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To cite this version:

Piermarco Cannarsa, Pierre Cardaliaguet. Regularity Results for Eikonal-Type Equations with Nonsmooth Coefficients. Nonlinear Differential Equations and Applications, 2012, 19 (6), pp.751–769. hal-00578730

HAL Id: hal-00578730

https://hal.science/hal-00578730v1

Submitted on 22 Mar 2011

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Regularity Results for Eikonal-Type Equations with Nonsmooth Coefficients

Piermarco Cannarsa* & Pierre Cardaliaguet †

March 22, 2011

Abstract

Solutions of the Hamilton-Jacobi equation $H(x, -Du(x)) = 1$, with $H(\cdot, p)$ Hölder continuous and $H(x, \cdot)$ convex and positively homogeneous of degree 1, are shown to be locally semiconcave with a power-like modulus. An essential step of the proof is the $\mathcal{C}^{1,\alpha}$-regularity of the extremal trajectories associated with the multifunction generated by D_pH.

Key words: viscosity solutions, semiconcave functions, differential inclusions, extremal trajectories

MSC Subject classifications: 49L25 34A60 26B25 49N60

1 Introduction

The importance of semiconcavity for the study of Hamilton-Jacobi equations and optimal control problems is by now widely acknowledged. Indeed, such a qualitative property ensures the upper semicontinuity and quasi-monotonicity of the superdifferential, provides upper bounds for the set where differentiability fails providing, at the same time, criteria for the propagation of singularities, and leads to stronger optimality conditions than the ones holding for a continuous (or Lipschitz continuous) function, see, for instance, [4] and the references therein.

Typically, a real-valued function u is semiconcave on the convex set $D \subset \mathbb{R}^N$ if there exists a modulus (i.e., nondecreasing upper semicontinuous function, vanishing at 0) $\omega : [0, \infty) \to [0, \infty)$ such that

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y) - C\lambda(1-\lambda)|x-y|\omega(|x-y|)$$

for all $x, y \in D$ and $\lambda \in [0, 1]$.

Semiconcavity results with a linear modulus hold for viscosity solutions of Hamilton-Jacobi equations with convex Hamiltonians which are sufficiently smooth with respect to the space variables, as well as value functions of optimal control problems with smooth dynamics and running cost (see, e.g., [8], [7],

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[3]; see also [4]). Known generalizations allow for Lipschitz continuous dependance with respect to space, provided the Hamiltonian is strictly convex and superlinear in the gradient variables (see [5], [9]).

In this paper we shall study the Dirichlet problem

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open subset of $\mathbb{R}^N$, $H(x, \cdot)$ is convex and positively homogeneous of degree 1, and $H(\cdot, p)$ is just Hölder continuous. Consequently, (1) fits none of the aforementioned settings. Nevertheless, our main result—Theorem 5.1 below—guarantees that the solution u of (1) is locally semiconcave in Ω with the power-like modulus $\omega(t) = Ct^\theta$, for some $\theta > 0$ depending on H.

The method of proof relies on the representation of $u(x)$ as the minimum time needed to reach $\partial\Omega$ along a trajectory of the differential inclusion

$$\begin{cases} x'(t) \in F(x(t)) \quad t \geq 0 \quad \text{a.e.} \\ x(0) = x\,, \end{cases} \tag{2}$$

where

$$F(x) = \text{co}\,\{D_pH(x, p) \ : \ p \in \mathbb{R}^N \backslash \{0\}\} \qquad \forall x \in \mathbb{R}^N\,.$$

An essential step of the analysis is the $\mathcal{C}^{1,\alpha}$-regularity of the extremal trajectories of (2), see Theorem 4.1. For time-dependent and isotropic Hamiltonians ($H = a(t, x)|p|$), such a regularity property—interesting in its own right—has already been observed in [2] for $N = 2$, and [6] for general N. However, the unexpected connection between Theorem 4.1 and the semiconcavity of the solution of (1) is, to our best knowledge, entirely new.

The main technical tools we borrow from convex analysis are recalled in detail in section 3, which makes this paper essentially self-contained.

2 Notation and assumptions

Let N be a positive integer. Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^N$, respectively, and set

$$B = \{x \in \mathbb{R}^N \ : \ |x| \leq 1\}\,.$$

More generally, for all $x \in \mathbb{R}^N$ and $\rho > 0$, $B(x, \rho)$ stands for the closed ball of radius ρ centered at x, that is, $B(x, \rho) = x + \rho B$.

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying the following assumptions for some positive constants C_0, r, R, with $r < R$, and $\alpha \in (0, 1/2)$.

Standing Assumptions (SA):

- For all $p \in \mathbb{R}^N$, the function $x \mapsto H(x, p)$ is 2α-Hölder continuous, and

$$|H(x, p) - H(y, p)| \leq C_0|x - y|^{2\alpha}|p| \qquad \forall x, y \in \mathbb{R}^N\,. \tag{3}$$

- For all $x \in \mathbb{R}^N$, the function $p \mapsto H(x, p)$ is convex on $\mathbb{R}^N$, positively homogeneous of degree one, and has linear growth, i.e.,

$$r|p| \ \leq \ H(x, p) \ \leq R|p| \qquad \forall p \in \mathbb{R}^N\,. \tag{4}$$

- For all $x \in \mathbb{R}^N$, the function $p \mapsto H(x,p)$ is continuously differentiable on $\mathbb{R}^N \backslash \{0\}$, and, for all $p, q \in \mathbb{R}^N \backslash \{0\}$,

$$\begin{aligned} -\frac{1}{2r} \left| D_p H(x,q) - D_p H(x,p) \right|^2 & \\ \leq \ \langle D_p H(x,q) - D_p H(x,p), \frac{p}{|p|} \rangle & \\ \leq -\frac{1}{2R} \left| D_p H(x,q) - D_p H(x,p) \right|^2 , & \end{aligned} \tag{5}$$

  where $D_p H(x,p)$ denotes the gradient of H in the p-variables at (x,p).

Hereafter, by a universal constant—briefly, a constant—we mean a *positive* real number that only depends on the parameters N, α, r, R, and C_0 introduced above. Generic constants appearing in computations will be denoted by C. A subscript $(C_1, C_2, \dots)$ will be added when necessary for future reference.

Example 2.1. Let $H(x,p) = |A(x)p|$, where $A : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is α-Hölder continuous on $\mathbb{R}^N$, $A(x)$ is invertible for all $x \in \mathbb{R}^N$, and

$$|A(x)| \leq C \quad \text{and} \quad |A(x)^{-1}| \leq C \qquad \forall x \in \mathbb{R}^N$$

for some constant C. Then H satisfies (SA) for a suitable choice of constants.

3 Preliminary results

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying our Standing Assumptions with fixed constants α, r, R, and C_0.

