



HAL
open science

Effective and big divisors on a projective symmetric variety

Alessandro Ruzzi

► **To cite this version:**

| Alessandro Ruzzi. Effective and big divisors on a projective symmetric variety. 2009. hal-00578614

HAL Id: hal-00578614

<https://hal.science/hal-00578614>

Preprint submitted on 21 Mar 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EFFECTIVE AND BIG DIVISORS ON A PROJECTIVE SYMMETRIC VARIETY

ALESSANDRO RUZZI

ABSTRACT. We describe the effective and the big cones of a projective symmetric variety. Moreover, we give a necessary and sufficient combinatorial criterion for the bigness of a nef divisor on a projective symmetric variety. When the divisor is G -stable, such criterion has an explicit geometric interpretation. Finally, we describe the spherical closure of a symmetric subgroup.

Keywords: Symmetric varieties, Big divisors.

Mathematics Subject Classification 2000: 14L30, 14C20, (14M17).

INTRODUCTION

Brion gave a description of the Picard group of a spherical variety in [Br89]. He also found necessary and sufficient conditions for the ampleness and global generation of a line bundle. From these conditions follows that a line bundle is nef if and only if it is globally generated. It is natural to ask what are the conditions on a line bundle to be effective, respectively big. It is known that the effective cone is closed, polyhedral and, if the variety is \mathbb{Q} -factorial, generated by the classes of the B -stable prime divisors. But in general it is hard to say which are the B -stable prime divisors whose classes generate an extremal ray of the effective cone. In the very special case of projective homogeneous varieties, the big cone coincides with the ample cone. More generally, the case of wonderful varieties is studied in [Br07]. For any normal projective variety, there is a very useful criterion for a nef divisor D to be big: D is big if and only if its volume $D^{\dim X}$ is strictly positive (see [La04], Theorem 2.2.16). In the case of a toric T -variety Z this criterion has a more combinatorial version. A T -stable, nef divisor D can be identified with a certain polytope with integral vertices, its moment polytope; moreover, the volume of D is equal to $(\dim Z)!$ times the volume of this polytope. In particular, D is big if and only if its moment polytope has positive volume. See also [FS09] for a partial study of the big cone of some toric varieties.

We are interested to study the bigness of Cartier \mathbb{Q} -divisors on symmetric varieties (over which acts a semisimple group). First, we describe explicitly the effective cone; we determine also when the classes of two B -stable prime divisors are proportional. When the variety is \mathbb{Q} -factorial, we find the conditions for the class of a B -stable prime divisor to generate an extremal ray of the effective cone (see Theorem 3.1, Theorem 3.2 and Corollary 3.1).

Given a \mathbb{Q} -factorial, projective symmetric variety X , we will give an explicit description of its big cone using the description of $Eff(X)$ and the fact the $Big(X)$ is the interior part of $Eff(X)$ (see Theorem 4.1). We will give also a combinatorial version of the cited criterion on the volume of a nef \mathbb{Q} -divisor D , stated in terms of the T -weights of the fibres of $\mathcal{O}(D)$ over the B -stable points (see Theorem 4.2). The idea of the proof is the following. Up to take a lifting and up to linear equivalence, one can reduce itself to study G -stable divisors on a projective toroidal symmetric variety, i.e. a projective variety such that the closure of none B -stable divisor of the open G -orbit contains a G -orbit. Each projective toroidal symmetric variety X

(with fixed open G -orbit) contains a projective toric variety Z^c which determines X uniquely. We prove that the restriction of a big \mathbb{Q} -divisor D of X to Z^c is always big. If the divisor is G -stable, then this condition is also sufficient (see Proposition 4.1). One can show that the subspace of $\text{Pic}(X)_{\mathbb{Q}}$ generated by the classes of G -stable divisors is a complement to the kernel of the restriction $\text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(Z^c)_{\mathbb{Q}}$. Then, we use the combinatorial description of $H^0(Z^c, \mathcal{O}(D)|_{Z^c})$ to prove Theorem 4.2.

We describe also the spherical closure of a symmetric subgroup H in Proposition 2.1. We will use such description to give a condition so that the class of a G -stable prime divisor is proportional to the class of a B -stable, but not G -stable, divisor (see Corollary 2.1 and Theorem 3.2). Given a symmetric space G/H , H is called a symmetric subgroup and its spherical closure is defined, after Luna, as the subgroup \overline{H}^{sph} of $N_G(H)$ which stabilizes all the B -stable prime divisors of G/H . A subgroup H of G is wonderful (resp. spherical) if G/H has a wonderful compactification (resp. is spherical). The definition of spherical closure is useful to associate in a natural way a wonderful subgroup of G to any spherical subgroup of G . Moreover, $N_G(H)/H$ is isomorphic to the group of G -equivariant automorphisms of G/H , so \overline{H}^{sph} can be thought as a group of automorphisms.

1. INTRODUCTION AND NOTATIONS

In this section we introduce the necessary notations. The reader interested to the embedding theory of spherical varieties can see [LuVu], [Kn91], [Br97a] or [T06]. In [Vu90], this theory is explained in the particular case of symmetric varieties.

1.1. First definitions. Let G be a connected semisimple algebraic group over algebraic closed field \mathbf{k} of characteristic zero and let θ be an involution of G . Let H be a closed subgroup of G such that $G^\theta \subset H \subset N_G(G^\theta)$, then we say that G/H is a symmetric space and that H is a symmetric subgroup. We can assume G simply connected (see [Vu90], §2.1). An equivariant embedding of G/H is the data of a G -variety X together with an equivariant open embedding $\varphi : G/H \hookrightarrow X$, in particular $\varphi(g \cdot x) = g \cdot \varphi(x)$ for each $x \in G/H$. A normal G -variety is called a spherical variety if it contains a dense orbit under the action of an arbitrarily chosen Borel subgroup of G . One can show that every normal equivariant embedding of G/H is spherical (see [dCoPr83], Proposition 1.3); we call it a symmetric variety. The most important example of symmetric space is the one of a semisimple (or more generally reductive) group G seen as $(G \times G)$ -variety.

We say that a subtorus of G is split if $\theta(t) = t^{-1}$ for all its elements t ; moreover it is a maximal split torus if has maximal dimension. We say that any maximal torus containing a maximal split torus is maximally split. Any maximally split torus is θ stable; moreover they are all conjugate under G^θ (see [Vu74], Proposition 2 (iv) and Proposition 6). We fix arbitrarily a maximally split torus T containing a maximal split torus T^1 . Let R_G be the root system of G with respect to T . We can choose a Borel subgroup B containing T such that, given any positive root α with respect to B , either $\theta(\alpha) = \alpha$ or $\theta(\alpha)$ is negative. Moreover, BH is dense in G (see [dCoPr83], Lemma 1.2 and Proposition 1.3).

1.2. Colored fans. We want to describe the Picard group of a symmetric variety. Before doing this, we need to introduce some details about the classification of the symmetric varieties by their colored fans (this classification holds more generally for spherical varieties).

Let $D(G/H)$ be the set of colors of G/H , namely the set of B -stable prime divisors of G/H . They are the irreducible components of $(G/H) \setminus (BH/H)$ because BH/H is affine and open. We say that a spherical variety is simple if it contains

one closed orbit. Let X be a simple symmetric variety with closed orbit Y . We define the set of colors of X as the subset $\mathcal{F}(X)$ of $D(G/H)$ consisting of the colors whose closure in X contains Y . To each prime divisor D of X , we can associate the normalized discrete valuation v_D of $\mathbb{C}(G/H)$ whose ring is the local ring $\mathcal{O}_{X,D}$. One can prove that D is G -stable if and only if v_D is G -invariant, i.e. $v_D(g \cdot f) = v_D(f)$ for each $g \in G$ and $f \in \mathbb{C}(G/H)$. Let \mathcal{N} be the set of all G -invariant valuations of $\mathbb{C}(G/H)$ taking values in \mathbb{Z} and let $\mathcal{N}(X)$ be the set of the valuations associated to the G -stable prime divisors of X . Observe that each irreducible component of $X \setminus (G/H)$ has codimension one, because G/H is affine. Let $S := T/T \cap H \simeq T \cdot (H/H)$. One can show that the group $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is isomorphic to the character group $\chi(S)$ of S (see [Vu90], §2.3); in particular, it is a free abelian group. We define the *rank* of G/H as the dimension of S . We can identify the dual group $\text{Hom}_{\mathbb{Z}}(\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*, \mathbb{Z})$ with the group $\chi_*(S)$ of one-parameter subgroups of S . The restriction map of valuations to $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is injective over \mathcal{N} (see [Kn91], Corollary 1.8), so we can identify \mathcal{N} with a subset of $\chi_*(S)$. We say that \mathcal{N} is the valuation monoid of G/H . Observe that \mathcal{N} is stable under addition (see, for example [Kn91], Lemma 5.1). For each color F , we define $\rho(F)$ as the restriction of v_F to $\chi(S)$. In general, the map $\rho : D(G/H) \rightarrow \chi_*(S)_{\mathbb{R}}$ is not injective. Let $\mathcal{C}(X)$ be the cone in $\chi_*(S)_{\mathbb{R}}$ generated by $\mathcal{N}(X)$ and $\rho(\mathcal{F}(X))$. A simple symmetric varieties is uniquely determined by its colored cone $(\mathcal{C}(X), \mathcal{F}(X))$ (see [Kn91], Theorem 3.1).

Let Y be a G -orbit of a symmetric variety X . The set $\{x \in X \mid \overline{G \cdot x} \supset Y\}$ is an open simple G -subvariety of X with closed orbit Y , because any spherical variety contains only finitely many G -orbits. Let $\{X_i\}$ be the set of open simple subvarieties of X and define the set of colors of X , $\mathcal{F}(X)$, as $\bigcup_{i \in I} \mathcal{F}(X_i)$. The family $\mathbf{F}(X) := \{(\mathcal{C}(X_i), \mathcal{F}(X_i))\}_{i \in I}$ is called the colored fan of X and determines completely X (see [Kn91], Theorem 3.3). Moreover X is complete if and only if \mathcal{N} is contained in the support $\bigcup_{i \in I} \mathcal{C}(X_i)$ of $\mathbf{F}(X)$ (see [Kn91], Theorem 4.2).

If one allows G to be reductive, then the toric varieties are a special case of symmetric varieties. If X is a toric variety, then $D(G/H)$ is empty and we need only to consider the fan $\mathbf{F}^f = \{(\mathcal{C}(X_i))\}_{i \in I}$ associated to the colored fan of X (actually the classification of spherical varieties by colored fans is a generalization of the classification of toric varieties by fans). One can show that, fixed any symmetric space G/H such that ρ is injective (for example if G/H is a group), the symmetric varieties with open orbit G/H are classified by the fans $\mathbf{F}^f = \{(\mathcal{C}(X_i))\}_{i \in I}$.

