-itD_m} f'_+ \right\|_{H^{k_0}} = 0.$$

(4.31) follows from (4.32) if we can prove  $\theta(t) = \vartheta(t) - \vartheta(0)$ . To prove the claim we substitute  $R$  in (2.7) using (2.36) and then replace  $(z, f)$  with the last coordinate system obtained from Theorem 3.25. Then we get

$$(4.33) \quad i\dot{f} - \mathcal{H}f - (\dot{\vartheta} + \omega_0)P_c(\omega_0)\Sigma_3 f = G$$

where  $G = G_1(z, \|f\|_2^2) + G_2(z, \|f\|_2^2)f + G_3$  where  $G_2 \in L_t^\infty B(H^{k_0, -S}, H^{k_0, S})$  for  $S$  a fixed large number, and  $G_3 \in (L_t^1 H_x^{k_0} + L_t^2 H_x^{k_0, -S})$ . The two equations (4.33) and (4.11) are equivalent. This implies  $G = \text{rhs}(4.11)$  and  $\dot{\vartheta} + \omega_0 = 2\partial_{\|f\|_2^2} H$ . This yields the claim  $\theta(t) = \vartheta(t) - \vartheta(0)$ .  $\square$

Step (ii). In the proof of Theorem 4.1 consists in introducing the variable

$$(4.34) \quad g = f + Y, \quad Y := \sum_{|\lambda^0 \cdot (\mu - \nu)| > m - \omega_0} z^\mu \bar{z}^\nu R_{\mathcal{H}}^+(\lambda^0 \cdot (\mu - \nu)) H_{\mu\nu}^0.$$

Substituting the new variable  $g$  in (4.11), the first line on the rhs of (4.11) cancels out. We have

$$(4.35) \quad \begin{aligned} i\dot{g} - \mathcal{H}g - 2\partial_{\|f\|_2^2} H P_c(\omega_0)\Sigma_3 g &= \text{second line of (4.11)} + \\ 2\partial_{\|f\|_2^2} H P_c(\omega_0)\Sigma_3 Y + \sum_{k=1}^n &[\partial_{z_k} Y \partial_{\bar{z}_k} (Z + \mathcal{R}) - \partial_{\bar{z}_k} Y \partial_{z_k} (Z + \mathcal{R})]. \end{aligned}$$

We have:

**Lemma 4.7.** *For  $\epsilon$  sufficiently small,  $\tau_1 > 1$  and for  $C_0 = C_0(\mathcal{H})$  a fixed constant, we have*

$$(4.36) \quad \|g\|_{L_t^2([0, T], L_x^{2, -\tau_1})} \leq C_0 \epsilon + O(\epsilon^2).$$

*Proof.* Set  $F = (\text{second line of (4.11)} - \varphi(t)[\Sigma_3, P_d]g)$ . Then, proceeding as in (4.25), we have

$$(4.37) \quad \begin{aligned} \|g\|_{L_t^2 L_x^{2, -\tau_1}} &\leq \|e^{-it\mathcal{H}_0} Y(0)\|_{L_t^2 L_x^{2, -\tau_1}} + \|e^{-it\mathcal{H}_0} f(0)\|_{L_t^2 L_x^{2, -\tau_1}} + C\|F\|_{L_t^1 H_x^k \cap L_t^2 H_x^{k, \tau_1}} \\ &+ \left\| \int_0^t e^{i(t'-t)\mathcal{H}_0} \text{second line of (4.35)}(t') dt' \right\|_{L_t^2 L_x^{2, -\tau_1}} \\ &+ \|V_1 - \varphi(t)\Sigma_3 P_d V_2^{-1}\|_{L_t^\infty B(L_x^2, L_x^{2, -\tau_1})} \|V_2 g(t)\|_{L_t^2}. \end{aligned}$$

We have  $\|e^{-it\mathcal{H}_0} f(0)\|_{L_t^2 L_x^{2, -\tau_1}} \lesssim \|f(0)\|_{L_{tx}^2} \lesssim \epsilon$ . We have by Lemma 2.19

$$(4.38) \quad \|e^{-it\mathcal{H}_0} Y(0)\|_{L_t^2 L_x^{2, -\tau_1}} \leq C \sum_{|\lambda^0 \cdot (\mu - \nu)| > m - \omega_0} \epsilon^{|\mu + \nu|}.$$

We have  $\|\text{second line of (4.11)}\|_{L_t^1 L_x^2 + L_t^2 L_x^{2, \tau_1}} \leq O(\epsilon^2)$ . Similarly  $\|\varphi(t)[\Sigma_3, P_d]g\|_{L_t^2 L_x^{2, -\tau_1}} \leq C\epsilon \|g\|_{L_t^2 L_x^{2, \tau_1}}$ . Hence  $\|F\|_{L_t^1 L_x^2 \cap L_t^2 L_x^{2, \tau_1}} \leq C\epsilon \|g\|_{L_t^2 L_x^{2, -\tau_1}} + O(\epsilon^2)$ . Now we sketch a bound for the second line of (4.37).

$$(4.39) \quad \begin{aligned} &\sum_{|\lambda^0 \cdot (\mu - \nu)| > m - \omega_0} \left\| \int_0^t e^{i(t'-t)\mathcal{H}_0} \partial_{\|f\|_2^2} H(t') z^\mu(t') \bar{z}^\nu(t') P_c(\omega_0) R_{\mathcal{H}^*}^+(\lambda^0 \cdot (\mu - \nu)) \Sigma_3 H_{\mu\nu}^0 dt' \right\|_{L_t^2 L_x^{2, -\tau_1}} \\ &\leq \sum_{|\lambda^0 \cdot (\mu - \nu)| > m - \omega_0} \left\| \int_0^t \langle t - t' \rangle^{-\frac{3}{2}} |\partial_{\|f\|_2^2} H(t') z^\mu(t') \bar{z}^\nu(t')| dt' \right\|_{L_t^2} \lesssim C_2 \epsilon^2, \end{aligned}$$

where we used Lemma 2.19 with  $\mathcal{H}$  replaced by  $\mathcal{H}^*$ . Of the other contributions to the second line of (4.37) we focus on the main ones. Specifically we consider for  $\mu_j \neq 0$

$$(4.40) \quad \left\| \int_0^t e^{i(t'-t)\mathcal{H}_0} P_c(\omega_0) \frac{z^\mu \bar{z}^\nu}{z_j} \partial_{\bar{z}_j} Z_0 R_{\mathcal{H}}^+(\lambda^0 \cdot (\mu - \nu)) H_{\mu\nu}^0 dt' \right\|_{L_t^2 L_x^{2, -\tau_1}} \leq C \left\| \frac{z^\mu \bar{z}^\nu}{z_j} \partial_{\bar{z}_j} Z_0 \right\|_{L_t^2}$$

$$(4.41) \quad \text{for } \lambda(\omega_0) \cdot (\mu - \nu) > m - \omega_0.$$

We need to show

$$(4.42) \quad \left\| \frac{z^\mu \bar{z}^\nu}{z_j} \partial_{\bar{z}_j} Z_0 \right\|_{L_t^2} = O(\epsilon^2).$$