For all $p \neq 0$, set $f_p(x) = D_p H(x,p)$ and define

$$F(x) = \mathrm{co}\,\{ f_p(x) \ : \ p \in \mathbb{R}^N \backslash \{0\} \} \qquad \forall x \in \mathbb{R}^N , \tag{6}$$

where 'co' stands for convex hull. Note that, for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \backslash \{0\})$,

$$H(x,p) = \max_{v \in F(x)} \langle v, p \rangle \qquad \text{and} \qquad f_p(x) = \mathrm{argmax}_{v \in F(x)} \langle v, p \rangle . \tag{7}$$

We begin by recovering some properties of $F(\cdot)$ that follow directly from (SA).

Lemma 3.1. *The set-valued map F is 2α-Hölder continuous, i.e.,*

$$F(x) \subset F(y) + C_0 |x-y|^{2\alpha} B \qquad \forall x, y \in \mathbb{R}^N , \tag{8}$$

and satisfies, for all $x \in \mathbb{R}^N$, the curvature *estimates*

$$B\Big(f_p(x) - r\frac{p}{|p|}, r \Big) \subset F(x) \quad \text{and} \quad F(x) \subset B\Big(f_p(x) - R\frac{p}{|p|}, R \Big) \qquad \forall p \neq 0 , \tag{9}$$

as well as the controllability *condition*

$$B(0,r) \ \subset \ F(x) \ \subset \ B(0,R) . \tag{10}$$

Remark 3.2. The first inclusion in (9)—which can be interpreted as an upper bound for the curvature of $\partial F(x)$—is equivalent to the inequality

$$\left| v - f_p(x) + r\frac{p}{|p|} \right| \geq r \qquad \forall v \in \partial F(x)\,,$$

which in turn can be recast as follows

$$-\frac{1}{2r}\,|v - f_p(x)|^2 \leq \left\langle v - f_p(x), \frac{p}{|p|} \right\rangle \qquad \forall v \in \partial F(x)\,. \tag{11}$$

The second inclusion in (9)—a lower bound for the curvature—can be rephrased as

$$\left| v - f_p(x) + R\frac{p}{|p|} \right| \leq R \qquad \forall v \in F(x)\,,$$

which is equivalent to

$$\left\langle v - f_p(x), \frac{p}{|p|} \right\rangle \leq -\frac{1}{2R}|v - f_p(x)|^2 \qquad \forall v \in F(x)\,,$$

or, since $F(x)$ is convex,

$$\left\langle v - f_p(x), \frac{p}{|p|} \right\rangle \leq -\frac{1}{2R}|v - f_p(x)|^2 \qquad \forall v \in \partial F(x)\,. \tag{12}$$

Proof. Note that, since $H(x,\cdot)$ is the support function of $F(x)$, inequality (3) directly implies (8) while (4) entails (10). Let us now check that the regularity condition (5) implies (9). For this we just have to note that, for any $v \in \partial F(x)$, there is some $q \neq 0$ such that $v = f_q(x)$, so that (5) becomes

$$-\frac{1}{r}\,|v - f_p(x)|^2 \leq \left\langle v - f_p(x), \frac{p}{|p|} \right\rangle \leq -\frac{1}{R}\,|v - f_p(x)|^2 \qquad \forall v \in \partial F(x)\,.$$

These two equalities are equivalent to (11) and (12). □

Next, we derive a regularity result for $f_p(\cdot)$, which is actually a consequence of the Hölder continuity of F in (8) combined with the lower curvature bound in (5).

Lemma 3.3. *For all $p \in \mathbb{R}^N \backslash \{0\}$ we have*

$$|f_p(x) - f_p(y)| \leq (C_0 + \sqrt{2C_0R})\;|x-y|^\alpha \qquad \forall x, y \in \mathbb{R}^N\,.$$

Proof. Let $x, y \in \mathbb{R}^N$. In view of (8) there are $v_y \in F(x)$ and $v_x \in F(y)$ such that

$$|f_p(y) - v_y| \leq C_0|x-y|^{2\alpha} \quad \text{and} \quad |f_p(x) - v_x| \leq C_0|x-y|^{2\alpha}\,.$$

Then, by (12),

$$\left\langle v_y - f_p(x), \frac{p}{|p|} \right\rangle \leq -\frac{1}{2R}|v_y - f_p(x)|^2 \text{ and } \left\langle v_x - f_p(y), \frac{p}{|p|} \right\rangle \leq -\frac{1}{2R}|v_x - f_p(y)|^2\,.$$

Adding up the above two inequalities yields

$$|v_y - f_p(x)|^2 + |v_x - f_p(y)|^2 \;\leq -2R\left\langle v_y - f_p(x) + v_x - f_p(y), \frac{p}{|p|} \right\rangle \;\leq\; 2C_0R|x-y|^{2\alpha}\,.$$

So,

$$|f_p(y) - f_p(x)| \le |f_p(y) - v_y| + |v_y - f_p(x)| \le (C_0 + \sqrt{2C_0R})\ |x-y|^\alpha\,,$$

and the proof is complete. □

Describing the way how $f_p(x)$ depends on p is the object of our next result.

Lemma 3.4. *For every $x \in \mathbb{R}^N$ we have*

$$\frac{1}{R}|f_p(x) - f_q(x)| \le \left|\frac{p}{|p|} - \frac{q}{|q|}\right| \le \frac{1}{r}|f_p(x) - f_q(x)|\,, \qquad \forall p, q \in \mathbb{R}^N\backslash\{0\}\,.$$

Proof. Let $x \in \mathbb{R}^N$ and let $p, q \in \mathbb{R}^N\backslash\{0\}$. Let us start with the first inequality. Recalling the second condition in (9) in its equivalent form (12), we have, since $f_q(x) \in \partial F(x)$,

$$\Big\langle f_q(x) - f_p(x), \frac{p}{|p|}\Big\rangle \le -\frac{1}{2R}|f_q(x) - f_p(x)|^2\,.$$

In a symmetric way we also have

$$\Big\langle f_p(x) - f_q(x), \frac{q}{|q|}\Big\rangle \le -\frac{1}{2R}|f_p(x) - f_q(x)|^2\,.$$

Adding the two inequalities easily gives the first inequality.

