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► **To cite this version:**

Ivan Nourdin, Giovanni Peccati. Poisson approximations on the free Wigner chaos. *Annals of Probability*, 2013, 41 (4), pp.2709-2723. 10.1214/12-AOP815 . hal-00578465v2

**HAL Id: hal-00578465**

**<https://hal.science/hal-00578465v2>**

Submitted on 4 Apr 2011

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# Poisson approximations on the free Wigner chaos

by Ivan Nourdin\* and Giovanni Peccati†

**Abstract:** We prove that an adequately rescaled sequence  $\{F_n\}$  of self-adjoint operators, living inside a fixed free Wigner chaos of even order, converges in distribution to a centered free Poisson random variable with rate  $\lambda > 0$  if and only if  $\varphi(F_n^4) - 2\varphi(F_n^3) \rightarrow 2\lambda^2 - \lambda$  (where  $\varphi$  is the relevant tracial operator). This extends to a free setting some recent limit theorems by Nourdin and Peccati (2009), and provides a non-central counterpart to a result by Kemp *et al.* (2011). As a by-product of our findings, we show that Wigner chaoses of order strictly greater than 2 do not contain non-zero free Poisson random variables. Our techniques involve the so-called ‘Riordan numbers’, counting non-crossing partitions without singletons.

**Key words:** Catalan numbers; Contractions; Free Brownian motion; Free cumulants; Free Poisson distribution; Free probability; Marchenko-Pastur; Non-central limit theorems; Non-crossing partitions; Riordan numbers; Semicircular distribution; Wigner chaos.

**2000 Mathematics Subject Classification:** 46L54; 60H05; 60H07; 60H30.

## 1 Introduction

### 1.1 Overview

Let  $W$  be a standard Brownian motion on  $\mathbb{R}_+$  and let  $q \geq 1$  be an integer. For every deterministic symmetric function  $f \in L^2(\mathbb{R}_+^q)$ , we denote by  $I_q^W(f)$  the multiple stochastic Wiener-Itô integral of  $f$  with respect to  $W$ . Random variables of the form  $I_q^W(f)$  compose the so-called  $q$ th *Wiener chaos* associated with  $W$ . The concept of Wiener chaos roughly represents an infinite-dimensional analogous of Hermite polynomials for the one-dimensional Gaussian distribution (see e.g. [14] for an introduction to this topic).

The following two results, proved respectively in [13] and [9], provide an exhaustive characterization of normal and Gamma approximations on Wiener chaos. As in [9], we denote by  $F(\nu)$  a centered random variable with the law of  $2G(\nu/2) - \nu$ , where  $G(\nu/2)$  has a Gamma distribution with parameter  $\nu/2$  (if  $\nu \geq 1$  is an integer, then  $F(\nu)$  has a centered  $\chi^2$  distribution with  $\nu$  degrees of freedom).

**Theorem 1.1** (A) *Let  $N \sim \mathcal{N}(0, 1)$ , fix  $q \geq 2$  and let  $I_q^W(f_n)$  be a sequence of multiple stochastic integrals with respect to the standard Brownian motion  $W$ , where each  $f_n$  is a symmetric element of  $L^2(\mathbb{R}_+^q)$  such that  $E[I_q^W(f_n)^2] = q! \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = 1$ . Then, the following two assertions are equivalent, as  $n \rightarrow \infty$ :*

(i)  $I_q^W(f_n)$  converges in distribution to  $N$ ;

(ii)  $E[I_q^W(f_n)^4] \rightarrow E[N^4] = 3$ .

(B) *Fix  $\nu > 0$ , and let  $F(\nu)$  have the centered Gamma distribution described above. Let  $q \geq 2$  be an even integer, and let  $I_q^W(f_n)$  be a sequence of multiple stochastic integrals, where each*

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$f_n$  is symmetric and verifies  $E[I_q^W(f_n)^2] = E[F(\nu)^2] = 2\nu$ . Then, the following two assertions are equivalent, as  $n \rightarrow \infty$ :

- (i)  $I_q^W(f_n)$  converges in distribution to  $F(\nu)$ ;
- (ii)  $E[I_q(f_n)^4] - 12E[I_q(f_n)^3] \rightarrow E[F(\nu)^4] - 12E[F(\nu)^3] = 12\nu^2 - 48\nu$ .