Let  $z^\alpha \bar{z}^\beta$  be a generic monomial of  $Z_0$ . Then  $\partial_{\bar{z}_j}(z^\alpha \bar{z}^\beta) = \beta_j \frac{z^\alpha \bar{z}^{\beta-1}}{\bar{z}_j}$ , with the nontrivial case for  $\beta_j \neq 0$ . By Definition 3.21 we have  $\lambda(\omega_0) \cdot (\alpha - \beta) = 0$ . **(H:11)** can be applied and implies  $|\alpha| = |\beta| \geq 2$ . Thus in particular one has

$$(4.43) \quad \lambda(\omega_0) \cdot \alpha \geq \lambda_j(\omega_0) \Rightarrow \lambda(\omega_0) \cdot (\mu + \alpha) - \lambda_j(\omega_0) > m - \omega_0 .$$

So the following holds:

$$(4.44) \quad \left\| \frac{z^\mu \bar{z}^\nu}{z_j} \frac{z^\alpha \bar{z}^\beta}{\bar{z}_j} \right\|_{L_t^2} \leq \left\| \frac{z^\nu z^\beta}{\bar{z}_j} \right\|_{L_t^\infty} \left\| \frac{z^\mu z^\alpha}{z_j} \right\|_{L_t^2} \leq CC_2 C_3 \epsilon^{|\nu|+|\beta|} \leq CC_2 C_3 \epsilon^2 .$$

We conclude that the second line in (4.37) is  $O(\epsilon^2)$ . The estimates omitted are easier than (4.40) and (4.42).  $\|V_2 g\|_{L_{t,x}^2}$  can be bounded as in Lemma 4.4.  $\square$

#### 4.2. The Fermi golden rule.

Step (iii). We proceed as in [Cuc10]. We recall Remark 3.22. In particular we will only consider finite sums

$$|\mu + \nu| < 2N + 3 .$$

We will have  $\lambda_j^0 = \lambda_j(\omega_0)$  and  $\lambda_j = \lambda_j(\|f\|_2^2)$  as in Section 3.4.2.  $|\lambda_j^0 - \lambda_j| \lesssim C_1^2 \epsilon^2$  by (4.6), so in the sequel we can assume that  $\lambda^0$  satisfies the same inequalities of  $\lambda$ . Set

$$R_{\mu\nu}^+ = R_{\mathcal{H}}^+(\lambda^0 \cdot (\mu - \nu)) .$$

We substitute (4.11) in  $i\dot{z}_j = \frac{\partial}{\partial \bar{z}_j} H^{(r)}$  obtaining

$$(4.45) \quad \begin{aligned} i\dot{z}_j &= \partial_{\bar{z}_j}(H_2 + Z_0) + \sum_{|\lambda \cdot (\mu - \nu)| > m - \omega_0} \nu_j \frac{z^\mu \bar{z}^\nu}{\bar{z}_j} \langle g, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{\mu\nu} \rangle + \partial_{\bar{z}_j} \mathcal{R} \\ &- \sum_{\substack{|\lambda \cdot (\alpha - \beta)| > m - \omega_0 \\ |\lambda \cdot (\mu - \nu)| > m - \omega_0}} \nu_j \frac{z^{\mu + \alpha} \bar{z}^{\nu + \beta}}{\bar{z}_j} \langle R_{\alpha\beta}^+ H_{\alpha\beta}^0, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{\mu\nu} \rangle . \end{aligned}$$

We rewrite this as

$$(4.46) \quad i\dot{z}_j = \partial_{\bar{z}_j}(H_2 + Z_0) + \mathcal{E}_j$$

$$(4.47) \quad - \sum_{\substack{\lambda \cdot \beta > m - \omega_0 \\ \lambda \cdot \nu > m - \omega_0 \\ \lambda \cdot \beta - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \beta_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \nu_j \frac{\bar{z}^{\nu + \beta}}{\bar{z}_j} \langle R_{0\beta}^+ H_{0\beta}^0, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle$$

$$(4.48) \quad - \sum_{\substack{\lambda \cdot \alpha > m - \omega_0 \\ \lambda \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \nu_j \frac{z^\alpha \bar{z}^\nu}{\bar{z}_j} \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle .$$

Here the elements in (4.47) will be eliminated through a new change of variables.  $\mathcal{E}_j$  is a reminder term defined by

$$\mathcal{E}_j := \text{rhs}(4.45) - (4.47) - (4.48) .$$

Set

$$(4.49) \quad \begin{aligned} \zeta_j &= z_j - \sum_{\substack{\lambda \cdot \beta > m - \omega_0 \\ \lambda \cdot \nu > m - \omega_0 \\ \lambda \cdot \beta - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \beta_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \frac{\nu_j}{\lambda^0 \cdot (\beta + \nu)} \frac{\bar{z}^{\nu + \beta}}{\bar{z}_j} \langle R_{0\beta}^+ H_{0\beta}^0, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle \\ &+ \sum_{\substack{\lambda \cdot \alpha > m - \omega_0 \\ \lambda \cdot \nu > m - \omega_0 \\ \lambda^0 \cdot \alpha \neq \lambda^0 \cdot \nu \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \frac{\nu_j}{\lambda^0 \cdot (\alpha - \nu)} \frac{z^\alpha \bar{z}^\nu}{\bar{z}_j} \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta\alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle \end{aligned}$$

Notice that in (4.49), by  $\lambda \cdot \nu > \omega_0 - m$ , we have  $|\nu| > 1$ . Then by (4.7)

$$(4.50) \quad \begin{aligned} \|\zeta - z\|_{L_t^2} &\leq C\epsilon \quad \sum_{\substack{\lambda \cdot \alpha > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} \|z^\alpha\|_{L_t^2} \leq CC_2 M \epsilon^2 \\ \|\zeta - z\|_{L_t^\infty} &\leq C^3 \epsilon^3 \end{aligned}$$

with  $C$  the constant in (4.5) and  $M$  the number of terms in the rhs. In the new variables (4.46) is of the form

$$(4.51) \quad \begin{aligned} i\dot{\zeta}_j &= \partial_{\bar{\zeta}_j} H_2(\zeta, f) + \partial_{\bar{\zeta}_j} Z_0(\zeta, f) + \mathcal{D}_j \\ &- \sum_{\substack{\lambda^0 \cdot \alpha = \lambda^0 \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \nu_j \frac{\zeta^\alpha \bar{\zeta}^\nu}{\bar{\zeta}_j} \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle. \end{aligned}$$

From these equations by  $\sum_j \lambda_j^0 (\bar{\zeta}_j \partial_{\bar{\zeta}_j} (H_2 + Z_0) - \zeta_j \partial_{\zeta_j} (H_2 + Z_0)) = 0$  we get

$$(4.52) \quad \begin{aligned} \partial_t \sum_{j=1}^n \lambda_j^0 |\zeta_j|^2 &= 2 \sum_{j=1}^n \lambda_j^0 \Im (\mathcal{D}_j \bar{\zeta}_j) - \\ &- 2 \sum_{\substack{\lambda^0 \cdot \alpha = \lambda^0 \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \lambda^0 \cdot \nu \Im \left( \zeta^\alpha \bar{\zeta}^\nu \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle \right). \end{aligned}$$