We now prove the second inequality, which is slightly more subtle. Recalling the first inclusion in (9) and the definition of $f_p(x)$, we conclude that

$$\Big\langle q, f_p(x) - r\frac{p}{|p|} + rb\Big\rangle \le \langle q, f_q(x)\rangle\,, \qquad \forall b \in B\,.$$

Hence,

$$-r\Big\langle \frac{q}{|q|}, \frac{p}{|p|}\Big\rangle + r \le \Big\langle f_q(x) - f_p(x), \frac{q}{|q|}\Big\rangle\,.$$

Thus, exchanging p and q,

$$-r\Big\langle \frac{p}{|p|}, \frac{q}{|q|}\Big\rangle + r \le \Big\langle f_p(x) - f_q(x), \frac{p}{|p|}\Big\rangle\,.$$

Adding the above inequalities together leads to

$$r\Big|\frac{p}{|p|} - \frac{q}{|q|}\Big|^2 = 2r\Big(1 - \Big\langle \frac{p}{|p|}, \frac{q}{|q|}\Big\rangle\Big) \le \Big\langle f_p(x) - f_q(x), \frac{p}{|p|} - \frac{q}{|q|}\Big\rangle\,. \tag{13}$$

Since $f_p(x)$ and $f_q(x)$ are boundary points, (11) yields

$$\Big\langle f_p(x) - f_q(x), \frac{p}{|p|}\Big\rangle \le \frac{1}{2r}\,|f_q(x) - f_p(x)|^2\ .$$

and

$$\Big\langle f_q(x) - f_p(x), \frac{q}{|q|}\Big\rangle \le \frac{1}{2r}\,|f_q(x) - f_p(x)|^2\ .$$

Therefore,

$$\Big\langle f_p(x) - f_q(x), \frac{p}{|p|} - \frac{q}{|q|}\Big\rangle \le \frac{1}{r}|f_p(x) - f_q(x)|^2\ . \tag{14}$$

The conclusion follows from (13) and (14). □

Let us now consider the *polar* of H, namely the function H^0 defined by

$$H^0(x,q) := \max\big\{\langle p,q\rangle \ : \ H(x,p)\le 1\big\} \qquad \forall (x,q)\in\mathbb{R}^N\times\mathbb{R}^N\,.$$

It is well-known that, for all $(x,q)\in\mathbb{R}^N\times\mathbb{R}^N$,

$$H^0(x,q)\le 1 \qquad \Longleftrightarrow \qquad q\in F(x)\,, \tag{15}$$

and

$$H^0\big(x,D_pH(x,p)\big) = H^0\big(x,f_p(x)\big) = 1 \qquad \forall (x,p)\in\mathbb{R}^N\times(\mathbb{R}^N\backslash\{0\})\,. \tag{16}$$

The duality between H and H^0 brings similar qualitative properties for these two functions. For instance, on account of (10), we have

$$\frac{|q|}{R}\le H^0(x,q)\le\frac{|q|}{r} \qquad \forall (x,q)\in\mathbb{R}^N\times\mathbb{R}^N\,. \tag{17}$$

Moreover, H^0 is also Hölder continuous with respect to x, with the same exponent as H.

Lemma 3.5. *For all $q\in\mathbb{R}^N$,*

$$|H^0(x,q)-H^0(y,q)|\le\frac{C_0}{r^2}|q||x-y|^{2\alpha} \qquad \forall x,y\in\mathbb{R}^N\,. \tag{18}$$

Proof. Let $x,y,q\in\mathbb{R}^N$. Take $p\in\mathbb{R}^N$, with $H(x,p)\le 1$, such that $H^0(x,q)=\langle p,q\rangle$. Then, by (4), $|p|\le 1/r$. Also, by (3),

$$H(y,p)\le 1+\frac{C_0}{r}|x-y|^{2\alpha}\,.$$

So,

$$H^0(y,q)\ge\Big\langle\frac{p}{1+\frac{C_0}{r}|x-y|^{2\alpha}},q\Big\rangle=\frac{H^0(x,q)}{1+\frac{C_0}{r}|x-y|^{2\alpha}}\,.$$

On the other hand, in view of (17),

$$\begin{aligned} H^0(x,q) &= \frac{H^0(x,q)}{1+\frac{C_0}{r}|x-y|^{2\alpha}}+\frac{\frac{C_0}{r}|x-y|^{2\alpha}}{1+\frac{C_0}{r}|x-y|^{2\alpha}}H^0(x,q)\\ &\le \frac{H^0(x,q)}{1+\frac{C_0}{r}|x-y|^{2\alpha}}+\frac{C_0}{r^2}|q||x-y|^{2\alpha}\,. \end{aligned}$$

Thus,

$$H^0(y,q)\ge H^0(x,q)-\frac{C_0}{r^2}|q||x-y|^{2\alpha}\,.$$

Hence, we obtain the conclusion exchanging the roles of x and y. □

We now turn to the analysis of the level set

$$F^0(x)=\big\{p\in\mathbb{R}^N \ : \ H(x,p)\le 1\big\} \qquad x\in\mathbb{R}^N\,.$$

Lemma 3.6. *Let $x \in \mathbb{R}^N$. Then, for every $p, p' \in \mathbb{R}^N$ with $H(x,p) = H(x,p') = 1$,*

$$|p - p'| \leq C\Big|\frac{p'}{|p'|} - \frac{p}{|p|}\Big| \tag{19}$$

for some constant C.

Proof. First of all, the reader be warned that, as x plays no role in this proof, the $x-$dependence in H will be omitted. For all $\theta, \theta' \in S^{N-1}$, we have

$$\Big|\frac{\theta}{H(\theta)} - \frac{\theta'}{H(\theta')}\Big| \leq \frac{|\theta - \theta'|}{H(\theta)} + \frac{|H(\theta) - H(\theta')|}{H(\theta)H(\theta')}\,.$$

Since H is Lipschitz continuous by (10), recalling $r \leq H(\theta), H(\theta') \leq R$ we conclude that

$$\frac{|\theta - \theta'|}{H(\theta)} + \frac{|H(\theta) - H(\theta')|}{H(\theta)H(\theta')} \leq C|\theta - \theta'|$$

for some constant C. Therefore,

$$\Big|\frac{\theta}{H(\theta)} - \frac{\theta'}{H(\theta')}\Big| \leq C|\theta - \theta'|\,.$$

Now, observe that the map $\theta \mapsto \theta/H(\theta)$ is a bijection between the unit sphere S^{N-1} and $\partial F^0(x)$. So, applying the above inequality to $\theta, \theta' \in S^{N-1}$ chosen such that $p = \theta/H(\theta)$ and $p' = \theta'/H(\theta')$ we obtain the conclusion. □

Lemma 3.7 (Lower curvature estimate for F^0)**.** *There is a constant R' such that $F^0(x)$ satisfies the lower curvature estimate of radius R' for all $x \in \mathbb{R}^N$, i.e.,*

$$F^0(x) \subset B\Big(\frac{p}{H(x,p)} - R'\frac{f_p(x)}{|f_p(x)|}, R'\Big) \qquad \forall x, p \in \mathbb{R}^N,\ p \neq 0$$

or, equivalently,

$$\Big\langle p' - p, \frac{f_p}{|f_p|}\Big\rangle \leq -\frac{1}{2R'}|p' - p|^2 \qquad \forall p, p' \in \partial F^0 \tag{20}$$

