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Alessandro Ruzzi

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FANO SYMMETRIC VARIETIES WITH LOW RANK

Alessandro Ruzzi

Abstract

The symmetric projective varieties of rank one are all smooth and Fano by a classic result of Akhiezer. We classify the locally factorial (respectively smooth) projective symmetric G -varieties of rank 2 which are Fano. When G is semisimple we classify also the locally factorial (respectively smooth) projective symmetric G -varieties of rank 2 which are only quasi-Fano. Moreover, we classify the Fano symmetric G -varieties of rank 3 obtainable from a wonderful variety by a sequence of blow-ups along G -stable varieties. Finally, we classify the Fano symmetric varieties of arbitrary rank which are obtainable from a wonderful variety by a sequence of blow-ups along closed orbits.

keywords: Symmetric varieties, Fano varieties.

MSC 2010: 14M17, 14J45, 14L30

A Gorenstein (projective) normal algebraic variety X over \mathbb{C} is called a Fano variety if the anticanonical divisor is ample. The Fano surfaces are classically called Del Pezzo surfaces. The importance of Fano varieties in the theory of higher dimensional varieties is similar to the importance of Del Pezzo surfaces in the theory of surfaces. Moreover Mori's program predicts that every uniruled variety is birational to a fibration whose general fiber is a Fano variety (with terminal singularities).

Let θ be an involution of a reductive group G (over \mathbb{C}) and let H be a closed subgroup of G such that $G^\theta \subset H \subset N_G(G^\theta)$. A symmetric variety is a normal G -variety with an open orbit isomorphic to G/H . The symmetric varieties are a generalization of the toric varieties. The toric smooth Fano varieties with rank at most four have been classified. By [AlBr04], Theorem 4.2 there is only a finite number of Fano smooth symmetric varieties with a fixed open orbit. In [Ru07] we have classified the smooth compact symmetric varieties with Picard number one and G semisimple, while in [Ru10] we have given an explicitly geometrical description of such varieties; they are automatically Fano.

In this work, we want to classify the Fano symmetric varieties with low rank (and G semisimple). First, we consider a special case of arbitrary rank. We say that a variety X is quasi \mathbb{Q} -Fano if $-K_X$ is a nef and big \mathbb{Q} -divisor. Fixed an open orbit G/H with G semisimple, there is a unique maximal compactification between the ones which have only one closed orbit. Such variety is called the standard compactification. If it is also smooth, it is called the wonderful compactification; this is the case, for example, if $H = N_G(G^\theta)$ (see [dCoPr83], Theorem 3.1). We prove that the standard symmetric varieties are all quasi \mathbb{Q} -Fano and we describe when they are Fano. We determine also the symmetric Fano varieties obtainable from a wonderful one by a sequence of blow-ups along closed orbits. In particular, we prove that such a variety must be either a wonderful one or the blow-up of a wonderful one along the unique closed orbit.

Next we consider the symmetric varieties of rank at most three. The rank of a symmetric variety X is defined as the rank of $\mathbb{C}(X)^{(B)}/\mathbb{C}^*$, where B is any fixed Borel subgroup of G . The symmetric varieties with rank one are all wonderful; moreover one can show that they are isomorphic, under the action of $\text{Aut}^0(X)$, either to a projective homogeneous variety G/P with P maximal, or to $\mathbb{P}^n \times \mathbb{P}^n$ (see [A83]). Thus they are all Fano.

We classify all the locally factorial (resp. smooth) Fano symmetric varieties of rank 2. When G is semisimple, we classify also the locally factorial (resp. smooth) symmetric varieties which are only quasi-Fano. In the proof of such result we obtain a classification of the toroidal Fano varieties of rank 2 with G semisimple (without assumption on the regularity).

Finally, we classify the smooth Fano symmetric varieties of rank three which are obtainable from a wonderful one by a sequence of blow-ups along G -subvarieties (in particular G is semisimple). This class of varieties is quite large; indeed any compact symmetric variety is dominated by a variety obtained from the wonderful one by a sequence of blow-ups along G -subvarieties of codimension two (see [dCoPr85], Theorem 2.4). This result on 3-rank varieties can be generalized to varieties obtainable from a generic wonderful varieties of rank 3 by a sequence of blow-ups along G -subvarieties (without suppose G/H symmetric).

1 Introduction and notations

In this section we introduce the necessary notations. The reader interested to the embedding theory of spherical varieties can see [Kn91], [Br97a] or [T06]. In [Vu90] is explained such theory in the particular case of symmetric varieties.

1.1 First definitions

Let G be a connected reductive algebraic group over \mathbb{C} and let θ be an involution of G . Given a closed subgroup H such that $G^\theta \subset H \subset N_G(G^\theta)$, we say that G/H is a *symmetric space* and that H is a *symmetric subgroup*. A normal G -variety is called a *spherical variety* if it contains a dense B -orbit (B is a chosen Borel subgroup of G). We say that a subtorus of G is split if $\theta(t) = t^{-1}$ for all its elements t ; moreover it is a *maximal split torus* if it has maximal dimension. A maximal torus containing a maximal split torus is *maximally split*; any maximally split torus is θ stable (see [T06], Lemma 26.5). We fix arbitrarily a maximal split torus T^1 and a maximally split torus T containing T^1 . Let R_G be the root system of G w.r.t. T . We can choose a Borel subgroup $T \subset B$ such that, for any positive root α , either $\theta(\alpha) = \alpha$ or $\theta(\alpha)$ is negative. Moreover, BH is dense in G (see [dCoPr83], Lemma 1.2 and Proposition 1.3). In particular, every normal equivariant embedding of G/H is spherical; we call it a *symmetric variety*.

We can assume that G is the direct product of a simply connected, semisimple group with a central split torus.

1.2 Colored fans

Now, we introduce some details about the classification of the symmetric varieties by their colored fans (this classification is defined more generally for

spherical varieties). Let $D(G/H)$ be the set of B -stable prime divisors of G/H ; its elements are called *colors*. We say that a spherical variety is *simple* if it contains one closed orbit. Let X be a simple symmetric variety with closed orbit Y . We define the set of *colors of X* as the subset $D(X)$ of $D(G/H)$ consisting of the colors whose closure in X contains Y . To each prime divisor D of X , we can associate the normalized discrete valuation v_D of $\mathbb{C}(G/H)$ whose ring is $\mathcal{O}_{X,D}$; D is G -stable if and only if v_D is G -invariant. Let N be the set of all G -invariant valuations of $\mathbb{C}(G/H)$ taking value in \mathbb{Z} and let $N(X)$ be the set of the valuations associated to the G -stable prime divisors of X . Observe that each irreducible component of $X \setminus (G/H)$ has codimension one, because G/H is affine. Let $S := T/T \cap H \simeq T \cdot (eH/H)$. One can show that the group $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is isomorphic to the character group $\chi(S)$ of S (see [Vu90], §2.3); in particular, it is a free abelian group. We define the *rank l* of G/H as the rank of $\chi(S)$. We can identify the dual group $\text{Hom}_{\mathbb{Z}}(\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*, \mathbb{Z})$ with the group $\chi_*(S)$ of one-parameter subgroups of S ; so we can identify $\chi_*(S)_{\mathbb{R}}$ with $\text{Hom}_{\mathbb{Z}}(\chi(S), \mathbb{R})$. The restriction map to $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is injective over N (see [Br97a], §3.1 Corollaire 3), so we can identify N with a subset of $\chi_*(S)_{\mathbb{R}}$. We say that N is the *valuation monoid* of G/H . For each color D , we define $\rho(D)$ as the restriction of v_D to $\chi(S)$. In general, the map $\rho : D(G/H) \rightarrow \chi_*(S)_{\mathbb{R}}$ is not injective. Let $C(X)$ be the cone in $\chi_*(S)_{\mathbb{R}}$ generated by $N(X)$ and $\rho(D(X))$. We say that the pair $(C(X), D(X))$ is the *colored cone* of X ; it determines univocally X (see [Br97a], §3.3 Théorème).

Let Y be an orbit of a symmetric variety X . The set $\{x \in X \mid \overline{G \cdot x} \supset Y\}$ is an open simple G -subvariety of X with closed orbit Y , because any spherical variety contains only a finite number of G -orbits. Let $\{X_i\}$ be the set of open simple subvarieties of X and define the set of *colors of X* , $D(X)$, as $\bigcup_{i \in I} D(X_i)$. The family $\mathcal{F}(X) := \{(C(X_i), D(X_i))\}_{i \in I}$ is called the *colored fan* of X and determines completely X (see [Br97a], §3.4 Théorème 1). Moreover X is compact if and only if $\text{cone}(N)$ is contained in the support $|\mathcal{F}(X)| := \bigcup_{i \in I} C(X_i)$ of the colored fan (see [Br97a], §3.4 Théorème 2).

Given a symmetric variety X we denote by Δ (or by Δ_X) the fan associated to the colored fan of X , by $\Delta(i)$ the set of i -dimensional cones in Δ and by $\Delta[p]$ the set of primitive generators of the 1-dimensional cones of Δ . The fan Δ is formed by all the faces of the cones C such that there is a colored cone $(C, F) \in \mathcal{F}(X)$. The toric varieties are a special case of symmetric varieties. If X is a toric variety, then $D(G/H)$ is empty and we need only to consider the fan Δ_X (actually the theory of colored fans is a generalization of the classification of toric varieties by fans).

1.3 Restricted root system

To describe the sets N and $\rho(D(G/H))$, we associate a root system to G/H . We can identify $\chi(T^1)_{\mathbb{R}}$ with $\chi(S)_{\mathbb{R}}$ because $[\chi(S) : \chi(T^1)]$ is finite. We call again θ the involution induced on $\chi(T)_{\mathbb{R}}$. The inclusion $T^1 \subset T$ induces an isomorphism of $\chi(T^1)_{\mathbb{R}}$ with the (-1) -eigenspace of $\chi(T)_{\mathbb{R}}$ under the action of θ (see [T06], §26). Denote by W_G the Weyl group of G (w.r.t. T). We can identify $\chi(T^1)_{\mathbb{R}}$ with its dual $\chi_*(T^1)_{\mathbb{R}}$ by the restriction (\cdot, \cdot) to $\chi(T^1)_{\mathbb{R}}$ of a fixed W_G -invariant non-degenerate symmetric bilinear form on $\chi(T)_{\mathbb{R}}$. Let R_G^0 be the set of roots fixed by θ and let R_G^1 be $R_G \setminus R_G^0$. Let $R_G^{i,+} := R_G^i \cap R_G^+$.

The set $R_{G,\theta} := \{\beta - \theta(\beta) \mid \beta \in R_G^1\}$ is a root system in $\chi(S)_{\mathbb{R}}$ (see [Vu90]),

§2.3 Lemme), which we call the *restricted root system* of (G, θ) ; we call the non zero $\beta - \theta(\beta)$ the *restricted roots*. Usually we denote by β (resp. by α) a root of R_G (resp. of $R_{G, \theta}$); often we denote by ϖ (resp. by ω) a weight of R_G (resp. of $R_{G, \theta}$). We denote by $\overline{R}_G = \{\beta_1, \dots, \beta_n\}$ the basis of R_G associated to B and by $\varpi_1, \dots, \varpi_n$ the fundamental weights of R_G . Let \overline{R}_G^i be $\overline{R}_G \cap R_G^i$. There is a permutation $\overline{\theta}$ of \overline{R}_G^1 such that, $\forall \beta \in \overline{R}_G^1$, $\theta(\beta) + \overline{\theta}(\beta)$ is a linear combination of roots in \overline{R}_G^0 . We denote by $\alpha_1, \dots, \alpha_s$ the elements of the basis $\overline{R}_{G, \theta} := \{\beta - \theta(\beta) \mid \beta \in \overline{R}_G^1\}$ of $R_{G, \theta}$. If $R_{G, \theta}$ is irreducible we order $\overline{R}_{G, \theta}$ as in [Bo68]. Let b_i be equal to $\frac{1}{2}$ if $2\alpha_i$ belongs to $R_{G, \theta}$ and equal to one otherwise; for each i we define α_i^\vee as the coroot $\frac{2b_i}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$. The set $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$ is a basis of the dual root system $R_{G, \theta}^\vee$. We call the elements of $R_{G, \theta}^\vee$ the *restricted coroots*. Let $\omega_1, \dots, \omega_s$ be the fundamental weights of $R_{G, \theta}$ w.r.t. $\{\alpha_1, \dots, \alpha_s\}$ and let $\omega_1^\vee, \dots, \omega_s^\vee$ be the fundamental weights of $R_{G, \theta}^\vee$ w.r.t. $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$. Let C^+ be the positive closed Weyl chamber of $\chi(S)_\mathbb{R}$ and let $C^- := -C^+$.

We say that a dominant weight $\varpi \in \chi(T)$ is a *spherical weight* if $V(\varpi)$ contains a non-zero vector fixed by G^θ . In this case, $V(\varpi)^{G^\theta}$ is one-dimensional and $\theta(\varpi) = -\varpi$, so ϖ belongs to $\chi(S)_\mathbb{R}$. One can show that set of dominant weights of $R_{G, \theta}$ is the set of spherical weights and that C^+ is the intersection of $\chi(S)_\mathbb{R}$ with the positive closed Weyl chamber of R_G . Suppose $\beta_j - \theta(\beta_j) = \alpha_i$, then ω_i is a positive multiple of $\varpi_j + \varpi_{\overline{\theta}(j)}$. More precisely, we have $\omega_i = \varpi_j + \varpi_{\overline{\theta}(j)}$ if $\overline{\theta}(j) \neq j$, $\omega_i = 2\varpi_j$ if $\overline{\theta}(j) = j$ and β_j is orthogonal to R_G^0 and $\omega_i = \varpi_j$ otherwise (see [ChMa03], Theorem 2.3 or [T06], Proposition 26.4). We say that a spherical weight is *regular* if it is strictly dominant as weight of the restricted root system.

1.4 The sets N and $D(G/H)$

The set N is equal to $C^- \cap \chi_*(S)$; in particular, it consists of the lattice vectors of the rational, polyhedral, convex cone $C^- = \text{cone}(N)$. The set $\rho(D(G/H))$ is equal to $\overline{R}_{G, \theta}^\vee$ and any fibre $\rho^{-1}(\alpha^\vee)$ contains at most 2 colors. For any simple spherical variety X , $N(X)$ is formed by the primitive generators of the 1-faces of $C(X)$ which are contained in $\text{cone}(N)$. When X is symmetric, also $\rho(X)$ can be recovered by $C(X)$: its elements generate the 1-faces of $C(X)$ which are not contained in C^- . We say that (G, θ) is *indecomposable* if the unique normal, connected, θ -stable subgroup of G is the trivial one. In this case the number of colors is at most equal to $\text{rank}(G/H) + 1$. If θ is indecomposable then there are three possibilities: i) G is simple; ii) $G = \dot{G} \times \dot{G}$ with \dot{G} simple and $\theta(x, y) = (y, x)$; iii) $G = \mathbb{C}^*$ and $\theta(t) = t^{-1}$. See [Wa72], §1.1 for a classification of the involution of a simple group. In [Wa72] (and in [Ru10], §1) are also indicated the Satake diagrams of the indecomposable involutions. The Satake diagram of any involution (G, θ) is obtained from the Dynkin diagram of G as follows: 1) the vertices corresponding to element of \overline{R}_G^0 (resp. of \overline{R}_G^1) are black (resp. white); 2) two simple roots $\beta_1, \beta_2 \in \overline{R}_G^1$ such that $\overline{\theta}(\beta_1) = \beta_2$ are linked by a double-headed arrow.

If $\sharp D(G/H) > \text{rank}(G/H)$ and (G, θ) is indecomposable, we have two possibilities: 1) $G^\theta = H = N_G(G^\theta)$; 2) $H = G^\theta$ and $[G^\theta : N_G(G^\theta)] = 2$. In the last case any element of $N_G(G^\theta) \setminus G^\theta$ exchanges two colors and $R_{G, \theta}$ has type A_1 , B_2 or C_n . We say that a simple restricted root α is *exceptional* if $\sharp \rho^{-1}(\alpha^\vee) = 2$

and 2α is a restricted root. In this case the irreducible factor of $R_{G,\theta}$ containing α is associated to an indecomposable factor of G/G^θ as in 1). We say that also (G, θ) and any symmetric variety (with open orbit G/H) are exceptional. We denote by D_α the sum of the colors in $\rho^{-1}(\alpha^\vee)$ and by D_ω the G -stable divisor corresponding to $(\mathbb{R}^{\geq 0}\omega, \emptyset) \in \mathbb{F}(X)$.