The results stated in Theorem 1.1 provide a drastic simplification of the so-called *method of moments* for probabilistic approximations, and have triggered a huge amount of applications and generalizations, involving e.g. Stein's method, Malliavin calculus, power variations of Gaussian processes, Edgeworth expansions, random matrices and universality results. See [10, 11], as well as the forthcoming monograph [12], for an overview of the most important developments. See [8] for a constantly updated web resource, with links to all available papers on the subject.

In [5], together with Kemp and Speicher, we proved an analogue of Part A of Theorem 1.1 in the framework of free probability (and free Brownian motion). Let  $(\mathcal{A}, \varphi)$  be a free probability space and let  $\{S(t) : t \geq 0\}$  be a free Brownian motion defined therein (see Section 3 for details). As shown in [3], one can define multiple integrals of the type  $I_q^S(f)$ , where  $f$  is a square-integrable complex kernel (to simplify the notation, throughout the paper we shall drop the suffixes  $q, S$ , and write simply  $I(f) = I_q^S(f)$ ). Random variables of the type  $I(f)$  compose the so-called *Wigner chaos* associated with  $S$ , playing in free stochastic analysis a role analogous to that of the classical Gaussian Wiener chaos (see for instance [3], where Wigner chaoses are used to develop a free version of the Malliavin calculus of variations). The following statement is the main result of [5].

**Theorem 1.2** *Let  $s$  be a centered semicircular random variable with unit variance (see Definition 2.3(i)), fix an integer  $q \geq 2$ , and let  $I(f_n)$  be a sequence of multiple integrals of order  $q$  with respect to the free Brownian motion  $S$ , where each  $f_n$  is a mirror symmetric (see Section 3) element of  $L^2(\mathbb{R}_+^q)$  such that  $\varphi[I_q(f_n)^2] = \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = 1$ . Then, the following two assertions are equivalent, as  $n \rightarrow \infty$ :*

- (i)  $I(f_n)$  converges in distribution to  $s$ ;
- (ii)  $\varphi[I(f_n)^4] \rightarrow \varphi[s^4] = 2$ .

The principal aim of this paper is to prove a free analogue of Part B of Theorem 1.1. As explained in Section 2, and somehow counterintuitively, the free analogue of Gamma random variables is given by free Poisson random variables (see Definition 2.3(ii), and also [6, p. 203]). The free Poisson law is also known as the *Marchenko-Pastur distribution*, arising in random matrix theory as the limit of the eigenvalue distribution of large sample covariance matrices (see e.g. Bai and Silverstein [1, Ch. 3], Hiai and Petz [4, pp. 101-103 and 130] and the references therein). The following statement is the main achievement of the present work.

**Theorem 1.3** *Let  $q \geq 2$  be an even integer. Let  $Z(\lambda)$  have a centered free Poisson distribution with rate  $\lambda > 0$ . Let  $I(f_n)$  be a sequence of multiple integrals of order  $q$  with respect to the free Brownian motion  $S$ , where each  $f_n$  is a mirror symmetric element of  $L^2(\mathbb{R}_+^q)$  such that  $\varphi[I_q(f_n)^2] = \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \varphi[Z(\lambda)^2] = \lambda$ . Then, the following two assertions are equivalent, as  $n \rightarrow \infty$ :*

- (i)  $I(f_n)$  converges in distribution to  $Z(\lambda)$ ;
- (ii)  $\varphi[I(f_n)^4] - 2\varphi[I(f_n)^3] \rightarrow \varphi[Z(\lambda)^4] - 2\varphi[Z(\lambda)^3] = 2\lambda^2 - \lambda$ .