Proof. Again, we shall drop the x-dependence in all the formulas below since it is of no interest for this proof. Recalling Remark 3.2 we conclude that it suffices to prove inequality (20) for some constant R'. Let then $p, p' \in \partial F^0$. Since H is positively homogeneous of degree 1, we have

$$\begin{aligned} H(p') - H(p) - \langle D_pH(p), p' - p\rangle &= \langle D_pH(p'), p'\rangle - \langle D_pH(p), p\rangle - \langle D_pH(p), p' - p\rangle \\ &= \langle D_pH(p') - D_pH(p), p'\rangle \end{aligned}$$

where $D_pH(p) = f_p$ and $D_pH(p') = f_{p'}$. From the lower curvature estimate on F given in (12) it follows that

$$\Big\langle f_p - f_{p'}, \frac{p'}{|p'|}\Big\rangle \leq -\frac{1}{2R}\,|f_{p'} - f_p|^2\ .$$

Thus, combining the above inequality with the previous identity, and using the fact that $H(p) = H(p') = 1$,

$$\langle f_p, p' - p\rangle \ \leq\ -\frac{|p'|}{2R}\,|f_{p'} - f_p|^2\ .$$

Now, apply Lemma 3.4 to obtain

$$\langle f_p, p' - p\rangle \;\le\; -\frac{r^2|p'|}{2R}\Big|\frac{p'}{|p'|} - \frac{p}{|p|}\Big|^2 . \tag{21}$$

Finally, let C be the constant given by Lemma 3.6. Then, (19) and (21) yield

$$\Big\langle \frac{f_p}{|f_p|}, p' - p\Big\rangle \;\le\; -\frac{r^2|p'|}{2R^2}\Big|\frac{p'}{|p'|} - \frac{p}{|p|}\Big|^2 \;\le\; -\frac{r^2}{2C^2R^3}\,|p' - p|^2 .$$

whence the conclusion follows with $R' = C^2R^3/r^2$. □

In particular, Lemma 3.7 ensures $F^0(x)$ is a strictly convex set for any $x \in \mathbb{R}^N$. Thus, since $H^0(x,\cdot)$ is the support function of $F^0(x)$, $D_qH^0(x,q)$ exists for any $x, q \in \mathbb{R}^N$ with $q \neq 0$ (see, for instance, [4, Theorem A.1.20]). In fact, we shall soon prove a stronger property: the map $q \to D_qH^0(x,q)$ is locally Lipschitz continuous in $\mathbb{R}^N\backslash\{0\}$. Before doing this, let us collect some technical remarks on the link between H^0 and H and their derivatives.

Lemma 3.8. *We have, for any $p, q \in \mathbb{R}^N\backslash\{0\}$,*

$$\big[q \in \partial F(x) \text{ and } p = D_qH^0(x,q)\big] \qquad \Leftrightarrow \qquad \big[p \in \partial F^0(x) \text{ and } q = D_pH(x,p)\big] \tag{22}$$

In particular,

$$D_qH^0\Big(x, \frac{f_p(x)}{|f_p(x)|}\Big) = p \qquad \forall p \in \partial F^0(x) . \tag{23}$$

Proof. We just need to show the implication

$$\big[q \in \partial F(x) \text{ and } p = D_qH^0(x,q)\big] \qquad \Rightarrow \qquad \big[p \in \partial F^0(x) \text{ and } q = D_pH(x,p)\big]$$

because $H^{00} = H$. Let $q \in \partial F(x)$ and $p = D_qH^0(x,q)$. Note that $H(x,p) = H^0(x,q) = 1$ and, in particular, $p \in \partial F^0(x)$. By definition, we have

$$H^0(x,q')H(x,p') \ge \langle p', q'\rangle \qquad \forall p', q' \in \mathbb{R}^N . \tag{24}$$

This inequality becomes an equality for $(p', q') = (p, q)$ because

$$\langle p, q\rangle = \langle D_qH^0(x,q), q\rangle = H^0(x,q) = 1 = H^0(x,q)H(x,p) .$$

Taking the derivative in (24) with respect to p then gives

$$H^0(x,q)D_pH(x,p) = D_pH(x,p) = q .$$

Next we turn to the proof of (23). Recall first that $f_p(x) = D_pH(x,p)$ for any $p \neq 0$. So, if $p \in \partial F^0(x)$, then (22) implies that

$$D_qH^0\Big(x, \frac{f_p(x)}{|f_p(x)|}\Big) = D_qH^0\Big(x, f_p(x)\Big) = p ,$$

since $D_qH^0(x,\cdot)$ is $0-$homogeneous. □

Lemma 3.9. *There is a constant C such that, for every $x \in \mathbb{R}^N$,*

$$\left|D_qH^0(x,q) - D_qH^0(x,q')\right| \le \frac{C}{|q| \vee |q'|}|q-q'| \qquad \forall q,q' \in \mathbb{R}^N \setminus \{0\}\,.$$

Proof. Let us fix $p, p' \in \partial F^0(x)$. Owing to Lemma 3.7 in its equivalent form (20), we deduce that

$$\Big\langle p'-p, \frac{f_p(x)}{|f_p(x)|}\Big\rangle \le -\frac{1}{2R'}\,|p'-p|^2$$

and

$$\langle p-p', \frac{f_{p'}(x)}{|f_{p'}(x)|}\rangle \le -\frac{1}{2R'}\,|p'-p|^2\ .$$

Adding up the last two inequalities,

$$|p'-p|^2 \le R'\Big\langle p'-p, \frac{f_{p'}(x)}{|f_{p'}(x)|} - \frac{f_p(x)}{|f_p(x)|}\Big\rangle \le R'|p'-p|\Big|\frac{f_{p'}(x)}{|f_{p'}(x)|} - \frac{f_p(x)}{|f_p(x)|}\Big|\,. \quad (25)$$

Now, recall that the map $p \mapsto f_p(x)/|f_p(x)|$ is a bijection from $\partial F^0(x)$ to S^{N-1} to deduce that for all $q, q' \in S^{N-1}$ there are $p, p' \in \partial F^0(x)$ such that $q = f_p(x)/|f_p(x)|$ and $q' = f_{p'}(x)/|f_{p'}(x)|$. Then, combining (23) and (25),

$$\begin{aligned}\left|D_qH^0\left(x,q'\right) - D_qH^0\left(x,q\right)\right| &= \left|D_qH^0\Big(x, \tfrac{f_{p'}(x)}{|f_{p'}(x)|}\Big) - D_qH^0\Big(x, \tfrac{f_p(x)}{|f_p(x)|}\Big)\right| \\ &= |p'-p| \\ &\le R'\left|\tfrac{f_{p'}(x)}{|f_{p'}(x)|} - \tfrac{f_p(x)}{|f_p(x)|}\right| = R'|q'-q|\,.\end{aligned}$$