1.5 Toroidal symmetric varieties

In this section we want to define a special class of varieties. We say that a spherical variety is *toroidal* if $D(X) = \emptyset$. Let (C, F) be a colored cone of X , we say that the blow-up of X along the subvariety associated to (C, F) is the blow-up of X along (C, F) . In the following of this section we suppose G semisimple. Then there is a special simple compactification of G/H because $N_G(H)/H$ is finite. This compactification, called the standard compactification X_0 , is associated to $(\text{cone}(N), \emptyset)$ and it is the maximal simple compactification of G/H in the dominant order. We define e_i as the primitive positive multiple of $-\omega_i^\vee$ (in $\chi_*(S)$), so $\Delta_{X_0}[p] = \{e_1, \dots, e_l\}$. The standard compactification is wonderful (i.e. it is also smooth) if and only if $\chi_*(S) = \bigoplus \mathbb{Z}e_i$. De Concini and Procesi have proved that X_0 is wonderful if $H = N_G(G^\theta)$, or equivalently $\chi_*(S) = \bigoplus \mathbb{Z}\omega_i^\vee$ (see [dCoPr83] Theorem 3.1).

The standard compactification X_0 contains an affine toric S -variety Z_0 , which is a quotient of an affine space by a finite group. The toroidal varieties are the symmetric varieties which dominates the standard compactification and are in one-to-one correspondence with the S -toric varieties which dominates Z_0 .

Let P be the stabilizer of the B -stable affine open set $U := X_0 \setminus \bigcup_{D \in D(G/H)} \overline{D}$. This open set is P -isomorphic to $R_u P \times Z_0$, where $R_u P = \prod_{\beta \in R_G^+} U_\beta$ is the unipotent radical of P and $\dim Z_0 = \text{rank } X_0$. To any toroidal variety X we associate the inverse image Z of Z_0 by the projection $X \rightarrow X_0$. Moreover, $X \setminus \bigcup_{D \in D(G/H)} \overline{D}$ is P -isomorphic to $R_u P \times Z$. The toroidal varieties are also in one-to-one correspondence with a class of compact toric varieties in the following way. To a symmetric variety X , we associate the closure Z^c of Z in X ; Z^c is also the inverse image of Z_0^c . The fan of Z is the fan Δ_X associated to the colored fan of X , while the fan of Z^c consists of the translates of the cones of Z by the Weyl group $W_{G,\theta} \cong N_{G^\theta}(T^1)/C_{G^\theta}(T^1)$ of $R_{G,\theta}$.

1.6 The Picard group

The class group of a symmetric variety is generated by the classes of the B -stable prime divisors modulo the relations $\text{div}(f)$ with $f \in \mathbb{C}(G/H)^{(B)}$. Indeed $Cl(BH/H) = Pic(BH/H)$ is trivial. Given $\omega \in \chi(S)$ we denote by f_ω the element of $\mathbb{C}(G/H)^{(B)}$ with weight ω and such that $f_\omega(H/H) = 1$.

A Weyl divisor $\sum_{D \in D(G/H)} a_D D + \sum_{E \in N(X)} b_E E$ is a Cartier divisor if and only if, for any $(C, F) \in \mathcal{F}(X)$, there is $h_C \in \chi(S)$ such that $h_C(E) = a_E \forall E \in C$ and $h_C(\rho(D)) = a_D \forall D \in F$. Let $PL'(X)$ be the set of functions on the support $|\mathcal{F}(X)|$ such that: 1) are linear on each colored cone; 2) are integer on $\chi_*(S) \cap |\mathcal{F}(X)|$. Let $L(X) \subset PL'(X)$ be the subset composed by the restrictions of linear functions and let $PL(X) := PL'(X)/L(X)$. The $\{h_C\}$, corresponding to any Cartier divisor, defines an element of $PL(X)$. If X is compact, there is an exact sequence (see [Br89], Théorème 3.1):

$$0 \rightarrow \bigoplus_{D \in \mathcal{D}(G/H) \setminus \mathcal{D}(X)} \mathbb{Z}D \rightarrow \text{Pic}(X) \rightarrow \text{PL}(X) \rightarrow 0.$$

A Cartier divisor is globally generated (resp. ample) if and only if the associated function is convex (resp. strictly convex) and $h_C(\rho(D)) \leq a_D$ (resp. $h_C(\rho(D)) < a_D$) $\forall (C, F) \in \mathcal{F}(X)$ and $\forall D \in \mathcal{D}(G/H) \setminus F$. Given any linearized line bundle L , the space $H^0(X, L)$ is a multiplicity free G -module and, if L is globally generated, the highest weights of $H^0(X, L)$ are the elements of $\chi(S) \cap \text{hull}(\{h_C\}_{\dim C=l})$, where $\text{hull}(\{x_1, \dots, x_m\})$ is the convex hull of x_1, \dots, x_m (see [Br89], §3). Thus, a Cartier divisor on a projective symmetric variety is nef if and only if it is globally generated. Moreover, a nef G -stable Cartier divisor on a projective symmetric variety is big if and only if the associated piecewise linear function h is such that $(\sum_{C \in \Delta(l)} h_C, R^\vee) \neq 0$ for each irreducible factor R^\vee of $R_{G,\theta}^\vee$ (see [Ru09] Theorem 4.2). In particular, when θ is indecomposable every non-zero nef G -stable divisor is big. When X is toroidal we have an exact split sequence

$$0 \rightarrow \text{Pic}(X_0) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(Z) \rightarrow 0.$$

A normal variety X is locally factorial if the Picard group is isomorphic to the class group, while X is \mathbb{Q} -factorial if $\text{Pic}(X)_\mathbb{Q} \cong \text{Cl}(X)_\mathbb{Q}$. A simple symmetric variety associated to a colored cone (C, F) is locally factorial if: i) C is generated by a subset of a basis of $\chi_*(S)$ and ii) ρ is injective over F (see [Br97b] for a general statement in the spherical case). When the variety is toroidal the local factoriality is equivalent to the smoothness.

An anticanonical divisor $-K_X$ of X is $\sum_{\alpha \in \overline{R}_{G,\theta}^\vee} a_\alpha D_\alpha + \sum_{E \in N(X)} E$ with $\sum a_\alpha \omega_\alpha = 2\rho - 2\rho_0$. Here $2\rho := 2\rho_{R_G} = \sum_{\alpha \in R_G^+} \omega_\alpha$ is the sum of all the positive roots of R_G , while $2\rho_0 := 2\rho_{R_G^0}$ is the sum of the positive roots in R_G^0 .

Let k (or k_X) be the piecewise linear function associated to $-K_X$. The anticanonical divisor $-K_X$ is linearly equivalent to a unique G -stable divisor $-\tilde{K}_X$. The piecewise linear function \tilde{k} (or \tilde{k}_X) associated to $-\tilde{K}_X$ is equal to $k - 2\rho + 2\rho_0$ over $N(X)$ and to 0 over $\rho(D(X))$. Indeed $-\tilde{K}_X$ is $-K_X + \text{div}(\prod_{\alpha \in \overline{R}_{G,\theta}} f_\alpha^{a_\alpha})$, where $f_\alpha \in \mathbb{C}(G/H)^{(B)}$ is an equation of D_α (of weight ω_α). In particular $\tilde{k} = k - 2\rho + 2\rho_0$ if X is toroidal.

2 Standard symmetric varieties

In the following, unless explicitly stated, we always suppose G semisimple (we will consider the general reductive case mainly in §5.3). Moreover, we often denote the normalizer $N_G(H)$ by $N(H)$. In this section we show that all the standard symmetric varieties are quasi \mathbb{Q} -Fano varieties; moreover we classify the Fano ones. When the rank of G/H is one, the standard compactification X_0 of G/H is the unique G -equivariant compactification. In such a case X_0 is an homogeneous projective variety w.r.t. $\text{Aut}^0(X)$ by [A83]; moreover it is wonderful and Fano, because either it is $\mathbb{P}^n \times \mathbb{P}^n$ or it has Picard number one.

First of all, we reduce ourselves to the indecomposable case. Write (G, θ) as a product $\prod (G_j, \theta)$ of indecomposable involutions and let X_j be standard compactification of $G_j/(G_j \cap H)$.

| G/H | θ | n, l | wonderful |
|--|----------|-------------------|-----------|
| $Spin_{2n+1}/(Spin_l \times Spin_{2n+1-l})$ | BI | $n \geq l \geq 4$ | no |
| $Spin_7/(Spin_3 \times Spin_4)$ | BI | $n = l = 3$ | no |
| $Sp_{2n}/N(GL_n)$ | CI | $n = l \geq 3$ | yes |
| $Spin_{2l}/N(Spin_l \times Spin_l)$ | DI | $n = l \geq 4$ | yes |
| $Spin_{2n}/H$ ($G^\theta \subset H \subsetneq N(G^\theta)$) | DI | $n = l \geq 6$ | no |
| $Spin_{2n}/(Spin_l \times Spin_{2n-l})$ | DI | $n > l \geq 4$ | no |
| $Spin_8/(Spin_3 \times Spin_5)$ | DI | $n = 4, l = 3$ | no |
| $E_6/N(C_4)$ | EI | | yes |
| $E_7/N(A_7)$ | EV | | yes |
| E_7/A_7 | EV | | no |
| E_8/D_8 | $EVIII$ | | yes |
| $F_4/(C_3 \times A_1)$ | FI | | yes |
| $G_2/(A_1 \times A_1)$ | G | | yes |

Figure 1: Non \mathbb{Q} -Fano standard indecomposable symmetric varieties

Lemma 2.1 *The variety X is (quasi) \mathbb{Q} -Fano if and only if all the X_i are (quasi) \mathbb{Q} -Fano.*

Proof. The weight \tilde{k}_X is equal to $\sum \tilde{k}_{X_i}$. \square

A standard symmetric variety is always \mathbb{Q} -factorial; in particular, K_X is a \mathbb{Q} -Cartier divisor. Moreover, if X is wonderful then also the X_j are wonderful and $X = \prod X_j$ (see [Ru09] Corollary 2.1). We have the following theorem:

Theorem 2.1 *Let X be a standard indecomposable symmetric variety. Let n be the rank of G and let l be the rank of X . Then:*

- *The anticanonical divisor of X is always a nef and big \mathbb{Q} -divisor.*
- *Suppose X wonderful. Then it is not a Fano variety if and only: i) if the involution induced on $\chi(S)_{\mathbb{R}}$ is $-id$; ii) $R_{G,\theta}$ is different from A_n and B_n ; iii) $H = N_G(G^\theta)$.*
- *The standard indecomposable varieties whose anticanonical divisor is not ample are compactifications of the symmetric spaces in Figure 1.*

Proof. We have to determine when $(\tilde{k}_{C^-}, \alpha_i) \leq 0$ for each $i \in \{1, \dots, l\}$. We can write $-2\rho + 2\rho_0$ as the sum of the spherical weights $-2\tilde{\rho} = -2 \sum_{\beta_j \in \overline{R}_G^1} \varpi_j$ and $2\tilde{\rho}_0 = -2 \sum_{\beta_j \in \overline{R}_G^0} \varpi_j + 2\rho_0$. Write $\beta_j - \theta(\beta_j) = \alpha_i$, so $(\tilde{k}_{C^-}, \alpha_i) = 2(k_{C^-}, \beta_j)$ and $(2\tilde{\rho}_0, \alpha_i) = 4(\rho_0, \beta_j) \leq 0$. Thus $(-2\rho + 2\rho_0)(\alpha_i^\vee) = -1$ if $\omega_i = 2\varpi_j$ and $(-2\rho + 2\rho_0)(\alpha_i^\vee) \leq -2$ otherwise. Suppose now H autonormalizing, i.e. $H = N(G^\theta)$; the case where $H \subsetneq N(H)$ is very similar. We want to study $k_{C^-} = -\sum_{i=1}^l \alpha_i$. By the expression of the Cartan matrix of $R_{G,\theta}$, $k_{C^-}(\alpha_i^\vee) \leq 1$ for each i . Therefore \tilde{k}_{C^-} is always anti-dominant. If \tilde{k}_{C^-} is not regular, then there is (a unique) $\alpha_i = 2\beta_j \in \overline{R}_{G,\theta}$ such that $k_X(\alpha_i^\vee) = 1$; in particular G is simple. By the classification of the involutions by their Satake diagrams, \tilde{k}_{C^-} is not regular if and only if $\theta = -id$ over $\chi(S)$ and $R_{G,\theta}$ is different by A_l and B_l . \square

3 Blows-ups along closed orbits

In this sections we want to prove a partial result in arbitrary rank. We restrict ourselves to the smooth toroidal case. For the toric varieties of rank 2 one can easily proves the following property (*):

Let Z be a smooth toric variety of rank 2 and let Z' be a smooth toric variety birationnally proper over Z . If the anticanonical bundle of Z' is ample, then also the anticanonical bundle of Z is ample.

This allows to prove easily that a smooth toric variety proper over \mathbb{A}^2 with ample anticanonical bundle is either \mathbb{A}^2 or its blow-up in the origin. We would like to use a similar property to classify the smooth toroidal Fano symmetric varieties. Unfortunately the previous property (*) is false already in rank three. Indeed, let \overline{Z} be the 3-dimensional toric variety whose fan $\overline{\Delta}$ has maximal cones $\text{cone}(e_1, e_2, e_3)$ and $\text{cone}(e_1, e_2, e_1 + e_2 - e_3)$, where $\{e_1, e_2, e_3\}$ is any basis of $\chi_*(S)$. The function associated to its anticanonical bundle is linear, so $-K_Z$ is nef but non-ample. Furthermore, the blow-up \overline{Z}' of \overline{Z} along $\text{cone}(e_1, e_2)$ is Fano. We can take $\overline{Z} \times \mathbb{A}^m$ as higher dimensional example.

In the previous example, we have considered a blow-up along a subvariety with strictly positive dimension. Now, we prove a property similar to (*) considering only blow-ups along compact orbits, i.e. S -fixed points. In the next section, we prove a much stronger statement when the rank is three.

Lemma 3.1 *Let Z be a smooth l -dimensional toric variety whose fan contains two l -dimensional cones σ_+ and σ_- such that: i) $\sigma_+ \cap \sigma_-$ has dimension $l - 1$ and ii) $\sigma_+ \cup \sigma_-$ is strictly convex. Assume moreover that the piecewise linear function associated to the anticanonical bundle of Z is not strictly convex on $\sigma_+ \cup \sigma_-$. Then the anticanonical bundle of any toric variety obtained from Z by a sequence of blow-ups centred in S -fixed points is not ample.*

We can reformulate the first hypothesis in a more combinatorial way. Indeed, we can write $\sigma_+ = \text{cone}(v_1, \dots, v_{l-1}, v_+)$ and $\sigma_- = \text{cone}(v_1, \dots, v_{l-1}, v_-)$ with $v_1, \dots, v_{l-1}, v_+, v_-$ primitive and $v_+ + v_- = \sum a_i v_i$, where the a_i are positive integers, not all zero.

Proof of Lemma 3.1. The anticanonical bundle of a variety with satisfies the hypotheses of the lemma is not ample. Thus, it is sufficient to show that the blow-up Z' of Z centered in any $\sigma \in \Delta(l)$ satisfies again the hypotheses of the lemma. We can suppose $\sigma = \sigma_+$ by symmetry. Then the fan of Z' contains σ_- and $\sigma' := \text{cone}(v_1, \dots, v_{l-1}, v_+ + \sum v_i)$. We have $(v_+ + \sum v_i) + v_- = \sum (a_i + 1)v_i$, so Z' satisfies the hypotheses w.r.t. σ_- and σ' . \square

Now we can classify the toric varieties with ample anticanonical bundle which are obtained from \mathbb{A}^l by a sequence of blow-ups centered in S -fixed points.

Proposition 3.1 *Let Z be a smooth toric variety with ample anticanonical bundle which is obtained from \mathbb{A}^l by a sequence of blow-ups centered in S -fixed points, then it is either \mathbb{A}^l or the blow-up of \mathbb{A}^l in the S -fixed point.*

Proof. One can easily see that the blow-up Z_1 of \mathbb{A}^l in the S -stable point has ample anticanonical bundle. The blow-up of Z_1 in the S -fixed point corresponding to $\text{cone}(e_1, \dots, \hat{e}_j, \dots, e_l, \sum e_i)$ satisfies the hypotheses of the previous lemma w.r.t. $\text{cone}(e_1, \dots, \hat{e}_h, \dots, e_l, \sum_{i=1}^l e_i)$ and $\text{cone}(e_1, \dots, \hat{e}_h, \dots, \hat{e}_j, \dots, e_l, \sum_{i=1}^l e_i, 2 \sum_{i=1}^l e_i - e_j)$, where $h \neq j$. \square

Thus, a symmetric variety obtained from a wonderful one by a sequence of blow-ups along closed orbits can be Fano only if it is either the wonderful variety or its blow-up along the closed orbit. We have already considered the wonderful case. Now, we prove that, when such blow-up is Fano, the rank of every indecomposable factor of $R_{G,\theta}$ is at most 3.