One should note that the techniques involved in our proofs are different from those adopted in the previously quoted references, as they are based on a direct enumeration of contractions. These contractions emerge when iteratively applying product formulae for multiple Wigner integrals – see also [7]. One crucial point is that the moments of a free Poisson random variable can be expressed in terms of the so-called *Riordan numbers*, counting the number of non-crossing partitions without singletons (see e.g. [2]). We also stress that one cannot expect to have convergence to a non-zero free Poisson inside a Wigner chaos of odd order, since random variables inside such chaoses have all odd moments equal to zero, while one has e.g. that  $\varphi[Z(\lambda)^3] = \lambda$  (see Remark 2.5(ii)).

As a consequence of Theorem 1.3, we will be able to prove the following result, stating that Wigner chaoses of order greater than 2 do not contain any non-zero Poisson random variable.

**Proposition 1.4** *Let  $q \geq 4$  be even, and let  $F$  be a non-zero random variable in the  $q$ th Wigner chaos. Then,  $F$  cannot have a free Poisson distribution.*

As pointed out in Remark 3.2 below, centered Poisson random variables with integer rate can be realized as elements of the second Wigner chaos. As a consequence, Proposition 1.4 implies that the second Wigner chaos contains random variables whose distribution is not shared by any element of higher chaoses. This result parallels the findings of [5], where it is proved that Wigner chaoses of order  $\geq 2$  do not contain any non-zero semicircular random variable. Note that, at the present time, it is not known in general whether two non-zero random variables belonging to two distinct Wigner chaoses have necessarily different laws.

**Remark 1.5** We are still far from understanding the exact structure of the free Wigner chaos. For instance, almost nothing is known about the regularity of the distributions associated with the elements of a fixed Wigner chaos. In particular, we ignore whether such laws may have atoms or are indeed absolutely continuous (as those in the classical Wiener chaos).

## 1.2 The free probability setting

Our main reference for free probability is the monograph by Nica and Speicher [6], to which the reader is referred for any unexplained notion or result. We shall also use a notation which is consistent with the one adopted in [5].

For the rest of the paper, we consider as given a so-called (tracial)  $W^*$ -probability space  $(\mathcal{A}, \varphi)$ , where:  $\mathcal{A}$  is a von Neumann algebra of operators (with involution  $X \mapsto X^*$ ), and  $\varphi$  is a unital linear functional on  $\mathcal{A}$  with the properties of being *weakly continuous*, *positive* (that is,  $\varphi(XX^*) \geq 0$  for every  $X \in \mathcal{A}$ ), *faithful* (that is, such that the relation  $\varphi(XX^*) = 0$  implies  $X = 0$ ), and *tracial* (that is,  $\varphi(XY) = \varphi(YX)$ , for every  $X, Y \in \mathcal{A}$ ).

As usual in free probability, we refer to the self-adjoint elements of  $\mathcal{A}$  as *random variables*. Given a random variable  $X$  we write  $\mu_X$  to indicate the *law* (or *distribution*) of  $X$ , which is defined as the unique Borel probability measure on  $\mathbb{R}$  such that, for every integer  $m \geq 0$ ,  $\varphi(X^m) = \int_{\mathbb{R}} x^m \mu_X(dx)$  (see e.g. [6, Proposition 3.13]).

We say that the unital subalgebras  $\mathcal{A}_1, \dots, \mathcal{A}_n$  of  $\mathcal{A}$  are *freely independent* whenever the following property holds: let  $X_1, \dots, X_m$  be a finite collection of elements chosen among the  $\mathcal{A}_i$ 's in such a way that (for  $j = 1, \dots, m-1$ )  $X_j$  and  $X_{j+1}$  do not come from the same  $\mathcal{A}_i$  and  $\varphi(X_j) = 0$  for  $j = 1, \dots, m$ ; then  $\varphi(X_1 \cdots X_m) = 0$ . Random variables are said to be freely independent if they generate freely independent unital subalgebras of  $\mathcal{A}$ .

