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# Scattering and correlations

Yves Colin de Verdière \*

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## Introduction

Let us consider the propagation of scalar waves with the speed  $v > 0$  given by the wave equation  $u_{tt} - v^2 \Delta u = 0$  outside a compact domain  $D$  in the Euclidean space  $\mathbb{R}^d$ . Let us put  $\Omega = \mathbb{R}^d \setminus D$ . We can assume for example Neumann boundary conditions. We will denote by  $\Delta_\Omega$  the (self-adjoint) Laplace operator with the boundary conditions. So our stationary wave equation is the Helmholtz equation

$$v^2 \Delta_\Omega f + \omega^2 f = 0 \tag{1}$$

with the boundary conditions. We consider a bounded interval  $I = [\omega_-^2, \omega_+^2] \subset ]0, +\infty[$  and the Hilbert subspace  $\mathcal{H}_I$  of  $L^2(\Omega)$  which is the image of the spectral projector  $P_I$  of our operator  $-v^2 \Delta_\Omega$ .

Let us compute the integral kernel  $\Pi_I(x, y)$  of  $P_I$  defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of  $\Pi_I(x, y)$  w.r. to  $\omega_+$ , we get a simple general and exact relation between the correlation of scattered waves and the Green's function confirming the calculations from [2] in the case where  $D$  is a disk. See also [1] for other relations between correlations and Green's functions used in passive imaging.

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## 1 $\Pi_I(x, y)$ from spectral theory

Using the resolvent kernel (Green's function)  $G(\omega, x, y) = [(\omega^2 + v^2\Delta_\Omega)^{-1}](x, y)$  for  $\text{Im}\omega > 0$  and the Stone formula, we have:

$$\Pi_I(x, y) = -\frac{2}{\pi} \text{Im} \left( \int_{\omega_-}^{\omega_+} G(\omega + i0, x, y) \omega d\omega \right)$$

Taking the derivative w.r. to  $\omega_+$  of  $\Pi_{[\omega_-^2, \omega_+^2]}(x, y)$ , we get

$$\frac{d}{d\omega} \Pi_{[\omega_-, \omega^2]}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) . \quad (2)$$

## 2 Short review of scattering theory

There are many references for scattering theory: for example [3, 4].

Let us define for  $\mathbf{k} \in \mathbb{R}^d$  the plane wave

$$e_0(x, \mathbf{k}) = e^{i\langle \mathbf{k} | x \rangle} .$$

We are looking for solutions

$$e(x, \mathbf{k}) = e_0(x, \mathbf{k}) + e^s(x, \mathbf{k})$$

of the Helmholtz equation (1) in  $\Omega$  where  $e^s$ , the scattered wave satisfies the so-called Sommerfeld radiation condition<sup>1</sup>:

$$e^s(x, \mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left( e^\infty\left(\frac{x}{|x|}, \mathbf{k}\right) + O\left(\frac{1}{|x|}\right) \right), \quad x \rightarrow \infty .$$

The complex function  $e^\infty(\hat{x}, \mathbf{k})$  is usually called the *scattering amplitude*.

It is known that the previous problem admits a unique solution. In more physical terms,  $e(x, \mathbf{k})$  is the wave generated by the full scattering process from the plane wave  $e_0(x, \mathbf{k})$ . Moreover we have a generalized Fourier transform:

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\mathbf{k}) e(x, \mathbf{k}) |d\mathbf{k}|$$

with

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} \overline{e(y, \mathbf{k})} f(y) |dy| .$$

From the previous generalized Fourier transform, we can get the kernel of any function  $\Phi(-v^2\Delta_\Omega)$  as follows:

$$[\Phi(-v^2\Delta_\Omega)](x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Phi(v^2k^2) e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}| . \quad (3)$$

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<sup>1</sup>As often, we denote  $k := |\mathbf{k}|$  and  $\hat{\mathbf{k}} := \mathbf{k}/k$

### 3 $\Pi_I(x, y)$ from scattering theory

Using Equation (3) with  $\Phi = 1_I$  the characteristic functions of some bounded interval  $I = [\omega_-, \omega^2]$ , we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}| .$$

Using polar coordinates and defining  $|d\sigma|$  as the usual measure on the unit  $(d-1)$ -dimensional sphere, we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} k^{d-1} dk \int_{\mathbf{k}^2=k^2} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

We will denote by  $\sigma_{d-1}$  the total volume of the unit sphere in  $\mathbb{R}^d$ :  $\sigma_0 = 2$ ,  $\sigma_1 = 2\pi$ ,  $\sigma_2 = 4\pi, \dots$ .

Taking the same derivative as before, we get:

$$\frac{d}{d\omega} \Pi_{[\omega_-, \omega^2]}(x, y) = (2\pi)^{-d} \frac{\omega^{d-1}}{v^d} \int_{vk=\omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

Let us look at  $e(x, \mathbf{k})$  as a random wave with  $k = \omega/v$  fixed. The point-point correlation of such a random wave  $C_\omega^{\text{scatt}}(x, y)$  is given by:

$$C_\omega^{\text{scatt}}(x, y) = \frac{1}{\sigma_{d-1}} \int_{vk=\omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

Then we have:

$$\frac{d}{d\omega} \Pi_{[\omega_-, \omega^2]}(x, y) = (2\pi)^{-d} \frac{\omega^{d-1} \sigma_{d-1}}{v^d} C_\omega^{\text{scatt}}(x, y) . \quad (4)$$

### 4 Correlation of scattered plane waves and Green's function: the scalar case

From Equations (2) and (4), we get:

$$(2\pi)^{-d} \frac{\omega^{d-1} \sigma_{d-1}}{v^d} C_\omega^{\text{scatt}}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) .$$