We have the following lemma, whose proof (we skip) is similar to Lemma 4.7 [Cuc11]:

**Lemma 4.8.** *Assume inequalities (4.7). Then for a fixed constant  $c_0$  we have*

$$(4.53) \quad \sum_j \|\mathcal{D}_j \bar{\zeta}_j\|_{L^1[0, T]} \leq (1 + C_2) c_0 \epsilon^2.$$

For the sum in the second line of (4.52) we get

$$(4.54) \quad \begin{aligned} &2 \sum_{r > m - \omega_0} r \Im \left\langle R_{\mathcal{H}}^+(r) \sum_{\lambda^0 \cdot \alpha = r} \zeta^\alpha H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 \sum_{\lambda^0 \cdot \nu = r} (\zeta^\nu)^* H_{0\nu}^0 \right\rangle = \\ &2 \sum_{r > m - \omega_0} r \Im \left\langle R_{\mathcal{H}}^+(r) \sum_{\lambda^0 \cdot \alpha = r} \zeta^\alpha H_{\alpha 0}^0, \Sigma_3 \left[ \sum_{\lambda^0 \cdot \alpha = r} \zeta^\alpha H_{\alpha 0}^0 \right]^* \right\rangle = 2 \sum_{r > m - \omega_0} r \Im \langle R_{\mathcal{H}}^+(r) \mathbf{H}_r, \Sigma_3 \mathbf{H}_r^* \rangle, \end{aligned}$$

where  $\mathbf{H}_r := \sum_{\lambda^0 \cdot \alpha = r} \zeta^\alpha H_{\alpha 0}^0$  and where we have used  $i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{\mu\nu}^0 = -\Sigma_3 i\beta \alpha_2 \Sigma_1 H_{\nu\mu}^0 = \Sigma_3 i\beta \alpha_2 C H_{\nu\mu}^0 = \Sigma_3 (H_{\nu\mu}^0)^*$  by (3.100).

**Lemma 4.9.** *Consider  $\mathbf{H}_r$  in (4.54). Assume  $m - \omega_0 < r < m + \omega_0$ . Then*

$$(4.55) \quad \Im \langle R_{\mathcal{H}}^+(r) \mathbf{H}_r, \Sigma_3 \mathbf{H}_r^* \rangle \geq 0.$$

*If we assume (H:3), in particular if  $m/3 < \omega_0 < m$ , then (4.55) holds for all  $\mathbf{H}_r$  in (4.54).*

*Proof.* We proceed as in Lemma 10.5 [Cuc10]. Set  $\mathbf{F}_r = \mathcal{Z}_+ \mathbf{H}_r$ , where for  $\mathcal{Z}_+$  see Theorem A.4 with  $\omega = \omega_0$ . Set  $\mathbf{F}_r = \begin{pmatrix} a \\ b \end{pmatrix}$ . Then

$$(4.56) \quad \begin{aligned} \Im \langle R_{\mathcal{H}}^+(r) \mathbf{H}_r, \Sigma_3 \mathbf{H}_r^* \rangle &= \lim_{\varepsilon \searrow 0} \Im \langle R_{\mathcal{H}}(r + i\varepsilon) \mathbf{H}_r, \Sigma_3 \mathbf{H}_r^* \rangle = \lim_{\varepsilon \searrow 0} \Im \langle R_{\mathcal{H}_{\omega_0, 0}}(r + i\varepsilon) \mathbf{F}_r, \Sigma_3 \mathbf{F}_r^* \rangle \\ &= \lim_{\varepsilon \searrow 0} \Im \langle R_{D_m}(r + \omega + i\varepsilon) a, a^* \rangle - \lim_{\varepsilon \searrow 0} \Im \langle R_{D_m}(r - \omega + i\varepsilon) b, b^* \rangle \\ &= \frac{1}{2} \lim_{\varepsilon \searrow 0} \varepsilon \|R_{D_m}(r + \omega + i\varepsilon) a\|_{L^2}^2 - \Im \langle R_{D_m}(r - \omega) b, b^* \rangle = \frac{1}{2} \lim_{\varepsilon \searrow 0} \varepsilon \|R_{D_m}(r + \omega + i\varepsilon) a\|_{L^2}^2 \geq 0. \end{aligned}$$

Here we exploited that  $a, b \in L^2(\mathbb{R}^3)$ , that  $r - \omega < m$  and so  $R_{D_m}(r - \omega)$  is a well defined selfadjoint operator in  $L^2(\mathbb{R}^3)$ , that  $R_{D_m}(z) - R_{D_m}(z^*) = 2iR_{D_m}(z)R_{D_m}(z^*)\Im z$  and that  $R_{D_m}(z^*) = (R_{D_m}(z))^*$ .

Let us consider  $r = \lambda \cdot \mu$  with  $\mu \in \mathbb{N}_0^n$ ,  $\lambda \cdot \mu > m - \omega_0$  and  $\lambda \cdot \mu - \lambda_k < m - \omega_0$  for all  $k$  s.t.  $\mu_k \neq 0$ . Suppose  $\lambda \cdot \mu > m + \omega_0$ . Then we get  $m - \omega_0 + \lambda_k > m + \omega_0 \Rightarrow \lambda_k > 2\omega_0$ . Let  $N_k \in \mathbb{N}$  such that

$N_k \lambda_k < m - \omega_0 < (N_k + 1) \lambda_k$  as in **(H:9)**. Then  $(2N_k + 1) \omega_0 < m$ . So, if we assume as in **(H:3)** that  $\omega_0 > m/3$ , we obtain  $\lambda \cdot \mu < m + \omega_0$ . This shows that the assumption  $\lambda \cdot \mu > m + \omega_0$  is absurd.  $\square$

*Remark 4.10.* Notice that to get the conclusions of Lemma 4.9 we can ease the constraint  $3\omega > m$  to  $(2N_k + 1)\omega > m$  for all  $k = 1, \dots, n$ .

Now we will assume the following hypothesis.

**(H:12')** We assume that for some fixed constant  $C > 0$ , for any vector  $\zeta \in \mathbb{C}^n$  we have:

$$(4.57) \quad \begin{aligned} & \sum_{\substack{\lambda^0 \cdot \alpha = \lambda^0 \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \lambda^0 \cdot \nu \Im \left( \zeta^\alpha \bar{\zeta}^\nu \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle \right) \\ & \geq C \sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} |\zeta^\alpha|^2. \end{aligned}$$

*Remark 4.11.* Notice that by Lemma 4.9 we have  $\text{rhs}(4.57) \geq 0$ . It is likely then that **(H:12')** is true generically in the class of non linearities we consider. But we do not try to prove this point.