This is the desired estimate for $q, q' \in S^{N-1}$. Next, let $q, q' \in \mathbb{R}^N\setminus\{0\}$. Then, since $D_qH^0(x,\cdot)$ is homogeneous of degree 0,

$$\left|D_qH^0(x,q') - D_qH^0(x,q)\right| \le R'\Big|\frac{q'}{|q'|} - \frac{q}{|q|}\Big|\,.$$

Finally, observe that

$$\Big|\frac{q'}{|q'|} - \frac{q}{|q|}\Big| \le \Big|\frac{q'}{|q'|} - \frac{q}{|q'|}\Big| + \Big|\frac{q}{|q'|} - \frac{q}{|q|}\Big| = \frac{|q'-q|}{|q'|} + \frac{|q|}{|q||q'|}|\,|q'|-|q|\,| \le 2\frac{|q'-q|}{|q'|}$$

to complete the proof. □

4 Regularity of extremal trajectories

In this section, we shall prove a regularity result for the extremal trajectories of the differential inclusion

$$x'(t) \in F(x(t)) \qquad t \ge 0\,, \tag{26}$$

where F is the multifunction introduced in (6), and H is a given function satisfying (SA). Alternatively, this analysis could be addressed to differential inclusions associated with a multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ that satisfies (8), (9), and (10)

as standing assumptions, in which case the Hamiltonian H should be defined as in (7).

A *trajectory* of the above differential inclusion is a locally absolutely continuous arc $x(\cdot) : [0, \infty) \to \mathbb{R}^N$ that satisfies (26) for a.e. $t \geq 0$. Given a closed subset K of $\mathbb{R}^N$, we denote by $\mathcal{R}(t)$, $t \geq 0$, the *reachable set* (from K) in time t, that is,

$$\mathcal{R}(t) = \{x(t) \ : \ x(\cdot) \text{ is a trajectory of (26) with } x(0) \in K\} \ .$$

A trajectory $\bar{x}(\cdot)$ of (26) is called extremal on the time interval $[0, t]$ if $\bar{x}(t) \in \partial\mathcal{R}(t)$. In this case, one can show that in fact $\bar{x}(s) \in \mathcal{R}(s)$ for every $s \in [0, t]$.

Due to the special structure of F, described by the properties (8), (9), and (10), we will be able to show that all extremal trajectories are $\mathcal{C}^{1,\alpha/2}$-smooth. More precisely, we have the following result.

Theorem 4.1. *Assume* (SA) *and let $\bar{x}$ be an extremal trajectory of* (26) *on some time interval* $[0, T]$. *Then*

$$|\bar{x}'(t_2) - \bar{x}'(t_1)| \leq C(t_2 - t_1)^{\alpha/2} \qquad \forall t_1, t_2 \in [0, T] \tag{27}$$

for some constant C.

Proof. Let $\bar{x}$ be an extremal trajectory on $[0, T]$. Then, by extremality, $x'(t) \in \partial F(\bar{x}(t))$ for almost all $t \in [0, T]$, so that we can set

$$\bar{p}(t) = D_q H^0(\bar{x}(t), \bar{x}'(t)) \qquad \text{a.e. in } [0, T] \ .$$

Using Lemma 3.8, we obtain the following relation between $\bar{x}$ and $\bar{p}$:

$$\bar{x}'(t) = D_p H\big(\bar{x}(t), \bar{p}(t)\big) \quad \text{ for a.e. } \quad t \in [0, T] \, .$$

Step 1. We first claim that, for any $0 \leq t_1 < t_2 \leq T$ we have

$$t_2 - t_1 \ \leq \ H^0\Big(\bar{x}(t_2), \int_{t_1}^{t_2} D_p H\big(\bar{x}(t_2), \bar{p}(t)\big) dt\Big) + C(t_2 - t_1)^{1+\alpha} \, . \tag{28}$$

Proof of (28): Let us set

$$q = \frac{\bar{x}(t_2) - \bar{x}(t_1)}{|\bar{x}(t_2) - \bar{x}(t_1)|} \ .$$

Let $\lambda : [t_1, t_2] \to \mathbb{R}$ be a solution of the Cauchy problem

$$\begin{cases} \lambda'(t) = \dfrac{1}{H^0(\bar{x}(t_1) + \lambda(t)q, q)} \, , & t \in [t_1, t_2] \\ \lambda(t_1) = 0 \, . \end{cases}$$

Then $x(t) := \bar{x}(t_1) + \lambda(t)q$ is a trajectory of (26) since, owing to (15),

$$H^0\big(x(t), x'(t)\big) = H^0\big(x(t), \lambda'(t)q\big) = \frac{H^0\big(\bar{x}(t_1) + \lambda(t)q, q\big)}{H^0\big(\bar{x}(t_1) + \lambda(t)q, q\big)} = 1$$

for all $t \in [t_1, t_2]$. Therefore, since $\bar{x}$ is an extremal trajectory, the point $\bar{x}(t_1) + \lambda(t_2)q$ belongs to the segment $[\bar{x}(t_1), \bar{x}(t_2)]$. So,

$$\lambda(t_2) - \lambda(t_1) \leq |\bar{x}(t_2) - \bar{x}(t_1)| \ .$$

Note that, owing to (18), the above inequality, and the boundedness of F,

$$\Big|\frac{1}{H^0(\bar{x}(t_1)+\lambda(t)q,q)}-\frac{1}{H^0(\bar{x}(t_2),q)}\Big|\le C|\bar{x}(t_2)-\bar{x}(t_1)|^{2\alpha}\le C(t_2-t_1)^{2\alpha}$$

for all $t\in[t_1,t_2]$ and some constant C. Hence,

$$\lambda(t_2)-\lambda(t_1)=\int_{t_1}^{t_2}\frac{dt}{H^0(\bar{x}(t_1)+\lambda(t)q,q)}\ge\frac{t_2-t_1}{H^0(\bar{x}(t_2),q)}-C(t_2-t_1)^{1+2\alpha}\,.$$

So, appealing to Lemma 3.3,

$$\begin{aligned}
t_2-t_1 &\le H^0(\bar{x}(t_2),q)|\bar{x}(t_2)-\bar{x}(t_1)|+C(t_2-t_1)^{1+2\alpha}\\
&= H^0(\bar{x}(t_2),\bar{x}(t_2)-\bar{x}(t_1))+C(t_2-t_1)^{1+2\alpha}\\
&= H^0\Big(\bar{x}(t_2),\textstyle\int_{t_1}^{t_2}D_pH(\bar{x}(t),\bar{p}(t))dt\Big)+C(t_2-t_1)^{1+2\alpha}\\
&\le H^0\Big(\bar{x}(t_2),\textstyle\int_{t_1}^{t_2}D_pH(\bar{x}(t_2),\bar{p}(t))dt\Big)+C(t_2-t_1)^{1+\alpha}\,,
\end{aligned}$$

where the above constants may change from line to line. We have thus proved (28).

