



**HAL**  
open science

## A progenerator for representations of $SL(n,q)$ in transverse characteristic

Cédric Bonnafé

► **To cite this version:**

Cédric Bonnafé. A progenerator for representations of  $SL(n,q)$  in transverse characteristic. 2011.  
hal-00578542

**HAL Id: hal-00578542**

**<https://hal.science/hal-00578542>**

Submitted on 21 Mar 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A PROGENERATOR FOR REPRESENTATIONS OF $\mathbf{SL}_n(\mathbb{F}_q)$ IN TRANSVERSE CHARACTERISTIC

CÉDRIC BONNAFÉ

ABSTRACT. Let  $G = \mathbf{GL}_n(\mathbb{F}_q)$ ,  $\mathbf{SL}_n(\mathbb{F}_q)$  or  $\mathbf{PGL}_n(\mathbb{F}_q)$ , where  $q$  is a power of some prime number  $p$ , let  $U$  denote a Sylow  $p$ -subgroup of  $G$  and let  $R$  be a commutative ring in which  $p$  is invertible. Let  $D(U)$  denote the derived subgroup of  $U$  and let  $e = \frac{1}{|D(U)|} \sum_{u \in D(U)} u$ . The aim of this note is to prove that the  $R$ -algebras  $RG$  and  $eRGe$  are Morita equivalent (through the natural functor  $RG\text{-mod} \rightarrow eRGe\text{-mod}$ ,  $M \mapsto eM$ ).

Let  $n$  be a non-zero natural number,  $p$  a prime number,  $q$  a power of  $p$  and let  $\mathbb{F}_q$  denote a finite field with  $q$  elements. Let  $G_n = \mathbf{SL}_n(\mathbb{F}_q)$ . We denote by  $U_n$  the group of  $n \times n$  unipotent upper triangular matrices with coefficients in  $\mathbb{F}_q$  (so that  $U_n$  is a Sylow  $p$ -subgroup of  $G_n$ ). Let  $D(U_n)$  denote its derived subgroup: then, with  $N = (n-1)(n-2)/2$ ,

$$D(U_n) = \left\{ \begin{pmatrix} 1 & 0 & a_1 & \cdots & \cdots & a_{n-2} \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & a_N \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \mid a_1, a_2, \dots, a_N \in \mathbb{F}_q \right\}.$$

We fix a commutative ring  $R$  in which  $p$  is invertible and we set

$$e_n = \frac{1}{|D(U_n)|} \sum_{u \in D(U_n)} u \in RD(U_n).$$

Then  $e_n$  is an idempotent of  $RG_n$ . The aim of this note is to prove the following result (recall that an idempotent  $i$  of a ring  $A$  is called *full* if  $A = AiA$ ):

**Theorem 1.** *If  $p$  is invertible in  $R$ , then  $e_n$  is a full idempotent of  $RG_n$ .*

*Proof.* First, let  $R_0 = \mathbb{Z}[1/p]$ , let  $\zeta$  be a primitive  $p$ -th root of unity in  $\mathbb{C}$  and let  $\hat{R}_0 = R_0[\zeta]$ . Let  $\mathfrak{J}_0 = R_0 G_n e_n R_0 G_n$  and  $\tilde{\mathfrak{J}}_0 = \hat{R}_0 G_n e_n \hat{R}_0 G_n$ . Since  $p$  is invertible in  $R$ , there is a unique morphism of rings  $R_0 \rightarrow R$  which extends to a morphism of rings

---

*Date:* March 21, 2011.

*1991 Mathematics Subject Classification.* According to the 2000 classification: Primary 20C10; Secondary 20C20, 20C33.

$R_0G_n \rightarrow RG_n$ . So if  $1 \in \mathfrak{I}_0$ , then  $1 \in \mathfrak{I}$ . Also, as  $(1, \zeta, \dots, \zeta^{p-2})$  is an  $R_0$ -basis of  $\hat{R}_0$ , it is also an  $R_0G_n$ -basis of  $\hat{R}_0G_n$ . Therefore, if  $1 \in \hat{R}_0G_n e_n \hat{R}_0G_n = \hat{R}_0 \otimes_{R_0} (R_0G_n e_n R_0G_n)$ , then  $1 \in \mathfrak{I}_0$ . Consequently, in order to prove Theorem 1, we may (and we shall) work under the following hypothesis:

**Hypothesis.** *From now on, and until the end of this proof, we assume that  $R = \mathbb{Z}[1/p, \zeta]$ .*

Now, let  $P_n$  denote the subgroup of  $\mathbf{SL}_n(\mathbb{F}_q)$  defined by

$$P_n = \left\{ \left( \begin{array}{c|c} M & \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} \\ \hline 0 & 1 \end{array} \right) \mid M \in \mathbf{SL}_{n-1}(\mathbb{F}_q) \text{ and } a_1, \dots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then  $U_n \subset P_n$ . We shall prove by induction on  $n$  that

( $\mathcal{P}_n$ )  $e_n$  is a full idempotent of  $RP_n$ .