By **(H:12')** we have

$$(4.58) \quad 2 \sum_{j=1}^n \lambda_j^0 \Im (\mathcal{D}_j \bar{\zeta}_j) \gtrsim \partial_t \sum_{j=1}^n \lambda_j^0 |\zeta_j|^2 + \sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} |\zeta^\alpha|^2.$$

Then, for  $t \in [0, T]$  and assuming Lemma 4.8 we have

$$\sum_{j=1}^n \lambda_j^0 |\zeta_j(t)|^2 + \sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} \|\zeta^\alpha\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_2 \epsilon^2.$$

By (4.50) this implies  $\|z^\alpha\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_2 \epsilon^2$  for all the above multi indexes. So, from  $\|z^\alpha\|_{L^2(0,t)}^2 \lesssim C_2^2 \epsilon^2$  we conclude  $\|z^\alpha\|_{L^2(0,t)}^2 \lesssim C_2 \epsilon^2$ .

Note that as the condition  $|\lambda \cdot (\mu - \nu)| > m - \omega$  implies that  $|\mu + \nu| \geq 2$ , (4.45) implies that  $\dot{z}$  is integrable so that it has a limit at infinity which is necessarily 0. This yields Theorem 4.1 and completes the proof of Theorem 1.1.

**4.3. Proof of Theorem 3.16.** We only sketch the proof, which is similar to that of Theorem 3.15. For a particular solution satisfying the hypotheses of Theorem 3.16 we need to prove the conclusions of Theorem 4.1. The argument is exactly the same of Section 4.1 until we reach subsection 4.2, that is the task of estimating  $z$ . Instead of (4.51) we have

$$(4.59) \quad \begin{aligned} i\dot{\zeta}_j &= \varepsilon_j \partial_{\bar{\zeta}_j} H_2(\zeta, f) + \varepsilon_j \partial_{\bar{\zeta}_j} Z_0(\zeta, f) + \varepsilon_j \mathcal{D}_j \\ & - \varepsilon_j \sum_{\substack{\lambda^0 \cdot \alpha = \lambda^0 \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \nu_j \frac{\zeta^\alpha \bar{\zeta}^\nu}{\bar{\zeta}_j} \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle. \end{aligned}$$

From these equations by  $\sum_j \lambda_j^0 (\bar{\zeta}_j \partial_{\bar{\zeta}_j} (H_2 + Z_0) - \zeta_j \partial_{\zeta_j} (H_2 + Z_0)) = 0$  we get

$$(4.60) \quad \begin{aligned} \partial_t \sum_{j=1}^n \varepsilon_j \lambda_j^0 |\zeta_j|^2 &= 2 \sum_{j=1}^n \lambda_j^0 \Im (\mathcal{D}_j \bar{\zeta}_j) - \\ & - 2 \sum_{\substack{\lambda^0 \cdot \alpha = \lambda^0 \cdot \nu > m - \omega_0 \\ \lambda \cdot \alpha - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0 \\ \lambda \cdot \nu - \lambda_k < m - \omega_0 \forall k \text{ s.t. } \nu_k \neq 0}} \lambda^0 \cdot \nu \Im \left( \zeta^\alpha \bar{\zeta}^\nu \langle R_{\alpha 0}^+ H_{\alpha 0}^0, i\beta \alpha_2 \Sigma_1 \Sigma_3 H_{0\nu}^0 \rangle \right). \end{aligned}$$

The estimate of the reminder term in Lemma 4.8 continues to hold. The last line of (4.60) is negative by (4.54). We assume it is strictly negative and that in particular (4.57) holds. Then we get

$$(4.61) \quad \sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} |\zeta^\alpha|^2 \lesssim -\partial_t \sum_{j=1}^n \varepsilon_j \lambda_j^0 |\zeta_j|^2 + 2 \sum_{j=1}^n \lambda_j^0 \Im(\mathcal{D}_j \bar{\zeta}_j).$$

When we integrate in  $(0, t)$  for  $t \leq T$  we get

$$\sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} \|\zeta^\alpha\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_2 \epsilon^2.$$

In the rhs we have used the hypothesis  $|z(t)| \leq \epsilon$  for all  $t \geq 0$  to bound the first summation in the rhs of (4.61). This yields Theorem 3.16.

**4.4. Proof of Theorem 3.18.** Also here we just sketch the proof, which is similar to [Cuc09]. The proof is by contradiction. If the statement of Theorem 3.18 is wrong, then for  $|z(0)| + \|f(0)\|_{H^{k_0}} \leq \delta$  with  $\delta > 0$  sufficiently small, we can assume  $|z(t)| \leq \epsilon$  for all  $t \geq 0$  for any preassigned  $\epsilon > 0$ . This implies that we can apply Theorem 3.16. When get

$$(4.62) \quad \sum_{\substack{\lambda^0 \cdot \alpha > m - \omega_0 \\ \lambda^0 \cdot \alpha - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \alpha_k \neq 0}} \|\zeta^\alpha\|_{L^2(0,t)}^2 \lesssim \sum_{j=1}^n \varepsilon_j \lambda_j^0 (|\zeta_j(0)|^2 - |\zeta_j(t)|^2) + 2 \int_0^t \sum_{j=1}^n \lambda_j^0 \Im(\mathcal{D}_j \bar{\zeta}_j).$$

Suppose  $\varepsilon_{j_0} = -1$ . Then take initial datum  $z_j(0) = 0$  for  $j \neq j_0$ ,  $z_{j_0} = \delta$  and  $f(0) = 0$ . By  $f(0) = 0$  and Lemma (4.4) for  $\psi(0) = 0$  we get for  $t \in \mathbb{R}^+$

$$(4.63) \quad \|f\|_{L_t^p B_{q,2}^{k_0 - \frac{3}{p}} \cap L_t^2 H_x^{k_0, -\tau_0} \cap L_t^\infty L_x^\infty} \lesssim \mathcal{Y}^2 + \|R_1\|_{L_t^1 H_x^{k_0}} + \|R_2\|_{L_t^2 H_x^{k_0, \tau_0}}$$

where

$$(4.64) \quad \mathcal{Y}^2 := \sum_{\substack{\lambda^0 \cdot \mu > m - \omega_0 \\ \lambda^0 \cdot \mu - \lambda_k^0 < m - \omega_0 \forall k \text{ s.t. } \mu_k \neq 0}} \|z^\mu\|_{L_t^2}.$$

Similarly

$$(4.65) \quad \|g\|_{L_t^2 L_x^{2, -\tau_1}} \lesssim \delta^2 + \epsilon \mathcal{Y}^2 + \|R_1\|_{L_t^1 H_x^{k_0}} + \|R_2\|_{L_t^2 H_x^{k_0, \tau_0}}.$$

Then, proceeding as in [Cuc09, Cuc11] one improves the rhs in (4.53). Indeed, see Lemma 4.9 [Cuc11], we have

$$\sum_j \|\mathcal{D}_j \bar{\zeta}_j\|_{L^1(\mathbb{R}^+)} \leq C \mathcal{Y} \|g\|_{L_t^2 H_x^{4, -s}} + C \epsilon \mathcal{Y}^2 + C \|R_1\|_{L_t^1 H_x^{k_0}} + C \|R_2\|_{L_t^2 H_x^{k_0, \tau_0}}.$$

Then, one can see that  $\|R_1\|_{L_t^1 H_x^{k_0}} + \|R_2\|_{L_t^2 H_x^{k_0, \tau_1}} \lesssim o(1)\delta$ , going through Lemma 4.3, where  $o(1) \rightarrow 0$  as  $\delta \rightarrow 0$ . Then from (4.62) we get

$$\mathcal{Y}^2 \lesssim -\delta + o(1)\delta,$$

which is absurd.