### 1.3 Plan

The rest of the paper is organized as follows. In Section 2 we provide a characterization of centered free Poisson distributions in terms of non-crossing partitions. Section 3 deals with free Brownian motion and Wigner chaos. Section 4 contains the proofs of the main results of the paper (that is, Theorem 1.3 and Proposition 1.4), whereas Section 5 is devoted to some auxiliary lemmas.

## 2 Semicircular and centered free Poisson distributions

The following definition contains most of the combinatorial objects that are used throughout the text.

- Definition 2.1** (i) Given an integer  $m \geq 1$ , we write  $[m] = \{1, \dots, m\}$ . A *partition* of  $[m]$  is a collection of non-empty and disjoint subsets of  $[m]$ , called *blocks*, such that their union is equal to  $[m]$ . The cardinality of a block is called *size*. A block is said to be a *singleton* if it has size one.
- (ii) A partition  $\pi$  of  $[m]$  is said to be *non-crossing* if one cannot find integers  $p_1, q_1, p_2, q_2$  such that: (a)  $1 \leq p_1 < q_1 < p_2 < q_2 \leq m$ , (b)  $p_1, p_2$  are in the same block of  $\pi$ , (c)  $q_1, q_2$  are in the same block of  $\pi$ , and (d) the  $p_i$ 's are not in the same block of  $\pi$  as the  $q_i$ 's. The collection of the non-crossing partitions of  $[m]$  is denoted by  $NC(m)$ ,  $m \geq 1$ .
- (iii) For every  $m \geq 1$ , the quantity  $C_m = |NC(m)|$ , where  $|A|$  indicates the cardinality of a set  $A$ , is called the  *$m$ th Catalan number*. One sets by convention  $C_0 = 1$ . Also, recall the explicit expression  $C_m = \frac{1}{m+1} \binom{2m}{m}$ .
- (iv) We define the sequence  $\{R_m : m \geq 0\}$  as follows:  $R_0 = 1$ , and, for  $m \geq 1$ ,  $R_m$  is equal to the number of partitions in  $NC(m)$  having no singletons.
- (v) For every  $m \geq 1$  and every  $j = 1, \dots, m$ , we define  $R_{m,j}$  to be the number of non-crossing partitions of  $[m]$  with exactly  $j$  blocks and with no singletons. Plainly,  $R_m = \sum_{j=1}^m R_{m,j}$ . Also, when  $m$  is even, one has that  $R_{m,j} = 0$  for every  $j > m/2$ ; when  $m$  is odd, then  $R_{m,j} = 0$  for every  $j > (m-1)/2$ .

**Example 2.2** One has that:

- $R_1 = R_{1,1} = 0$ , since  $\{\{1\}\}$  is the only partition of  $[1]$ , and such a partition is composed of exactly one singleton;
- $R_2 = R_{2,1} = 1$ , since the only partition of  $[2]$  with no singletons is  $\{\{1, 2\}\}$ ;
- $R_3 = R_{3,1} = 1$ , since the only partition of  $[3]$  with no singletons is  $\{\{1, 2, 3\}\}$ ;

- $R_4 = 3$ , since the only non-crossing partitions of  $[4]$  with no singletons are  $\{\{1, 2, 3, 4\}\}$ ,  $\{\{1, 2\}, \{3, 4\}\}$  and  $\{\{1, 4\}, \{2, 3\}\}$ . This implies that  $R_{4,1} = 1$  and  $R_{4,2} = 2$ .

The integers  $\{R_m : m \geq 0\}$  are customarily called the *Riordan numbers*. A detailed analysis of the combinatorial properties of Riordan numbers is provided in the paper by Bernhart [2]; however, it is worth noting that the discussion to follow is self-contained, in the sense that no previous knowledge of the combinatorial properties of the sequence  $\{R_m\}$  is required.