1.3. Restricted root system. To describe the sets N and $\rho(D(G/H))$, we need to associate a root system to G/H . We can identify $\chi(T^1)_{\mathbb{R}}$ with $\chi(S)_{\mathbb{R}}$ because $\chi(S)$ has finite index in $\chi(T^1)$. We call again θ the involution induced on $\chi(T)_{\mathbb{R}}$. The inclusion $T^1 \subset T$ induces an isomorphism of $\chi(T^1)_{\mathbb{R}}$ with the (-1) -eigenspace of $\chi(T)_{\mathbb{R}}$ under the action of θ (see [T06], §26). Denote by W_G the Weyl group of G (with respect to T). We can identify $\chi(T^1)_{\mathbb{R}}$ with its dual $\chi_*(T^1)_{\mathbb{R}}$ by the restriction (\cdot, \cdot) of the Killing form to $\chi(T^1)_{\mathbb{R}}$. The set $R_{G,\theta} := \{\beta - \theta(\beta) \mid \beta \in R_G\} \setminus \{0\}$ is a root system in $\chi(S)_{\mathbb{R}}$ (see [Vu90], §2.3 Lemme), which we call the restricted root system of (G, θ) ; we call the non zero $\beta - \theta(\beta)$ the restricted roots. We denote by $\alpha_1, \dots, \alpha_s$ the elements of the basis $\overline{R}_{G,\theta} := \{\beta - \theta(\beta) \mid \beta \in R_G \text{ simple}\} \setminus \{0\}$ of $R_{G,\theta}$. We denote by $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$ the dual basis of the restricted coroot system $R_{G,\theta}^\vee$, i.e. the dual root system of $R_{G,\theta}$. Let C^- be the negative Weyl chamber of $R_{G,\theta}$ (in $\chi_*(S)_{\mathbb{Q}}$). Given a dominant weight λ of G , we denote by $V(\lambda)$ the irreducible representation of highest weight λ . See [ChMa03], Theorem 2.3 or [T06], Proposition 26.4 for an explicit description of the dominant weights of $R_{G,\theta}$. They are called spherical weights and are also dominant weights of R_G . Let $W_{G,\theta}$ be the

Weyl group of $R_{G,\theta}$; it is isomorphic to $N_{H^0}(T_1)/Z_{H^0}(T^1)$ and to $N_G(T_1)/Z_G(T^1)$ (see Proposition 26.2 in [T06]).

1.4. The sets \mathcal{N} and $D(G/H)$. The set \mathcal{N} is equal to $C^- \cap \chi_*(S)$; in particular, it consists of the lattice vectors of the rational, polyhedral, convex cone $C^- = \text{cone}(N)$. The set $\rho(D(G/H))$ is equal to $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$ and, for each i , the fibre $\rho^{-1}(\alpha_i^\vee)$ contains at most 2 colors. In particular, the number of colors of a symmetric space is at least its rank. We say that (G, θ) is indecomposable if any normal, connected, θ -stable subgroup of G is trivial. In this case the number of colors of G/H is at most equal to rank of G/H plus one. If $|D(G/H)| > \text{rank}(G/H)$ and (G, θ) is indecomposable, we have two possibilities: 1) $G^\theta = H = N_G(G^\theta)$; 2) H is equal to G^θ and has index two in $N_G(G^\theta)$. In the last case any element of $N_G(G^\theta) \setminus G^\theta$ exchanges two colors. Because of the simply-connectedness of G , we can always write G as a direct product $\prod G_i$ of θ -stable, semisimple, normal subgroup G_i such that each (G_i, θ) is indecomposable. Moreover, the finite cover G/G^θ of G/H is the direct product $\prod_{i \in I} G_i/G_i^\theta$.

We say that a simple restricted root α is exceptional if $\rho^{-1}(\alpha^\vee)$ contains two colors and 2α is a restricted root. Moreover, we say that a symmetric variety is exceptional if there is an exceptional root. If moreover θ is indecomposable, then $G^\theta = H = N_G(G^\theta)$. Furthermore, ρ is injective if H is semisimple. Let $D(G/H)^H$ be the set of colors F such that $\rho^{-1}(\rho(F)) = \{F\}$. One can show that $\rho^{-1}(\rho(F)) = \{F\}$ if and only if the equation of $\pi^{-1}(F)$ in G is H -invariant, where $\pi: G \rightarrow G/H$ is the projection. We denote by F_α the sum of the colors in $\rho^{-1}(\alpha^\vee)$. If $\alpha^\vee \notin \rho(D(G/H)^H)$, we write $\rho^{-1}(\alpha^\vee) = \{F_\alpha^+, F_\alpha^-\}$, so $F_\alpha = F_\alpha^+ + F_\alpha^-$.

1.5. Toroidal symmetric varieties. Before to describe the Picard group, we want to define a special class of varieties. We say that a spherical variety is toroidal if $\mathcal{F}(X) = \emptyset$. There is a special toroidal completion of any symmetric space G/H because $N_G(H)/H$ is finite. This completion, called the standard completion X_0 , is the simple symmetric variety associated to $(\text{cone}(\mathcal{N}), \emptyset)$ and it is the unique simple completion of G/H which dominates all the other simple completions. When X_0 is smooth then it is a wonderful variety (by the definition of Luna). In particular, X_0 is smooth (and wonderful) if $H = N_G(G^\theta)$. This case is been defined and studied by De Concini and Procesi in [dCoPr83].

X_0 contains an affine toric S -variety Z_0 , which is a quotient of an affine space by a finite group. The toroidal varieties are the symmetric varieties which dominate the standard completion; they are in one-to-one correspondence with the S -toric variety which dominates Z_0 . The correspondence is obtained in the following way. The open set $U := X_0 \setminus \bigcup_{D(G/H)} \overline{F}$ is a B -stable affine set; let P be its stabilizer. U is P -isomorphic to $R_u P \times Z_0$, where $R_u P = \prod_{\beta > 0, \theta(\beta) \neq \beta} U_\beta$ is the unipotent radical of P and $\dim Z_0 = \text{rank } X_0$. To any toroidal variety X we associate the inverse image Z of Z_0 by the projection $X \rightarrow X_0$. Moreover, $X \setminus \bigcup_{D(G/H)} \overline{F}$ is P stable and is P -isomorphic to $R_u P \times Z$. The toroidal varieties are also in one-to-one correspondence with a class of complete toric variety in the following way. To a symmetric variety X , we associate the closure Z^c of Z in X ; Z^c is also the inverse image of Z_0^c . The T -variety Z^c can be covered by finitely many $N_{G^\theta}(T^1)$ -translated of Z ; thus Z^c is a S -toric variety, in particular it is normal. The fan of Z is the fan $\mathbf{F}^f(X)$ associated to the colored fan $\mathbf{F}(X)$, while the fan of Z^c consists of the translates of the cones of Z by the Weyl group $W_{G,\theta}$ of $R_{G,\theta}$.

Given a symmetric variety X there is a unique minimal toroidal variety X^{dec} , called the decoloration of X , which dominates X . If $\mathbf{F}(X) = \{(C_i, F_i)\}_{i \in I}$, the colored fan of X^{dec} is $\{(C_i \cap \text{cone}(N), \emptyset)\}_{i \in I}$.

1.6. Big divisors. Before to describe the Picard group of a symmetric variety, we define some general notions about the line bundle. The reader can see [La04] for more details. Let X be a (normal) projective algebraic variety over an algebraically closed field of characteristic zero. Let $CDiv(X)$ the group of Cartier divisors. Given two Cartier divisors D_1 and D_2 on X , we say that they are *numerically equivalent* if $D_1 \cdot C = D_2 \cdot C$ for each curve C on X (here \cdot is the intersection product). In such a case we say also that $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are numerically equivalent.

We define two generalization of an ample bundle. By Nakai's criterion (see [La04], Theorem 1.2.23), a Cartier divisor D is ample if and only if the intersection product $D^{\dim Y} \cdot Y$ is strictly positive for each subvariety Y (with $\dim Y > 0$). A first generalization is obtained weakening such property. Indeed, we say that a Cartier divisor D is nef (or numerically effective) if $D \cdot C \geq 0$ for each curve C in X . Remark that in definition we have used only subvarieties of dimension one. But, by Kleiman's Theorem (see [La04], Theorem 1.4.9), D is nef if and only if $D^{\dim Y} \cdot Y \geq 0$ for each subvariety Y (with $\dim Y > 0$). To define the second generalization of ample divisor we need to define the Itaka dimension of a divisor. Given a Cartier divisor D , let $E(D) := \{m \geq 0 : H^0(X, \mathcal{O}(mD)) \neq 0\}$; given any $m \in E(D)$ we have a rational map $\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}(mD)))$. If $E(D)$ is empty we define the *Itaka dimension* $\kappa(D)$ of D as $-\infty$. Otherwise we define $\kappa(D) := \max_{m \in E(D)} \{\dim \phi_m(X)\}$. Remarks that $\kappa(D)$ is equal at most to the dimension of X . When D is the canonical divisor, $\kappa(D)$ is also called the Kodaira dimension of X . We say that a Cartier divisor on X is big if and only if its Itaka dimension is equal to the dimension of X . Clearly an ample divisor is big.

We have some equivalent conditions for the bigness of a divisor. First, we recall a lemma.

Lemma 1.1 (see [La04], Corollary 2.1.38). . *Let D be a Cartier divisor on X and let $\kappa = \kappa(D)$. Then there are strictly positive constants a and A such that*

$$a \cdot m^\kappa \leq \dim H^0(X, \mathcal{O}(mD)) \leq A \cdot m^\kappa$$

for all sufficiently large $m \in E(D)$.

Proposition 1.1 (see [La04], Lemma 2.2.3 and Corollary 2.2.7). . *Let D be a Cartier divisor on X . The following conditions are equivalent:*

- D is big;
- there is a constant $C > 0$ such that $\dim H^0(X, \mathcal{O}(mD)) \geq C \cdot m^{\dim X}$ for all sufficiently large $m \in E(D)$;
- for each ample divisor A , there is $m > 0$ such that $mD - A$ is linearly equivalent to an effective divisor;
- for each ample divisor A , there is $m > 0$ such that $mD - A$ is numerically equivalent to an effective divisor.

When D is nef is a lot easier to verify if it is big:

Proposition 1.2 (see [La04], Theorem 2.2.16 and Corollary 1.4.41). *Let D be a nef divisor on X and let n be the dimension of X . Then D is big if and only if $\text{vol}(D) = D^n$ is strictly positive.*

One can define a \mathbb{Q} -divisor (respectively \mathbb{R} -divisor) as an element of $CDiv(X)_{\mathbb{Q}}$ (resp. of $CDiv(X)_{\mathbb{R}}$). All the previous definitions can easily extend to \mathbb{Q} -divisors. For example, we say that a \mathbb{Q} -divisor D is big if there is $m > 0$ such that mD is a big divisor. One can also extends such definitions to \mathbb{R} -divisors, but we do not do it here because it is a little more technical. Often one works in the quotient $\text{Pic}(X)_{\mathbb{R}} / \equiv$ of the real Picard group by the numerical equivalence for the following two reasons: such space is finite-dimensional (see [La04], Proposition 1.1.14) and the previous

definitions depends only by the numerical class of a divisor (see [La04], Corollary 1.2.20 and Corollary 2.2.7). Often, we will work with \mathbb{Q} -divisors by simplicity.