Lemma 3.2 *Let $X_{1,\dots,l}$ be the blow-up of the wonderful compactification of G/H and suppose that $R_{G,\theta}$ contains an irreducible factor of rank at least three. If $X_{1,\dots,l}$ is Fano then it is indecomposable, has rank 3 and $H \subsetneq N(G^\theta)$.*

Proof. The weights $\{\tilde{k}_C\}$ associated to $-\tilde{K}_{X_{1,\dots,l}}$ are $\lambda_i = -2\rho + 2\rho_0 - (l-2)e_i^* + \sum_{j \neq i} e_j^*$ with $i = 1, \dots, l$. First, suppose G/H indecomposable and write $e_i^* = -x_i \alpha_i$. If $H \subsetneq N(G^\theta)$, then $R_{G,\theta}$ has type A_1 , B_2 or C_l .

We consider two cases. First, suppose that there is $\beta_h \in \overline{R}_G^1$ orthogonal to R_G^0 . Write $\beta_h - \theta(\beta_h) = \alpha_j$, so $0 > (\lambda_j, \alpha_j^\vee) = (-2\rho + 2\rho_0, \alpha_j^\vee) + ((l-2)x_j \alpha_j^\vee, \alpha_j^\vee) - (\sum_{i \neq j} x_i \alpha_i^\vee, \alpha_j^\vee) \geq -2 + 2(l-2)x_j + 0$. Observe that $x_j^{-1} \leq 2$, so $l = 3$ and $H \subsetneq N(G^\theta)$.

If there is not such a root, we have the following possibilities: 1) θ has type A_{III} and G/G^θ is SL_{2l+2}/Sp_{2l+2} ; 2) θ has type C_{II} and G/G^θ is $Sp_{2n}/(Sp_{2l} \times Sp_{2n-2l})$; 3) θ has type D_{III} and G/G^θ is SO_{4l+2}/GL_{2l+1} . Then there are $\beta_3, \beta_5 \in R_G^0$ orthogonal to $\overline{R}_G^0 \setminus \{\beta_3, \beta_5\}$ and $\beta_4 \in R_G^1$ such that $\alpha_2 = \beta_3 + 2\beta_4 + \beta_5$. Moreover, $(\beta_3, \beta_3) = (\beta_4, \beta_4) = (\beta_5, \beta_5)$, $\alpha_2^\vee = \frac{1}{(\beta_2, \beta_2)} \alpha_2$ and $x_i = 1$ if $i < l$. Thus $0 > (\lambda_2, \alpha_2^\vee) = (-2\rho + 2\rho_0, \alpha_2^\vee) + (l-2)(\alpha_2, \alpha_2^\vee) - (\alpha_1, \alpha_2^\vee) - x_3(\alpha_3, \alpha_2^\vee) \geq -4 + 2(l-2) + 1 + x_3$, so we have again $l = 3$ and $H \subsetneq N(G^\theta)$.

Finally, suppose θ decomposable. Let $(G, \theta) = (G_1, \theta_1) \times (G_2, \theta_2)$ with $l' := \text{rank}(G_1/G_1^\theta) \geq 3$ and define the weight λ'_i for G_1 in an analogous way to the λ_i . We have $\lambda_i = \lambda'_i - (l-l')e_i^* + \omega$ where ω is orthogonal to $\overline{R}_{G_1,\theta}$. By the previous part of the proof there is always an i with $\lambda'_i(\alpha_i^\vee) \geq -1$, so $\lambda_i(\alpha_i^\vee) \geq \lambda'_i(\alpha_i^\vee) + \frac{1}{2}(\alpha_i, \alpha_i^\vee) \geq 0$, a contradiction. \square

By an explicit analysis of the indecomposable involutions of rank at most three we obtain:

Theorem 3.1 *Let G/H be a symmetric space of rank $l (> 1)$ associated to an involution θ and let X be a compact symmetric variety obtained from the wonderful compactification of G/H by a sequence of blow-ups along closed orbits.*

1. *If X is a Fano variety then either it is the wonderful variety X_0 or it is the blow-up $X_{1,\dots,l}$ of X along the closed orbit.*
2. *If there is an indecomposable factor of (G, θ) of rank at least 3 then $X_{1,\dots,l}$ is not Fano.*
3. *If (G, θ) has rank at least 6 and has an indecomposable factor of rank 2, then $X_{1,\dots,l}$ is not Fano.*
4. *If $X_{1,\dots,l}$ is Fano, the possibilities for the indecomposable factors of G/H are as in Figure 2 (we indicate also the eventual conditions on rank G/H so that such a factor can appear).*

| G'/H' | $\theta G'$ | $\text{rank } G/H$ | $\text{rank } G'/H'$ |
|--|-------------|--------------------|----------------------|
| SL_3 | A_2 | $l = 2$ | 2 |
| $Spin_5$ | B_2 | $l = 2$ | 2 |
| $SL_6/N(Sp_6)$ | AII | $l = 3$ | 2 |
| $SL_{n+1}/S(GL_2 \times GL_{n-1}), n \geq 4$ | $AIII$ | $l = 2$ | 2 |
| $Sp_{2n}/Sp_4 \times Sp_{2n-4}, n \geq 5$ | CII | $l \leq 3$ | 2 |
| $Sp_8/Sp_4 \times Sp_4$ | CII | $l = 3$ | 2 |
| $Sp_8/N(Sp_4 \times Sp_4)$ | CII | $l = 2$ | 2 |
| SO_{10}/GL_5 | $DIII$ | $l \leq 3$ | 2 |
| $E_6/N(D_5 \times \mathbb{C}^*)$ | $EIII$ | $l \leq 4$ | 2 |
| $E_6/N(F_4)$ | EIV | $l \leq 5$ | 2 |
| PSL_2 | A_1 | $l = 2$ | 1 |
| SL_2 | A_1 | $l \leq 3$ | 1 |
| $SL_{n+1}/N(SO_n)$ | AI | $l = 2$ | 1 |
| SL_{n+1}/SO_n | AI | $l = 2$ | 1 |
| $SL_6/N(Sp_6)$ | AII | $l \leq 3$ | 1 |
| SL_6/Sp_6 | AII | $l \leq 5$ | 1 |
| $SL_{n+1}/N(S(GL_1 \times GL_n))$ | AIV | $l \leq n + 1$ | 1 |
| $SO_{2n+1}/N(SO_1 \times SO_{2n})$ | BII | $l \leq n + 1$ | 1 |
| $SO_{2n+1}/(SO_1 \times SO_{2n})$ | BII | $l \leq 2n$ | 1 |
| $Sp_{2n}/(Sp_2 \times Sp_{2n-2})$ | CII | $l \leq 2n$ | 1 |
| $SO_{2n}/N(SO_1 \times SO_{2n-1})$ | DII | $l \leq n$ | 1 |
| $SO_{2n}/(SO_1 \times SO_{2n-1})$ | DII | $l \leq 2n - 1$ | 1 |
| F_4/B_4 | FII | $l \leq 12$ | 1 |

Figure 2: Fano $X_{1,\dots,l}$

4 Regular Fano varieties of rank 3

In this section we suppose X_0 wonderful; recall that $\{e_1, e_2, e_3\}$ is the basis of $\chi_*(S)$ which generates C^- . We classify all the Fano symmetric varieties obtainable by a wonderful symmetric variety of rank three from X_0 by a succession of blow-ups along G -subvarieties. This class of varieties contains many varieties; indeed each compact symmetric variety is dominated by a smooth toroidal variety obtained by a succession of blow-ups along G -subvarieties of codimension two. We begin proving a result similar to Lemma 3.1.

Let \tilde{Z} be the toric variety whose fan $\tilde{\Delta}$ has maximal cones $\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v_- = x_1v_1 + x_2v_2 - v_+)$, where $\{v_1, v_2, v_+\}$ is a basis of $\chi_*(S)$, $x_1 + x_2 > 0$ and $x_1 \geq x_2 \geq 0$. The anticanonical bundle of \tilde{Z} is ample if and only if $x_1 = x_1 + x_2 = 1$. In this case, \tilde{Z} is the blow up of \mathbb{A}^3 along a stable subvariety of codimension 2. Moreover, the anticanonical bundle of \tilde{Z} is nef, but non-ample if and only if $x_1 + x_2 = 2$. We have two possibilities: either $v_+ + v_- = v_1 + v_2$ or $v_+ + v_- = 2v_1$. In the first case we have a variety isomorphic to the variety \bar{Z} of the previous section. This is the more problematic case, so we will study it in a second time.

Lemma 4.1 *Let Z be a smooth 3-dimensional toric variety whose fan contains two maximal cones $\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v_-)$ such that $v_+ + v_- = x_1v_1 + x_2v_2$, where x_1 and x_2 are integers with $x_1 \geq x_2 \geq 0$. Suppose moreover*

that $x_1 \geq 2$. Then the anticanonical bundle of any toric variety Z' obtained from Z by a sequence of blow-ups along S -subvarieties is not ample.

Proof. Remark that the anticanonical bundle of Z is not ample. We say that a variety satisfies weakly the hypotheses of the lemma if $x_1 + x_2 \geq 2$ (instead of $x_1 \geq 2$). We use the following trivial observation: $x_1 + x_2 > 2$ implies $x_1 \geq 2$. One can try to prove this lemma by induction as the Lemma 3.1. Unfortunately we can prove only the following weaker statement.

Lemma 4.2 *Let Z be a toric variety which satisfies weakly the hypotheses of Lemma 4.1 and let Z' be the blow-up of Z along a cone τ .*

1. *If $\tau \neq \text{cone}(v_1, v_2)$, then Z' satisfies the hypotheses of Lemma 4.1.*
2. *If Z satisfies the hypotheses of Lemma 4.1, then Z' satisfies weakly the hypotheses of Lemma 4.1.*

Proof. We can suppose $\tau \subset \text{cone}(v_1, v_2, v_-)$ by symmetry. If $\tau \neq \text{cone}(v_1, v_2)$, we have three possibilities: $\tau = \text{cone}(v_1, v_-)$, $\tau = \text{cone}(v_2, v_-)$ and $\tau = \text{cone}(v_1, v_2, v_-)$. We always have $\Delta_{Z'}[p] = \Delta_Z[p] \cup \{v' := v_- + b_1 v_1 + b_2 v_2\}$ with $b_1, b_2 \in \{0, 1\}$. Moreover, $\Delta_{Z'}$ contains the cones $\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v')$ and we have $v' + v_+ = (x_1 + b_1)v_1 + (x_2 + b_2)v_2$ with $(x_1 + b_1) + (x_2 + b_2) > 2$, so Z_1 satisfies the hypotheses of the lemma.

Finally let $\tau = \text{cone}(v_1, v_2)$. The fan of Z' contains the cones $\text{cone}(v_1, v_1 + v_2, v_+)$ and $\text{cone}(v_1, v_1 + v_2, v_-)$. We have $v_+ + v_- = (x_1 - x_2)v_1 + x_2(v_1 + v_2)$ with $(x_1 - x_2) + x_2 = x_1 \geq 2$. \square

Now, we consider the general case. We have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow \dots \leftarrow Z_i \leftarrow \dots \leftarrow Z_r = Z'$ where Z_{i+1} is the blow-up of Z_i along the cone τ_i . Let $\Delta_i = \Delta_{Z_i}$ and let j be the maximal index such that Z_j satisfies the hypotheses (w.r.t. $\text{cone}(w_1, w_2, w_+)$ and $\text{cone}(w_1, w_2, w_-)$). By the previous lemma Z_{j+1} satisfies weakly the hypotheses, in particular its anticanonical bundle is not ample. By the maximality of j , Z_{j+1} does not satisfy the hypotheses, so Z_{j+1} contains a variety isomorphic to \overline{Z} . Let $\overline{\Delta}$ be the fan of such variety. If Δ_r contains $\overline{\Delta}$ then $-K_{Z'}$ is not ample. Otherwise there is a minimal h such that $\overline{\Delta}$ is not contained in Δ_{h+1} . We claim that Z_{h+1} satisfies the hypotheses of the lemma, a contradiction.

By the previous lemma $\tau_j = \text{cone}(w_1, w_2)$. We know that Z_{j+1} satisfies weakly the hypotheses w.r.t. $\sigma_+ = \text{cone}(w_1, w_1 + w_2, w_+)$ and $\sigma_- = \text{cone}(w_1, w_1 + w_2, w_-)$. Moreover, the open subvariety of Z_{j+1} with maximal cones σ_+ and σ_- is isomorphic to \overline{Z} . By the Lemma 4.2 2) we can suppose $\tau_h = \text{cone}(w_1, w_1 + w_2)$.

The fan of Z_{h+1} contains two cones $\text{cone}(w_1 + w_2, w_+, 2w_1 + w_2)$ and $\text{cone}(w_1 + w_2, w_+, w')$, with $w' = w_2 + x_1(w_1 + w_2) + x_2 w_+$ and $x_1, x_2 \geq 0$. Therefore Z_{h+1} satisfies the hypotheses of the lemma w.r.t. these cones. Indeed $(2w_1 + w_2) + w' = (2 + x_1)(w_1 + w_2) + x_2 w_+$. \square

Now we want to study the varieties which contain an open subvariety isomorphic to \overline{Z} . Observe that these varieties are never Fano varieties. Let Z be such a variety and let Z' be the blow-up of Z along the subvariety of \overline{Z} associated to $\text{cone}(e_1, e_2)$. We prove that, if Z' satisfies the hypotheses of Lemma 4.1, then there are not Fano varieties obtainable from Z by a sequence of blow-ups.

Lemma 4.3 *Let Z be a smooth 3-dimensional toric variety whose fan contains $\text{cone}(v_1, v_2, v_3)$ and $\text{cone}(v_1, v_2, v_1 + v_2 - v_3)$ for suitable v_1, v_2, v_3 . Let Z' be the blow-up of Z along the stable subvariety corresponding to $\text{cone}(v_1, v_2)$ and let Z'' be a toric variety obtained from Z by a sequence of blow-ups along S -subvarieties. If the anticanonical bundle of Z'' is ample, then Z'' is obtainable from Z' by a sequence of blow-ups along S -subvarieties.*

Proof. We cannot proceed as in the previous lemma, because we do not know the other cones of Δ_Z . We have again a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow \dots \leftarrow Z_i \leftarrow \dots \leftarrow Z_h = Z''$ where $\pi_{i+1} : Z_{i+1} \rightarrow Z_i$ is the blow-up along τ_i . First of all, there is a (minimal) cone τ_j contained in $\text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$, because otherwise the anticanonical bundle of Z'' is not ample. By Lemma 4.1, τ_j is $\text{cone}(v_1, v_2)$. We want to reorder the cones associated to the subvarieties along which we are blowing-up. Clearly this operation is not well defined in general.

We consider the following sequence of blow-ups: $Z = Z'_0 \leftarrow Z'_1 \leftarrow \dots \leftarrow Z'_i \leftarrow \dots \leftarrow Z'_{j+1}$, where $\pi'_0 : Z'_1 \rightarrow Z'_0$ is the blow-up along τ_j and $\pi'_{i+1} : Z'_{i+1} \rightarrow Z'_i$ is the blow-up along τ_{i-1} for each $i \geq 1$. Let $\Delta'_i = \Delta_{Z'_i}$. We show that these blow-ups are well defined and that $Z'_{j+1} = Z_{j+1}$.

The cone τ_{i-1} belongs to Δ'_i for each $1 \leq i \leq j$ because τ_i is contained in $|\Delta| \setminus \text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$ for each $i \leq j$. Moreover, the elements of $\Delta'_1(3)$ not contained in $\text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$ are exactly the elements of $\Delta_0(3) \setminus \{\text{cone}(v_1, v_2, v_3), \text{cone}(v_1, v_2, v_1 + v_2 - v_3)\}$.

Z is the union of the following two open S -subvarieties: U_1 whose fan has maximal cones $\text{cone}(v_1, v_2, v_3)$ and $\text{cone}(v_1, v_2, v_1 + v_2 - v_3)$; U_2 whose fan has maximal cones $\Delta(3) \setminus \{\text{cone}(v_1, v_2, v_3), \text{cone}(v_1, v_2, v_1 + v_2 - v_3)\}$.