Hence, we have

**Theorem 1** *For the scalar wave equation  $u_{tt} - v^2 \Delta u = 0$  outside a bounded domain in  $\mathbb{R}^d$ , we have the following expression of the correlation of scattered wave of frequency  $\omega$  in terms of the Green's function:*

$$C_\omega^{\text{scatt}}(x, y) = -\frac{2^{d+1} \pi^{d-1} v^d}{\sigma_{d-1} \omega^{d-2}} \text{Im}(G(\omega + i0, x, y)) .$$

For later use, we put

$$\gamma_d = \frac{2^{d+1} \pi^{d-1}}{\sigma_{d-1}} . \quad (5)$$

## 5 The case of elastic waves

We will consider the elastic wave equation in the domain  $\Omega$ :

$$\hat{H}\mathbf{u} - \omega^2\mathbf{u} = 0,$$

with self-adjoint boundary conditions. We will assume that, *at large distances*, we have

$$\hat{H}\mathbf{u} = -a \Delta\mathbf{u} - b \operatorname{grad} \operatorname{div} \mathbf{u} .$$

where  $a$  and  $b$  are constants:

$$a = \frac{\mu}{\rho}, \quad b = \frac{\lambda + \mu}{\rho}$$

with  $\lambda, \mu$  the Lamé's coefficients and  $\rho$  the density of the medium. We will denote  $v_P := \sqrt{a+b}$  (resp.  $v_S := \sqrt{a}$ ) the speeds of the  $P$ - (resp.  $S$ -) waves near infinity.

### 5.1 The case $\Omega = \mathbb{R}^d$

We want to derive the spectral decomposition of  $\hat{H}$  from the Fourier inversion formula. Let us choose, for  $\mathbf{k} \neq 0$ , by  $\hat{\mathbf{k}}, \hat{\mathbf{k}}_1, \dots, \hat{\mathbf{k}}_{d-1}$  an orthonormal basis of  $\mathbb{R}^d$  with  $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$  such that these vectors depends in a measurable way of  $\mathbf{k}$ . Let us introduce  $P_P^{\mathbf{k}} = \hat{\mathbf{k}}\hat{\mathbf{k}}^*$  the orthogonal projector onto  $\hat{\mathbf{k}}$  and  $P_S^{\mathbf{k}} = \sum_{j=1}^{d-1} \hat{\mathbf{k}}_j\hat{\mathbf{k}}_j^*$  so that  $P_P + P_S = \operatorname{Id}$ . Those projectors correspond respectively to the polarizations of  $P$ - and  $S$ -waves.

We have

$$\begin{aligned} \Pi_I(x, y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega & \left( v_P^{-d} \int_{v_P k = \omega} e^{i\mathbf{k}(x-y)} P_P^{\mathbf{k}} d\sigma + \right. \\ & \left. v_S^{-d} \int_{v_S k = \omega} e^{i\mathbf{k}(x-y)} P_S^{\mathbf{k}} d\sigma \right) . \end{aligned}$$

using the plane waves

$$e_P^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}$$

and

$$e_{S,j}^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}_j$$

we get the formula<sup>2</sup>:

$$\begin{aligned} \Pi_I(x, y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega & \left( v_P^{-d} \int_{v_P k = \omega} |e_P^O(x, \mathbf{k})\rangle \langle e_P^O(y, \mathbf{k})| d\sigma + \right. \\ & \left. v_S^{-d} \sum_{j=1}^{d-1} \int_{v_S k = \omega} |e_{S,j}^O(x, \mathbf{k})\rangle \langle e_{S,j}^O(y, \mathbf{k})| d\sigma \right) . \end{aligned}$$

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<sup>2</sup>We use the “bra-ket” notation of quantum mechanics:  $|e\rangle\langle f|$  is the operator  $x \rightarrow \langle f|x\rangle e$  where the brackets are linear w.r. to the second entry and anti-linear w.r. to the first one

## 5.2 Scattered plane waves

There exists scattered plane waves

$$\begin{aligned} e_P(x, \mathbf{k}) &= e_P^O(x, \mathbf{k}) + e_P^s(x, \mathbf{k}) \\ e_{S,j}(x, \mathbf{k}) &= e_{S,j}^O(x, \mathbf{k}) + e_{S,j}^s(x, \mathbf{k}) \end{aligned}$$

satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of  $\hat{H}$ .

## 5.3 Correlations of scattered plane waves and Green's function

Following the same path as for scalar waves, we get an identity which holds now for the full Green's tensor  $\text{Im}\mathbf{G}(\omega + iO, x, y)$ :

**Theorem 2** *For the elastic wave equation, we have the following expression of the imaginary part of the Green's function in terms of the correlation of the scattered S- and P- waves:*

$$\begin{aligned} \text{Im}\mathbf{G}(\omega + iO, x, y) &= -\gamma_d^{-1}\omega^{d-2} \left( \frac{1}{\sigma_{d-1}v_P^d} \int_{v_P k=\omega} |e_P(x, \mathbf{k})\rangle\langle e_P(y, \mathbf{k})|d\sigma + \right. \\ &\quad \left. \frac{1}{\sigma_{d-1}v_S^d} \sum_{j=1}^{d-1} \int_{v_S k=\omega} |e_{S,j}(x, \mathbf{k})\rangle\langle e_{S,j}(y, \mathbf{k})|d\sigma \right), \end{aligned}$$

with  $\gamma_d$  defined by Equation (5).

This formula expresses the fact that the correlation of scattered plane waves randomized with the appropriate weights ( $v_P^{-d}$  versus  $v_S^{-d}$ ) is proportional to the Green's tensor. Let us insist on the fact that this is true everywhere in  $\Omega$  even in the domain where  $a$  and  $b$  are not constants.

## References

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