We define the following cones in $Pic(X)_{\mathbb{Q}}/\cong$:

- $Amp(X)$ is the cone generated by the classes of ample divisors;
- $Nef(X)$ is the cone generated by the classes of nef divisors;
- $Big(X)$ is the cone generated by the classes of big divisors;
- $Eff(X)$ is the cone generated by the classes of effective divisors;
- the pseudo-effective cone $PE(X)$ is the closure of $Big(X)$.

Such cone are related in the following way:

Proposition 1.3. *All the previous cone are convex. Moreover:*

- $Amp(X)$ and $Big(X)$ are open;
- $Nef(X)$ and $PE(X)$ are closed;
- [see citeLa04, Theorem 1.4.23] $Nef(X)$ is the closure of $Amp(X)$ and $Amp(X)$ is the interior part of $Nef(X)$;
- [see citeLa04, Theorem 2.2.26] $PE(X)$ is the closure of $Big(X)$ and $Big(X)$ is the interior part of $PE(X)$;
- $Big(X) \subset Eff(X) \subset PE(X)$.

We define $Eff(X)$ in the same way when X is only complete (and normal). Finally we say that X is \mathbb{Q} -factorial if its rational Picard group coincides with the rational class group; clearly the smooth varieties are \mathbb{Q} -factorial.

When X is a spherical variety, two Cartier divisors are numerically equivalent if and only if they are linearly equivalent (see for example [Br93], Corollaire 1.3), so we can omit the quotient by \cong . Moreover, any effective divisor is linearly equivalent to a B -stable effective divisor (see for example [Br93], Théorème 1.3). Thus, if X is \mathbb{Q} -factorial, the effective cone $Eff(X)$ is the closed polyhedral cone generated by the class of the B -stable prime divisors (there are a finite number of them); in particular $Eff(X)$ is equal to the pseudo-effective cone.

1.7. The Picard group. The class group of a symmetric variety is generated by the classes of the B -stable prime divisors with the relations $div(f)$, where $f \in \mathbb{C}(G/H)^{(B)}$. Indeed $Cl(BH/H) = Pic(BH/H)$ is trivial. Given $\omega \in \chi(S)$ we denote by f_{ω} the element of $\mathbb{C}(G/H)^{(B)}$ with weight ω and such that $f_{\omega}(H/H) = 1$. We denote by v_E the image of $E \in \mathcal{N}(X)$ in $\chi_*(S)$; vice versa, given an element ω of the image of $\mathcal{N}(X)$ in $C^- \cap \mathcal{N}$ we denote by E_{ω} the corresponding elements of $\mathcal{N}(X)$.

A Weil divisor $D = \sum_{F \in D(G/H)} a_F F + \sum_{E \in \mathcal{N}(X)} b_E E$ is a Cartier divisor if and only if, for each colored cone $(\mathcal{C}, \mathcal{F})$, there is $h_{\mathcal{C}} \in \chi(S)$ such that $h_{\mathcal{C}}(E) = a_E$ for each $E \in \mathcal{C}$ and $h_{\mathcal{C}}(\rho(F)) = a_F$ for each $F \in \mathcal{F}$. The $h_{\mathcal{C}}$ define a piecewise linear function on the support of $\mathbf{F}(X)$ (see [Br89], Proposition 3.1). We denote such function by h^D or by h ; sometimes we will use also the notation $h_{\mathcal{C}}^D$ instead of $h_{\mathcal{C}}$.

Let $\widetilde{PL}(X)$ be the set of functions h on the support $\bigcup \mathcal{C}$ of $\mathbf{F}(X)$ such that: 1) h is linear on each colored cone; 2) h takes integral values at all the point of $\chi_*(S) \cap (\bigcup \mathcal{C})$. Let $PL(X)$ be the quotient of $\widetilde{PL}(X)$ by the subset of restrictions of linear functions. If X is complete, there is the following exact sequence (see [Br89], Theorem 3.1):

$$0 \rightarrow \bigoplus_{F \in D(G/H)} \mathbb{Z}F \rightarrow Pic(X) \rightarrow PL(X) \rightarrow 0.$$

A Cartier divisor is globally generated (resp. ample) if and only if the associated function is convex (resp. strictly convex) and $h_{\mathcal{C}}(\rho(F)) \leq a_F$ (resp. $h_{\mathcal{C}}(\rho(F)) < a_F$) for each colored cone $(\mathcal{C}, \mathcal{F})$ and each $F \in D(G/H) \setminus \mathcal{F}$ (see

[Br89], Proposition 3.3). In particular a Cartier divisor is nef if and only if it is globally generated. Suppose for simplicity that X is toroidal. Given any Cartier divisor $D = \sum_{D(G/H)} n_F F + \sum_{N(X)} h(E)E$, then $H^0(X, \mathcal{O}(D))$ is a multiplicity free representation of G and its highest weights are in one-to-one correspondence with the points of $\chi(S) \cap P(D)$ where $P(D)$ is the polytope in $\chi(S)_{\mathbb{R}}$ intersection of the following half-spaces (see [Br89], Theorem 3.3): i) $\{m : m(E) + h(E) \geq 0\}$ for each $E \in \mathcal{N}(X)$; ii) $\{m : m(F) + n_F \geq 0\}$ for each $F \in D(G/H)$. If D is globally generated then the $h_{\mathcal{C}}$ belongs to $\chi(S) \cap P(D)$. If moreover X is toroidal and D is G -stable, then $P(D)$ is the intersection of the positive Weyl chamber C^+ with the convex hull $Q(D) = \text{convex}(wh_{\mathcal{C}})$ of the points $wh_{\mathcal{C}}$, where (\mathcal{C}, \emptyset) varies in the set of maximal colored cone and w varies in $W_{G, \theta}$ (see [Bi90], Corollary 4.1). Furthermore, the integral points in $Q(D)$ are the weights of a basis of semi-invariant vectors of $H^0(Z^c, \mathcal{O}(D)|_{Z^c})$ (see [Bi90], Proposition 4.1) and the volume of $\mathcal{O}(D)|_{Z^c}$ is equal to $(\text{rank } G/H)! \text{vol}(Q(D))$.

Remark 1. Let $\varphi : X \rightarrow X'$ be a G -equivariant, birational morphism of symmetric varieties and let L be any line bundle over X' , then $H^0(X', L)$ is isomorphic to $H^0(X, \varphi^*(L))$; in particular, L is big if and only if $\varphi^*(L)$ is big. Moreover, L is nef if and only if $\varphi^*(L)$ is nef, because of the previous description of nef divisors.

When X is toroidal we have the following split exact sequence (see [Br89], Proposition 3.2):

$$0 \rightarrow \text{Pic}(X_0) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(Z) \rightarrow 0,$$

where the maps are induced respectively by the projection $X \rightarrow X_0$ and by the inclusion $Z \hookrightarrow X$. Given any simple X , its Picard group is isomorphic to $\bigoplus_{F \in D(G/H) \setminus \mathcal{F}(X)} \mathbb{Z}[F]$; in particular $\text{Pic}(X_0) = \bigoplus_{F \in D(G/H)} \mathbb{Z}[F]$.

A (complete) symmetric variety is \mathbb{Q} -factorial if and only if each colored cone is simplicial and ρ is injective over $\mathcal{F}(X)$ (see [Br93], Proposition 4.2). Recall that a cone is said simplicial if it is generated by a number of vectors equal to its dimension. In particular the standard completion of any symmetric space is \mathbb{Q} -factorial. The conditions for the smoothness are much more complicated (see [Ru07], Theorem 2.2). Notice that the most part of this section is true for any spherical variety: in particular the descriptions of the class group and of the Picard group holds in general. Also the previous condition for the \mathbb{Q} -factoriality is stated in [Br93] in a more general form which holds for all spherical varieties.

2. SPHERICAL CLOSURE

We define (after Luna) the spherical closure \overline{H}^{sph} of H as the subgroup of $N_G(H)$ which stabilizes all the colors of G/H . The standard completion X_0^{sph} of G/\overline{H}^{sph} is wonderful and the standard completion X_0 of G/H is a ramified cover over X_0^{sph} . Moreover the projection induces an isomorphism between their rational Picard groups. Indeed we can identify $D(G/H)$ with $D(G/\overline{H}^{sph})$.

Proposition 2.1. *Let G/H a symmetric space, then we can write $(G, \theta) = \prod_{i=1}^n (G_i, \theta)$ so that G/H is a direct product $\prod G_i/(G_i \cap H)$, where the (G_i, θ) with $i > 1$ are indecomposable and $|D(G_i/(G_i \cap H))| > \text{rank}(G_i/(G_i \cap H))$ if and only if $i > 1$. Moreover $\overline{H}^{sph} = N_{G_1}((G_1 \cap H)) \times \prod_{i=2}^n (G_i \cap H)$.*

To prove such proposition we will use §1.4. In particular, we will use the following fact: if there is $n \in N(H)$ and $D_1, D_2 \in D(G/H)$ such that $nD_1 = D_2$, then $\rho(D_1) = \rho(D_2)$.

Proof. First, we reduce to the non-exceptional case. Write $(G, \theta) = \prod_{i=1}^n (G_i, \theta)$ with (G_1, θ) non exceptional and the other (G_i, θ) indecomposable and exceptional. For each $i > 1$, we have $G_i^\theta = G_i \cap H = N_{G_i}(G_i^\theta)$ and $|D(G_i/(G_i \cap H))| = \text{rank}(G_i/(G_i \cap H)) + 1$; in particular the $G_i \cap H$ with $i > 1$ are spherically closed in G_i and G/H is the direct product $\prod G_i/(G_i \cap H)$.

Suppose now X non-exceptional and write $(G, \theta) = \prod (G_i, \theta)$ with the (G_i, θ) indecomposable. Let $H_i := G_i \cap H$. We can think of $D(G_i/G_i^\theta)$ as a subset of $D(G/G^\theta)$ by associating $F \times \prod_{j \neq i} G_j/G_j^\theta \in D(G/G^\theta)$ to any $F \in D(G_i/G_i^\theta)$. We can suppose that: 1) $|D(G_i/G_i^\theta)| > \text{rank}(G_i/G_i^\theta)$ if and only if $i > r$; 2) there is $h \in H$ which exchanges two colors of G_i/G_i^θ if and only if $r < i \leq r + m$. Observe that if $i > r$ there is always an element of $N_G(H)$ which exchanges two colors of G_i/G_i^θ . Let $G_0 = \prod_{i=1}^{r+m} G_i$, then \overline{H}^{sph} is contained in $N' := N_{G_0}(H_0) \times \prod_{i>r+m} H_i$ because any element of $N_G(H) \setminus N'$ exchanges two colors of some G_i/G_i^θ with $i > r + m$ (which correspond to two distinct colors of G/H). Moreover, the number of colors of G_0/H_0 is equal to its rank. But $|D(G_0/H_0)| \geq |D(G_0/N_{G_0}(H_0))| \geq \text{rank}(G_0/N_{G_0}(H_0)) = \text{rank}(G_0/H_0)$, thus the spherical closure of H_0 in G_0 is $N_{G_0}(H_0)$. Therefore $\overline{H}^{sph} \supset N_{G_0}(H_0) \times \prod_{i>r+m} H_i$. \square

Corollary 2.1. *If H is spherically closed, then G/H is a direct product of indecomposable symmetric spaces $\prod G_i/H_i$. Moreover, the wonderful completion of G/H is the product of the wonderful completions of the G_i/H_i .*

Remark 2. Let X be a symmetric variety with open orbit G/H . Then there is a unique minimal symmetric variety X^{sph} with open orbit G/\overline{H}^{sph} and with an equivariant proper morphism $X \rightarrow X^{sph}$ that extend the canonical projection $G/H \rightarrow G/\overline{H}^{sph}$. Indeed, we can identify $D(G/H)$ with $D(G/\overline{H}^{sph})$, respectively $(\mathbb{C}(G/H)^B/\mathbb{C}^*)_{\mathbb{Q}}$ with $(\mathbb{C}(G/\overline{H}^{sph})^B/\mathbb{C}^*)_{\mathbb{Q}}$. Thus the colored fan of X defines a colored fan associated to an embedding of G/\overline{H}^{sph} . It is easy to show that this variety satisfies the requested properties s (see also [Kn91], §4).