The blow-up π'_0 induces an isomorphism between U_2 and its inverse image, because $\text{cone}(v_1, v_2)$ is not contained in any maximal cone of U_2 . In the same way π_j induces an isomorphism between the inverse image of U_2 in Z_j and its inverse image in Z_{j+1} . So the inverse image of U_2 in Z_{j+1} is isomorphic to the the inverse image of U_2 in Z'_{j+1} . Moreover $\pi'_j \circ \dots \circ \pi'_2$ induces an isomorphism between $(\pi'_1)^{-1}(U_1)$ and its inverse image. In the same way $\pi_{j-1} \circ \dots \circ \pi_1$ induces an isomorphism between U_1 and its inverse image. So the inverse image of U_1 in Z_{j+1} is isomorphic to the the inverse image of U_1 in Z'_{j+1} . The lemmas follows because there is at most one morphism between two toric S -varieties extending the identity automorphism of S . \square

We now restrict the possible Fano symmetric varieties with rank three (and fixed G/H) which are obtainable as before to a finite explicit list.

Proposition 4.1 *The toric varieties obtainable from \mathbb{A}^3 by a sequence of blow-ups and with ample anticanonical bundle are, up to isomorphisms:*

1. \mathbb{A}^3 ;
2. a 2-blow-up of \mathbb{A}^3 ;
3. the 3-blow-up of \mathbb{A}^3 ;
4. the variety whose fan has maximal cones: $\text{cone}(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\text{cone}(e_1, e_3, e_1 + e_2 + e_3)$, $\text{cone}(e_2, e_3, e_1 + e_2 + e_3)$ and $\text{cone}(e_2, e_1 + e_2, e_1 + e_2 + e_3)$. This variety is obtainable from \mathbb{A}^3 by two consecutive blow-ups along subvarieties of codimension two;

5. the variety whose fan has maximal cones: $\text{cone}(e_1, e_3, e_1 + e_2 + 2e_3)$, $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\text{cone}(e_1, e_2, e_1 + e_2 + e_3)$, $\text{cone}(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\text{cone}(e_2, e_3, e_1 + e_2 + 2e_3)$. This variety is obtainable from \mathbb{A}^3 by a 3-blow up followed by a 2-blow up.

Proof. We proceed as follows: the anticanonical bundle of \mathbb{A}^3 is ample, so we consider all the possible blow-ups of \mathbb{A}^3 . Let Z be a blow-up of \mathbb{A}^3 : 1) if Z satisfies the hypotheses of Lemma 4.1 we know that there are not toric varieties with ample anticanonical bundle and obtainable from Z by a sequence of blow-ups; 2) if Z satisfies the hypotheses of the Lemma 4.3 we study the variety Z' of such lemma; 3) finally, if the anticanonical bundle Z is ample, we reiterate the procedure. Observe that a priori it is possible that Z belongs to none of the previous cases. In the following, if two blow-ups of a given variety are isomorphic, we examine only one of them. Given a toric variety Z , let Δ be its fan and let k be the piecewise linear function associated to $-K_Z$. Suppose that all the maximal cones in Δ are 3-dimensional. Remember that $-K_Z$ is ample if and only if, given any cone $C \in \Delta(3)$ and any $v \in \Delta[p]$ with $v \notin C$, $(k_C)(v) < 1$.

Let $Z_0 = \mathbb{A}^3$; it has ample canonical bundle. Up to isomorphisms there are two blow-ups of \mathbb{A}^3 : the blow-up Z_1 along $\text{cone}(e_1, e_2)$ and the blow-up Z_2 along $\text{cone}(e_1, e_2, e_3)$.

One can show that the anticanonical bundle of Z_1 is ample because $\Delta_1(2) = \{\text{cone}(e_1, e_3, e_1 + e_2), \text{cone}(e_2, e_3, e_1 + e_2)\}$ and $\Delta[p] = \{e_1, e_2, e_3, e_1 + e_2\}$. The blow-ups of Z_1 are, up to isomorphisms: i) the blow-up Z_{11} along $\text{cone}(e_1, e_3)$; ii) the blow-up Z_{12} along $\text{cone}(e_1, e_1 + e_2)$; iii) the blow-up Z_{13} along $\text{cone}(e_1 + e_2, e_2, e_3)$; iv) the blow-up Z_{14} along $\text{cone}(e_1 + e_2, e_3)$.

The variety Z_{11} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_3, e_1 + e_2, e_1 + e_3)$ and $\text{cone}(e_3, e_1 + e_2, e_2)$. Hence we have to study the blow-up Z_{11b} of Z_{11} along $\text{cone}(e_3, e_1 + e_2)$. This variety satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1 + e_2, e_1 + e_3, e_1)$ and $\text{cone}(e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3)$. Hence we have to study the the blow-up Z_{11c} of Z_{11b} along $\text{cone}(e_1 + e_2, e_1 + e_3)$. Z_{11c} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, e_2)$ and $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$. Hence we have to consider the blow-up Z_{11d} of Z_{11c} along $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3)$. Z_{11d} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_2, e_1 + e_2 + e_3, e_3)$ and $\text{cone}(e_2, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$. Thus there are not toric varieties with ample anticanonical bundle and obtained from Z_{11} by a sequence of blow-ups.

Z_{12} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_3, e_1 + e_2, 2e_1 + e_2)$ and $\text{cone}(e_3, e_1 + e_2, e_2)$. Z_{13} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_3, e_1 + e_2, e_1)$ and $\text{cone}(e_3, e_1 + e_2, e_1 + 2e_2 + e_3)$.

The anticanonical bundle of Z_{14} is ample because $\Delta_{14}(2) = \{\text{cone}(e_1 + e_2, e_1, e_1 + e_2 + e_3), \text{cone}(e_1, e_3, e_1 + e_2 + e_3), \text{cone}(e_2, e_3, e_1 + e_2 + e_3), \text{cone}(e_1 + e_2, e_2, e_1 + e_2 + e_3)\}$ and $\Delta[p] = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\}$. The blow-ups of Z_{14} are: i) the blow-up Z_{141} of Z_{14} along $\text{cone}(e_1, e_1 + e_2)$; ii) the blow-up Z_{142} of Z_{14} along $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3)$; iii) the blow-up Z_{143} of Z_{14} along $\text{cone}(e_3, e_1 + e_2 + e_3)$; iv) the blow-up Z_{144} of Z_{14} along $\text{cone}(e_2, e_3, e_1 + e_2 + e_3)$; v) the blow-up Z_{145} of Z_{14} along $\text{cone}(e_2, e_1 + e_2, e_1 + e_2 + e_3)$; vi) the blow-up Z_{146} of Z_{14} along $\text{cone}(e_1, e_3)$; vii) the blow-up Z_{147} of Z_{14} along $\text{cone}(e_1, e_1 + e_2 + e_3)$.

The variety Z_{141} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, e_2)$ and $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, 2e_1 + e_2)$. The variety Z_{142} satisfies

the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + e_3, e_3)$. The variety Z_{143} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2)$ and $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$. The variety Z_{144} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_3, e_1 + e_2 + e_3, e_1)$ and $\text{cone}(e_3, e_1 + e_2 + e_3, e_1 + 2e_2 + 2e_3)$. The variety Z_{145} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_2, e_1 + e_2 + e_3, e_3)$ and $\text{cone}(e_2, e_1 + e_2 + e_3, 2e_1 + 3e_2 + e_3)$.

The variety Z_{146} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2)$. Hence we have to study the blow-up of Z_{146} along $\text{cone}(e_1, e_1 + e_2 + e_3)$. This variety is Z_{11c} , so there are no toric varieties with ample anticanonical bundle which are obtained from Z_{146} by a sequence of blow-ups.

The variety Z_{147} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, e_2)$ and $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3, 2e_1 + e_2 + e_3)$. Hence we have to study the blow-up Z_{147b} of Z_{147} along $\text{cone}(e_1 + e_2, e_1 + e_2 + e_3)$. This variety satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_2, e_1 + e_2 + e_3, e_3)$ and $\text{cone}(e_2, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$. Observe that we have classified the toric varieties with anticanonical bundle which are obtained from Z_1 by a sequence of blow-ups.

The anticanonical bundle of Z_2 is ample because $\Delta_2(2) = \{\text{cone}(e_2, e_3, e_1 + e_2 + e_3), \text{cone}(e_1, e_3, e_1 + e_2 + e_3), \text{cone}(e_1, e_2, e_1 + e_2 + e_3)\}$ and $\Delta_2(p) = \{e_1, e_2, e_3, e_1 + e_2 + e_3\}$. The blow-ups of Z_2 are, up to isomorphisms: i) the variety Z_{14} ; ii) the blow-up Z_{21} of Z_2 along $\text{cone}(e_1, e_2, e_1 + e_2 + e_3)$; iii) the blow-up Z_{22} of Z_2 along $\text{cone}(e_3, e_1 + e_2 + e_3)$.

Z_{21} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_2, e_1 + e_2 + e_3, e_3)$ and $\text{cone}(e_2, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$.

The anticanonical bundle of Z_{22} is ample because $\Delta_{22}(2) = \{\text{cone}(e_1, e_3, e_1 + e_2 + 2e_3), \text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3), \text{cone}(e_1, e_2, e_1 + e_2 + e_3), \text{cone}(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3), \text{cone}(e_2, e_3, e_1 + e_2 + 2e_3)\}$ and $\Delta_{22}[p] = \{e_1, e_2, e_3, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3\}$.

The blow-ups of Z_{22} are, up to isomorphisms: i) the variety Z_{143} which satisfies the hypotheses of the Lemma 4.1; ii) the blow-up Z_{221} of Z_{22} along $\text{cone}(e_3, e_1 + e_2 + 2e_3)$; iii) the blow-up Z_{222} of Z_{22} along $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$; iv) the blow-up Z_{223} of Z_{22} along $\text{cone}(e_1, e_1 + e_2 + 2e_3)$; v) the blow-up Z_{224} of Z_{22} along $\text{cone}(e_1, e_3)$; vi) the blow-up Z_{225} of Z_{22} along $\text{cone}(e_1, e_1 + e_2 + e_3)$; vii) the blow-up Z_{226} of Z_{22} along $\text{cone}(e_1, e_3, e_1 + e_2 + 2e_3)$; viii) the blow-up Z_{227} of Z_{22} along $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$; ix) the blow-up Z_{228} of Z_{22} along $\text{cone}(e_1, e_2, e_1 + e_2 + e_3)$.

Z_{221} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_1 + e_2 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_1 + e_2 + 3e_3)$. Z_{222} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_3)$ and $\text{cone}(e_1, e_1 + e_2 + 2e_3, 2e_1 + 2e_2 + 3e_3)$. Z_{223} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, e_2)$ and $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, 2e_1 + e_2 + 2e_3)$.

Z_{224} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_1 + e_2 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_1 + e_3)$. Hence we have to study the blow-up Z_{224b} of Z_{224} along $\text{cone}(e_1, e_1 + e_2 + 2e_3)$. This variety satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + e_3, e_2)$ and $\text{cone}(e_1, e_1 + e_2 + e_3, 2e_1 + e_2 + 2e_3)$. Z_{225} satisfies the hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(e_1, e_1 + e_2 + 2e_3, 2e_1 + e_2 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_3)$. Hence we have to study the blow-up Z_{225b} of Z_{225} along $\text{cone}(e_1, e_1 + e_2 + 2e_3)$. Z_{225b} satisfies the

| G/H | θ | \bar{m}_1 | m_1 | |
|------------------------------------|----------|-------------|--------|---|
| PSL_2 | A_1 | 1 | 0 | |
| SL_2 | A_1 | 2 | 1 | |
| $SL_2/N(SO_2)$ | AI | 0 | 0 | h |
| SL_2/SO_2 | AI | 1 | 0 | h |
| $SL_2/N(Sp_2)$ | AII | 2 | 1 | |
| SL_2/Sp_2 | AII | 4 | 3 | |
| $SL_n/S(GL_1 \times GL_{n-1})$ | AIV | n | $n-1$ | e |
| $SO_{2n+1}/N(SO_1 \times SO_{2n})$ | BII | $n-1$ | $n-1$ | |
| $SO_{2n+1}/(SO_1 \times SO_{2n})$ | BII | $2n-1$ | $2n-2$ | |
| $Sp_{2n}/(Sp_2 \times Sp_{2n-2})$ | CII | $2n-1$ | $2n-2$ | |
| $SO_{2n}/S(O_1 \times O_{2n-1})$ | DII | $n-1$ | $n-2$ | |
| $SO_{2n}/(SO_1 \times SO_{2n-1})$ | DII | $2n-2$ | $2n-3$ | |
| F_4/B_4 | FII | 11 | 10 | |

Figure 3: Weights of rank 1 symmetric spaces

hypotheses of the Lemma 4.3 w.r.t. $\text{cone}(2e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, e_1 + e_2 + e_3)$ and $\text{cone}(2e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, 2e_1 + e_2 + 2e_3)$. Hence we have to consider the blow-up Z_{225c} of Z_{225b} along $\text{cone}(2e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$. This variety satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, e_2)$ and $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, 3e_1 + 2e_2 + 3e_3)$.

Z_{226} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + 2e_3, e_1 + e_2 + e_3)$ and $\text{cone}(e_1, e_1 + e_2 + 2e_3, 2e_1 + e_2 + 3e_3)$. Z_{227} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, e_2)$ and $\text{cone}(e_1 + e_2 + e_3, e_1 + e_2 + 2e_3, 3e_1 + 2e_2 + 3e_3)$. Z_{228} satisfies the hypotheses of the Lemma 4.1 w.r.t. $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\text{cone}(e_1, e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3)$. \square

Let G/H be a symmetric space of rank three such that the standard compactification X_0 of G/H is wonderful. We introduce the following notations:

- we denote by X_{ij} the blow-up of X_0 along $(\text{cone}(e_i, e_j), \emptyset)$;
- we denote by X_{123} the blow-up of X_0 along the closed G -orbit;
- we denote by $X_{123,ij}$ the blow-up of X_{123} along $(\text{cone}(e_i, e_j), \emptyset)$;
- we denote by $X_{123,i}$ the blow-up of X_{123} along $(\text{cone}(e_i, e_1 + e_2 + e_3), \emptyset)$.

By [Ru09] Corollary 2.1, if $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and X_0 is wonderful, then $H = H_1 \times H_2$, where $H_i := H \cap G_i$. Given an 1-rank symmetric space G_i/H_i , let $\psi_i(r) := -2\rho + 2\rho_0 - re_i^*$, $m_i := \max\{r : \psi_i(r)(\alpha_i^\vee) < 0\}$ and $\bar{m}_i := \max\{r : \psi_i(r)(\alpha_i^\vee) \leq 0\}$. In Figure 3 are written the value of m_i and \bar{m}_i for the various G_i/H_i . Moreover, we indicate by e (resp. by h) when G_i/H_i is exceptional (resp. hermitian non-exceptional). By an explicitly analysis we can prove the following theorems.