## APPENDIX A. RESOLVENT ESTIMATES

### A.1. Resonances for $\mathcal{H}_\omega$ .

**Definition A.1.** We will say that a point  $\lambda \in \mathbb{R}$  with  $|\lambda| \geq m - \omega$  is a resonance if one of the following two equations admits a nontrivial solution:

$$(A.1) \quad (1 + R_{\mathcal{H}_{\omega,0}}^+(\lambda) V_\omega) u = 0, \quad u \in L^{2, -\tau}(\mathbb{R}^3, \mathbb{C}^8) \text{ for some } \tau > 1/2 \text{ and } u(-x) \equiv \beta \Sigma_3 u(x);$$

$$(A.2) \quad (1 + R_{\mathcal{H}_{\omega,0}}^-(\lambda) V_\omega) u = 0, \quad u \in L^{2, -\tau}(\mathbb{R}^3, \mathbb{C}^8) \text{ for some } \tau > 1/2 \text{ and } u(-x) \equiv \beta \Sigma_3 u(x).$$



## A.2. Estimates on the resolvent.

**Lemma A.2.** *We assume (H:6)–(H:8). Then for any  $\tau > 1$  there exists a constant  $C_1 = C_1(\tau, \omega)$  upper semicontinuous  $\omega$  s.t. for any  $u_0(x) \in L^2(\mathbb{R}^3, \mathbb{C}^8)$  and any  $\varepsilon > 0$  we have*

$$(A.3) \quad \|\langle x \rangle^{-\tau} R_{\mathcal{H}_\omega}(\lambda \pm i\varepsilon) P_c(\mathcal{H}_\omega) u_0\|_{L_{\lambda,x}(\mathbb{R}^4)} \leq C_1 \|P_c(\mathcal{H}_\omega) u_0\|_{L^2(\mathbb{R}^3)}.$$

*Proof.* Notice that by Lemma 2.12 for any  $\tau > 1$ , any  $u_0(x) \in L^2(\mathbb{R}^3, \mathbb{C}^8)$  and any  $\varepsilon > 0$  we have

$$(A.4) \quad \|\langle x \rangle^{-\tau} R_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon) u_0\|_{L_{\lambda,x}(\mathbb{R}^4)} \leq C(\tau) \|u_0\|_{L^2(\mathbb{R}^3)}.$$

Let  $u_0 = P_c(\mathcal{H}_\omega) u_0$ ,  $A(x) = \langle x \rangle^{-\tau}$  and  $B(x) \in \mathcal{S}(\mathbb{R}^3, B(\mathbb{C}^8, \mathbb{C}^8))$  s.t.  $B^* A = V_\omega$ . Then

$$(A.5) \quad AR_{\mathcal{H}_\omega}(z) u_0 = (1 + AR_{\mathcal{H}_{\omega,0}}(z) B^*)^{-1} AR_{\mathcal{H}_{\omega,0}}(z) u_0.$$

Pick  $\delta_0 > 0$  sufficiently small so that by (H:6) for any  $\lambda_j(\omega) \in \sigma_d(\mathcal{H}_\omega)$  we have  $|\lambda_j(\omega)| < m - \omega - \delta_0$ . Then by (A.4) and (A.5), Lemma A.2 is a consequence of the Lemma A.3 below.  $\square$

**Lemma A.3.** *Let  $A(x), B(x)$  be as above in (A.5). Then, if we assume (H:3), (H:6) and (H:7), there exists a constant  $C_2 = C_2(\tau, \omega)$  upper semicontinuous in  $\omega$  such that for any  $\varepsilon > 0$  we have*

$$(A.6) \quad \sup_{\lambda \in (\mathbb{R} \setminus [-m+\omega+\delta_0, m-\omega-\delta_0])} \|(1 + AR_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon) B^*)^{-1}\|_{B(L_x^2, L_x^2)} \leq C_2.$$

For any  $\tau > 1$  the following limit

$$(A.7) \quad R_{\mathcal{H}_\omega}^\pm(\lambda) = \lim_{\varepsilon \searrow 0} R_{\mathcal{H}_\omega}(\lambda \pm i\varepsilon)$$

exist in  $B(H_x^{1,\tau}, L_x^{2,-\tau})$  and the convergence is uniform for  $\lambda$  in compact sets.

*Proof.* First of all we prove (A.6) in low energies. We have

$$(A.8) \quad \sup_{\lambda \in ([-\mu_1, \mu_1] \setminus [-m+\omega+\delta_0, m-\omega-\delta_0])} \|(1 + AR_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon) B^*)^{-1}\|_{B(L_x^2, L_x^2)} < \infty \quad \forall \text{ fixed } \mu_1 > 0.$$

We know:  $z \rightarrow AR_{\mathcal{H}_{\omega,0}}(z) B^*$  is a holomorphic map with domain  $\mathbb{C} \setminus \mathbb{R}$  and values in  $B(L_x^2, L_x^2)$ ;  $(1 + AR_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon) B^*)^{-1}$  is defined for all  $z \in \mathbb{C} \setminus \mathbb{R}$ . Furthermore,  $\lim_{\varepsilon \searrow 0} AR_{\mathcal{H}_{\omega,0}}(\lambda \pm i\varepsilon) B^*$ , by (ii) Lemma 2.12, exists in  $B(L_x^2, L_x^2)$  and the convergence is uniform for  $\lambda$  in compact sets. Then we apply Lemma 7.5 [Ber82] and conclude that, outside closed sets  $\Gamma^\pm \subset \mathbb{R}$  with 0 Lebesgue measure in  $\mathbb{R}$ , the map  $z \rightarrow (1 + AR_{\mathcal{H}_{\omega,0}}(z) B^*)^{-1}$  extends in a continuous map defined in  $\{z : \Im z > 0\} \cup (\mathbb{R} \setminus \Gamma^+)$  (resp.  $\{z : \Im z < 0\} \cup (\mathbb{R} \setminus \Gamma^-)$ ) with values in  $B(L_x^2, L_x^2)$ . Given  $\lambda \in \Gamma^+$  there exists  $\psi \in L_x^2 \setminus \{0\}$  with  $\psi = -AR_{\mathcal{H}_{\omega,0}}^+(\lambda) B^* \psi$ . But then, by standard arguments  $u := R_{\mathcal{H}_{\omega,0}}^+(\lambda) B^* \psi$  is a nonzero solution of (A.1). But by section A.1 and by hypotheses (H7)–(H8), it follows that the intersection  $\Gamma^+$  with  $\mathbb{R} \setminus (-m + \omega + \delta_0, m - \omega - \delta_0)$  is empty. Given  $\lambda \in \Gamma^-$  there exists  $\psi \in L_x^2 \setminus \{0\}$  with  $\psi = -AR_{\mathcal{H}_{\omega,0}}^-(\lambda) B^* \psi$ . Then  $v := R_{\mathcal{H}_{\omega,0}}^-(\lambda) B^* \psi$  is a nonzero solution in  $L^{2,-\tau}$ , for  $\tau > 1/2$ , of  $(1 + R_{\mathcal{H}_{\omega,0}}^-(\lambda) V_\omega) v = 0$ . But then, by Lemma 2.4 we have  $(1 + R_{\mathcal{H}_{\omega,0}}^+(-\lambda) V_\omega) \Sigma_1 C v = 0$ . Once again, no such nonzero  $v$  can exist for  $\lambda \in \mathbb{R} \setminus (-m + \omega + \delta_0, m - \omega - \delta_0)$ .