3. EFFECTIVE CONE OF A COMPLETE SYMMETRIC VARIETY

First we determine $Eff(X)$ when X is \mathbb{Q} -factorial, then we consider the case where X is projective but possibly not \mathbb{Q} -factorial.

Theorem 3.1. *Let X be a \mathbb{Q} -factorial complete symmetric variety. Then*

- (1) *$Eff(X)$ is a closed polyhedral cone whose extremal rays are generated by:*
 - *the colors which are not contained in $D(G/H)^H$;*
 - *the G -stable prime divisors E whose classes are not proportional to any of the $[F_{\alpha_i}^+ + F_{\alpha_i}^-]$.*
- (2) *given any extremal ray r of $Eff(X)$, there exist a unique prime divisor D which belongs to one of the two previous families and such that $[D] \in r$.*

Let $\widetilde{Div}^G(X)$ be the abelian group freely generated by the G -stable prime divisors and let $Div^G(X)$ be its image in $Pic(X)$. These groups are isomorphic; indeed there is no non-trivial relation between these divisors, because any rational function which is a G -eigenvector is constant. If X is toroidal, we can identify $\widetilde{Div}^G(X)_{\mathbb{R}}$ with $\widetilde{Div}^T(Z)_{\mathbb{R}}$ by the restriction.

To prove the theorem we use the explicit knowledge of the relations of $Cl(X)$. In particular, the principal divisor associated to any function in $\mathbb{C}(X)^{(B)}$ is a linear combination of the F_{α} and of the G -stable prime divisors. Moreover, any $[F_{\alpha}]$ belongs to $Div^G(X)_{\mathbb{R}}$ because $div(f_{\omega_{\alpha}}) = F_{\alpha} + \sum_{E \in \mathcal{N}(X)} v_E(\omega_{\alpha})E$. Thus the class of each F_{α} belongs to $\sigma := Eff(X) \cap Div^G(X)_{\mathbb{R}}$. This will implies that $Eff(X)$ is generated by the others B -stable prime divisors.

Proof of Theorem 3.1. Exactly as in [Br07], Lemma 2.3.1, we can prove that σ is $\text{cone}([E], E \in \mathcal{N}(X))$; thus σ is simplicial because $\mathcal{N}(X)$ is a basis of $\widehat{\text{Div}}^G(X)_{\mathbb{R}}$. Moreover, σ contains all the $[F_{\alpha}]$ because $\text{div}(f_{\omega_{\alpha}}) = F_{\alpha} + \sum_{E \in \mathcal{N}(X)} (\omega_{\alpha}, v_E)E$ and the ω_E are antidominant. Therefore, the theorem is proved if ρ is injective.

In the general case, $\text{Pic}(X)_{\mathbb{R}} = \bigoplus_{E \in \mathcal{N}(X)} \mathbb{R}[E] \oplus \bigoplus_{\alpha_i^{\vee}: |\rho^{-1}(\alpha_i^{\vee})|=2} \mathbb{R}[F_{\alpha_i}^+]$. Indeed, $\text{Pic}(X)_{\mathbb{R}} (= \text{Cl}(X)_{\mathbb{R}})$ is generated by the $[E]$ with $E \in \mathcal{N}(X)$, the $[F_{\alpha}]$ and the $[F_{\alpha^+}]$ with $|\rho^{-1}(\alpha_i^{\vee})| = 2$ (see §1.4 for the definition of F_{α}^+ and F_{α^+}). Moreover, the relations are freely generated by the following ones ($*_{\omega_{\alpha}}$): $[F_{\alpha}] = \sum_{E \in \mathcal{N}(X)} (\omega_{\alpha}, -v_E)[E]$ with $\alpha \in \overline{R}_{G,\theta}$. Indeed, $\text{Cl}(X)$ is generated by the B -stable prime divisors with relations $\text{div}(f_{\omega}) = 0$ for any $\omega \in \chi(S)$; moreover the relation ($*_{\omega_{\alpha}}$) is the one corresponding to the fundamental spherical weight ω_{α} (see also §1.7). Observe that ρ is injective over $\mathcal{F}(X)$ because X is \mathbb{Q} -factorial. Hence, $\text{Eff}(X)$ is generated by $\mathcal{N}(X)$ and by $D(G/H) \setminus D(G/H)^H$. Each $[F_{\alpha}^+]$ generates an extremal ray of $\text{Eff}(X)$ because for any divisor $F_{\alpha}^+ + \text{div}(f_{\omega}) = \sum_{E \in \mathcal{N}(X)} n_E E + \sum_{\beta \in \rho(D(G/H)^H)} n_{\beta} F_{\beta}^+ + \sum_{\beta \notin \rho(D(G/H)^H)} n_{\beta} F_{\beta}^-$ such that $n_{\alpha}^+ = 0$, we have $n_{\alpha}^- = -1$. We can argue similarly for the F_{α}^- . \square

Given any symmetric variety X , let $X^{\leq 1}$ be the open G -subvariety composed of orbits of codimension at most 1. We have an equivariant morphism $q : X^{\leq 1} \rightarrow X_0^{\text{sph}}$ which can be extended to X if and only if X is toroidal. Here X_0^{sph} is the wonderful completion of $G/\overline{H}^{\text{sph}}$.

Theorem 3.2. *Let X be a \mathbb{Q} -factorial complete symmetric variety. Then:*

- (1) *The class of a G -stable prime divisor E belongs to the cone generated by the classes of colors if and only if it is proportional to some $[F_{\alpha}]$.*
- (2) *The class of a G -stable prime divisor E is proportional to $[F_{\alpha}]$ if and only if the irreducible factor R of $R_{G,\theta}$ containing α is orthogonal to $v_{E'}$ for any $E' \in \mathcal{N}(X)$ different from E .*
- (3) *The class of a G -stable prime divisor E is proportional to some $[F_{\alpha}]$ if and only if there is a G -equivariant morphism $\varphi : X^{\leq 1} \rightarrow X'$ such that X' is a wonderful symmetric G -variety, $\varphi(E) \subsetneq X'$ and $\varphi(E') = X'$ for each G -stable prime divisor E' of X different from E .*
- (4) *If a such morphism exist, we can identify the restricted root system of X' with a product R of irreducible factors of $R_{G,\theta}$. Then $[E]$ is proportional to $[F_{\alpha}]$ for each simple α in R . We can also suppose that the stabilizer of $\varphi(H/H)$ is generated by $\overline{H}^{\text{sph}}$ and some normal, θ -stable, connected subgroups of G . If moreover X is toroidal, φ can be extended to X , $-v_E$ is a fundamental spherical weight and X' is a product of wonderful symmetric varieties of rank one.*
- (5) *[Lemma 2.3.2 of [Br07]] The class of a G -stable prime divisor E generates an extremal ray of $\text{Eff}(X)_{\mathbb{R}}$ which does not contain the class of a color if and only if $\dim H^0(X, \mathcal{O}(mE)) = 1$ for each positive integer m .*
- (6) *[Lemma 2.3.3 of [Br07]] The class of a color F generates an extremal ray of $\text{Eff}(X)_{\mathbb{R}}$ which does not contain the class of any $E \in \mathcal{N}(X)$ if and only if there is a G -equivariant morphism $\varphi : X \rightarrow G/P$, where $P \supseteq H$ is a maximal parabolic subgroup, such that F is the preimage of the Schubert divisor (the unique B -stable prime divisor) in G/P .*

Before to prove the theorem, we do some remarks.

Remark 3. Observe that given an irreducible factor R of $R_{G,\theta}$, there is always an $E \in \mathcal{N}(X)$ with $(v_E, R) \neq 0$ because of the completeness of X . The statement of the third point is very similar to that of Lemma 2.3.4 in [Br07].

Remark 4. Write $R_{G,\theta}$ as a product $\prod R_i$ of irreducible factors, then X_0^{sph} is a product $\prod X_{R_i}$ by Corollary 2.1. In the proof we show that to check if $[E]$ belongs to $\mathbb{R}^{\geq 0}[F_\alpha]$ with $\alpha \in R_{i_0}$ is sufficient to check if $\pi_{i_0} \circ q : X^{\leq 0} \rightarrow X_{R_0}$ satisfies the conditions of point (3).

Remark 5. In the setting of wonderful varieties, there are never two colors whose classes are proportional.

Remark 6. In [Br07] the Lemma 2.3.1 and 2.3.2 are stated for wonderful varieties, but their proof holds for any \mathbb{Q} -factorial complete spherical variety whose open orbit is sober, i.e. $N_G(H)/H$ is finite. In the setting of symmetric varieties, one can explicitly construct the morphism of point (6). Let F be a color as in the point (6) and let α^\vee be $\rho(F)$. Then, by Theorem 3.1, $\rho^{-1}(\alpha)$ contains two colors and ω_α is the sum $\omega_1 + \omega_2$ of two (possibly equal) fundamental weights of G . We can also suppose that $G^\theta = P(\omega_1) \cap P(-\omega_1)$ and $\omega_2 = -\varpi_\circ \omega_1$, where ϖ_\circ is the longest element of W_G . Also the other color in $\rho^{-1}(\alpha)$ satisfies the conditions of (6) and the corresponding applications are the following:

$$\begin{aligned} G/G^\theta &\rightarrow G/P(\omega_1) \subset \mathbb{P}(V(\omega_1)) \\ g &\rightarrow g \cdot [v_{\omega_1}] \end{aligned}$$

and

$$\begin{aligned} G/G^\theta &\rightarrow G/P(-\omega_1) \subset \mathbb{P}(V(\omega_2)) \\ g &\rightarrow g \cdot [v_{-\omega_1}] \end{aligned}$$

where v_χ is a weight vector of weight χ . Furthermore, $G/P(-\omega_1)$ is isomorphic to $G/P(\omega_2)$. There are some difference according to whether α is exceptional or not. If α is exceptional then ω_1 is different from ω_2 ; in particular the stabilizer of F_α^+ is different from the stabilizer of F_α^- . Moreover $\mathbb{P}(V(\omega_1))$, resp. $\mathbb{P}(V(\omega_2))$, contains a unique point fixed by G^θ .