It is clear that Theorem 1 follows immediately from ( $\mathcal{P}_n$ ).

As  $e_1 = 1$  and  $e_2 = 1$ , it follows that ( $\mathcal{P}_1$ ) and ( $\mathcal{P}_2$ ) hold. So assume that  $n \geq 3$  and that ( $\mathcal{P}_{n-1}$ ) holds. Let  $I_n$  denote the identity  $n \times n$  matrix and let

$$V_n = \left\{ \left( \begin{array}{c|c} I_{n-1} & \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} \\ \hline 0 & 1 \end{array} \right) \mid a_1, \dots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then  $V_n \simeq (\mathbb{F}_q^+)^{n-1}$  and  $P_n = \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes V_n \simeq \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes (\mathbb{F}_q^+)^{n-1}$ . We set  $V'_n = D(U_n) \cap V_n$ , so that  $V'_n \simeq (\mathbb{F}_q^+)^{n-2}$  is normalized by  $P_{n-1}$ . Then

$$D(U_n) = D(U_{n-1}) \ltimes V'_n.$$

We now define

$$f_n = \frac{1}{|V'_n|} \sum_{v \in V'_n} v,$$

so that

$$e_n = e_{n-1} f_n.$$

By the induction hypothesis, there exists  $g_1, h_1, \dots, g_l, h_l$  in  $P_{n-1}$  and  $r_1, \dots, r_l$  in  $R$  such that

$$1 = \sum_{i=1}^l r_i g_i e_{n-1} h_i.$$

Therefore, as  $P_{n-1}$  normalizes  $V'_n$ , it centralizes  $f_n$  and so

$$f_n = \left( \sum_{i=1}^l r_i g_i e_{n-1} h_i \right) f_n = \sum_{i=1}^l r_i g_i e_{n-1} f_n h_i = \sum_{i=1}^l r_i g_i e_n h_i.$$

So  $f_n \in RP_n e_n RP_n$ .

Let  $\mu_p$  denote the subgroup of  $R^\times$  generated by  $\zeta$ . If  $\chi \in \text{Hom}(V_n, \mu_p)$ , we define  $b_\chi$  to be the associated primitive idempotent of  $RV_n$ :

$$b_\chi = \frac{1}{|V_n|} \sum_{v \in V_n} \chi(v)^{-1} v \quad \in \quad RV_n.$$

Then, as  $V_n$  is an elementary abelian  $p$ -group, we get

$$f_n = \sum_{\substack{\chi \in \text{Hom}(V_n, \mu_p) \\ \text{Res}_{V'_n} \chi = 1}} b_\chi.$$

We fix a non-trivial element  $\chi_0 \in \text{Hom}(V_n, \mu_p)$  whose restriction to  $V'_n$  is trivial. Then

$$b_{\chi_0} = b_{\chi_0} f_n \quad \text{and} \quad b_1 = b_1 f_n,$$

so  $b_1$  and  $b_{\chi_0}$  belong to  $RP_n e_n RP_n$ .

But  $\mathbf{SL}_{n-1}(\mathbb{F}_q) \subset P_n$  has only two orbits for its action on  $\text{Hom}(V_n, \mu_p)$ : the orbit of 1 and the orbit of  $\chi_0$ . Therefore,  $b_\chi \in RP_n e_n RP_n$  for all  $\chi \in \text{Hom}(V_n, \mu_p)$ . Consequently,

$$1 = \sum_{\chi \in \text{Hom}(V_n, \mu_p)} b_\chi \in RP_n e_n RP_n,$$

as desired. □

**Finite reductive groups.** Let  $\mathbb{F}$  be an algebraic closure of  $\mathbb{F}_q$ , let  $\mathbf{G}$  be a connected reductive group over  $\mathbb{F}$  and let  $F : \mathbf{G} \rightarrow \mathbf{G}$  be an isogeny such that some power  $F^\delta$  is a Frobenius endomorphism relative to an  $\mathbb{F}_q$ -structure. We denote by  $\mathbf{U}$  an  $F$ -stable maximal unipotent subgroup of  $\mathbf{G}$  (it is the unipotent radical of an  $F$ -stable Borel subgroup). Define

$$e = \frac{1}{|\mathbf{D}(\mathbf{U})^F|} \sum_{u \in \mathbf{D}(\mathbf{U})^F} u \quad \in \quad \mathbf{RG}^F.$$