Step 2. Let us fix $0\le t_1<t_2\le T$ and let $\bar{t}$ be such that

$$H^0(\bar{x}(t_2),\bar{x}(\bar{t})-\bar{x}(t_1))=H^0(\bar{x}(t_2),\bar{x}(t_2)-\bar{x}(\bar{t}))\,. \tag{29}$$

Define

$$a=\bar{x}(\bar{t})-\bar{x}(t_1)\quad\text{and}\quad b=\bar{x}(t_2)-\bar{x}(\bar{t})\,. \tag{30}$$

We claim that

$$H^0(\bar{x}(t_2),a)+H^0(\bar{x}(t_2),b)\le H^0(\bar{x}(t_2),a+b)+C(t_2-t_1)^{1+\alpha} \tag{31}$$

and

$$|2\bar{t}-t_1-t_2|\le C(t_2-t_1)^{1+\alpha}\,. \tag{32}$$

Proof of (31) and (32): Again by Lemma 3.3, and then using Jensen's inequality, we obtain

$$\begin{aligned}
H^0(\bar{x}(t_2),a) &= H^0\Big(\bar{x}(t_2),\textstyle\int_{t_1}^{\bar{t}}D_pH(\bar{x}(s),\bar{p}(s))ds\Big)\\
&\le H^0\Big(\bar{x}(t_2),\textstyle\int_{t_1}^{\bar{t}}D_pH(\bar{x}(t_2),\bar{p}(s))ds\Big)+C(t_2-t_1)^{1+\alpha}\\
&\le \textstyle\int_{t_1}^{\bar{t}}H^0\Big(\bar{x}(t_2),D_pH(\bar{x}(t_2),\bar{p}(s))\Big)ds+C(t_2-t_1)^{1+\alpha}\\
&\le \bar{t}-t_1+C(t_2-t_1)^{1+\alpha}\,.
\end{aligned} \tag{33}$$

Applying (28) between t_1 and $\bar{t}$ gives

$$\bar{t}-t_1\le H^0\Big(\bar{x}(\bar{t}),\int_{t_1}^{\bar{t}}D_pH(\bar{x}(\bar{t}),\bar{p}(t))dt\Big)+C(\bar{t}-t_1)^{1+\alpha}\,. \tag{34}$$

Now, in order to bound the above right-hand side observe that

$$\big|D_pH(\bar{x}(\bar{t}),\bar{p}(t))-D_pH(\bar{x}(t),\bar{p}(t))\big|\le C|\bar{x}(\bar{t})-\bar{x}(t)|^{\alpha}\le C(\bar{t}-t_1)^{\alpha}$$

in view of Lemma 3.3, and

$$\big|H^0\big(\bar{x}(\bar{t}),a\big)-H^0\big(\bar{x}(t_2),a\big)\big|\le C\,|a|\,|\bar{x}(\bar{t})-\bar{x}(t_2)|^{2\alpha}\le C(\bar{t}_2-t_1)^{1+2\alpha}$$

owing to Lemma 3.5. Therefore, (34) leads to

$$\begin{aligned}\bar{t}-t_1 &\le H^0\Big(\bar{x}(\bar{t}),\underbrace{\int_{t_1}^{\bar{t}}D_pH(\bar{x}(t),\bar{p}(t))dt}_{a}\Big)+C(\bar{t}-t_1)^{1+\alpha}\\ &\le H^0(\bar{x}(t_2),a)+C(t_2-t_1)^{1+\alpha}\,.\end{aligned}\tag{35}$$

On account of (33) and (35), we have

$$H^0(\bar{x}(t_2),a)-C(t_2-t_1)^{1+\alpha}\le\bar{t}-t_1\le H^0(\bar{x}(t_2),a)+C(t_2-t_1)^{1+\alpha}\,.$$

In the same way,

$$H^0(\bar{x}(t_2),b)-C(t_2-t_1)^{1+\alpha}\le t_2-\bar{t}\le H^0(\bar{x}(t_2),b)+C(t_2-t_1)^{1+\alpha}\,.$$

Combining the above two inequalities with the choice of $\bar{t}$ made in (29) gives (32). Moreover, adding up the above inequalities and recalling (28), we get

$$H^0(\bar{x}(t_2),a)+H^0(\bar{x}(t_2),b)\le(t_2-t_1)+C(t_2-t_1)^{1+\alpha}\le H^0(\bar{x}(t_2),a+b)+C(t_2-t_1)^{1+\alpha}\,,$$

which yields (31).

Step 3. We now claim that, for any $0\le t_1<t_2\le T$, we have

$$\Big|\bar{x}\Big(\frac{t_1+t_2}{2}\Big)-\frac{\bar{x}(t_2)+\bar{x}(t_1)}{2}\Big|\le C(t_2-t_1)^{1+\alpha}\,.\tag{36}$$

Proof of (36): Having fixed $0\le t_1<t_2\le T$, we will use the same notation for $\bar{t}$, a, and b as in (29) and (30). Moreover, since $x(t_2)$ is fixed in the reasoning below, as we often did before we will omit the $x(t_2)$-dependance of H^0 and all the other maps appearing in this proof.

Let us set, for any $q\in\mathbb{R}^N\backslash\{0\}$, $g_q=D_qH^0(q)$. We use below repetitively the following remark:

$$\text{for any } q\neq 0,\text{ if } p=g_q,\text{ then } f_p=q/H^0(q).$$

Indeed, since $q/H^0(q)\in\partial F$ and $p=D_qH^0(q/H^0(q))$ (because D_qH^0 is $0-$homogeneous), Lemma 3.8 implies that $q/H^0(q)=D_pH(p)=f_p$.

We first show that

$$\frac{1}{2R'}|g_q-g_{q'}|^2\le\langle g_q-g_{q'},\frac{q}{|q|}\rangle\qquad\forall q,q'\in\mathbb{R}^N\backslash\{0\}\,,\tag{37}$$

where R' is the constant appearing in Lemma 3.7. For this, let us consider the lower curvature estimate (20) in Lemma 3.7 with $p=g_q$ and $p'=g_{q'}$: because of the remark above and since $p,p'\in\partial F^0$, we have

$$\Big\langle g_{q'}-g_q,\frac{q}{|q|}\Big\rangle\le-\frac{1}{2R'}|g_{q'}-g_q|^2$$

which is exactly (37).