Instead, if α is non-exceptional then $\omega_1 = \omega_2$; in particular, the stabilizer of F_α^+ is equal to the stabilizer of F_α^- . Moreover, $\mathbb{P}(V(\omega_1))$ contains two points fixed by G^θ , namely $[v_{\omega_1}]$ and $[v_{-\omega_1}]$. In this case there is an element n of $N_G(H) \setminus H$ which exchanges F_α^+ with F_α^- ; moreover n exchanges v_{ω_1} with $v_{-\omega_1}$.

Proof of Theorem 3.2. We have already showed the first point in the proof of the previous theorem. First, we will prove the point (2). Then we will use it to prove the points (3) and (4). We will use also the Corollary 2.1 to find an explicit candidate for the application φ (see also Remark 4). Because $\text{div}(\omega_\alpha) = F_\alpha + \sum_{E \in \mathcal{N}(X)} (\omega_\alpha, v_E)E$, $[E]$ is proportional to $[F_\alpha]$ if and only if $(\omega_\alpha, v_{E'}) = 0$ for any $E' \neq E$. But the $-v_{E'}$ are dominant coweights. Thus, if $[E] \in \mathbb{R}^{\geq 0}[F_\alpha]$ and α' belongs to the irreducible factor of $R_{G,\theta}$ containing α , then $[E] \in \mathbb{R}^{\geq 0}[F_{\alpha'}]$.

Given any irreducible factor R of $R_{G,\theta}$, we can write, by Corollary 2.1, $X_0^{sph} = X_1 \times X_2$ where the X_i are wonderful varieties and the restricted root system of X_2 is equal to R . Given any $v \in C^-$ and $\alpha \in R$, $(\omega_\alpha, v) = 0$ if and only if, for any $\alpha' \in R$, $-\omega_{\alpha'}^\vee$ is not contained in the face of C^- whose relative interior contains v . Thus, $[E]$ is proportional to $[F_\alpha]$ if and only if the following condition (*) holds: if $K \in \mathcal{N}(X_0^{sph})$ contains the image of a G -stable prime divisor of $X^{\leq 1}$ different from E , then K has the form $K' \times X_2$ with $K' \in \mathcal{N}(X_1)$. Hence, if $[E]$ is proportional to $[F_\alpha]$, then the projection on X_2 of any $q(E')$ with $E' \neq E$ is the whole X_2 .

Vice versa suppose that exists a morphism φ as in the statement. By the Corollary 2.1 and by the description of morphisms between spherical varieties, we can suppose that the stabilizer of $\varphi(H/H)$ is generated by \overline{H}^{sp} and some normal, θ -stable connected subgroups of G (we may have to compose or to lift φ with a finite equivariant morphism). We need also the following property: if H' is a symmetric subgroup of G which contains H and does not contain any normal connected subgroup of G , then $H' \subset N_G(H)$ (see [dCoPr83], Lemma 1.7). Therefore, φ is the composite of q with a projection $X_0^{sph} = X_1 \times X_2 \rightarrow X_2$. Now, the hypotheses on φ implies the condition (*).

Finally, if X is toroidal then $-\omega_\alpha^\vee$ is contained in $\mathcal{N}(X)$, so $[F_\alpha]$ can be proportional only to $E_{-\omega_\alpha^\vee}$. \square

Remark 7. Let X be any projective symmetric variety (possibly non \mathbb{Q} -factorial) and let $D = \sum_{F \in D(G/H)} a_F F + \sum_{E \in N(X)} b_E F$ be an effective Cartier divisor on X , so $a_F, b_E \geq 0$. Up to exchange some F_α^+ with F_α^- , there is an effective divisor $D' = D_1 + D_2$ linearly equivalent to D and such that: i) D_1 is G -stable and effective; ii) $D_2 = \sum a_\alpha^+ F_\alpha^+$ with $a_\alpha^+ \geq 0$ for each α . Moreover, D is nef (resp. big) if and only if D_1 is nef (resp. big). Indeed, we can suppose X toroidal by taking the pullback of these line bundles to a desingularization X' of X^{dec} . Then D is nef (resp. big) if and only if D' is nef (resp. big). Moreover, $h^{D'} = h^{D_1}$ and the coefficients of D' with respect to the F_α^- are all zero (and lesser than the coefficients of D' with respect to the F_α^+). Finally, if $D' = \sum_{F \in D(G/H)^H} c_F F + \sum d_\alpha^+ F_\alpha^+ + \sum d_\alpha^- F_\alpha^- + \sum_{E \in N(X)} f_E E$, the dimension of $H^0(X, \mathcal{O}(D'))$ and the combinatorial conditions on the nefness of D_1 depend only on the linear functions on the linear functions $h_C^{D_1}$, on the $c_F (= 0)$ and on the $\min\{d_\alpha^+, f_\alpha^+\} (= 0)$.

Corollary 3.1. *Let X be a projective symmetric variety. Then $Eff(X)$ is the intersection of $Pic(X)_\mathbb{R}$ with the polyhedral cone of $Cl(X)_\mathbb{R}$ whose extremal rays are generated by the classes of the colors not in $D(G/H)^H$ and by the classes of the G -stable prime divisors which are not linearly equivalent to a multiple of $[F_\alpha]$ with $F_\alpha \notin D(G/H)^H$. Moreover, the statement of Theorem 3.2 holds again.*

Proof of Corollary 3.1. To prove the corollary it is sufficient to show that there is a \mathbb{Q} -factorial complete symmetric variety X' and an equivariant morphism $\psi : X' \rightarrow X$ which induces an isomorphism between $(X')^{\leq 1}$ and $X^{\leq 1}$; in particular ψ induces an isomorphism between $Cl(X)$ and $Cl(X')$. Moreover $\psi^* Eff(X) = Eff(X') \cap \psi^* Pic(X)_\mathbb{R} \subset Cl(X)_\mathbb{R}$ and $Pic(X)_\mathbb{R} \cong \psi^* Pic(X)_\mathbb{R} \subset Pic(X')_\mathbb{R} = Cl(X')_\mathbb{R} \cong Cl(X)_\mathbb{R}$. Observe that the Theorem 3.2 depends only by $X^{\leq 1}$.

Now, we will construct X' . The procedure will be more complicated if X is neither non-exceptional nor toroidal. We need to define a new variety \tilde{X} , isomorphic to X in codimension 1: let $\mathbf{F}(\tilde{X}) := \{(\mathcal{C}, \tilde{\mathcal{F}}) : (\mathcal{C}, \mathcal{F}) \in \mathbf{F}(X)\}$, where $\tilde{\mathcal{F}} := \rho(\rho^{-1}(\mathcal{F}))$, and let \tilde{X} be the corresponding variety. Remark that if X is non-exceptional or toroidal, then it is equal to \tilde{X} . We have a morphism $p : X \rightarrow \tilde{X}$ which is an isomorphism between $X^{\leq 1}$ and $\tilde{X}^{\leq 1}$, thus it is sufficient to find a variety $\varphi : X' \rightarrow \tilde{X}$ over \tilde{X} such that: 1) φ factorizes by p and 2) φ is an isomorphism in codimension 1.

First, we define the fan $\mathbf{F}^f(X')$ associated to X' . The idea is the following: the cones in $\mathbf{F}^f(\tilde{X}) (= \mathbf{F}^f(X))$ are generated by some faces of an appropriate polytope in $\chi_*(S)$ (which is the polar polytope of the moment polytope of an ample bundle D over \tilde{X}); we triangularize the faces of such polytope and define $\mathbf{F}^f(X')$ as the family of cones generated by the simplices obtained from the previous faces.

To define the previous polytope we need an ample Cartier divisor D over \tilde{X} , such that: i) the interior of $P(D)$ contains 0 and ii) D is linearly equivalent to a

G -stable divisor. We have defined \tilde{X} to assure the existence of such a divisor. Now, we will find it; let D' an ample Cartier divisor on X . As in the Remark 7 we can write $D' = D_1 + D_2 + D_3$ where: 1) D_1 is linearly equivalent to a G -stable divisor; 2) $D_2 + D_3$ is a positive linear combination of the F_α^+ . Moreover, we can suppose that D_2 is $\sum_{\alpha \in I} a_\alpha^+ F_\alpha^+$, where I is the set of α such that F_α^+ belongs to $\mathcal{F}(X)$ (and D_3 is $\sum_{\alpha \notin I} a_\alpha^+ F_\alpha^+$). One can easily show that $D'' := D_1 + D_2 + \sum_{\alpha \in I} a_\alpha^+ F_\alpha^-$ is an ample divisor over \tilde{X} . Indeed, $h^{\tilde{X}, D''} = h^{X, D'} = h^{X, D_1 + D_2}$ and the minimum of the coefficients of D'' w.r.t. the colors in $\rho^{-1}(\alpha^\vee)$ is lesser than the corresponding minimum for D' . Remark that we have used the fact that $X^{\leq 1}$ is isomorphic to $\tilde{X}^{\leq 1}$, so $Cl(X) \cong Cl(\tilde{X})$ (but $Pic(X)$ can be non-isomorphic to $Pic(\tilde{X})$). Then $D^{(3)} := D'' + \sum_{\alpha \notin \rho(\mathcal{F}(\tilde{X}))} F_\alpha^+$ is ample over \tilde{X} and is linearly equivalent to a G -stable effective divisor $D^{(4)}$ such that $h^{D^{(4)}}(v_E) > 0$ for each $E \in \mathcal{N}(\tilde{X})$. Indeed, none irreducible factor of $R_{G, \theta}^\vee$ can be contained in $span\{\rho(\mathcal{F}(X))\}$ because the colored cones are strictly convex. Therefore we can choose D as $nD^{(4)} + \sum_{\alpha \notin \rho(\mathcal{F}(\tilde{X}))} F_\alpha^+ + \sum_{\alpha \in \rho(\mathcal{F}(\tilde{X}))} (F_\alpha^+ - \sum_{\mathcal{N}(\tilde{X})} v_E(\omega_\alpha) E)$ with $n \gg 0$.

Let P be the polar polytope of $P(D)$, i.e. $P = \{n \in \chi_*(S)_\mathbb{R} : m(n) \geq -1 \forall m \in P(D)\}$. Then, given any cone \mathcal{C} in the fan $\mathbf{F}^f(\tilde{X})$, \mathcal{C} is generated by an appropriate face of P . Observe that there are faces of P which are associated to none colored cone of \tilde{X} . Let A' be the set of vertices of P and set $A = A' \cup \{0\}$. We need to give a triangulation of P with vertices in A . Given a polytope Q generated by the points $S = \{q_1, \dots, q_m\}$, a *subdivision* of Q with vertices in S is a finite collection $\{Q_1, \dots, Q_r\}$ of polytopes such that: i) Q is the union $\bigcup Q_i$; ii) the vertices of each Q_i are drawn from S ; iii) if $i \neq j$ then $Q_i \cap Q_j$ is a common (possibly empty) face of the boundaries of Q_i and Q_j . If all the Q_i are simplexes, the subdivision is called a *triangulation*. Before to define the desired triangulation of P , we need to define an elementary construction step.