Having considered the low energy case (A.8), we consider for  $\mu_1$  any fixed large real number:

$$(A.9) \quad \sup_{|\lambda| \geq \mu_1} \|(1 + AR_{\mathcal{H}_{\omega,0}}^\pm(\lambda) B^*)^{-1}\|_{B(L_x^2, L_x^2)} \leq C_3.$$

For definiteness we will consider  $\lambda \geq \mu_1$ . We consider the expansion

$$(A.10) \quad \sum_{\ell=0}^{\infty} \left( AR_{\mathcal{H}_{\omega,0}}^\pm(\lambda) B^* \right)^\ell.$$

We start now the implementation of the high energy argument in [EGS09]. We have

$$(A.11) \quad R_{\mathcal{H}_{\omega,0}}^\pm(\lambda) = \begin{pmatrix} R_{D_m}^\pm(\lambda + \omega) & 0 \\ 0 & R_{D_m}^\pm(\lambda - \omega) \end{pmatrix} = R_0^\pm(\lambda) \mathcal{A}(\lambda, \nabla)$$

where

$$(A.12) \quad \begin{aligned} R_0^\pm(\lambda) &:= \begin{pmatrix} R_{-\Delta+m^2}^\pm((\lambda+\omega)^2)I_2 & 0 \\ 0 & R_{-\Delta+m^2}^\pm((\lambda-\omega)^2)I_2 \end{pmatrix} \\ \mathcal{A}(\lambda, \nabla) &:= \begin{pmatrix} \mathcal{A}_1(\lambda, \nabla) & 0 \\ 0 & \mathcal{A}_2(\lambda, \nabla) \end{pmatrix}, \quad \mathcal{A}_j(\lambda, \nabla) := \begin{pmatrix} \lambda - (-1)^j \omega + m & -i\sigma \cdot \nabla \\ -i\sigma \cdot \nabla & \lambda - (-1)^j \omega - m \end{pmatrix}. \end{aligned}$$

For definiteness let us consider  $R_{\mathcal{H}_{\omega,0}}^+$ . Let now  $\chi_0, \psi_0 \in C_0^\infty(\mathbb{R})$  by cutoffs supported near 0 and let  $\chi_1 := 1 - \chi_0$  and  $\psi_1 := 1 - \psi_0$ . We can choose them so that

$$(A.13) \quad \chi_1(|x-y|) = (\psi_0(|x|)\psi_1(|y|) + \psi_1(|x|)\psi_0(|y|) + \psi_1(|x|)\psi_1(|y|))\chi_1(|x-y|)$$

We split for a fixed large number  $M_0 > 0$

$$(A.14) \quad \begin{aligned} R_{-\Delta+m^2}^+((\lambda - (-1)^j \omega)^2, x, y) &= \sum_{\ell=0}^1 R_{\ell j}(\lambda, x, y), \\ R_{\ell j}(\lambda, x, y) &:= \frac{e^{i\sqrt{(\lambda - (-1)^j \omega)^2 + m^2}|x-y|}}{4\pi|x-y|} \chi_\ell\left(\frac{|x-y|}{M_0}\right). \end{aligned}$$

By (A.13)–(A.14) we have  $c_{M_0}$  with  $\lim_{M_0 \rightarrow +\infty} c_{M_0} = 0$  s.t. for  $j = 0, 1$

$$(A.15) \quad \|AR_{1j}(\lambda)B^*\|_{B(L_x^2, L_x^2)} \leq c_{M_0}.$$

Then by  $\|AR_{-\Delta}^\pm(\lambda)B^*\|_{B(L_x^2, L_x^2)} \leq C$ , for fixed  $C'$

$$(A.16) \quad \|AR_{0j}(\lambda)B^*\|_{B(L_x^2, L_x^2)} \leq C'.$$

We have

$$(A.17) \quad R_{0j}(\lambda, x, y) = \lambda R_{-\Delta}^+ \left( \sqrt{\left(1 - (-1)^j \frac{\omega}{\lambda}\right)^2 + \frac{m^2}{\lambda^2}}, \lambda x, \lambda y \right) \chi_0\left(\frac{|x-y|}{M_0}\right).$$

Key to the fact that (A.9) follows quite directly from [EGS09] is the fact that we can write

$$(A.18) \quad R_{-\Delta}^+ \left( \sqrt{\left(1 - (-1)^j \frac{\omega}{\lambda}\right)^2 + \frac{m^2}{\lambda^2}}, x, y \right) \chi_0\left(\frac{|x-y|}{\lambda M_0}\right) = \frac{e^{i|x-y|}}{|x-y|} a_{\lambda,j}(|x-y|) + \frac{b_{\lambda,j}(|x-y|)}{|x-y|},$$

with

$$(A.19) \quad \begin{aligned} |a_{\lambda,j}^{(k)}(r)| &\leq C(M_0, k)r^{-k} \quad \forall k \geq 0, \quad a_{\lambda,j}^{(k)}(r) = 0 \quad \forall 0 < r < 1 \\ |b_{\lambda,j}^{(k)}(r)| &\leq C(M_0, k) \quad \forall k \geq 0, \quad b_{\lambda,j}^{(k)}(r) = 0 \quad \forall r > 2. \end{aligned}$$

Notice that (A.18)–(A.19) are formulas of the same type of (3.2)–(3.4) [EGS09]. As a consequence for any fixed small  $\delta_0 > 0$  there are  $\ell_0 = \ell(\delta_0)$  and  $\mu_1 = \mu_1(\delta_0)$  such that for  $\lambda \geq \mu_1$  we have

$$(A.20) \quad \left\| \left( A\chi_0\left(\frac{|\cdot|}{\lambda M_0}\right) R_{\mathcal{H}_{\omega,0}}^+(\lambda)B^* \right)^{\ell_0} \right\|_{B(L_x^2, L_x^2)} \leq \delta_0.$$

Then for  $\ell$  large

$$(A.21) \quad \left\| \left( A\chi_0\left(\frac{|\cdot|}{\lambda M_0}\right) R_{\mathcal{H}_{\omega,0}}^+(\lambda)B^* + A\chi_1\left(\frac{|\cdot|}{\lambda M_0}\right) R_{\mathcal{H}_{\omega,0}}^+(\lambda)B^* \right)^\ell \right\|_{B(L_x^2, L_x^2)} \leq 2^\ell (2C')^\ell \delta_0^{\frac{\ell}{\ell_0}}.$$