Theorem 4.1 *Let G/H be an indecomposable symmetric space of rank 3 such that its standard compactification X_0 is wonderful. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is X_0 , X_{12} , X_{13} , X_{23} or $X_{123,13}$. More precisely, the Fano ones are those which appears in Figure 4.*

| G/H | θ | X |
|--|----------|---|
| PSL_4 | A_3 | X_0, X_{13} |
| SO_7 | B_3 | X_0 |
| PSp_6 | C_3 | X_0, X_{13} |
| Sp_6 | C_3 | X_0, X_{13} |
| $SL_4/N(SO_4)$ | AI | X_0 |
| $SL_8/N(Sp_8)$ | AII | $X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$ |
| $SL_{n+1}/S(L_3 \times L_{n-2}), n \geq 6$ | AIII | X_0, X_{13} |
| $SL_6/N(S(L_3 \times L_3))$ | AIII | X_0 |
| $SL_6/S(L_3 \times L_3)$ | AIII | X_0 |
| $SO_{2n+1}/N(SO_3 \times SO_{2n-2})$ | BI | X_0 |
| $Sp_6/N(GL_3)$ | CI | \emptyset |
| Sp_6/GL_3 | CI | X_0 |
| $Sp_{12}/N(Sp_6 \times Sp_6)$ | CII | $X_0, X_{12}, X_{13}, X_{23}$ |
| $Sp_{12}/(Sp_6 \times Sp_6)$ | CII | $X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$ |
| $Sp_{2n}/(Sp_6 \times Sp_{2n-6}), n > 6$ | CII | $X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$ |
| $SO_{2n}/N(SO_3 \times SO_{2n-3})$ | DI | X_0 |
| $SO_{12}/N(GL_6)$ | DIII | X_0, X_{12} |
| $SO_{12}/N(GL_6)$ | DIII | X_0, X_{12} |
| SO_{14}/GL_7 | DIII | $X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$ |
| $E_7/N(E_6 \times \mathbb{C}^*)$ | EVII | X_0, X_{12} |
| $E_7/(E_6 \times \mathbb{C}^*)$ | EVII | X_0, X_{12} |

Figure 4: Fano indecomposable symmetric varieties of rank 3

Theorem 4.2 *Let G/H be a symmetric space such that X_0 is wonderful and such that $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ with $\text{rank } G_i/G_i^\theta = i$. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,12}$ or $X_{123,3}$. More precisely, the classification of such varieties is as in Figure 5. In the second column, we indicate the conditions on m_1 so that X is Fano.*

Theorem 4.3 *Let G/H be a symmetric space such that X_0 is wonderful. Suppose that $(G, \theta) = (G_1, \theta) \times (G_2, \theta) \times (G_3, \theta)$ with $\text{rank } G_i/G_i^\theta = 1$ and let x_r be the number of factors G_i such that $\psi_i(r)$ is antidominant and regular. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ or $X_{123,3}$. More precisely, we have the following classification (depending on G/H):*

- If $x_1 \leq 1$, then the smooth Fano compactifications of G/H are X_0, X_{12}, X_{13} and X_{23} . In particular, there are four of them.
- If $x_1 = 2$, then there are five Fano varieties. Let $i < j$ be the indices such that $\psi_i(1)$ and $\psi_j(1)$ are anti-dominant and regular. The smooth Fano compactifications of G/H are $X_0, X_{12}, X_{13}, X_{23}$ and $X_{123,ij}$.

| G_2/H_2 | m_1 | X |
|--|--------------|--|
| PSL_3 | – | $X_0, X_{12}, X_{13}, X_{23}$ |
| SO_5 | – | X_0, X_{12} |
| $Spin_5$ | – | $X_0, X_{12}, X_{13}, X_{23}$ |
| G_2 | – | X_0, X_{13} |
| $SL_3/N(SO_3)$ | – | X_0 |
| $SL_6/N(Sp_6)$ | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}$ |
| $SL_{n+1}/S(GL_2 \times GL_{n-1})$ ($n \geq 5$) | – | $X_0, X_{12}, X_{13}, X_{23}$ |
| | $m_1 \geq 1$ | $X_{123,13}$ |
| $SL_5/S(GL_2 \times GL_3)$ | – | $X_0, X_{12}, X_{13}, X_{23}$ |
| $SL_4/N(S(GL_2 \times GL_2))$ | – | X_0 |
| $SL_4/S(GL_2 \times GL_2)$ | – | X_0, X_{12} |
| $SO_5/S(O_2 \times O_3)$ | – | X_0 |
| $SO_5/(SO_2 \times SO_3)$ | – | X_0 |
| $SO_{2n+1}/S(O_2 \times O_{2n-1})$ | – | X_0, X_{13} |
| $SO_{2n+1}/(SO_2 \times SO_{2n-1})$ | – | X_0, X_{13} |
| $Sp_{2n}/Sp_4 \times Sp_{2n-4}$ ($n \geq 5$) | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}$ |
| | $m_1 \geq 2$ | $X_{123,2}$ |
| $Sp_8/N(Sp_4 \times Sp_4)$ | – | $X_0, X_{12}, X_{13}, X_{23}$ |
| $Sp_8/Sp_4 \times Sp_4$ | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}$ |
| $SO_{2n}/S(O_2 \times O_{2n-2})$ | – | X_0, X_{13} |
| $SO_{2n}/SO_2 \times SO_{2n-2}$ | – | X_0, X_{13} |
| $SO_8/N(GL_4)$ | – | X_0, X_{12} |
| SO_8/GL_4 | – | X_0, X_{12} |
| SO_{10}/GL_5 | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}$ |
| $E_6/D_5 \times \mathbb{C}^*$ | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}, X_{123,1}$ |
| | $m_1 \geq 2$ | $X_{123,2}, X_{123,3}$ |
| $E_6/N(F_4)$ | – | $X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$ |
| | $m_1 \geq 1$ | $X_{123}, X_{123,12}, X_{123,13}, X_{123,1}$ |
| | $m_1 \geq 2$ | $X_{123,2}, X_{123,3}$ |
| $G_2/(A_1 \times A_1)$ | – | $\#$ |

Figure 5: Fano decomposable symmetric varieties of rank 3

- If (x_1, x_2) is equal to $(3, 0)$ or to $(3, 1)$, then there are eight Fano varieties: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}$ and $X_{123,23}$.
- If $(x_1, x_2) = (3, 2)$, then there are nine Fano varieties. Suppose that the $\psi_j(2)$ with $j \neq i$ are anti-dominant and regular, then the Fano varieties are: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}$ and $X_{123,i}$.
- If $(x_1, x_2) = (3, 3)$, then there are eleven Fano varieties: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ and $X_{123,3}$.

Remark that the first part of the proof holds in a more general context.

Corollary 4.1 *Given any wonderful G -variety X_0 of rank 3 (even non symmetric) and any Fano variety X obtained from X_0 by a succession of blow-ups along G -stable subvarieties, then X is one of the following varieties: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ or $X_{123,3}$.*

5 (Quasi) Fano symmetric varieties of rank 2

5.1 Fano symmetric varieties

In this section we consider the quasi-Fano locally factorial symmetric varieties with rank 2 (and G only reductive). Remark that to individuate univocally a projective 2-rank symmetric variety with ρ injective over $\varrho^{-1}(\varrho(D(X)))$ is sufficient to give $\Delta[p]$. We begin classifying the Fano varieties with G semi-simple. First, we consider two special cases: i) G/H is indecomposable, while X is neither simple nor toroidal; ii) X is toroidal.

Lemma 5.1 *Let X be a locally factorial projective symmetric variety. Suppose that G/H is indecomposable and that X is neither simple nor toroidal, then $R_{G,\theta} = A_2$, $H = G^\theta$ and $\mathcal{F}(X)$ contains $(\text{cone}(-\alpha_1^\vee - \alpha_2^\vee), \emptyset)$.*

Proof. We do a case-to-case analysis. 1) Suppose $R_{G,\theta} = A_2$ and $H = N(G^\theta)$. Then there is $\text{cone}(\alpha_1^\vee, v)$ in Δ , with $v := -\frac{x}{3}\alpha_1^\vee - \frac{y}{3}\alpha_2^\vee \in \text{int}(C^-)$ primitive. We can write $-\omega_2^\vee$ as a positive integral combination of α_1^\vee and v ; thus $y \in \{1, 2\}$. Moreover, $0 < (-v, \alpha_2) = -\frac{x}{3} + \frac{2y}{3}$, so $0 < x < 2y$. If $y = 1$, then $v = -\frac{1}{3}(\omega_1^\vee + \omega_2^\vee) \notin \chi_*(S)$. If $y = 2$, we have three possibilities: i) $v = -\omega_2^\vee$ which is not regular; ii) $v = -\frac{2}{3}(\omega_1^\vee + \omega_2^\vee) \notin \chi_*(S)$; iii) $v = -\alpha_1^\vee - \frac{2}{3}\alpha_2^\vee \notin \chi_*(S)$.

2) Suppose $R_{G,\theta} = A_2$ and $H = G^\theta$. Then there is $\text{cone}(\alpha_1^\vee, v)$ in Δ , with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. Hence $y = 1$ because $\{\alpha_1^\vee, v\}$ is a basis of $\chi_*(S)$. Moreover, $0 < (-v, \alpha_2) = -x + 2$, so $v = -\alpha_1^\vee - \alpha_2^\vee$ as in the statement.

3) Suppose that $R_{G,\theta} = B_2$ and that $\text{cone}(\alpha_1^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - \frac{y}{2}\alpha_2^\vee$ primitive. We can write $-\omega_2^\vee = -\alpha_1^\vee - \alpha_2^\vee$ as a positive integral combination of α_1^\vee and v ; thus $y \in \{1, 2\}$. Moreover, $0 < (-v, \alpha_2) = -x + y$, so $0 < x < y$. Therefore $v = -\omega_2^\vee$ is not regular.

4) Suppose that $R_{G,\theta} = B_2$ and that $\text{cone}(\alpha_2^\vee, v) \in \Delta$. If $H = N(G^\theta)$, then $\{\alpha_2^\vee, v\}$ cannot be a basis of $\chi_*(S) = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee$.

5) Suppose $R_{G,\theta} = B_2$ and $H = G^\theta$. Suppose also that $\text{cone}(\alpha_2^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. Then $x = 1$ because $\{\alpha_2^\vee, v\}$ is a basis of $\chi_*(S)$. Moreover, $0 < (-v, \alpha_1) = 2 - 2y$, so $0 < y < 1$.

- 6) Suppose $R_{G,\theta} = BC_2$ and $\text{cone}(\alpha_1^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $y = 1$. Moreover, $0 < (-v, \alpha_2) = -x + 1$, so $0 < x < 1$.
- 7) Suppose $R_{G,\theta} = BC_2$ and $\text{cone}(\alpha_2^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $x = 1$ and $0 < (-v, \alpha_1) = 2 - y$, so $v = -\omega_1^\vee$ is not regular.
- 8) Suppose $R_{G,\theta} = G_2$ and $\text{cone}(\alpha_1^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $y = 1$ and $0 < (-v, \alpha_2) \leq -3x + 2$, so $0 < x < \frac{2}{3}$.
- 9) Suppose $R_{G,\theta} = G_2$ and $\text{cone}(\alpha_2^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $x = 1$. Moreover, $0 < (-v, \alpha_1) = 2 - y$ and $0 < (-v, \alpha_2) = -3 + 2y$; thus $3 < 2y < 4$. \square

In the next two lemmas, we do not make any hypothesis on the regularity of X .

Lemma 5.2 *Let X be a Fano symmetric variety with G semisimple. Then $\mathcal{F}(X)$ contains at most three colored 1-cones.*

Proof. Suppose by contradiction that $\mathcal{F}(X)$ contains $(\sigma_i := \text{cone}(v_i, v_{i+1}), \emptyset)$, with $i = 1, 2, 3$ and v_1, v_2, v_3, v_4 primitive. We can write v_2 as a positive linear combination $xv_1 + yv_4$ of v_1 and v_4 . Then $k_{\sigma_2}(v_1) < k(v_1) = 1$, $k_{\sigma_2}(v_4) < k(v_4) = 1$ and $1 = k_{\sigma_2}(v_2) = xk_{\sigma_2}(v_1) + yk_{\sigma_2}(v_4) \leq 0$. \square

Lemma 5.3 *Let X be a Fano non-simple toroidal symmetric variety. Then X_0 is smooth and $\Delta[p]$ is either $\{e_1, e_1 + re_2, e_2\}$ or $\{e_1, re_1 + e_2, e_2\}$. These varieties are smooth if and only if $r = 1$.*

Proof. By the previous lemma, we have $\Delta[p] = \{e_1, e_2, v\}$ for an appropriate v . First suppose that X_0 is smooth, i.e. $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and write $v = x_1e_1 + x_2e_2$. For each i , let $\sigma_i = \text{cone}(e_i, v)$ and $\{i, i^c\} = \{1, 2\}$, so k_{σ_i} is $e_i^* + \frac{1-x_i}{x_{i^c}}e_{i^c}^*$. If $x_1 \geq 2$ and $x_2 \geq 2$, then $\frac{1-x_1}{x_2}$ and $\frac{1-x_2}{x_1}$ are strictly negative integers, so $x_1 \geq x_2 + 1 \geq x_1 + 2$, a contradiction.

Suppose now that X_0 is singular. Then $R_{G,\theta}$ is either A_2 or $A_1 \times A_1$. In the first case, $e_i = -3\omega_i^\vee$ and the strictly positive integer $k(-\omega_1^\vee - \omega_2^\vee)$ is lesser than $\frac{1}{3}k(-3\omega_1^\vee) + \frac{1}{3}k(-3\omega_2^\vee) = \frac{2}{3}$, a contradiction.

Finally suppose $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_2^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. Then, there is i such that $-\omega_1^\vee - \omega_2^\vee = av + b(-2\omega_i^\vee)$ with $a, b \geq 0$. The integer $a + b = k(-\omega_1 - \omega_2)$ is strictly lesser than $\frac{1}{2}k(-2\omega_1) + \frac{1}{2}k(-2\omega_2) = 1$, a contradiction. \square

Remark that the previous two lemmas apply also to a toroidal symmetric variety with $-K_X$ ample, $|\mathcal{F}(X)|$ convex and generated by a basis of $\chi_*(S)$ (without supposing X compact). Now, we state the main result of this section.

Theorem 5.1 *Let G/H be a symmetric space of rank 2 (with G semisimple).*

- *If a (projective) symmetric variety X is Fano then $\rho^{-1}(\rho(D(X))) = D(X)$. If moreover X is locally factorial, then $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$.*
- *If $R_{G,\theta}$ is irreducible and X is a Fano locally factorial equivariant compactification of G/H , we have exactly the following possibilities for $\Delta[p]$:*
 1. $\Delta[p] = \{e_1, \alpha_1^\vee\}$ if $\sharp\rho^{-1}(\alpha_1^\vee) = 1$ and $R_{G,\theta}$ is not G_2 ;
 2. $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $\sharp\rho^{-1}(\alpha_2^\vee) = 1$ and $R_{G,\theta}$ is not B_2 ;
 3. $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $R_{G,\theta} = B_2$ and $H = G^\theta$;

4. $\Delta[p] = \{-\omega_1^\vee, -\omega_2^\vee\}$ if $G/H \neq G_2/(SL_2 \times SL_2)$ and $H = N(G^\theta)$;
5. $\Delta[p] = \{-2\omega_1^\vee, -\omega_2^\vee\}$ if $R_{G,\theta} = B_2$ and $H = G^\theta$;
6. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $H = G^\theta$ and $R_{G,\theta} = A_2$;
7. $\Delta[p] = \{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$ (and $\Delta[p] = \{\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee, -3\omega_2^\vee\}$) if $H = G^\theta$ and $R_{G,\theta} = A_2$;
8. $\Delta[p] = \{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$ (and $\Delta[p] = \{\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee, -3\omega_2^\vee\}$) if $H = G^\theta$, $R_{G,\theta} = A_2$ and $\theta \neq -id$ over $\chi_*(T)$;
9. $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$ if $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $-2\rho + 2\rho_0 + e_i^* \in \text{int}(C^-)$ for each i .

The previous varieties are singular in the following cases:

- $\Delta[p] = \{e_i, \alpha_i^\vee\}$ and $i = 1, 2$ if $R_{G,\theta} = A_2$;
- $\Delta[p] = \{e_1, \alpha_1^\vee\}$ if $R_{G,\theta} = B_2$, $H = G^\theta$ (and $\sharp D(G/H) = 2$);
- $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $R_{G,\theta} = BC_2$ (and $\sharp D(G/H) = 2$).

- Suppose $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and let m_i be as in section §4. If $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, we have the following locally factorial Fano varieties:

1. $\Delta[p] = \{e_1, e_2\}$;
2. $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$;
3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2\}$ if $r \leq m_2 + 1$ and $\sharp \rho^{-1}(\alpha_1^\vee) = 1$;
4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, e_1 - r\alpha_2^\vee\}$ if $r \leq m_1 + 1$ and $\sharp \rho^{-1}(\alpha_2^\vee) = 1$;
5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2, -(r+1)\alpha_1^\vee + e_2\}$ if $r \leq m_2$ and $\sharp \rho^{-1}(\alpha_1^\vee) = 1$;
6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, e_1 - r\alpha_2^\vee, e_1 - (r+1)\alpha_2^\vee\}$ if $r \leq m_1$ and $\sharp \rho^{-1}(\alpha_2^\vee) = 1$;
7. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $\sharp D(G/H) = 2$.

Only the first two are smooth.

- If G/H is decomposable but $\chi_*(S) = \mathbb{Z}(\frac{1}{2}\alpha_1^\vee + \frac{1}{2}\alpha_2^\vee) \oplus \mathbb{Z}\alpha_2^\vee$, we have the following locally factorial Fano compactifications of G/H :

1. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$;
2. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_2+1}{2}$;
3. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_1+1}{2}$;
4. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee, -\frac{2r+3}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_2-1}{2}$;
5. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+3}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_1-1}{2}$.

The first is smooth, while the other ones are smooth if and only if $r = 0$.

Remark that, supposing $R_{G,\theta} = A_1 \times A_1$, the m_i have the same value if $\chi_*(S) = \mathbb{Z}(\omega_1^\vee + \omega_2^\vee) \oplus \mathbb{Z}2\omega_2^\vee$ or if $H = G^\theta$. Moreover, $\sharp D(G/H) = 2$ if $\chi_*(S) = \mathbb{Z}(\omega_1^\vee + \omega_2^\vee) \oplus \mathbb{Z}2\omega_2^\vee$, because any $n \in H \setminus G^\theta$ exchanges the colors of G/G^θ associated to the same coroot.

Proof. The first property holds because $-K_X$ is linearly equivalent to a G -stable divisor. Let $a_i := (2\rho - 2\rho_0)(\alpha_i^\vee)$ for each i ; recall that $a_1, a_2 \geq 1$ (see §2). In the following we always suppose $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$.