The next result follows immediately from Theorem 1:

**Theorem 2.** *Assume that  $(\mathbf{G}, F)$  is split of type A. Then  $e$  is a full idempotent of  $\mathbf{RG}^F$ .*

**Corollary 3.** *If  $(\mathbf{G}, F)$  is split of type A, then the functors*

$$\begin{array}{ccc} \mathbf{RG}^F\text{-mod} & \longrightarrow & e\mathbf{RG}^F e\text{-mod} \\ M & \longmapsto & eM \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{RG}^F e\text{-mod} & \longrightarrow & \mathbf{RG}^F\text{-mod} \\ N & \longmapsto & \mathbf{RG}^F e \otimes_{e\mathbf{RG}^F e} N \end{array}$$

*are mutually inverse equivalences of categories. In particular,  $\mathbf{RG}^F$  and  $e\mathbf{RG}^F e$  are Morita equivalent, and  $\mathbf{RG}^F e$  is a progenerator for  $\mathbf{RG}^F$ .*

*Proof.* This follows from Theorem 2 and, for instance, [3, Example 18.30]. □

**Examples.** Theorem 2 and Corollary 3 can be applied for instance in the case where  $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$ ,  $\mathbf{SL}_n(\mathbb{F}_q)$  or  $\mathbf{PGL}_n(\mathbb{F}_q)$ .

**Comments.** (1) It is natural to ask whether Theorem 2 (or Corollary 3) can be generalized to other finite reductive groups. In fact, it cannot be generalized: indeed, if for instance  $R = \mathbb{C}$ , then saying that  $e$  is a full idempotent of  $R\mathbf{G}^F$  means that every irreducible character of  $\mathbf{G}^F$  is an irreducible component of an Harish-Chandra induced of some Gelfand-Graev character. But, if  $\mathbf{G}$  is quasi-simple and  $(\mathbf{G}, F)$  is not split of type  $A$ , then  $\mathbf{G}^F$  contains a unipotent character which does not belong to the principal series: this character cannot be an irreducible component of an Harish-Chandra induced of a Gelfand-Graev character.

(2) In [1], a crucial step for the proof of a special case of the geometric version of Broué's abelian defect conjecture was [1, Theorem 4.1], where R. Rouquier and the author have proved the above Theorem 2 in the case where  $R$  is the integral closure of  $\mathbb{Z}_\ell$  in a sufficiently large algebraic extension of  $\mathbb{Q}_\ell$  (here,  $\ell$  is a prime number different from  $p$ ). The proof was essentially based on the classification, due to Dipper [2, 4.15 and 5.23], of simple modules for  $G_n$  in characteristic  $\ell$ , and especially of cuspidal ones, which involves Deligne-Lusztig theory.

The interest of the proof given here is that it does not rely on any classification of simple modules, and is based on elementary methods: as a by-product of this elementariness, Theorem 2 and Corollary 3 are valid over any commutative ring (in which  $p$  is invertible, which is a necessary condition if one wants the idempotent  $e_n$  to be well-defined).

#### REFERENCES

- [1] C. BONNAFÉ & R. ROUQUIER, Coxeter orbits and modular representations, *Nagoya Math. J.* **183** (2006), 1-34.
- [2] R. DIPPER, On quotients of Hom-functors and representations of finite general linear groups II, *J. Algebra* **209** (1998), 199-269.
- [3] T.-Y. LAM, *Lectures on Modules and Rings*, Graduate Texts in Mathematics **189**, Springer, 1999, xxiv + 557 pages.

CNRS - UMR 5149, INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER,  
UNIVERSITÉ MONTPELLIER 2, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX,  
FRANCE

*E-mail address:* `cedric.bonnafe@math.univ-montp2.fr`

*URL:* `http://ens.math.univ-montp2.fr/~bonnafe/`