Let Q be a n -dimensional polytope in \mathbb{R}^n , let F be a $n - 1$ -dimensional face of Q , let H be the unique hyperplane containing F and let v be a point in \mathbb{R}^d . The polytope Q is contained in exactly one of the closed halfspaces determined by H . If v is contained in the opposite open halfspace, then F is said to be *visible* from v . If Q is a k -dimensional polytope in \mathbb{R}^n with $k < n$ and $v \in Aff(Q)$, then the above definition can be modified in the obvious way so that everything is considered relative to the ambient space $aff(Q)$. Suppose $S = \{Q_1, \dots, Q_m\}$ is a subdivision of a n -dimensional polytope $Q = conv(V)$ in \mathbb{R}^n and let $v \in V$. The result of *pushing* v is, by definition, the subdivision S' of Q obtained by modifying the $Q_i \in S$ as follows:

- If $v \notin Q_i$, then $Q_i \in S'$.
- If $v \in Q_i$ and $conv(vert(Q_i) \setminus \{v\})$ is $(n - 1)$ -dimensional (i.e. Q_i is a pyramid with apex v), then $Q_i \in S'$.
- If $v \in Q_i$ and $Q'_i := conv(vert(Q_i) \setminus \{v\})$ is n -dimensional, then $Q'_i \in S'$. Also, if F is any $(n - 1)$ -dimensional face of Q'_i that is visible from v , then $conv(F \cup \{v\}) \in S'$.

Let $Q = convex(V)$ and order the point of $V = \{v_1, \dots, v_m\}$ in an arbitrary way; then the subdivision obtained by starting with the trivial one and pushing the points of V in that order is a triangulation (see [Le97], §14.2). Returning to our problem, let P and A be as in the first part of the proof and order the points of A so that 0 is the first point. Let T be the subdivision of P obtained from the trivial subdivision by pushing the points of A in the chosen order. This triangulation induces a triangulation of the (proper) faces of P . Let $\{T_i\}_{i \in I}$ be the set of $(s - 1)$ -dimensional simplices obtained in such way, where $s = rank \chi_*(S)$, and let $I' \subset I$ be the family of simplices whose relative interior intersects C^- . Given $i \in I'$, let C_i

be the cone generated by T_i . We want to define $\mathbf{F}(X')$ so that $\mathbf{F}^f(X')$ is composed by the faces of all the \mathcal{C}_i with $i \in I'$. Such set is a fan by the definition of subdivision of a polytope; moreover its support is the same of the one of $\mathbf{F}(\tilde{X})$.

For each $i \in I'$, we define \mathcal{F}_i as follows. For each α^\vee in T_i , choose a color E_α in $\mathcal{F} \cap \rho^{-1}(\alpha^\vee)$, where $(\mathcal{C}, \mathcal{F})$ is the (s -dimensional) colored cone of X containing \mathcal{C}_i . Finally, define \mathcal{F}_i as the set of such E_α . If α^\vee is contained in two different simplices, say T_i and T_j , then we choose the same E_α for both \mathcal{F}_i and \mathcal{F}_j . Remark that we need to work with the colored cones of X because we want that X' dominates not only \tilde{X} , but also X . The previous choices are possible because of the combinatorial conditions for the ampleness of a Cartier divisor D' on X . Indeed, suppose by contradiction that there are a simple restricted root α and two colored cones of X , say $(\mathcal{C}_1, \mathcal{F}_1)$, $(\mathcal{C}_2, \mathcal{F}_2)$, such that $F_\alpha^+ \in \mathcal{F}_1 \setminus \mathcal{F}_2$ and $F_\alpha^- \in \mathcal{F}_2 \setminus \mathcal{F}_1$. Write $D' = \sum_{D(G/H)^H} a_F F + \sum b_\beta^+ F_\beta^+ + \sum b_\beta^- F_\beta^- + \sum_{\mathcal{N}(X)} c_E E$ and let h be the convex function associated to D' . Then $b_\alpha^+ = b_\alpha^- = h(\alpha)$ because $F_\alpha^+, F_\alpha^- \in \mathcal{F}(X)$. Moreover $b_\alpha^+ = h_{\mathcal{C}_1}(\rho(F_\alpha^+)) = h_{\mathcal{C}_1}(\rho(F_\alpha^-)) < h(\rho(F_\alpha^-)) = h(\alpha)$, a contradiction. Thus $\{(\mathcal{C}_i, \mathcal{F}_i) : i \in I'\}$ is a colored fan and the associated symmetric variety satisfies the requested properties. \square

4. BIGNESS OF \mathbb{Q} -DIVISORS ON A PROJECTIVE SYMMETRIC VARIETY

First we describe the big cone of any \mathbb{Q} -factorial, projective symmetric variety. Then we will prove two criterions for a nef (G -stable) divisor to be big.

4.1. The big cone.

Theorem 4.1. *Let X be a projective, \mathbb{Q} -factorial symmetric variety. Then $Big(X)$ is the union of the following cones (whose closure is simplicial): $\bigoplus_{E \in \mathcal{N}(X)} \mathbb{R}^{>0}[E] \oplus \bigoplus_{\alpha^\vee \notin \rho(D(G/H)^H)} \mathbb{R}^{\geq 0}[F_\alpha^\bullet]$, where the $F_\alpha^\bullet \in \rho^{-1}(\alpha^\vee)$ are chosen in all the ways possible.*

To prove such theorem we will use Theorem 3.1 plus the explicit expression of the relations (ω_α) .

Proof. Let I be $\overline{R}_{G,\theta}^\vee \setminus \rho(D(G/H)^H)$. First, we prove that all the cones in the statement are contained in $Big(X)$. It is sufficient to prove that $\dot{\sigma} := \bigoplus_{E \in \mathcal{N}(X)} \mathbb{Q}^{>0}[E]$ is contained in $Big(X)$ because the sum of a big divisor with an effective one is big. Given any element $[D]$ of $\dot{\sigma}$, there are r_α such that $[D'] := [D] - \sum_I r_\alpha [F_\alpha]$ belongs to $\dot{\sigma}$, because $\dot{\sigma}$ is open and all the $[F_\alpha]$ belongs to the closure σ of $\dot{\sigma}$. Thus $[D] = [D'] + \sum_I r_\alpha [F_\alpha]$ is big by Theorem 3.1.

Now, let D be a big divisor. Up to exchanging some F_α^+ with the corresponding F_α^- , we can write $D = \sum_{E \in \mathcal{N}(X)} n_E E + \sum_{\alpha \in \overline{R}_{G,\theta}} n_\alpha F_\alpha + \sum_{\alpha \in I} n_\alpha^+ F_\alpha^+$ with positive coefficients. This divisor is linear equivalent to an effective divisor $D' = \sum_{E \in \mathcal{N}(X)} m_E E + \sum_{\alpha \in I} m_\alpha^+ F_\alpha^+$, so it is sufficient to show that all the m_E are strictly positive. There are two cases: i) the class of any G -stable prime divisor generates an extremal ray of $Eff(X)$; ii) there is some E linearly equivalent to a multiple of F_α with $\alpha \in I$ (see Theorem 3.1). In the first case, all the m_E (and all the m_α^+) are strictly positive because $Big(X)$ is an open cone of dimension equal to $|\mathcal{N}(X)| + |I|$ and all the vectors in the sum generate an extremal ray of $Eff(X)$. In the second case, some $[E]$ is equal to some $t[F_\alpha^+] + t[F_\alpha^-]$, so we can't use the same argument. Let J be the set of G -stable prime divisors which generate an extremal ray of $Eff(X)$ and let $K \subset I$ be subset of roots such that $[F_\alpha]$ is proportional to some $[E]$ with $E \in \mathcal{N}(X)$. Then we can write $[D] = \sum_{E \in J} m_E [E] + \sum_{I \setminus K} m_\alpha^+ [F_\alpha^+] + \sum_K r_\alpha^+ [F_\alpha^+] + \sum_K r_\alpha^- [F_\alpha^-]$ and, as before, the coefficients must be strictly positive. Given $E' \notin J$ and $\alpha \in K$ such that

$[E'] = t_\alpha[F_\alpha]$ with $t_\alpha > 0$, we have $r_\alpha^+ = t_\alpha m_{E'} + m_\alpha^+$ and $r_\alpha^- = t_\alpha m_\alpha^-$. Therefore $m'_{E'}$ is strictly positive for all E' . \square

4.2. Bigness of nef divisors. Now, we want to study the bigness of a fixed (nef) Cartier divisor.

Remark 8. We want to observe that to study the bigness of a fixed (Cartier) \mathbb{Q} -divisor we can reduce to the case of a smooth toroidal symmetric variety with H wonderful. These are the smooth symmetric varieties proper over a wonderful one.

First of all, we can reduce to the smooth toroidal case because of the Remark 1. Suppose now X toroidal and let X^{sph} be the completion of G/H^{sph} with the same colored fan of X (see Remark 2). Then X^{sph} dominates the wonderful completion X_0^{sph} of G/H^{sph} . We have unique equivariant morphisms $\varphi : X \rightarrow X^{sph}$ and $X_0 \rightarrow X_0^{sph}$ (which send H/H to $\overline{H}^{sph}/\overline{H}^{sph}$) and the pushforwards of such morphisms are isomorphisms between their rational Picard groups. In general the pushforwards are defined between the (rational) class groups; in our case the pushforward define an isomorphism between the rational class groups which restricts to an isomorphism between the rational Picard groups.

Indeed, $\dim Cl(X)_{\mathbb{Q}} = \dim Cl(X^{sph})_{\mathbb{Q}}$ and $\dim Pic(X)_{\mathbb{Q}} = \dim Pic(X^{sph})_{\mathbb{Q}}$ because $D(G/H^{sph}) \cong D(G/H)$ and X^{sph} “has” the same colored fan of X (forgetting the lattice in $(\mathbb{C}(G/H)^B/\mathbb{C}^*)_{\mathbb{Q}}$); in other words the rational Picard group does not depend on the lattice $\chi_*(S)$. If X^{sph} is smooth, then $\phi_* \circ \phi^* : Cl(X^{sph})_{\mathbb{Q}} \rightarrow Cl(X^{sph})_{\mathbb{Q}}$ is $(deg \phi)Id$ and $\phi^*(Pic(X^{sph})_{\mathbb{Q}}) \subset Pic(X)_{\mathbb{Q}}$, so the claim holds. In the general case we take a desingularization $\psi : \overline{X}^{sph} \rightarrow X^{sph}$ of X^{sph} and we define \overline{X} as the completion of G/H with colored fan $\mathbf{F}(\overline{X}^{sph})$, $\varphi : \overline{X} \rightarrow X$, resp. $\overline{\varphi} : \overline{X} \rightarrow \overline{X}^{sph}$, as the obvious maps. Then we regard to the subspace $\psi^*(Pic(X^{sph})_{\mathbb{Q}})$ of $Pic(\overline{X}^{sph})_{\mathbb{Q}}$ (isomorphic to $Pic(X^{sph})_{\mathbb{Q}}$) and use the following facts: i) $(\overline{\varphi}^* \circ \psi^*)(Pic(X^{sph})_{\mathbb{Q}}) = (\varphi^* \circ \phi^*)(Pic(X^{sph})_{\mathbb{Q}})$ is contained in $\varphi^*(Pic(X)_{\mathbb{Q}})$ and ii) $\phi_* \circ \varphi_* = \psi_* \circ \overline{\varphi}_*$.