Given a random variable  $X$ , we denote by  $\{\kappa_m(X) : m \geq 1\}$  the sequence of the *free cumulants* of  $X$ . We recall (see [6, p. 175]) that the free cumulants of  $X$  are completely determined by the following relation: for every  $m \geq 1$

$$\varphi(X^m) = \sum_{\pi=\{b_1, \dots, b_j\} \in NC(m)} \prod_{i=1}^j \kappa_{|b_i|}(X), \quad (2.1)$$

where  $|b_i|$  indicates the size of the block  $b_i$  of the non-crossing partition  $\pi$ . It is clear from (2.1) that the sequence  $\{\kappa_m(X) : m \geq 1\}$  completely determines the moments of  $X$  (and viceversa).

**Definition 2.3** (i) The centered *semicircular distribution* of parameter  $t > 0$ , denoted by  $S(0, t)(dx)$ , is the probability distribution given by

$$S(0, t)(dx) = (2\pi t)^{-1} \sqrt{4t - x^2} dx, \quad |x| < 2\sqrt{t}.$$

We recall the classical relation:

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m} S(0, t)(dx) = C_m t^m,$$

where  $C_m$  is the  $m$ th Catalan number (so that e.g. the second moment of  $S(0, t)$  is  $t$ ). Since the odd moments of  $S(0, t)$  are all zero, one deduces from the previous relation and (2.1) (e.g. by recursion) that the free cumulants of a random variable  $s$  with law  $S(0, t)$  are all zero, except for  $\kappa_2(s) = \varphi(s^2) = t$ .

(ii) The *free Poisson distribution* with rate  $\lambda > 0$ , denoted by  $P(\lambda)(dx)$  is the probability distribution defined as follows: (a) if  $\lambda \in (0, 1]$ , then  $P(\lambda) = (1 - \lambda)\delta_0 + \lambda\tilde{\nu}$ , and (b) if  $\lambda > 1$ , then  $P(\lambda) = \tilde{\nu}$ , where  $\delta_0$  stands for the Dirac mass at 0. Here,  $\tilde{\nu}(dx) = (2\pi x)^{-1} \sqrt{4\lambda - (x - 1 - \lambda)^2} dx$ ,  $x \in ((1 - \sqrt{\lambda})^2, (1 + \sqrt{\lambda})^2)$ . If  $X_\lambda$  has the  $P(\lambda)$  distribution, then [6, Proposition 12.11] implies that

$$\kappa_m(X_\lambda) = \lambda, \quad m \geq 1. \quad (2.2)$$

From now on, we will denote by  $Z(\lambda)$  a random variable having the law of  $X_\lambda - \lambda 1$  (centered free Poisson distribution), where 1 is the unity of  $\mathcal{A}$ . Plainly,  $\kappa_1[Z(\lambda)] = \varphi[Z(\lambda)] = 0$ .

Note that both  $S(0, t)$  and  $P(\lambda)$  are compactly supported, and therefore are uniquely determined by their moments (by the Weierstrass theorem). Definition 2.3-(ii) is taken from [6, Definition 12.12]. The choice of the denomination “free Poisson” comes from the following

two facts: (1)  $P(\lambda)$  can be obtained as the limit of the free convolution of Bernoulli distributions (see [6, Proposition 12.11]), and (2) the classical Poisson distribution of parameter  $\lambda$  has (classical) cumulants all equal to  $\lambda$  (see e.g. [14, Section 3.3]). As recalled in the Introduction, the free Poisson law is also called the ‘‘Marchenko-Pastur distribution’’.

The following statement contains a characterization of the moments of  $Z(\lambda)$ , and shows that, when  $\lambda$  is integer, then  $Z(\lambda)$  is the free equivalent of a classical centered  $\chi^2$  random variable with  $\lambda$  degrees of freedom. This last fact could alternatively be deduced from [6, Proposition 12.13], but here we prefer to provide a self-contained argument.