I) Suppose X simple. If X is toroidal (i.e. $X = X_0$) have to exclude the following cases by §2: i) $G/H = G_2/(SL_2 \times SL_2)$; ii) $R_{G,\theta} = A_2$ and $H = G^\theta$; iii) $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. In the first case X_0 is not Fano, while in the last two cases X_0 is not smooth. If $\rho(D(X)) = \{\alpha_1^\vee\}$, then θ is indecomposable because $C(X)$ is strictly convex. Moreover, $\Delta[p] = \{\alpha_1^\vee, e_1 = -x_1\alpha_1^\vee - x_2\alpha_2^\vee\}$ and $k_X = a_1\omega_1 - \frac{x_1a_1+1}{x_2}\omega_2$; so $k_X(\alpha_2^\vee) \leq 0 < a_2$. Thus we have only to verify if X is locally factorial. We have to exclude two cases: 1) $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -\omega_1^\vee)\}$ and $R_{G,\theta} = G_2$; 2) $\Delta(2) = \{\text{cone}(\alpha_2^\vee, -\omega_2^\vee)\}$, $R_{G,\theta} = B_2$ and $H = N(G^\theta)$.

II) Suppose now that X is not simple. In the toroidal case $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$ and X_0 is smooth by Lemma 5.3. This variety is Fano if and only if $-2\rho + 2\rho_0 + e_i^*$ is antidominant and regular for each i . III) Assume moreover that X is not toroidal. Suppose first θ indecomposable. Then, by Lemma 5.1, $R_{G,\theta} = A_2$, $H = G^\theta$ and $\Delta(2)$ contains $\text{cone}(\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee)$, up to reindexing. If $\rho(D(X))$ contains also α_2^\vee then $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee), \text{cone}(\alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee)\}$. Moreover, $\tilde{k}_X(\alpha_1^\vee) = \tilde{k}_X(\alpha_2^\vee) = 0$ and $\tilde{k}_X(-\alpha_1^\vee - \alpha_2^\vee) \geq 1$, so \tilde{k}_X is strictly convex.

If $\rho(D(X)) = \{\alpha_1^\vee\}$, then $\Delta[p]$ contains $-3\omega_1^\vee$. Remark that $\{-\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$ is a basis of $\chi_*(S)$. By the Lemma 5.3 we have two possibilities for $\Delta[p]$: $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$ or $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$. In the first case there are not other conditions because the weights of k are $a_1\omega_1 - (a_1+1)\omega_2$ and $-\omega_2$. In the last case we have to impose that $a_1, a_2 > 1$. Indeed the weights of k are $a_1\omega_1 - (a_1+1)\omega_2, -\alpha_2$ and $-\omega_1 + \omega_2$. Moreover $(-\alpha_2)(\alpha_1^\vee) < a_1$ and $(-\omega_1 + \omega_2)(\alpha_2^\vee) < a_2$. In §2, we have seen that $a_1 \leq 1, a_2 \leq 1$ if and only if $\theta = -id$.

IV) Suppose now θ decomposable. First suppose $D(X) = D(G/H)$. Then $\Delta(2)$ contains $\text{cone}(\alpha_1^\vee, x(-m\alpha_1^\vee - \alpha_2^\vee))$ and $\text{cone}(\alpha_2^\vee, x(-\alpha_1^\vee - r\alpha_2^\vee))$ with $x \in \{1, \frac{1}{2}\}$. Remark that $\mathbb{R}^+(-m\alpha_1^\vee - \alpha_2^\vee) = \mathbb{R}^+(-\alpha_1^\vee - \frac{1}{m}\alpha_2^\vee)$ and $x(-\alpha_1^\vee - r\alpha_2^\vee) \notin \text{cone}(\alpha_1^\vee, x(-m\alpha_1^\vee - \alpha_2^\vee))$ so $r \leq \frac{1}{m}$. Therefore $\Delta(2) = \{\text{cone}(\alpha_1^\vee, x(-\alpha_1^\vee - \alpha_2^\vee)), \text{cone}(\alpha_2^\vee, x(-\alpha_1^\vee - \alpha_2^\vee))\}$.

V) Suppose now that $\rho(D(X))$ contains exactly one coroot, say α_1^\vee . Suppose X_0 smooth and let $\sigma_0 := \text{cone}(\alpha_1^\vee, -m\alpha_1^\vee + e_2)$ be in $\Delta(2)$. The case with X_0 singular is very similar. We can apply the Lemma 5.3 to the maximal open toroidal subvariety X' of X (whose colored fan has support $\text{cone}(-\alpha_1^\vee, -m\alpha_1^\vee + e_2)$). There are two possibilities for $\Delta_{X'}$: its maximal cones are either $\{\sigma_1 := \text{cone}(-\alpha_1^\vee, -m\alpha_1^\vee + e_2)\}$ or $\{\sigma_1 := \text{cone}(-m\alpha_1^\vee + e_2, -(m+1)\alpha_1^\vee + e_2), \sigma_2 := \text{cone}(-\alpha_1^\vee, -(m+1)\alpha_1^\vee + e_2)\}$. In the first case the unique non-trivial condition is $k_{\sigma_1}(\alpha_2^\vee) < a_2$, which is equivalent to the following one: $\tilde{k}_{\sigma_1}(\alpha_2^\vee) = (\psi_2(m-1))(\alpha_2^\vee) < 0$. In the second case the unique non-trivial condition is $\tilde{k}_{\sigma_2}(\alpha_2^\vee) = (\psi_2(m))(\alpha_2^\vee) < 0$.

Now, we verify the smoothness of such varieties. First, we explain the conditions for a projective locally factorial symmetric variety X with rank two to be smooth (see [Ru07], Theorems 2.1 and 2.2). Let Y be a open simple G -subvariety of X whose closed orbit is compact; then the associated colored cone (C, F) has dimension two. Write $C = \text{cone}(v_1, v_2)$ and let $C^\vee = \text{cone}(v_1^*, v_2^*)$ be the dual cone; we take v_1 and v_2 primitive. If X is smooth then ρ is injective

over F and $\rho(F)$ does not contain any exceptional root. Suppose such conditions verified and let R' the root subsystem of $R_{G,\theta}$ generated by the simple roots α such that $\alpha^\vee \in \rho(F)$ and $\sharp\rho^{-1}(\alpha^\vee) = 1$. If there is not such a root, then Y is smooth. Otherwise Y it is smooth if and only if: i) R' has type A_1 ; ii) up to reindexing, $\frac{1}{2}(2v_1^* - v_2^*)$ is the fundamental weight of R' .

Suppose now that $C = \text{cone}(\alpha_1^\vee, -r\alpha_1^\vee - \alpha_2^\vee)$ with $-r\alpha_1^\vee - \alpha_2^\vee$ primitive. If X is Fano, then $\sharp\rho^{-1}(\alpha_1^\vee) = 1$. If moreover θ is indecomposable, then $R_{G,\theta} = A_2$, $H = G^\theta$ and $r = 1$. Furthermore R' is A_1 and $C^\vee = \text{cone}(v_1^* = \omega_1 - \omega_2, v_2^* = -\omega_2)$. Hence $\frac{1}{2}(2v_1^* - v_2^*) = \frac{1}{2}\alpha_1$; thus Y is smooth. If $G/G^\theta = G_1/G_1^\theta \times G_2/G_2^\theta$, then $R' = R_{G_1,\theta}$, $H = G_1^\theta \times (H \cap G_2)$ and $C^\vee = \text{cone}(v_1^* = \omega_1 - r\omega_2, v_2^* = -\omega_2)$. Hence $R_{G_1,\theta}$ has to be A_1 and $\frac{1}{2}(2v_1^* - v_2^*) = \omega_1 + (\frac{1}{2} - r)\omega_2$ has to be $\frac{1}{2}\alpha_1$. Thus Y is not smooth. Suppose now that $R_{G,\theta} = A_1 \times A_1$, $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$ and $C = \text{cone}(\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee)$. Then R' is A_1 and $C^\vee = \text{cone}(v_1^* = \omega_1 - (2r+1)\omega_2, v_2^* = -2\omega_2)$; moreover, $\frac{1}{2}(2v_1^* - v_2^*) = \omega_1 - 2r\omega_2$; thus Y is smooth if and only if $r = 0$. The other cases are similar. \square

Using the Lemma 5.3, one can easily prove the following proposition:

Proposition 5.1 *Let G/H be a symmetric space of rank 2 with X_0 smooth. If θ is indecomposable, the Fano toroidal compactifications of G/H are as in figure 6. If θ is decomposable, let m_i be as in §4. Then, the Fano toroidal compactifications of G/H are the following ones:*

- $\Delta[p] = \{e_1, e_2\}$;
- $\Delta[p] = \{e_1, e_2, re_1 + e_2\}$ with $r \leq m_2 + 1$;
- $\Delta[p] = \{e_1, e_2, e_1 + re_2\}$ with $r \leq m_1 + 1$.

5.2 Smooth quasi Fano varieties

Now, we want to classify the smooth (resp. locally factorial) quasi Fano symmetric varieties with rank two and G semisimple. A Gorenstein (projective) variety is called quasi Fano if its anticanonical divisor is big and nef.

Theorem 5.2 *Let G/H be a symmetric space of rank 2 (with G semisimple).*

- *The nefness of the anticanonical bundle of a compactification of G/H depends only by the fan associated to the colored fan (and not by the whole colored fan).*
- *The fans of the locally factorial quasi Fano (but non-Fano) compactifications of an indecomposable symmetric space of rank 2 (with G semisimple) are those in Figure 7 (we have also to request that ρ is injective over $D(X)$). Such a variety is singular if and only if $\rho(D(X))$ contains an exceptional root.*
- *If $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$, let m_i and \bar{m}_i as in §4. Supposing $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, let $v_j(i) := -i\alpha_j^\vee + e_{j^c}$ and $w_j(x, y) := -(xy+1)\alpha_j^\vee + ye_{j^c}$. We have the following locally factorial quasi Fano compactifications of G/H , which are not Fano (we always suppose ρ injective over $D(X)$):*

1. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if i) $s = 0, 1$, ii) $r+s \leq \bar{m}_2 + 1$, iii) either $r+s > m_2 + 1$ or $\sharp\rho^{-1}(\alpha_1^\vee) = 2$;

| G/H | θ | $\Delta[p]$ | $\Delta[p]$ |
|---|----------|---|--|
| PSL_3 | A_2 | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| SO_3 | B_2 | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $Spin_3$ | B_2 | $\{-2\omega_1^\vee, -\omega_2^\vee\}$ | $\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ |
| G_2 | G_2 | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SL_3/N(SO_3)$ | AI | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SL_6/N(Sp_6)$ | $AIII$ | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ |
| | | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ |
| $SL_{n+1}/S(GL_2 \times GL_{n-1})$ | $AIII$ | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}, r \leq n-3$ |
| $SL_4/N(S(GL_2 \times GL_2))$ | $AIII$ | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SL_4/S(GL_2 \times GL_2)$ | $AIII$ | $\{-\omega_1^\vee, -2\omega_2^\vee\}$ | |
| $SO_{2n+1}/S(O_2 \times O_{2n-1})$ | BI | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SO_{2n+1}/SO_2 \times SO_{2n-1}$ | BI | $\{-2\omega_1^\vee, -\omega_2^\vee\}$ | |
| $Sp_{2n}/(Sp_4 \times Sp_{2n-4})$ $n \geq 5$ | CII | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}, r \leq 2n-6$ |
| | | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ | |
| $Sp_8/N(Sp_4 \times Sp_4)$ | CII | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| $Sp_8/(Sp_4 \times Sp_4)$ | CII | $\{-\omega_1^\vee, -2\omega_2^\vee\}$ | $\{-\omega_1^\vee, -2\omega_2^\vee, -2\omega_1^\vee - 2\omega_2^\vee\}$ |
| | | $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ | $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee\}$ |
| $SO_{2n}/S(O_2 \times O_{2n-2})$ | DI | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SO_{2n}/SO_2 \times SO_{2n-2}$ | DI | $\{-2\omega_1^\vee, -\omega_2^\vee\}$ | |
| $SO_8/N(SO_4)$ | $DIII$ | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| SO_8/SO_4 | $DIII$ | $\{-\omega_1^\vee, -2\omega_2^\vee\}$ | |
| SO_{10}/SO_5 | $DIII$ | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| | | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ | $\{-\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ |
| $E_6/D_5 \times \mathbb{C}^*$ | $EIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - r\omega_2^\vee\}, r \leq 3$ | $\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}, r \leq 4$ |
| | | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| $E_6/N(F_4)$ | EIV | $\{-\omega_1^\vee, -\omega_2^\vee\}$ | |
| | | $\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}, r \leq 4$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - r\omega_2^\vee\}, r \leq 4$ |
| $G_2/(A_1 \times A_1)$ | G | \nexists | |

Figure 6: Fano toroidal indecomposable symmetric varieties with rank 2

2. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if *i*) $s = 0, 1$, *ii*) $r+s \leq \bar{m}_1 + 1$, *iii*) either $r+s > m_1 + 1$ or $\#\rho^{-1}(\alpha_2^\vee) = 2$;
3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if *i*) $s \geq 2$ and *ii*) $r+s \leq \bar{m}_2 + 1$;
4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if *i*) $s \geq 2$ and *ii*) $r+s \leq \bar{m}_1 + 1$;
5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2, w_1(r, 1), \dots, w_1(r, s)\}$ if $r \leq \bar{m}_2$ and $2 \leq s \leq \bar{m}_1 + 1$;
6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -r\alpha_2^\vee + e_1, w_2(r, 1), \dots, w_2(r, s)\}$ if $r \leq \bar{m}_1$ and $2 \leq s \leq \bar{m}_2 + 1$;
7. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $\#D(G/H) > 2$;
8. $\Delta[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + (s-1)e_2, e_1 + se_2, e_2\}$ if $2 \leq s \leq \bar{m}_1 + 1$;
9. $\Delta[p] = \{e_2, e_1 + e_2, 2e_1 + e_2, \dots, (s-1)e_1 + e_2, se_1 + e_2, e_1\}$ if $2 \leq s \leq \bar{m}_2 + 1$.

These varieties are smooth if either they are toroidal or if, $\forall \alpha^\vee \in \rho(D(X))$, $\#\rho^{-1}(\alpha^\vee) = 2$ and $2\alpha \notin R_{G,\theta}$.

- If G/H is decomposable but $\chi_*(S) = \mathbb{Z}(\frac{1}{2}\alpha_1^\vee + \frac{1}{2}\alpha_2^\vee) \oplus \mathbb{Z}\alpha_2^\vee$, let $v_j(i) := -\frac{2i+1}{2}\alpha_j^\vee - \frac{1}{2}\alpha_{j^c}^\vee$ and $w_j(x, y) := -\frac{2xy+y+2}{2}\alpha_j^\vee - \frac{y}{2}\alpha_{j^c}^\vee$. We have the following locally factorial quasi Fano compactifications of G/H , which are not Fano:

1. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if *i*) $r \geq 0$, *ii*) $s \geq 2$ and *iii*) $r+s \leq \frac{\bar{m}_2+1}{2}$;
2. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if *i*) $r \geq 0$, *ii*) $s \geq 2$ and *iii*) $r+s \leq \frac{\bar{m}_1+1}{2}$;
3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if *i*) $r \geq 0$, *ii*) $s = 0, 1$ and *iii*) $\frac{\bar{m}_2+1}{2} < r+s \leq \frac{\bar{m}_2+1}{2}$;
4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if *i*) $r \geq 0$, *ii*) $s = 0, 1$ and *iii*) $\frac{\bar{m}_1+1}{2} < r+s \leq \frac{\bar{m}_1+1}{2}$;
5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee, w_1(r, 1), \dots, w_1(r, s)\}$ if *i*) $r \leq \frac{\bar{m}_2-1}{2}$ and *ii*) $2 \leq s \leq \bar{m}_1 + 1$;
6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee, w_2(r, 1), \dots, w_2(r, s)\}$ if *i*) $r \leq \frac{\bar{m}_1-1}{2}$ and *ii*) $2 \leq s \leq \bar{m}_2 + 1$;
7. $\Delta[p] = \{-\alpha_1^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$.

The last variety is smooth, while the other ones are smooth if and only if $r = 0$.

The idea of the proof is to utilize Theorem 5.1 thanks to the following lemma. Remark that, when θ is indecomposable, if $-K_X$ is nef it is also big (see §1.6).