A line bundle $\mathcal{O}(D)$ on X is big if and only if $\phi_*(\mathcal{O}(D))$ is big. Indeed if $\chi_*(T/T \cap \overline{H}^{sph}) \subset \frac{1}{m} \chi_*(T/T \cap H)$, then $|P(rD) \cap \chi_*(T/T \cap H)| \leq |P(rD) \cap \chi(T/T \cap \overline{H}^{sph})| \leq |P(mrD) \cap \chi_*(T/T \cap H)|$ for each positive integer r (see §1.6 and [Br89], §3). The last inequality holds because the multiplication by m defines an inclusion of $P(rD) \cap \chi_*(T/T \cap \overline{H}^{sph})$ in $P(mrD) \cap m \chi_*(T/T \cap \overline{H}^{sph}) (\subset P(mrD) \cap \chi_*(T/T \cap H))$.

Let X be a toroidal symmetric variety and suppose X_0 smooth. Using the results of [Br89], one can show that a G -stable divisor D on X is ample (resp. nef) if and only if $\mathcal{O}(D)|_{Z^c}$ is ample (resp. nef). Moreover, this holds if and only if the restrictions of $\mathcal{O}(D)$ to Z and to the closed orbits are ample (resp. nef). These last conditions can be stated as appropriate conditions on the function $h = \{h_{\mathcal{C}}\}$ and on the weights $h_{\mathcal{C}}$, where the (\mathcal{C}, \emptyset) are the maximal colored cones of $\mathbf{F}(X)$: 1) D is nef if and only if h is convex and the $-h_{\mathcal{C}}$ are spherical weights; 2) D is ample if and only if h is strictly convex and the $-h_{\mathcal{C}}$ are regular spherical weights (i.e. they are strictly dominant weights of $R_{G,\theta}$). We want to prove a similar condition for the bigness of any nef line bundle. Observe that, given a closed G -orbit $\mathcal{O}_{\mathcal{C}}$ of X associated to a (maximal) colored cone (\mathcal{C}, \emptyset) , the weight of the fiber of $\mathcal{O}(D)$ over the B -stable point of $\mathcal{O}_{\mathcal{C}}$ is $-h_{\mathcal{C}}$ because D is G -stable (see [Bi90], §2). Suppose by simplicity X toroidal. First we prove that a nef G -stable divisor is big if and only if its restriction to the associate complete toric variety Z^c is big (see Proposition 4.1). Then, we use the fact the such restriction is big if and only if $vol(Q(D))$ is strictly

positive (see §1.7 for the definition of $Q(D)$). For example, if X is wonderful, θ is indecomposable and D is a G -stable divisor with associated function h one can easily prove the following description (recall that there is a unique maximal colored cone (\mathcal{C}, \emptyset)):

- D is ample if and only if $-h_{\mathcal{C}}$ is spherical and regular (i.e. it is a strongly dominant weight of $R_{G,\theta}$);
- D is nef if and only if $-h_{\mathcal{C}}$ is spherical;
- D is big and nef if and only if $-h_{\mathcal{C}}$ is spherical and non-zero.

When X is only toroidal, but θ is again indecomposable, we will prove that D is nef and big if and only if the sum $-\sum_{\{\mathcal{C} \in \mathbf{F}^f(X) \text{ maximal}\}} h_{\mathcal{C}}$ is spherical and non-zero (see Theorem 4.2).

Remark 9. Given a complete symmetric variety X , let $p : X^{dec} \rightarrow X$ be the decoloration of X and let Z^c be the complete toric variety associate to X^{dec} . If X is projective then X^{dec} and Z^c are projective. Indeed, let D be an ample divisor on X , then $D' := p^*D + \sum_{F \in D(G/H)} F$ is ample on X^{dec} . Indeed, $h^{p^*D} = h^{D'}$ is equal to the restriction of h^D on C^- . Thus, $(h_{\mathcal{C}}^{D'})(F) \leq a_F < a_F + 1$ for each $(\mathcal{C}, \emptyset) \in \mathbf{F}(X^{dec})$ and $F \in D(G/H)$ (here a_F is the coefficient of D with respect to F ; see also §1.7).

Proposition 4.1. *Let X be a projective symmetric variety and let D be a B -stable, Cartier \mathbb{Q} -divisor on X . If D is big then its restriction $\mathcal{O}(p^*D)|_{Z^c}$ to the associated complete toric variety Z^c is big. Moreover, if D is G -stable and $\mathcal{O}(p^*D)|_{Z^c}$ is big then D is big.*

Proof. By the previous discussion we can suppose X is smooth and toroidal (see also proof of the Corollary 3.1). First, we describe the restriction $i^* : \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(Z^c)_{\mathbb{Q}}$, then we use the Proposition 1.1 applied respectively to X and Z^c . Under the previous assumption, we can define a linear map $i^* : \text{Div}^B(X)_{\mathbb{Q}} \rightarrow \text{Div}^T(Z^c)_{\mathbb{Q}}$ such that $[i^*(D)] = i^*([D])$ in the following way: $i^* : \text{Div}^B(X) \rightarrow \text{Pic}(X)_{\mathbb{Q}} \cong \text{Pic}^G(X)_{\mathbb{Q}} \rightarrow \text{Pic}^T(Z^c)_{\mathbb{Q}} \cong \text{Div}^T(Z^c)_{\mathbb{Q}}$. Here $i : Z^c \hookrightarrow X$ is the inclusion and $\text{Pic}^G(X)$ is the group of G -linearized line bundles. See [Od88], Proposition 2.1 for the last isomorphism. First we want to study the kernel and the image of i^* by using some techniques similar to ones in [Vu90] and in [Bi90].

Given a G -stable prime divisor E on X we define $E|Z$ as the closure of $E \cap Z$ in Z^c , so $E|Z^c := i^*(E)$ is $\sum_{w \in W_{G,\theta}} wE|Z$ and has support $E \cap Z^c$. Hence, $E|Z^c$ and $E|Z$ are effective divisors on Z^c . Moreover, the T -stable prime divisors on Z^c are the $wE|Z$ with $E \in \mathcal{N}(X)$ and $w \in W_{G,\theta}$ (actually w is a fixed representant in $N_{H^0}(T^1)$ of the corresponding element in $W_{G,\theta}$). Let $\pi : G \rightarrow G/H$ be the projection. The kernel of $i^* : \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(Z^c)_{\mathbb{Q}}$ is generated by the $[F_{\alpha}^+ - F_{\alpha}^-]$ with $\alpha \notin \rho(D(G/H)^H)$. Indeed, let ω_{α}^{\pm} be the T -weight of an equation of $\pi^{-1}(F_{\alpha}^{\pm})$, then, for any $t \in T^1$, $(-\omega_{\alpha}^-)(t) = \theta(\omega_{\alpha}^+)(t) = \omega_{\alpha}^+(\theta(t)) = \omega_{\alpha}^+(t^{-1}) = (-\omega_{\alpha}^+)(t)$, so $2\omega_{\alpha}^+|T^1 = 2\omega_{\alpha}^-|T^1 = \omega_{\alpha}|T^1$, where ω_{α} is the fundamental spherical weight corresponding to α (see [Ru07], pages 6-8 and [Vu90] §3.3-3.4). In particular, there is $Y_{\alpha} \in \text{Div}^G(X)$ with $i^*(2F_{\alpha}^+) = i^*(2F_{\alpha}^-) = i^*(Y_{\alpha})$. Observe that $\text{Div}^G(X)_{\mathbb{Q}}$ is a complement to $\ker(i^*)$, so i^* is injective over $\text{Div}^G(X)_{\mathbb{Q}}$ and $i^*(\text{Pic}(X)_{\mathbb{Q}}) = i^*(\text{Div}^G(X)_{\mathbb{Q}})$.

Let $(\text{Div}^T(Z^c)_{\mathbb{Q}})^{W_{G,\theta}}$ be the subgroup of $W_{G,\theta}$ -invariants in $\text{Div}^T(Z^c)_{\mathbb{Q}}$; we can identify this subgroup with its image in $\text{Pic}(Z^c)_{\mathbb{Q}}$. Moreover, this image is $i^*(\text{Pic}(X)_{\mathbb{Q}}) = i^*(\text{Div}^G(X)_{\mathbb{Q}})$.

Now, we prove the first statement; suppose D big. Then we can assume, up to linear equivalence, that $mD = A + M$ where $m \gg 0$, A is an ample, B -stable divisor and M is an effective, B -stable divisor. Then $mi^*D = i^*A + i^*M$, with i^*A ample. By Theorem 3.1 we can write, up to linear equivalence, $M =$

$\sum_{E \in \mathcal{N}(X)} a_E E + \sum b_\alpha^+ F_\alpha^+ + \sum b_\alpha^- F_\alpha^-$ with positive coefficients. Thus $i^*(M) = \sum_{E \in \mathcal{N}(X)} a_E E|Z^c + \frac{1}{2} \sum (b_\alpha^+ + b_\alpha^-) i^*(Y_\alpha)$ is effective, so i^*D is big.

Vice versa, suppose that $i^*(D)$ is big and that D is G -stable. Fix an ample, G -stable divisor A on X , then, for $m \gg 0$, $mi^*(D) - i^*(A)$ is linearly equivalent to an effective, T -stable divisor M' on Z^c . We claim that we can choose $i^*(mD - A)$ as M' . Remark that $i^*(mD - A)$ is $W_{G,\theta}$ -invariant.

Let D_1 and D_2 be T -stable divisors on Z^c such that: 1) D_1 is $W_{G,\theta}$ invariant; 2) D_2 is linear equivalent to 0; 3) $D_1 + D_2$ is effective. We claim that D_1 is effective. Indeed, suppose by contradiction D_1 non-effective and write $D_1 = \sum_{E \in \mathcal{N}(X), w \in W_{G,\theta}} a_{E,w} wE|Z$, $D_2 = \sum_{E \in \mathcal{N}(X), w \in W_{G,\theta}} b_{E,w} wE|Z$. Then $D_2 \neq 0$ and there is a strictly negative $a_{E,w}$. Notice that there is (E, w') such that $b_{E,w'} \leq 0$ because D_2 is principal. Thus $a_{E,w'} + b_{E,w'} = a_{E,w} + b_{E,w'} < 0$, a contradiction. Thus there is $M \in \text{Div}^G(X)_\mathbb{Q}$ with $i^*M = M'$ and $mD = A + M$. Moreover, the coefficient of M with respect to any E is equal to the coefficient of M' with respect to any E is equal to the coefficient of M' with respect to $E|Z$, which we know to be positive. \square

Remark 10. If D is in the kernel of i^* , then it is not big. Moreover, if ρ is not injective, there are non-big divisors D with $i^*(D)$ big. Suppose by simplicity X toroidal and let $\alpha_i^\vee \notin \rho(D(G/H)^H)$, then $\sum_{j \neq i} F_{\alpha_j} + 3F_{\alpha_i^+} - F_{\alpha_i^-}$ is not big, but it is equal to $\sum_j F_{\alpha_j} + 2(F_{\alpha_i^+} - F_{\alpha_i^-})$, where $\sum_j F_j$ is big and $(F_{\alpha_i^+} - F_{\alpha_i^-}) \in \ker i^*$.