**Proposition 2.4** *Let the notation of Definition 2.1 and Definition 2.3 prevail. Then, for every real  $\lambda > 0$  and every integer  $m \geq 1$ ,*

$$\varphi[Z(\lambda)^m] = \sum_{j=1}^m \lambda^j R_{m,j}. \quad (2.3)$$

*Let  $p = 1, 2, \dots$  be an integer, then  $Z(p)$  has the same law as  $\sum_{i=1}^p (s_i^2 - 1)$ , where  $s_1, \dots, s_p$  are  $p$  freely independent random variables with the  $S(0, 1)$  distribution, and 1 is the unit of  $\mathcal{A}$ .*

*Proof.* From (2.2), one deduces that  $\kappa_m[Z(\lambda)] = \lambda$  for every  $m \geq 2$ . Since  $\kappa_1[Z(\lambda)] = 0$ , we infer from (2.1) that

$$\varphi[Z(\lambda)^m] = \sum_{\pi=\{b_1, \dots, b_j\} \in NC(m)} \lambda^j \mathbf{1}_{\{\pi \text{ has no singletons}\}},$$

which immediately yields (2.3). To prove the last part of the statement, consider first the case  $p = 1$ , write  $s = s_1$  and fix an integer  $m \geq 2$ . In order to build a non-crossing partition of  $[m]$ , say  $\pi$ , one has to perform the following three steps: (a) choose an integer  $j \in \{0, \dots, m\}$ , denoting the number of singletons of  $\pi$ , (b) choose the  $j$  singletons of  $\pi$  among the  $m$  available integers (this can be done in exactly  $\binom{m}{j}$  distinct ways), (c) build a non-crossing partition of the remaining  $m - j$  integers with blocks at least of size 2 (this can be done in exactly  $R_{m-j}$  distinct ways). Since  $C_0 = R_0 = 1$  and  $C_1 = 1 = R_0 + R_1$ , it follows that Catalan and Riordan numbers are linked by the following relation: for every  $m \geq 0$

$$C_m = \sum_{j=0}^m \binom{m}{j} R_{m-j} = \sum_{j=0}^m \binom{m}{j} R_j, \quad (2.4)$$

where the last equality follows from  $\binom{m}{j} = \binom{m}{m-j}$ . By inversion, one therefore deduces that

$$R_m = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} C_j, \quad m \geq 0.$$

Therefore

$$\begin{aligned} \varphi[(s^2 - 1)^m] &= \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} \varphi(s^{2j}) = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} C_j \\ &= R_m = \sum_{j=1}^m R_{m,j} = \varphi[Z(1)^m], \end{aligned}$$



from which we infer that  $s^2 - 1 \stackrel{\text{law}}{=} Z(1)$ , yielding the desired conclusion when  $p = 1$ . Let us now consider the general case, that is,  $p \geq 2$ . By free independence, for any  $m \geq 2$ , we have that

$$\kappa_m \left( \sum_{i=1}^p (s_i^2 - 1) \right) = p \times \kappa_m(s_1^2 - 1) = p \times \kappa_m(Z(1)) = p = \kappa_m(Z(p)).$$

This implies that  $\sum_{i=1}^p s_i^2 - 1 \stackrel{\text{law}}{=} Z(p)$ , and the proof of Proposition 2.4 is concluded.  $\square$

**Remark 2.5** (i) Relation (2.4) is well known – see e.g. [2, Section 5] for an alternate proof based on “difference triangles”.

(ii) Using the last two points of Example 2.2, we deduce from (2.3) that  $\varphi[Z(\lambda)^3] = \lambda R_{3,1} = \lambda$ , while  $\varphi[Z(\lambda)^4] = \lambda R_{4,1} + \lambda^2 R_{4,2} = \lambda + 2\lambda^2$ .

### 3 Free Brownian motion and Wigner chaos

Our main reference for the content of this section is the paper by Biane and Speicher [3].