Proof. The first property holds because the inequalities in the conditions for the nefness of a Cartier divisor are not strict. I) First suppose ρ non-injective. We have to consider the varieties whose fan is as in Theorem 5.1, but which are not Fano because $\#\rho^{-1}(\rho(D(X))) \neq \#D(X)$. In such cases ρ has to be

injective over $D(X)$, so that X is locally factorial. If θ is indecomposable we have the following possibilities: i) $R_{G,\theta} = BC_2$ and $\Delta(2) = \{\text{cone}(\alpha_2^\vee, -\omega_2)\}$; ii) $R_{G,\theta} = B_2$, $H = G^\theta$ and $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -2\omega_1)\}$. If θ is decomposable, we have to consider all the possibilities listed in Theorem 5.1 which correspond to non-toroidal varieties with $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. We have also to substitute the conditions of type $\psi_i(m) < 0$ with the corresponding conditions $\psi_i(m) \leq 0$.

Lemma 5.4 *Let X be a projective symmetric variety with $-K_X$ nef, then there is a symmetric variety X' below X such that the piecewise linear function associated to $-K_{X'}$ is strictly convex (over the colored fan of X') and coincides with the function associated to $-K_X$. If X is toroidal, we can choose X' toroidal.*

Proof of Lemma 5.4. Let Δ' be the fan whose maximal cones are the maximal cones over which k_X is linear. Given any cone $C \in \Delta'$, define F_C as $\{D \in D(G/H) : \rho(D) \in C\}$. We claim that $\{(C, F_C)\}_{C \in \Delta', \text{Int } C \cap C^- \neq \emptyset}$ is a colored fan associated to a symmetric variety which satisfies the conditions of the lemma. Remark that $|\Delta'| = |\mathcal{F}(X)|$. We have only to prove that the maximal cones of Δ' are strictly convex. Let $C = \text{cone}(\alpha_1^\vee, \dots, \alpha_r^\vee, -\varpi_1^\vee, \dots, -\varpi_s^\vee) \in \Delta'(l)$ (with $\varpi_1^\vee, \dots, \varpi_s^\vee$ dominant) and suppose by contradiction that it contains the line generated by $v = \sum_{i=1}^r a_i \alpha_i^\vee + \sum_{j=1}^s b_j (-\varpi_j^\vee)$. Then C contains also $\mathbb{R}v'$, where $v' = \sum b_j (-\varpi_j^\vee)$; indeed $-v' = -v + \sum a_i \alpha_i^\vee$. Write $-v' = \sum_{\alpha \in \bar{R}_{G,\theta}} c_\alpha \alpha^\vee$; then C contains all the α^\vee such that $c_\alpha \neq 0$, because $-v' \in C \cap C^+$ and any spherical weight is a positive rational combination of the simple restricted roots. Thus, if $v' \neq 0$, $k_C(-v') = \sum c_\alpha k_C(\alpha^\vee) \geq 0$, while $k_C(v') = \sum b_j k_C(-\varpi_j^\vee) > 0$. Suppose now that $v = \sum_{i=1}^r a_i \alpha_i^\vee$ with $a_j \neq 0$. Then C contains $\mathbb{R}\alpha_j^\vee$. Write $-\alpha_j^\vee = \sum_{i=1}^r a'_i \alpha_i^\vee + \sum_{i=1}^s b'_i (-\varpi_i^\vee)$ with positive coefficients, then there is j_0 such that $b_{j_0} \neq 0$. So $k_C(-\alpha_j^\vee) \geq b_{j_0} > 0$ and $k_C(\alpha_j^\vee) > 0$, a contradiction. \square .

II) Suppose X toroidal and let X' be as in Lemma 5.4. If $X' = X$ and is simple, then $G/H = G_2/(SL_2 \times SL_2)$. If $X' \neq X$ and is simple, then it must be singular. Otherwise, given any $w = x_1 e_1 + x_2 e_2 \in \Delta_X[p] \setminus \Delta_{X'}[p]$, we have $x_i \in \mathbb{Z}^{>0}$ and $x_1 + x_2 = k(w) = 1$. If $R_{G,\theta} = A_2$ and $H = G^\theta$, then $k_X = -\frac{1}{3}(\omega_1 + \omega_2) \notin \chi(S)$, so X is not locally factorial. Thus $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. Let $w = -x_1 \alpha_1^\vee - x_2 \alpha_2^\vee$ be in $\Delta_X[p] \setminus \Delta_{X'}[p]$; then $2x_1, 2x_2 \in \mathbb{Z}^{>0}$ and $x_1 + x_2 = k(w)$ is 1. Thus $\Delta_X[p] = \{-\alpha_1^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$. In this case $-K_X$ is nef but non-ample.

If X' is not simple, then the standard compactification of G/H has to be smooth because of Lemma 5.3. So it is sufficient to prove the following lemma.

Lemma 5.5 *The smooth toric varieties birationally proper over \mathbb{A}^2 with nef anticanonical divisor are, up to isomorphisms, \mathbb{A}^2 and the varieties Z_m , where Z_m is the variety whose fan has maximal cones $\{\text{cone}(e_1, e_1 + e_2), \text{cone}(e_1 + e_2, e_1 + 2e_2), \dots, \text{cone}(e_1 + (m-1)e_2, e_1 + me_2), \text{cone}(e_1 + me_2, e_2)\}$.*

Proof of Lemma 5.5. The piecewise linear k_m function associated to the anticanonical bundle of Z_m is linear on $\text{cone}(e_1, e_1 + me_2)$. It is easy to see that thus such function is convex.

Now we show that, given any Z as in the hypotheses, it is isomorphic to Z_m for an appropriate m . Notice, that any smooth toric variety birationally proper over \mathbb{A}^2 is obtained by a sequence of blow-ups. Thus there is nothing

to prove if $\sharp\Delta[p] < 4$. Suppose now that $\sharp\Delta[p] \geq 4$; we claim that up to isomorphisms, Δ contains $\text{cone}(e_1, e_1 + e_2)$ and $\text{cone}(e_1 + e_2, e_1 + 2e_2)$. We know that $\tau = \mathbb{R}^{\geq 0}(e_1 + e_2)$ is contained in Δ .

First of all, we determine the restrictions of k to the cones containing τ and afterwards we will determine the cones themselves. Let $\sigma = \text{cone}(e_1 + e_2, b_1e_1 + b_2e_2) \in \Delta(2)$ be a maximal cone containing τ and write $k_\sigma = x_1e_1^* + x_2e_2^*$, so $(k_\sigma)(e_1 + e_2) = x_1 + x_2 = 1$ and $x_i = (k_\sigma)(e_i) \leq 1$ for each i . Hence k_σ is e_i^* for an appropriate i and $b_i = (k_\sigma)(b_1e_1 + b_2e_2) = 1$. Because of the non-singularity of Z the only possibilities for σ are $\text{cone}(e_1 + e_2, e_1)$, $\text{cone}(e_1 + e_2, e_2)$, $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ and $\text{cone}(e_1 + e_2, 2e_1 + e_2)$. The fan Δ does not contain both $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ and $\text{cone}(e_1 + e_2, 2e_1 + e_2)$; otherwise $k_{\text{cone}(e_1 + e_2, e_1 + 2e_2)}(2e_1 + e_2) = 2 > 1 = k(2e_1 + e_2)$. Observe that if Δ contains $\text{cone}(e_1 + e_2, 2e_1 + e_2)$, then Z is isomorphic to a variety whose fan contains $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ by the isomorphism induced by the automorphism of $\chi_*(S)$ that exchanges e_1 and e_2 . So the claim is proved.

Because of the non-singularity of Z , Δ contains a cone $\sigma = \text{cone}(e_1 + me_2, e_2)$ for a suitable integer m ; we want to show that Z is Z_m . Let Z' be the open toric subvariety of Z whose fan Δ' is $\Delta \setminus \{\text{cone}(e_1 + me_2, e_2), \text{cone}(e_2)\}$.

We claim that, for each integer $r \geq 1$, there is a unique variety \tilde{Z}'_r with the two following properties: 1) the fan $\tilde{\Delta}'_r$ of \tilde{Z}'_r has support $\text{cone}(e_1, e_1 + re_2)$; 2) \tilde{Z}'_r is an open subvariety of a toric variety \tilde{Z}_r with nef anticanonical bundle and birationally proper over \mathbb{A}^2 . In particular, the anticanonical divisor of \tilde{Z}'_r is nef. The open subvariety Z'_r of Z_r whose fan is $\Delta_r \setminus \{\text{cone}(e_1 + re_2, e_2), \text{cone}(e_2)\}$, satisfies these properties. So it is sufficient to prove the claim.

We show the claim for induction on r . We have already verified the basis of induction. Let \tilde{Z}'_r be a variety that satisfies the hypotheses of the claim and let σ' be the unique cone in $\tilde{\Delta}'_r(2)$ which contains $e_1 + re_2$. Because of the inductive hypothesis it is sufficient to show that $\sigma' = \text{cone}(e_1 + re_2, e_1 + (r-1)e_2)$.

Let k be the function associated to the anticanonical bundle of a fixed \tilde{Z}_r . Let $k_{\sigma'} = x_1e_1^* + x_2e_2^*$, then $1 = k_{\sigma'}(e_1 + re_2) = x_1 + rx_2$ and $x_i = k_{\sigma'}(e_i) \leq 1$ for each i , so the unique possibilities for $k_{\sigma'}$ are e_1^* and $-(r-1)e_1^* + e_2^*$. Write $\sigma' = \text{cone}(e_1 + re_2, v)$ with $v = c_1e_1 + c_2e_2$ primitive. If $k_{\sigma'} = -(r-1)e_1^* + e_2^*$, then $c_2 = (r-1)c_1 + 1$ because $(-(r-1)e_1^* + e_2^*)(v) = 1$. Because of the non-singularity of Z we have $c_1 - 1 = \pm 1$, so there are two possibilities: either $\sigma' = \text{cone}(e_1 + re_2, e_2)$ or $\sigma' = \text{cone}(e_1 + re_2, 2e_1 + (2r-1)e_2)$. We exclude the first one because e_2 does not belong to $|\tilde{\Delta}'_r|$. We exclude also the second one because $k_{\text{cone}(e_1 + re_2, 2e_1 + (2r-1)e_2)}(2e_1 + (2r-1)e_2) = 2 > k(v)$. If $k_{\sigma'} = e_1^*$, then $c_1 = k_{\sigma'}(v) = 1$. Because of the smoothness of Z we have $c_2 - r = \pm 1$. Again, we exclude $e_1 + (r+1)e_2$ because it does not belong to $|\tilde{\Delta}'_r|$. Thus $\sigma' = \text{cone}(e_1 + re_2, e_1 + (r-1)e_2)$. \square

Suppose $\Delta_X[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + re_2, e_2\}$. Then $-K_X$ is nef if and only if $-2\rho + 2\rho_0 + e_1^*$ and $-2\rho + 2\rho_0 - (r-1)e_1^* + e_2^*$ are antidominant. In such a case, if θ is decomposable, then the sum of these weights is regular, so $-K_X$ is big. If $r = 1$ and the previous weights are regular, then $-K_X$ is ample. Remark that if $r = 1$ and θ is decomposable, such weights are always regular.

III) If X is not toroidal and θ is indecomposable, let X' be as in Lemma 5.4. By Lemma 5.1 and Theorem 5.1, $R_{G,\theta} = A_2$, $H = G^\theta$ and $\Delta_{X'}[p]$ contains properly $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$, up to reindexing. By Lemma 5.5 we have two

possibilities for $\Delta_{X'}[p]$: i) $\{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -5\omega_1^\vee - 2\omega_2^\vee, \dots, -(r+3)\omega_1^\vee - r\omega_2^\vee, -3\omega_1^\vee\}$ and ii) $\{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -7\omega_1^\vee - \omega_2^\vee, \dots, -(3r+1)\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$.

Recall that $a_i = (2\rho - 2\rho_0)(\alpha_i^\vee)$. In the first case k is linear over $\text{cone}(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee)$, $\sigma := \text{cone}(-\omega_1^\vee - \omega_2^\vee, -(r+3)\omega_1^\vee - r\omega_2^\vee)$ and $\text{cone}(-(r+3)\omega_1^\vee - r\omega_2^\vee, -3\omega_1^\vee)$. The unique non-trivial condition is $k_\sigma(\alpha_1^\vee) = r \leq a_1$. In the second case k is linear over $\text{cone}(\omega_1^\vee, -\omega_1^\vee - \omega_2^\vee)$, $\text{cone}(-\omega_1^\vee - \omega_2^\vee, -(3r+1)\omega_1^\vee - \omega_2^\vee)$ and $\sigma := \text{cone}(-(3r+1)\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee)$. The unique non-trivial condition is $k_\sigma(\alpha_2^\vee) = 2r - 1 \leq a_2$ (or equivalently $\tilde{k}_\sigma(\alpha_2^\vee) \leq 0$). If $r = 1$, then $\theta = -id$ over $\chi_*(T)$ so that $-K_X$ is not ample.

IV) Finally, suppose X non toroidal and θ decomposable. Suppose also X_0 smooth; the case X_0 singular is very similar. By the proof of Theorem 5.1, $\rho(D(X))$ cannot be $\rho(D(G/H))$. If $\rho(D(X)) = \alpha_1^\vee$, then, by the local factoriality of X , $e_1 = -\alpha_1^\vee$ and there is $r \in \mathbb{Z}^{>0}$ such that $\sigma := \text{cone}(\alpha_1^\vee, v' = -r\alpha_1^\vee + e_2) \in \Delta$. We apply Lemma 5.4 to the maximal open toroidal subvariety of X (whose colored fan has support $\text{cone}(-\alpha_1^\vee, v')$). Then, by Lemma 5.5, $\Delta[p]$ has to be $\{-\alpha_1^\vee, v', -\alpha_1^\vee + v', -\alpha_1^\vee + 2v', \dots, -\alpha_1^\vee + sv', \alpha_1^\vee\}$, $\{-\alpha_1^\vee, v', -\alpha_1^\vee + v', -2\alpha_1^\vee + v', \dots, -s\alpha_1^\vee + v', \alpha_1^\vee\}$ or $\{-\alpha_1^\vee, v', \alpha_1^\vee\}$.

In the first case, k is linear over $\text{cone}(\alpha_1^\vee, v')$, $\sigma_1 := \text{cone}(v'_2 - \alpha_1^\vee + sv')$ and $\sigma_2 := \text{cone}(-\alpha_1^\vee + sv', -\alpha_1^\vee)$. The non-trivial conditions are $\tilde{k}_{\sigma_1}(\alpha_1^\vee) \leq 0$ and $\tilde{k}_{\sigma_2}(\alpha_2^\vee) \leq 0$ (or equivalently $s \leq \tilde{m}_1 + 1$ and $r \leq \tilde{m}_2$). If $s = 1$, then $r > m_2$ because $-K_X$ is not ample. In the second case we can suppose $s \geq 2$; k is linear over $\text{cone}(\alpha_1^\vee, v')$, $\sigma_1 := \text{cone}(v', -s\alpha_1^\vee + v')$ and $\sigma_2 := \text{cone}(-s\alpha_1^\vee + v', -\alpha_1^\vee)$. The unique non-trivial condition is $\tilde{k}_{\sigma_2}(\alpha_2^\vee) \leq 0$, or, equivalently, $r + s - 1 \leq \tilde{m}_2$. In the first two cases $\tilde{k}_{\sigma_1}(\alpha_2^\vee) < 0$ and $\tilde{k}_{\sigma_2}(\alpha_1^\vee) < 0$, so $-\tilde{K}_X$ is big. In the last case, we proceed as in the proof of Theorem 5.1, obtaining $m_2 + 1 < r \leq \tilde{m}_2 + 1$. Moreover, $\tilde{k}_{\text{cone}(\alpha_1^\vee, v')}(\alpha_2^\vee) < 0$ and $\tilde{k}_{\text{cone}(-\alpha_1^\vee, v')}(\alpha_1^\vee) < 0$, so $-K_X$ is big.

We can study the smoothness of all the previous varieties as in the proof of Theorem 5.1. Remark that if $H = G^\theta$, θ is decomposable, $\sharp\rho^{-1}(\alpha_1^\vee) = 2$ and $(C(Y), D(Y)) = (\text{cone}(\alpha_1^\vee, -r\alpha_1^\vee - \alpha_2^\vee), F)$, then Y is smooth if and only if $\sharp F = 1$ and α_1 is not exceptional (but Y cannot be an open subvarieties of a Fano variety). The symmetric varieties in the statement are all projective. Indeed, if $\Delta_X[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + (s-1)e_2, e_1 + se_2, e_2\}$, then the following divisor is ample: $-pK_X + \sum_{i=1}^s i^2 D_{e_1 + ie_2} + m \sum_{D(G/H) \setminus D(X)} D$ with $p, m \gg 0$. Indeed the piecewise linear function associated to $\sum_{i=1}^s i^2 D_{e_1 + ie_2}$ is strictly convex on $\{C \in \Delta_X : C \subset \text{cone}(e_1, e_1 + s_2)\}$. Moreover, k_X is strictly convex on the fan with maximal cones $\text{cone}(e_1, e_1 + s_2)$ and $\text{cone}(e_1 + s_2, e_2)$. The other cases are similar to this one.