For  $\delta_0$  sufficiently small, (A.21) implies (A.9).  $\square$

**A.3. Wave operators and similarity.** We start stating a corollary of Lemma A.2.

**Theorem A.4.** *Assume the hypotheses of Lemma A.2. Let  $A, B^* \in \mathcal{S}(\mathbb{R}^3, B(\mathbb{C}^8, \mathbb{C}^8))$  s.t.  $V_\omega = B^*A$ . Then there are isomorphisms  $\mathcal{W}_\pm: L^2(\mathbb{R}^3, \mathbb{C}^8) \rightarrow L_c^2(\mathcal{H}_\omega)$  and  $\mathcal{Z}_\pm: L_c^2(\mathcal{H}_\omega) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^8)$ , inverses of each other, defined as follows: for  $u \in L^2(\mathbb{R}^3, \mathbb{C}^8)$ ,  $v \in L_c^2(\mathcal{H}_\omega)$ ,*

$$(A.22) \quad \begin{aligned} \langle \mathcal{W}_\pm u, v^* \rangle &= \langle u, v^* \rangle \mp \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \langle AR_{\mathcal{H}_{\omega,0}}(\lambda \pm i\epsilon)u, (BR_{\mathcal{H}_\omega^*}(\lambda \pm i\epsilon)v)^* \rangle d\lambda; \\ \langle \mathcal{Z}_\pm v, u^* \rangle &= \langle v, u^* \rangle \pm \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \langle AR_{\mathcal{H}_\omega}(\lambda \pm i\epsilon)v, (BR_{\mathcal{H}_{\omega,0}^*}(\lambda \pm i\epsilon)u)^* \rangle d\lambda. \end{aligned}$$

For any Borel function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  we have

$$(A.23) \quad \beta(\mathcal{H}_{\omega,0}) = \mathcal{Z}_+ \beta(\mathcal{H}_{\omega}) P_c(\mathcal{H}_{\omega}) \mathcal{W}_+ = \mathcal{Z}_- \beta(\mathcal{H}_{\omega}) P_c(\mathcal{H}_{\omega}) \mathcal{W}_-$$

In particular,  $\mathcal{W}_{\pm}$  (resp.  $\mathcal{Z}_{\pm}$ ) define isomorphisms  $H^k(\mathbb{R}^3, \mathbb{C}^8) \rightarrow P_c(\mathcal{H}_{\omega}) H^k(\mathbb{R}^3, \mathbb{C}^8)$  (resp. and viceversa) for all  $k$ . We also have

$$(A.24) \quad \begin{aligned} \mathcal{W}_{\pm} u &= \lim_{t \rightarrow \pm\infty} e^{it\mathcal{H}_{\omega}} e^{-it\mathcal{H}_{\omega,0}} u \text{ for all } u \in L^2(\mathbb{R}^3, \mathbb{C}^8); \\ \mathcal{Z}_{\pm} v &= \lim_{t \rightarrow \pm\infty} e^{it\mathcal{H}_{\omega,0}} e^{-it\mathcal{H}_{\omega}} v \text{ for all } v \in L_c^2(\mathcal{H}_{\omega}). \end{aligned}$$

*Proof.* The proof follows by Lemma A.2 by means of the argument for Theorem 1.5 [Kat66]. (A.24) follows by Theorem 3.9 [Kat66].  $\square$

**Lemma A.5.** For any  $\tau > 1$  and any  $u \in H^{2,\tau}$  we have:

$$(A.25) \quad P_c(\mathcal{H}_{\omega}) u = \frac{1}{2\pi i} \int_{\sigma_e(\mathcal{H}_{\omega})} (R_{\mathcal{H}_{\omega}}^+(\lambda) - R_{\mathcal{H}_{\omega}}^-(\lambda)) u d\lambda.$$

*Proof.* By Corollary A.4 and by the spectral Theorem, see p.81 volume II [Tay96], we have

$$P_c(\mathcal{H}_{\omega}) u = \mathcal{W} \mathcal{Z} u = \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \mathcal{W} \lim_{a \nearrow \infty} \int_{\sigma_e(\mathcal{H}_{\omega}) \cap [-a,a]} (R_{\mathcal{H}_{\omega,0}}(\lambda + i\varepsilon) - R_{\mathcal{H}_{\omega,0}}(\lambda - i\varepsilon)) \mathcal{Z} u d\lambda.$$

Hence by Theorem A.4

$$\begin{aligned} P_c(\mathcal{H}_{\omega}) u &= \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \lim_{a \nearrow \infty} \int_{\sigma_e(\mathcal{H}_{\omega}) \cap [-a,a]} (R_{\mathcal{H}_{\omega}}(\lambda + i\varepsilon) - R_{\mathcal{H}_{\omega}}(\lambda - i\varepsilon)) u d\lambda \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \lim_{a \nearrow \infty} \int_{\sigma_e(\mathcal{H}_{\omega}) \cap [-a,a]} (R_{\mathcal{H}_{\omega}}(\lambda + i\varepsilon) - R_{\mathcal{H}_{\omega}}(\lambda - i\varepsilon)) \mathcal{H}_{\omega} u \frac{d\lambda}{\lambda^2}. \end{aligned}$$

In the last formula the two limits commute. So

$$(A.26) \quad P_c(\mathcal{H}_{\omega}) u = \frac{1}{2\pi i} \lim_{a \nearrow \infty} \int_{\sigma_e(\mathcal{H}_{\omega}) \cap [-a,a]} (R_{\mathcal{H}_{\omega}}^+(\lambda) - R_{\mathcal{H}_{\omega}}^-(\lambda)) u d\lambda.$$

The rhs of (A.26) is what we mean in the rhs of (A.25).  $\square$

## REFERENCES

- [BC09] G. Berkolaiko and A. Comech. On spectral stability of solitary waves of nonlinear Diracequation on a line, arXiv:0910.0917, 2009.
- [BC09] D. Bambusi and S. Cuccagna. On dispersion of small energy solutions of the nonlinear klein gordon equation with a potential, arXiv:0908.4548, 2009.
- [BCDM88] M. Balabane, T. Cazenave, A. Douady, and F. Merle. Existence of excited states for a nonlinear Dirac field. *Comm. Math. Phys.*, 119(1):153–176, 1988.
- [BCV90] M. Balabane, T. Cazenave, and L. Vázquez. Existence of standing waves for Dirac fields with singular nonlinearities. *Comm. Math. Phys.*, 133(1):53–74, 1990.
- [Bec08] M. Beceanu. A centre-stable manifold for the focussing cubic NLS in  $\mathbb{R}^{1+3}$ . *Comm. Math. Phys.*, 280(1):145–205, 2008.
- [Ber82] A.-M. Berthier. Spectral theory and wave operators for the Schrödinger equation, volume 71 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass., 1982.
- [BG87] A.-M. Berthier and V. Georgescu. On the point spectrum of Dirac operators. *J. Funct. Anal.*, 71(2):309–338, 1987.
- [Bou06] N. Boussaid. Stable directions for small nonlinear Dirac standing waves. *Comm. Math. Phys.*, 268(3):757–817, 2006.
- [Bou08a] N. Bournaveas. Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension. *Discrete Contin. Dyn. Syst.*, 20(3):605–616, 2008.
- [Bou08b] N. Boussaid. On the asymptotic stability of small nonlinear Dirac standing waves in a resonant case. *SIAM J. Math. Anal.*, 40(4):1621–1670, 2008.
- [BP92] V. S. Buslaev and G. S. Perel'man. Scattering for the nonlinear Schrödinger equation: states that are close to a soliton. *Algebra i Analiz*, 4(6):63–102, 1992.
- [BP95] V. S. Buslaev and G. S. Perel'man. On the stability of solitary waves for nonlinear Schrödinger equations. In Nonlinear evolution equations, volume 164 of Amer. Math. Soc. Transl. Ser. 2, pages 75–98. Amer. Math. Soc., Providence, RI, 1995.
- [Bre84] P. Brenner. On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations. *Mathematische Zeitschrift*, 186(3):383–391, 1984.
- [CK01] M. Christ and A. Kiselev. Maximal functions associated to filtrations. *J. Funct. Anal.*, 179(2):409–425, 2001.