Next we note that

$$|a-b| \leq CH^0(a)|g_a - g_b| \tag{38}$$

Indeed let us apply the first inequality in Lemma 3.4 to $p = g_a$ and $q = g_b$. Since $f_p = a/H^0(a)$ and $f_q = b/H^0(b)$ and since $H^0(a) = H^0(b)$ by (29), we have

$$|a-b| \leq RH^0(a)\left|\frac{g_a}{|g_a|} - \frac{g_b}{|g_b|}\right| \leq CH^0(a)|g_a - g_b| \ .$$

In order to estimate the right-hand side of inequality (38), let us observe that, in view of (31),

$$\begin{array}{rcl} 0 & \leq & H^0(a+b) - H^0(a) - H^0(b) + C(t_2-t_1)^{1+\alpha} \\ & = & \langle g_{a+b}, a+b\rangle - \langle g_a, a\rangle - \langle g_b, b\rangle + C(t_2-t_1)^{1+\alpha} \end{array}$$

so that

$$0 \leq \langle g_{a+b} - g_a, a\rangle + \langle g_{a+b} - g_b, b\rangle + C(t_2-t_1)^{1+\alpha} \ .$$

Plugging inequality (37) into this inequality leads to

$$|a|\,|g_{a+b} - g_a|^2 + |b|\,|g_{a+b} - g_b|^2 \leq C(t_2-t_1)^{1+\alpha} \ ,$$

i.e., since $|a| \geq (\bar{t} - t_1)/C$ and $|b| \geq (t_2 - \bar{t})/C$ and (32) holds,

$$|g_{a+b} - g_a| \leq C(t_2-t_1)^{\alpha/2} \ , \ |g_{a+b} - g_b| \leq C(t_2-t_1)^{\alpha/2} \ .$$

So, recalling that $H^0(a) \leq C(\bar{t} - t_1) \leq C(t_2 - t_1)$, we get from (38),

$$|a-b| \ \leq \ CH^0(a)(|g_{a+b} - g_a(x)| + |g_{a+b} - g_b(x)|) \ \leq \ C(t_2-t_1)^{1+\alpha/2} \ .$$

From the definition of a and b, this means that

$$|2\bar{x}(\bar{t}) - \bar{x}(t_1) - \bar{x}(t_2)| \leq C(t_2-t_1)^{1+\alpha/2} \ .$$

Using again (32) and the Lipschitz continuity of $\bar{x}$ then easily yields to (36).

Conclusion. In view of (36), Theorem 2.1.10 of [4] states that each component of $\bar{x}$ is semi-convex and semi-concave with a modulus m of the form $m(\rho) = C\rho^{\alpha/2}$. Then, from Theorem 3.3.7 of [4], $\bar{x}$ is of class $\mathcal{C}^{1,\alpha/2}$ and (27) holds. □

5 The semiconcavity result

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying our Standing Assumptions with constants α, r, R, and C_0, and let $\Omega \subset \mathbb{R}^N$ be an open set.

In this section, we will apply the previous analysis to study the regularity of the solution to the Dirichlet problem

$$\left\{ \begin{array}{ll} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{array} \right. \tag{39}$$

The existence, uniqueness, and Lipschitz continuity of the viscosity solution u of the above problem is well-known, as well as the representation formula

$$u(x) = \inf\big\{t \geq 0 \ : \ \exists\, x(\cdot) \text{ trajectory of (26) with } x(0) = x\,,\ x(t) \in \partial\Omega\big\} \quad (40)$$

(see, e.g., [1]).

We recall that a function $v : \Omega \to \mathbb{R}$ is locally θ-semiconcave, with $\theta \in (0, 1]$, if for every compact convex set $\mathcal{O} \subset \Omega$ there is a constant $C_{\mathcal{O}}$ such that

$$v(\lambda x + (1-\lambda)y) \geq \lambda v(x) + (1-\lambda)v(y) - C_{\mathcal{O}}\lambda(1-\lambda)|x-y|^{1+\theta}$$

for all $x, y \in \mathcal{O}$ and $\lambda \in [0, 1]$. We are now ready for our main result.

Theorem 5.1. *Assume* (SH). *Then the solution u of (39) is locally θ-semiconcave in Ω for every $\theta \in (0, \frac{\alpha}{4+\alpha})$.*

Proof. The strategy of the proof is the following. Fix

$$\beta \in \Big(\frac{2}{2+\alpha}, \frac{4}{4+\alpha}\Big)\,, \quad (41)$$

and observe that

$$0 < \theta := \beta\,\frac{2+\alpha}{2} - 1 < \frac{\alpha}{4+\alpha}\,.$$

Let $\mathcal{O} \subset\subset \Omega$ be an open convex set. We are going to show that

$$u(x+h) + u(x-h) - 2u(x) \ \leq\ C|h|^{\beta(2+\alpha)/2} \quad (42)$$

for all $h \in \mathbb{R}^N$ sufficiently small (in this proof, C denotes a generic constant depending only on α, r, R, C_0, and $\mathcal{O}$). Since u is continuous, owing to [4, Theorem 2.1.10] the above inequality implies that u is locally θ-semiconcave in Ω.

Step 1. Let $\bar{x} \in \mathcal{O}$ and let $\bar{x}(\cdot)$ be a solution of the minimization problem in (40)—an optimal trajectory for short. Since $\bar{x}(\cdot)$ is extremal on $[0, u(\bar{x})]$, Theorem 4.1 implies that $\bar{x}(\cdot)$ is of class $\mathcal{C}^{1,\alpha/2}$ and satisfies

$$|\bar{x}'(t_2) - \bar{x}'(t_1)| \leq C|t_2 - t_1|^{\alpha/2} \qquad \forall t_1, t_2 \in [0, u(\bar{x})]\,.$$

Setting $\bar{v} = \bar{x}'(0)$, from the above inequality we obtain

$$|\bar{x}(t) - \bar{x} - t\bar{v}| \leq Ct^{1+\alpha/2} \qquad \forall t \in [0, u(\bar{x})]\,. \quad (43)$$

Step 2. Let $h \in \mathbb{R}^N$ be small enough, and set $\bar{t} = |h|^\beta$. We will now build a trajectory $x_+(\cdot)$ such that

$$\begin{cases} x_+(0) = \bar{x} + h\,, \\ x_+(t) \in [\bar{x} + h, \bar{x}(\bar{t})] & \forall t \in [0, \tau_+]\,, \\ x_+(t) = \bar{x}(t + \bar{t} - \tau_+) & \forall t \in [\tau_+, u(\bar{x}) + \tau_+ - \bar{t}]\,. \end{cases}$$