5.3 Symmetric Fano varieties with G reductive

In this section we consider the 2-rank locally factorial Fano symmetric varieties over which acts a group G which is only reductive. If G is a torus, then X is the projective space. So, we can suppose $G = G' \times \mathbb{C}^*$ with $H \cap \mathbb{C}^* = (\mathbb{C}^*)^\theta = \{\pm id\}$. Write $\chi_*(\mathbb{C}^*/\{\pm id\}) = \mathbb{Z}f$.

If $R_{G, \theta} = BC_1$, then $H = G^\theta$ and $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\alpha^\vee$. Instead, if $R_{G, \theta} = A_1$ there are three possibilities: 1) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\alpha^\vee$ and $H = G^\theta$; 2) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\frac{1}{2}\alpha^\vee$ and $H = N_{G'}(G^\theta) \times \{\pm 1\}$; 3) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\frac{\alpha^\vee + f}{2}$. In the last case H is generated by G^θ and by $n_1 n_2$, where $n_1 \in N_{G'}((G')^\theta) \setminus (G')^\theta$ and

| G/H | θ | $\Delta[p]$ |
|--|----------|--|
| PSL_3 | (A_2) | $\#$ |
| SL_3 | (A_2) | $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee, -5\omega_i^\vee - 2\omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2$ |
| SO_5 | (B_2) | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| $Spin_5$ | (B_2) | $\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, -2\omega_1^\vee - 2\omega_2^\vee\}$ |
| G_2 | (G_2) | $\#$ |
| $SL_3/N(SO_3)$ | AI | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| SL_3/SO_3 | AI | $\{\alpha_i^\vee, -3\omega_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee\}, i = 1, 2$ |
| $SL_6/N(Sp_6)$ | AII | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -2\omega_i^\vee - \omega_{i^c}^\vee\}, i = 1, 2$ |
| SL_6/Sp_6 | AII | $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(r+3)\omega_i^\vee - r\omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2; r = 2, 3, 4$ $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee, -7\omega_i^\vee - \omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2$ |
| $SL_{n+1}/S(GL_2 \times GL_{n-1})$ $n \geq 4$ | $AIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq n-2$ $\{\alpha_2^\vee, -\omega_2^\vee\}$ |
| $SL_4/N(S(GL_2 \times GL_2))$ | $AIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ |
| $SL_4/S(GL_2 \times GL_2)$ | $AIII$ | $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ $\{\alpha_2^\vee, -2\omega_2^\vee\}$ |
| $SO_{2n+1}/S(O_2 \times O_{2n-1})$ | BI | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r \leq n-2; n \geq 3$ |
| $SO_5/S(O_2 \times O_3)$ | BI | $\#$ |
| $SO_{2n+1}/$ $(SO_2 \times SO_{2n-1})$ | BI | $\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, \dots, -2r\omega_1^\vee - \omega_2^\vee\}, r \leq n-1$ $\{\alpha_1^\vee, -\omega_1^\vee\}$ |
| $Sp_{2n}/(Sp_4 \times Sp_{2n-4}),$ $n > 4$ | CII | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq n-5$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ |
| $Sp_8/N(Sp_4 \times Sp_4)$ | CII | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ |
| $Sp_8/(Sp_4 \times Sp_4)$ | CII | $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee, \dots, -r\omega_1^\vee - 2\omega_2^\vee\}, 2 \leq r \leq 3$ $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee\}$ |
| $SO_{2n}/S(O_2 \times O_{2n-2})$ | DI | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r \leq n-2$ |
| $SO_{2n}/(SO_2 \times SO_{2n-2})$ | DI | $\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, \dots, -2r\omega_1^\vee - \omega_2^\vee\}, r \leq n-2$ $\{\alpha_1^\vee, -2\omega_1^\vee\}$ |
| $SO_8/N(SO_4)$ | $DIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ |
| SO_8/SO_4 | $DIII$ | $\{-\omega_1^\vee, -2\omega_2^\vee, \dots, -\omega_1^\vee - 2r\omega_2^\vee\}, r = 1, 2$ $\{\alpha_2^\vee, -2\omega_2^\vee\}$ |
| SO_{10}/SO_5 | $DIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r = 2, 3$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ $\{\alpha_2^\vee, -\omega_2^\vee\}$ |
| $E_6/(D_5 \times \mathbb{C}^*)$ | $EIII$ | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq 5$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -\omega_1^\vee - r\omega_2^\vee\}, 2 \leq r \leq 3$ $\{\alpha_2^\vee, -\omega_2^\vee\}$ |
| $E_6/N(F_4)$ | EIV | $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -r\omega_i^\vee - \omega_{i^c}^\vee\}, i = 1, 2; r = 2, 3, 4$ |
| E_6/F_4 | EIV | $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(r+3)\omega_i^\vee - r\omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2; r = 2, \dots, 8$ $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(3r+1)\omega_i^\vee - \omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2; r = 2, 3, 4$ |
| $G_2/(A_1 \times A_1)$ | G | $\{-\omega_1^\vee, -\omega_2^\vee\}$ |

Figure 7: quasi-Fano indecomposable symmetric varieties with rank 2

$n_2 \in \mathbb{C}^*$ has order four; in particular $[G^\theta : H] = 2$. Let e be the primitive positive multiple of $-\alpha^\vee$ and let $\{e^*, f^*\}$ be the dual basis of $\{e, f\}$.

Theorem 5.3 *Let G/H be a symmetric space of rank two, such that G is neither semisimple nor abelian. As before, write $\psi(r) = -2\rho + 2\rho_0 - re^*$ and $m_1 := \max\{r : \psi(r) < 0\}$. The Fano locally factorial compactifications of G/H are the following ones:*

1. $\Delta[p] = \{f, -f, e + rf\}$ if $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$, $r \in \mathbb{Z}$, $r \leq m_1 + 1$ and $-r \leq m_1 + 1$;
2. $\Delta[p] = \{f, -f, e + rf, e + (r + 1)f\}$ if $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$, $r \in \mathbb{Z}$, $r \leq m_1$ and $-r \leq m_1 + 1$;
3. $\Delta[p] = \{\alpha^\vee, -f, -\alpha^\vee + f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\alpha^\vee - f\}$) if $\chi_*(S) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}f$ and $\sharp D(G/H) = 1$;
4. $\Delta[p] = \{\alpha^\vee, -f, -\alpha^\vee, -\alpha^\vee + f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\alpha^\vee, -\alpha^\vee - f\}$) if $\chi_*(S) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}f$ and $\sharp D(G/H) = 1$;
5. $\Delta[p] = \{f, -f, -\frac{1}{2}\alpha^\vee + \frac{2r+1}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$, $r \in \mathbb{Z}$, $r \leq \frac{m_1+1}{2}$ and $-r \leq \frac{m_1+3}{2}$;
6. $\Delta[p] = \{f, -f, -\frac{1}{2}\alpha^\vee + \frac{2r+1}{2}f, -\frac{1}{2}\alpha^\vee + \frac{2r+3}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$, $r \in \mathbb{Z}$, $r \leq \frac{m_1-1}{2}$ and $-r \leq \frac{m_1+3}{2}$;
7. $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f, -\alpha^\vee\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$;
8. $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$;
9. $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$) if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$;
10. $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$) if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee + f}{2})$.

The only singular varieties are the ones in the cases 3) and 4).

Observe that X cannot be simple because the valuation cone is not strictly convex. We begin with a lemma similar to Lemma 5.2.

Lemma 5.6 *Let X be a Fano locally factorial symmetric variety with G as before. If X is toroidal, there are at most four colored 1-cones. Otherwise, there are at most three colored 1-cones.*

Proof of Lemma 5.6. First suppose X toroidal. Then $k(\pm f) = 1$. Let $\sigma \in \Delta(2)$ be a cone which does not contain neither f nor $-f$, then $k_\sigma(\pm f) \leq 0$. So k_σ is a multiple of e^* . If there is another cone $\sigma' \in \Delta(2)$ with the same properties (and such that $\dim(\sigma \cap \sigma') = 1$), then $k_\sigma = k_{\sigma'}$. If X is not toroidal, we can study the maximal open toroidal subvariety X' of X as in Lemma 5.2 (because $|\mathcal{F}(X')|$ is strictly convex). \square

Proof of Theorem 5.3. We have to request, as in Theorem 5.1, that $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$, but in this case $\sharp\rho(D(X)) \leq 1$. If X is not toroidal, then

$e = -\alpha^\vee$, because there is an appropriate $\text{cone}(\alpha^\vee, v)$ in $\Delta(2)$. Suppose first that $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$. We have to consider the following cases: 1) If $\Delta[p] = \{f, -f, v\}$, then $v = e + rf$ because $\{f, v\}$ is a basis of $\chi_*(S)$. We have to impose that $\tilde{k}_{\text{cone}(f, e+rf)}(\alpha^\vee) = \psi(r-1)(\alpha^\vee) < 0$ and that $\tilde{k}_{\text{cone}(-f, e+rf)}(\alpha^\vee) = \psi(-r-1)(\alpha^\vee) < 0$. 2) Suppose $\Delta[p] = \{f, -f, v_1, v_2\}$. As before $v_i = e + x_i f$; moreover $x_1 = x_2 \pm 1$ because $\{v_1, v_2\}$ is a basis of $\chi_*(S)$. Suppose $x_1 = x_2 + 1$. We have to impose $\tilde{k}_{\text{cone}(f, e+(x_2+1)f)}(\alpha^\vee) = \psi(x_2)(\alpha^\vee) < 0$ and $\tilde{k}_{\text{cone}(-f, e+x_2f)}(\alpha^\vee) = \psi(-x_2-1)(\alpha^\vee) < 0$.

3) Suppose $\Delta[p] = \{\alpha^\vee, -f, v\}$. Then $v = -\alpha^\vee + rf = -m\alpha^\vee \pm f$ because X is smooth. But $-f$ is not contained in $\text{cone}(\alpha^\vee, v)$, thus $v = -\alpha^\vee + f$. 4) Suppose $\Delta[p] = \{\alpha^\vee, -f, v_1, v_2\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \sigma := \text{cone}(v_1, v_2), \text{cone}(-f, v_2)\}$. Then $v_1 = -m\alpha^\vee + f$ and $v_2 = -\alpha^\vee + rf$. Observe that $\mathbb{R}^{\geq 0}v_1 = \mathbb{R}^{\geq 0}(-\alpha^\vee + \frac{1}{m}f)$, so $r \leq 0$. Furthermore $v_2 - rv_1 = -(1 - rm)\alpha^\vee = \pm\alpha^\vee$, so mr is 0 or 2. Thus $v_2 = -\alpha^\vee$. Moreover $h_\sigma(-f) = m - 1 < 1$, so $v_1 = -\alpha^\vee + f$.

5) Suppose $\Delta[p] = \{\alpha^\vee, v_1, v_2\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \text{cone}(v_1, v_2), \text{cone}(\alpha^\vee, v_2)\}$. Thus $v_1 = -r\alpha^\vee + f$ and $v_2 = -m\alpha^\vee - f$ with $r, m > 0$. Moreover, $v_1 + v_2 = -(r+m)\alpha^\vee = \pm\alpha^\vee$, a contradiction. 6) Suppose $\Delta[p] = \{\alpha^\vee, v_1, v_2, v_3\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \sigma := \text{cone}(v_1, v_2), \text{cone}(v_2, v_3), \text{cone}(\alpha^\vee, v_3)\}$. Write $v_2 = -x\alpha^\vee + yf$. By the smoothness of X , $v_1 = -r\alpha^\vee + f$, $v_3 = -m\alpha^\vee - f$, $x = -my \pm 1$ and $x = +ry \pm 1$. The last two conditions, plus $x > 0$ imply that $v_2 = -\alpha^\vee$. Moreover $k_\sigma(v_3) = m + r - 1 < 1$, a contradiction.

Now, suppose $\chi_*(S) = \mathbb{Z}\frac{f+\alpha^\vee}{2} \oplus \mathbb{Z}f$. 1) The toroidal case can be studied as before. 2) Suppose that $\Delta(2)$ contains two cones $\sigma_\pm := \text{cone}(v_\pm, \alpha^\vee)$. Let $u = \frac{1}{2}\alpha^\vee + \frac{1}{2}f$. We have $v_+ = -\frac{2m+1}{2}\alpha^\vee + \frac{1}{2}f = -(2m+1)u + (m+1)f$ and $v_- = -\frac{2r+1}{2}\alpha^\vee - \frac{1}{2}f = -(2r+1)u + rf$ with $m, r \geq 0$. First, suppose that there is another $v = -xu + yf$ in $\Delta[p]$; here $x > 0$. By the smoothness, we have $x(m+1) = y(2m+1) \pm 1$ and $xr = y(2r+1) \pm 1$. Thus $x[(m+1)(2r+1) - r(2m+1)] = \pm 1 \mp 1$, so $x(m+r+1) = 2$, $x(m+1) = y(2m+1) + 1$ and $xr = y(2r+1) - 1$. If $x = 1$, then the previous three equations are not compatible. Instead, if $x = 2$ then $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\alpha^\vee, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$. Next, suppose that $\Delta[p] = \{\alpha^\vee, v_+, v_-\}$; in particular $\{v_+, v_-\}$ is a basis. Thus $v_+ + v_- = -(m+r+1)\alpha^\vee = \pm\alpha^\vee$, so $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$.

3) Finally, suppose that $\Delta[p]$ contains α^\vee and $-f$. Then Δ contains $\sigma = \text{cone}(\alpha^\vee, v_1)$ and $\text{cone}(-f, v_2)$, with $v_1 = -\frac{2m+1}{2}\alpha^\vee + \frac{1}{2}f$, $m \geq 0$ and $v_2 = -\frac{1}{2}\alpha^\vee - \frac{2r+1}{2}f$. If $v_1 = v_2$, then $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f\}$. Otherwise, by Lemma 5.6, Δ contains $\text{cone}(v_1, v_2)$. Furthermore, $(2r+1)v_1 + v_2 = \pm\alpha^\vee$, hence $(2r+1)(2m+1)$ is 1 or -3. But $r \geq 0$ because $v_2 \notin \sigma$, so $v_1 = -\frac{1}{2}\alpha^\vee + f$ and $v_2 = -\frac{1}{2}\alpha^\vee - f$.

References

- [A83] D. N. Akhiezer, Equivariant completions of homogeneous algebraic varieties by homogeneous divisor, *Ann. global Anal. Geom.* **1**, (1983), no. 1, 49-78.
- [AlBr04] V. Alexeev; M. Brion, Boundedness of spherical Fano varieties. The Fano Conference, 69-80, Univ. Torino, Turin, 2004.

- [Bo68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV, V, VI, Actualités Scientifiques et Industrielles 1337*, Hermann Paris 1968.
- [Br89] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphérique, *Duke Math. J.* **58**, (1989), no. 2, 397-424.
- [Br97a] M. Brion, Variétés sphériques, Notes de cours, 1997.
- [Br97b] M. Brion, Curves and divisors in spherical varieties, in Algebraic groups and Lie Groups *Austral. Math. Soc. Lect. Ser.* **9**, (1997), 21-34.
- [ChMa03] R. Chirivì and A. Maffei, The ring of sections of a complete symmetric variety, *J. Algebra*, **261**, (2003), no. 2, 310-326.
- [dCoPr83] C. De Concini and C. Procesi, Complete symmetric varieties, in Invariant Theory, "Lecture notes in Math" **996**, 1-44, Springer-Verlag, New York, 1983.
- [dCoPr85] C. De Concini and C. Procesi, Complete symmetric varieties II, Intersection theory, *Adv. Stud. Pure Math.* **6** (1985), 481-513.
- [Kn91] F. Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225-249, Manoj Prakashan, Madras, 1991.
- [Ru07] A. Ruzzi, Smooth projective symmetric varieties with Picard number equal to one, *Internat. J. Math.* **22**, (2011), n.2, 145-177.
- [Ru09] A. Ruzzi, Effective and Big Divisors on a Projective Symmetric Variety, arXiv: math.arXiv:0906.5226.
- [Ru10] A. Ruzzi, Geometrical description of smooth projective symmetric varieties with Picard number one, *Transform. groups* **15** (2010), no. 1, 201-226.
- [T06] D. A. Timashev, Homogeneous spaces and equivariant embeddings, preprint, 2006, arXiv: math.AG/0602228.
- [Vu90] T. Vust, Plongements d'espace symétriques algébriques: une classification, *Ann. Scuola Norm. Pisa Cl. Sci. (4)* **17**, (1990), n. 2, 165-195.
- [Wa72] G. Warner, Harmonic analysis on semi-simple Lie group, Springer-Verlag, New York (1972).