- [CL82] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. Comm. Math. Phys., 85(4):549–561, 1982.
- [CM08] S. Cuccagna and T. Mizumachi. On asymptotic stability in energy space of ground states for nonlinear Schrödinger equations. Comm. Math. Phys., 284(1):51–77, 2008.
- [CP06] M. Chugunova and D. Pelinovsky. Block-diagonalization of the symmetric first-order coupled-mode system. SIAM J. Appl. Dyn. Syst., 5(1):66–83 (electronic), 2006.
- [CPV05] S. Cuccagna, D. Pelinovsky, and V. Vougalter. Spectra of positive and negative energies in the linearized NLS problem. Comm. Pure Appl. Math., 58(1):1–29, 2005.
- [Cuc01] S. Cuccagna. Stabilization of solutions to nonlinear Schrödinger equations. Comm. Pure Appl. Math., 54(9):1110–1145, 2001.
- [Cuc08] S. Cuccagna. On asymptotic stability in energy space of ground states of NLS in 1D. J. Differential Equations, 245(3):653–691, 2008.
- [Cuc09] S. Cuccagna. On instability of excited states of the nonlinear Schrödinger equation. Phys. D, 238(1):38–54, 2009.
- [Cuc10] S. Cuccagna. The hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states, arXiv:0910.3797, 2010.
- [Cuc11] S. Cuccagna. On scattering of small energy solutions of non autonomous hamiltonian nonlinear schrödinger equations, J. Differential Equations, 250(5):2347–2371, 2011.
- [CV86] T. Cazenave and L. Vázquez. Existence of localized solutions for a classical nonlinear Dirac field. Comm. Math. Phys., 105(1):35–47, 1986.
- [EGS09] M. B. Erdoğan, M. Goldberg, and W. Schlag. Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions. Forum Math., 21(4):687–722, 2009.
- [ELS08] M.J. Esteban, M. Lewin, and E. Séré. Variational methods in relativistic quantum mechanics. Bull. Amer. Math. Soc. (N.S.), 45(4):535–593, 2008.
- [ES95] M.J. Esteban and É. Séré. Stationary states of the nonlinear Dirac equation: a variational approach. Comm. Math. Phys., 171(2):323–350, 1995.
- [GM01] V. Georgescu and M. Măntoiu. On the spectral theory of singular Dirac type Hamiltonians. J. Operator Theory, 46(2):289–321, 2001.
- [GSS87] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. J. Funct. Anal., 74(1):160–197, 1987.
- [GSS90] M. Grillakis, J., and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II. J. Funct. Anal., 94(2):308–348, 1990.
- [Gua08] Meijiao Guan. Solitary wave solutions for the nonlinear Dirac equations, arXiv:0812.2273, 2008.
- [His00] P. D. Hislop. Exponential decay of two-body eigenfunctions: a review. In Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory (Berkeley, CA, 1999), Conf. 4 Electron. J. Differ. Equ. Conf., pages 265–288 (electronic), San Marcos, TX, 2000. Southwest Texas State Univ.
- [IM99] A. Iftimovici and M. Măntoiu. Limiting absorption principle at critical values for the Dirac operator. Lett. Math. Phys., 49(3):235–243, 1999.
- [Kat66] T. Kato. Wave operators and similarity for some non-selfadjoint operators. Math. Ann., 162:258–279, 1965/1966.
- [KK10] A. Komech and A. Komech. On global attraction to quantum stationary states. Diracequation with mean field interaction, February 2010.
- [Mer88] F. Merle. Sur la non-existence de solutions positives d'équations elliptiques surlinéaires. C. R. Acad. Sci. Paris Sér. I Math., 306(6):313–316, 1988.
- [MM08] Y. Martel and F. Merle. Asymptotic stability of solitons of the gKdV equations with general nonlinearity. Math. Ann., 341(2):391–427, 2008.
- [NS10] K. Nakanishi and W. Schlag. Global dynamics above the ground state energy for the cubic nls equation in 3d, July 2010.
- [Oun00] H. Ounaies. Perturbation method for a class of nonlinear Dirac equations. Differential Integral Equations, 13(4-6):707–720, 2000.
- [OI93] P. J. Olver. Applications of Lie groups to differential equations. Second edition. Graduate Texts in Mathematics, 107. Springer-Verlag, New York, 1993.
- [PS10] D. Pelinovsky and A. Stefanov. Asymptotic stability of small gap solitons in the nonlinear dirac equations, arXiv:1008.4514, 2010.
- [Ran] A.F. Ranada. Classical nonlinear Dirac field models of extended particles. Quantum theory, groups, fields and particles, 271–291, 1982.
- [RS78] M. Reed and B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [Sh83] J.Shatah. Stable standing waves of nonlinear Klein-Gordon equations. Comm. Math. Phys., 91:313–327, 1983.
- [SS85] J.Shatah and W.Strauss. Instability of nonlinear bound states. Comm. Math. Phys., 100:173–190, 1985.
- [Sog95] C.D. Sogge. Lectures on nonlinear wave equations. International Press Boston, 1995.
- [Sol70] M. Soler. Classical, stable, nonlinear spinor field with positive rest energy. Physical Review D, 1(10):2766–2769, 1970.
- [SS00] H.. Smith and C. Sogge. Global Strichartz estimates for nontrapping perturbations of the Laplacian. Comm. Partial Differential Equations, 25(11-12):2171–2183, 2000.
- [SU08] Y. Saitō and T. Umeda. The zero modes and zero resonances of massless Dirac operators. Hokkaido Math. J., 37(2):363–388, 2008.
- [SV86] W.. Strauss and L. Vázquez. Stability under dilations of nonlinear spinor fields. Phys. Rev. D (3), 34(2):641–643, 1986.

- [SW89] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering theory for nonintegrable equations. In Integrable systems and applications (Île d'Oléron, 1988), volume 342 of Lecture Notes in Phys., pages 312–327. Springer, Berlin, 1989.
- [SW92] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data. J. Differential Equations, 98(2):376–390, 1992.
- [Tay96] M. Taylor. Partial differential equations., volumes 115–117 of Applied Mathematical Sciences. Springer-Verlag, New York, 1996.
- [Tha92] B. Thaller. The Dirac equation. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.
- [Wei85] M.I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. SIAM J. Math. Anal., 16(3):472–491, 1985.
- [Wei86] M.I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math., 39(1):51–67, 1986.

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