Notice that $x_+(u(\bar{x}) + \tau_+ - \bar{t}) = \bar{x}(u(\bar{x})) \in \mathbb{R}^N \backslash \Omega$, so that

$$u(\bar{x} + h) \leq u(\bar{x}) + \tau_+ - \bar{t}\,. \quad (44)$$

Proof of Step 2. In order to construct the line segment part, let us set $q_+ = \bar{x}(\bar{t}) - (\bar{x} + h)$ and observe that, in view of (43),

$$q_+ = \bar{t}\bar{v} + O(\bar{t}^{1+\alpha/2}) - h\,. \tag{45}$$

Then, $q_+ \neq 0$ since $\bar{t} >> |h|$ and $|\bar{v}| \geq 1/r$. Let $\lambda(\cdot)$ be a solution of the Cauchy problem

$$\begin{cases} \lambda'(t) = \dfrac{1}{H^0(\bar{x} + h + \lambda(t)q_+, q_+)}\,, & t \geq 0 \\ \lambda(0) = 0\,. \end{cases}$$

Since $\lambda(\cdot)$ is strictly increasing there is a unique time τ_+ such that $\lambda(\tau_+) = 1$. Now, set

$$x_+(t) = \bar{x} + h + \lambda(t)q_+ \qquad t \in [0, \tau_+]\,.$$

Then $x_+(\cdot)$ is a solution of the differential inclusion (26) on $[0, \tau_+]$ because

$$H^0(x_+(t), x_+'(t)) = 1 \qquad \forall t \in [0, \tau_+]\,.$$

Moreover $x_+(\tau_+) = \bar{x}(\bar{t})$. Thus, defining

$$x_+(t) = \bar{x}(t + \bar{t} - \tau_+) \qquad \forall t \in [\tau_+, u(\bar{x}) + \tau_+ - \bar{t}]$$

completes the construction of $x_+(\cdot)$.

Step 3. We will now prove the estimate

$$\tau_+ \leq \bar{t} + \big\langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_+ - \bar{t}\bar{v}\big\rangle + C\,\frac{|h|^2}{\bar{t}}\,. \tag{46}$$

Proof of Step 3. To begin with, let us note that any $\xi \in [\bar{t}\bar{v}, q_+]$ satisfies $|\xi| \geq \bar{t}/C$. Indeed, if $\xi = \mu q_+ + (1-\mu)\bar{t}\bar{v}$ for some $\mu \in [0,1]$, then, by (45), $\xi = \bar{t}\bar{v} + O(\bar{t}^{1+\alpha/2}) - h$ with $|h| = \bar{t}^{1/\beta}$ and $\beta \in (0,1)$. So, for $|h|$ small enough, we have the desired claim: $|\xi| \geq \bar{t}/C$. Then, by Lemma 3.9 we conclude that the map $\xi \mapsto D_q H^0(\bar{x}, \xi)$ is Lipschitz on $[\bar{t}\bar{v}, q_+]$ with constant $C/\bar{t}$. So, for all $t \in [0, \tau_+]$,

$$\begin{aligned} &H^0(x_+(t), q_+) \;\leq H^0(\bar{x}, q_+) + C|x_+(t) - \bar{x}|^{2\alpha}|q_+| \\ &\leq H^0(\bar{x}, \bar{t}\bar{v}) + \langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_+ - \bar{t}\bar{v}\rangle + (C/\bar{t})|q_+ - \bar{t}\bar{v}|^2 + C|x_+(t) - \bar{x}|^{2\alpha}|q_+|\,. \end{aligned}$$

Since $x_+(t) \in [\bar{x} + h, \bar{x}(\bar{t})]$, we have

$$|x_+(t) - \bar{x}| \leq \max\{|h|, |\bar{x}(\bar{t}) - \bar{x}|\} \leq C\bar{t}\,.$$

Also, on account of (45) and (41),

$$|q_+| \leq C\bar{t} \qquad \text{and} \qquad |q_+ - \bar{t}\bar{v}| = |O(\bar{t}^{1+\alpha/2}) - h| \leq C|h|\,.$$

Noting that $H^0(\bar{x}, \bar{v}) = 1$ because $\bar{v} \in \partial F(\bar{x})$, the above inequality yield, by the homogeneity of $H^0(\bar{x}, \cdot)$,

$$H^0(x_+(t), q_+) \;\leq\; \bar{t} + \big\langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_+ - \bar{t}\bar{v}\big\rangle + C\Big(\frac{|h|^2}{\bar{t}} + \bar{t}^{1+2\alpha}\Big)\,,$$

where $|h|^2/\bar{t} > \bar{t}^{1+2\alpha}$ because $\bar{t} = |h|^\beta$ with $\beta > 1/(1+\alpha/2)$. So,

$$H^0(x_+(t), q_+) \;\le\; \bar{t} + \big\langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_+ - \bar{t}\bar{v}\big\rangle + C|h|^2/\bar{t}\,.$$

Then

$$1 = \int_0^{\tau_+} \lambda'(t)dt = \int_0^{\tau_+} \frac{dt}{H^0(x_+(t), q_+)} \ge \frac{\tau_+}{\bar{t} + \big\langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_+ - \bar{t}\bar{v}\big\rangle + C|h|^2/\bar{t}}\,,$$

which in turn yields (46).

Conclusion. Repeating the above reasoning with $q_- = \bar{x}(\bar{t}) - (\bar{x} - h)$, we can build a solution $x_-(\cdot)$ to (26) such that $x_-(0) = \bar{x} - h$, $x_-(t) \in [\bar{x} - h, \bar{x}(\bar{t})]$ on the time interval $[0, \tau_-]$, and $x_-(t) = \bar{x}(t + \bar{t} - \tau_-)$ on $[\tau_-, u(\bar{x}) + \tau_- - \bar{t}]$. Therefore,

$$u(\bar{x} - h) \le u(\bar{x}) + \tau_- - \bar{t}\,, \tag{47}$$

where τ_- can be estimated as above:

$$\tau_- \le \bar{t} + \big\langle D_q H^0(\bar{x}, \bar{t}\bar{v}), q_- - \bar{t}\bar{v}\big\rangle + C\,\frac{|h|^2}{\bar{t}}\,. \tag{48}$$

Hence, by (44), (47), (46), (48), and (43) we obtain

$$\begin{aligned} u(\bar{x}+h) + u(\bar{x}-h) - 2u(\bar{x}) \;&\le\; \tau_+ - \bar{t} + \tau_- - \bar{t} \\ &\le\; 2\big\langle DH^0(\bar{x}, \bar{t}\bar{v}), \bar{x}(\bar{t}) - \bar{x} - \bar{t}\bar{v}\big\rangle + C\,\tfrac{|h|^2}{\bar{t}} \\ &\le\; C|h|^{\beta(2+\alpha)/2} \end{aligned}$$

since $\beta < 4/(4+\alpha)$. We have thus attained (42), which completes the proof. □

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