**Definition 3.1** ( $L^p$  spaces) (i) For  $1 \leq p \leq \infty$ , we write  $L^p(\mathcal{A}, \varphi)$  to indicate the  $L^p$  space obtained as the completion of  $\mathcal{A}$  with respect to the norm  $\|a\|_p = \varphi(|a|^p)^{1/p}$ , where  $|a| = \sqrt{a^*a}$ , and  $\|\cdot\|_\infty$  stands for the operator norm.

(ii) For every integer  $q \geq 2$ , the space  $L^2(\mathbb{R}_+^q)$  is the collection of all complex-valued functions on  $\mathbb{R}_+^q$  that are square-integrable with respect to the Lebesgue measure. Given  $f \in L^2(\mathbb{R}_+^q)$ , we write

$$f^*(t_1, t_2, \dots, t_q) = \overline{f(t_q, \dots, t_2, t_1)},$$

and we call  $f^*$  the *adjoint* of  $f$ . We say that an element of  $L^2(\mathbb{R}_+^q)$  is *mirror symmetric* if

$$f(t_1, \dots, t_q) = f^*(t_1, \dots, t_q),$$

for almost every vector  $(t_1, \dots, t_q) \in \mathbb{R}_+^q$ . Notice that mirror symmetric functions constitute a Hilbert subspace of  $L^2(\mathbb{R}_+^q)$ .

(iii) Given  $f \in L^2(\mathbb{R}_+^q)$  and  $g \in L^2(\mathbb{R}_+^p)$ , for every  $r = 1, \dots, \min(q, p)$  we define the  $r$ th *contraction* of  $f$  and  $g$  as the element of  $L^2(\mathbb{R}_+^{p+q-2r})$  given by

$$\begin{aligned} f \frown^r g(t_1, \dots, t_{p+q-2r}) & \\ = \int_{\mathbb{R}_+^r} f(t_1, \dots, t_{q-r}, y_r, y_{r-1}, \dots, y_1) g(y_1, y_2, \dots, y_r, t_{q-r+1}, t_{p+q-2r}) dy_1 \cdots dy_r. & \end{aligned} \quad (3.5)$$

One also writes  $f \frown^0 g(t_1, \dots, t_{p+q}) = f \otimes g(t_1, \dots, t_{p+q}) = f(t_1, \dots, t_q) g(t_{q+1}, \dots, t_{p+q})$ . In the following, we shall use the notations  $f \frown^0 g$  and  $f \otimes g$  interchangeably. Observe that, if  $p = q$ , then  $f \frown^p g = \langle f, g^* \rangle_{L^2(\mathbb{R}_+^q)}$ .



A free Brownian motion  $S$  on  $(\mathcal{A}, \varphi)$  consists of: (i) a filtration  $\{\mathcal{A}_t : t \geq 0\}$  of von Neumann sub-algebras of  $\mathcal{A}$  (in particular,  $\mathcal{A}_u \subset \mathcal{A}_t$ , for  $0 \leq u < t$ ), (ii) a collection  $S = \{S(t) : t \geq 0\}$  of self-adjoint operators such that:

- $S(t) \in \mathcal{A}_t$  for every  $t$ ;
- for every  $t$ ,  $S(t)$  has a semicircular distribution  $S(0, t)$ , with mean zero and variance  $t$ ;
- for every  $0 \leq u < t$ , the ‘increment’  $S(t) - S(u)$  is freely independent of  $\mathcal{A}_u$ , and has a semicircular distribution  $S(0, t - u)$ , with mean zero and variance  $t - u$ .