Let D be a B -stable nef Cartier divisor on a projective symmetric variety. By the Remark 7 we can write $D \sim D_1 + D_2$ with: i) D_1 is G -stable, effective and nef; ii) D_2 is a positive linear combination of the F_α^+ , up to exchanging some F_α^+ with the corresponding F_α^- . Moreover D is big if and only if D_1 is big. Remark, however, that the previous choice of F_α^+ in $\rho(\alpha)$ depends on D and that, if ρ is not injective, we can always find another nef divisor for which such choice does not hold, for example $D_1 + F_\alpha^-$. Thus to study the bigness of any fixed B -stable nef Cartier divisor, we can reduce ourselves to study the bigness of an opportune G -stable nef Cartier divisor. Observe that if none F_α^+ belongs to $\mathcal{F}(X)$ (for example, if X is toroidal), then $h^{D_1+D_2} = h^{D_1}$. We say that D satisfies (*) if it is equal to $D_1 + D_2$, with D_1, D_2 as before. Remark the, given any B -stable and nef D , we can rename the F_α^+ so that D satisfies (*).

Theorem 4.2. *Let X be a projective symmetric variety. Let D be a nef, B -stable Cartier \mathbb{Q} -divisor on X which satisfies (*) and let h be the piecewise linear function over the support of $\mathbf{F}(X)$ associated to D_1 . Write $h = \{h_C\}_{\{(C,\mathcal{F})\}}$ where the $\{(C,\mathcal{F})\}$ are the maximal colored cones. Then D is big if and only if $(\sum h_C, R^\vee) \neq 0$ for each irreducible factor R^\vee of $R_{G,\theta}^\vee$. Moreover, if X is toroidal or if D is G -stable, then h is also the piecewise linear function associated to D .*

The idea of the proof is the following: first, we reduce to study the pullback of $\mathcal{O}(D)$ to Z^c by the previous proposition. Then, we study $\text{vol}(W_{G,\theta} \cdot (\sum h_C))$ to verify if such line bundle is big.

Proof. We can suppose X toroidal and \mathbb{Q} -factorial. Let s be the rank of G/H (equal to the dimension of Z^c). In this way the support of h can change, but the weight $\sum h_C$ doesn't change. Moreover, we can suppose that D is G -stable by Remark 7. By Proposition 4.1, D is big if and only if $i^*(D)$ is big, where $i : Z^c \rightarrow X$ is the inclusion. Observe that $i^*(D)$ is nef and globally generated. Denoted by $Q(D)$ the polytope $\text{convex}(W_{G,\theta}\{h_C\})$ (as in §1.7), $i^*(D)$ is big if and only if $\text{vol}(i^*(D)) := i^*(D)^s = s! \text{vol}(Q(D))$ is strictly positive. Observe that the h_C are antidominant because D is nef and G -stable.

First, suppose that D is big and suppose by contradiction that there is an irreducible factor R^\vee of $R_{G,\theta}^\vee$ such that $(\sum h_{\mathcal{C}}, R^\vee) = 0$. Then $(h_{\mathcal{C}}, R^\vee) = 0$ for each \mathcal{C} because the $h_{\mathcal{C}}$ are antidominant. Let α^\vee be any simple coroot in R^\vee , then $(wh_{\mathcal{C}}, \alpha^\vee) = 0$ for all \mathcal{C} and for all $w \in W_{G,\theta}$. So $Q(D)$ is contained in the hyperplane $(\alpha^\vee, \cdot) = 0$, thus it has volume 0.

Now, suppose that it is verified the condition $(\sum h_{\mathcal{C}}, R^\vee) \neq 0$ for each irreducible factor R^\vee of $R_{G,\theta}$. Let s be rank of X and let n be the number of s -dimensional colored cones, then $v_1 = \frac{1}{n} \sum_{(\mathcal{C}, \mathcal{F}): \dim \mathcal{C}=s} h_{\mathcal{C}}$ belongs to $Q(D)$, so $\text{convex}(W_{G,\theta}v_1) \subset Q(D)$. Thus it is sufficient to prove that $\text{vol}(\text{convex}(W_{G,\theta}v_1)) > 0$. Write $\overline{R}_{G,\theta} = I_1 \sqcup J_1$ with $(v_1, \alpha) \neq 0$ if and only if $\alpha \in I_1$. If J_1 is not empty, then, by hypothesis, there are $\alpha \in I_1$ and $\beta \in J_1$ such that $(\alpha, \beta) \neq 0$. Thus $(s_\alpha v_1, \beta) = -(\beta, \alpha^\vee)(v_1, \alpha) < 0$; moreover $s_\alpha v_1 \in Q(D)$. Hence $v_2 = \frac{2}{3}v_1 + \frac{1}{3}s_\alpha v_1$ is antidominant and belongs to $\text{convex}(W_{G,\theta}v_1)$. Write $\overline{R}_{G,\theta} = I_2 \sqcup J_2$ with $(v_2, \alpha) \neq 0$ if and only if $\alpha \in I_2$. Notice that $I_1 \subsetneq I_2$; in particular J_2 contains no irreducible factor of $R_{G,\theta}$. By induction we can find m such $\overline{R}_{G,\theta} = I_m$ and v_m belongs to $\text{convex}(W_{G,\theta}v_1)$; in particular v_m is strictly antidominant. So $\text{vol}(\text{convex}(W_{G,\theta}v_1)) \geq \text{vol}(\text{convex}(W_{G,\theta}v_m)) > 0$. \square

Remark that, if (G, θ) is indecomposable, we obtain that any non-trivial, G -stable, nef, Cartier \mathbb{Q} -divisor is big. This fact can also be proved directly, because in this case the morphism associated to the divisor has to be birational.

5. FINAL REMARKS

The results of this work cannot be extended to a general spherical variety. In particular, Theorem 3.1 is false if H has infinite index in $N_G(H)$, i.e. G/H is not sober. This means that the valuation cone $\text{cone}(\mathcal{N})$ is not strictly convex. First, the class of two distinct G -stable prime divisors can generate the same halfline. For example $\mathbb{P}^1 = \mathbb{C}^* \cup \{0\} \cup \{\infty\}$, seen as toric variety, has two G -stable prime divisors and Picard number 1. Furthermore, it is not clear how to extend the Theorem 3.2 to the horospherical varieties. Consider for example $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^2 \oplus \mathbb{C})$ as completion of SL_2/U , where U is the group of upper triangular matrices with diagonal entries equal to 1. This variety has one color, one G -stable divisor, rank 1 and Picard number 1. In particular, one can show that, given an equivariant morphism $SL_3/U \rightarrow SL_3/H$ onto a spherical space, H is U , SL_3 , a Borel subgroup or the semidirect product of U with a cyclic group (indeed, there is a Borel subgroup such that $U = [B, B] \subset H \subset B$ and $B/U \cong \mathbb{C}^*$). We have take these examples from [Br07], §4.1.

Also the Theorem 4.2 does not hold for a general spherical variety. We can generalize it in two ways: i) substituting $R_{G,\theta}$ with the spherical root system; ii) substituting $R_{G,\theta}^\vee$ with the image of colors. But, in the first case the horospherical varieties have not spherical roots and the statement would be trivially satisfied by all the divisors. In the second case, the flag manifolds have rank zero, so the image of ρ has to be 0. Therefore none divisor can satisfy the statement. We hope that the Theorem 4.2 can be generalized to any sober spherical variety. To this aim, it would be useful to prove such theorem using the definition of big divisors based on $\dim H^0(X, mL)$. Unfortunately, we have not succeeded in doing it, even for the symmetric case.

ACKNOWLEDGMENTS

The author has been supported by the M.P.I.M of Bonn.

REFERENCES

- [Bi90] E. Bifet, *On complete symmetric varieties*, Adv. Math. **80** (1990), no. 2, 225-249.
- [Br89] M. Brion, *Groupe de Picard et nombres caractéristiques des variétés sphérique*, Duke Math. J. **58**, (1989), no. 2, 397-424.
- [Br93] M. Brion, *Variétés sphériques et théorie de Mori*, Duke Math. J. **72**, (1993), no. 2, 369-404.
- [Br97a] M. Brion, *Variétés sphériques*, Notes de cours, 1997.
- [Br97b] M. Brion, *Curves and divisors in spherical varieties*, in *Algebraic groups and Lie Groups* Austral. Math. Soc. Lect. Ser. **9**, (1997), 21-34.
- [Br07] M. Brion, *The total coordinate ring of a wonderful variety*, J. Algebra. **313**, (2007), no. 1, 61-99.
- [ChMa03] R. Chirivì and A. Maffei, *The ring of sections of a complete symmetric variety*, J. Algebra, **261**, (2003), no. 2, 310-326.
- [dCoPr83] C. De Concini and C. Procesi, *Complete symmetric varieties*, in *Invariant Theory*, Lecture notes in Math **996**, 1-44, Springer-Verlag, New York, 1983.
- [FS09] O. Fujino and H. Sato, *Smooth projective toric varieties whose nontrivial nef line bundles are big*, Proc. Japan Acad. Ser. A Math. Sci. **85**, (2009), no. 7, 89-94.
- [Kn91] F. Knop, *The Luna-Vust theory of spherical embedding*, in *Proceeding of the Hyderabad Conference on Algebraic Groups*, Manoj-Prakashan, (1991), pp. 225-249.
- [La04] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.
- [Le97] C. W. Lee, *Subdivisions and triangulations of polytopes*, in Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, (1997), 271-290.
- [LuVu] D. Luna and T. Vust, *Plongements d'espaces homogènes*, Comment. Math. Helv. **58** (1983), no. 2, 186-245.
- [Od88] T. Oda, *Convex bodies and algebraic geometry*, Ergeb. Math., 3. Folge, vol. 15, **9**, Berlin Heidelberg New York: Springer (1988).
- [Ru07] A. Ruzzi, *Smooth projective symmetric varieties with Picard number equal to one*, Internat. J. Math. **22**, (2011), n.2, 145-177.
- [T06] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, preprint, 2006, arXiv: math.AG/0602228.
- [Vu74] T. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes*, Bull. Soc. Math. France. **102** (1974), 317-333.
- [Vu90] T. Vust, *Plongements d'espace symétriques algébriques: une classification*, Ann. Scuola Norm. Pisa Cl. Sci. (4) **17**, (1990), n. 2, 165-195.
- Current address:* DIPARTIMENTO DI MATEMATICA "GUIDO CASTELNUOVO", "SAPIENZA" UNIVERSITÀ DI ROMA, PIAZZALE ALDO MORO 5, 00185 ROMA, ITALY
- E-mail address:* ruzzi@mat.uniroma1.it