For every integer  $q \geq 1$ , the collection of all random variables of the type  $I_q^S(f) = I(f)$ ,  $f \in L^2(\mathbb{R}_+^q)$ , is called the  $q$ th Wigner chaos associated with  $S$ , and is defined according to [3, Section 5.3], namely:

- first define  $I(f) = (S(b_1) - S(a_1)) \dots (S(b_q) - S(a_q))$ , for every function  $f$  having the form

$$f(t_1, \dots, t_q) = \prod_{i=1}^q \mathbf{1}_{(a_i, b_i)}(t_i), \quad (3.6)$$

where the intervals  $(a_i, b_i)$ ,  $i = 1, \dots, q$ , are pairwise disjoint;

- extend linearly the definition of  $I(f)$  to ‘simple functions vanishing on diagonals’, that is, to functions  $f$  that are finite linear combinations of indicators of the type (3.6);
- exploit the isometric relation

$$\langle I(f_1), I(f_2) \rangle_{L^2(\mathcal{A}, \varphi)} = \varphi(I(f_1)^* I(f_2)) = \varphi(I(f_1^*) I(f_2)) = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}_+^q)}, \quad (3.7)$$

where  $f_1, f_2$  are simple functions vanishing on diagonals, and use a density argument to define  $I(f)$  for a general  $f \in L^2(\mathbb{R}_+^q)$ .

Observe that relation (3.7) continues to hold for every pair  $f_1, f_2 \in L^2(\mathbb{R}_+^q)$ . Moreover, the above sketched construction implies that  $I(f)$  is self-adjoint if and only if  $f$  is mirror symmetric. Finally, we recall the following fundamental multiplication formula, proved in [3]. For every  $f \in L^2(\mathbb{R}_+^q)$  and  $g \in L^2(\mathbb{R}_+^p)$ , where  $q, p \geq 1$ ,

$$I(f)I(g) = \sum_{r=0}^{\min(q,p)} I(f \frown^r g). \quad (3.8)$$

**Remark 3.2** Let  $\{e_i : 1, \dots, p\}$  be an orthonormal system in  $L^2(\mathbb{R}_+)$ . Then, the random variables  $s_i = I(e_i)$ ,  $i = 1, \dots, p$ , have the  $S(0, 1)$  distribution and are freely independent. Moreover, the product formula (3.8) implies that

$$\sum_{i=1}^p (s_i^2 - 1) = I\left(\sum_{i=1}^p e_i \otimes e_i\right),$$

and therefore that the double integral  $I(\sum_{i=1}^p e_i \otimes e_i)$  has a centered free Poisson distribution with rate  $p$ .

## 4 Proof of the main results

### 4.1 Proof of Theorem 1.3

In the free probability setting (see e.g. [6, Definition 8.1]) convergence in distribution is equivalent to the convergence of moments, so that  $I(f_n)$  converges in distribution to  $Z(\lambda)$  if and only if  $\varphi(I(f_n)^m) \rightarrow \varphi(Z(\lambda)^m)$ , for every  $m \geq 1$ . In particular, convergence in distribution implies  $\varphi(I(f_n)^4) - 2\varphi(I(f_n)^3) \rightarrow \varphi(Z(\lambda)^4) - 2\varphi(Z(\lambda)^3) = 2\lambda^2 - \lambda$ .

Now assume that  $\varphi[I(f_n)^4] - 2\varphi[I(f_n)^3] \rightarrow 2\lambda^2 - \lambda$ . We have to show that, for every  $m \geq 2$ ,  $\varphi[I(f_n)^m] \rightarrow \varphi[Z(\lambda)^m]$ . Iterative applications of the product formula (3.8) yield

$$I(f_n)^m = \sum_{(r_1, \dots, r_{m-1}) \in A_m} I((\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n),$$

where

$$A_m = \{(r_1, \dots, r_{m-1}) \in \{0, 1, \dots, q\}^{m-1} : r_2 \leq 2q - 2r_1, r_3 \leq 3q - 2r_1 - 2r_2, \dots, r_{m-1} \leq (m-1)q - 2r_1 - \dots - 2r_{m-2}\}.$$

We deduce that

$$\varphi[I(f_n)^m] = \sum_{(r_1, \dots, r_{m-1}) \in B_m} (\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n,$$

with  $B_m = \{(r_1, \dots, r_{m-1}) \in A_m : 2r_1 + \dots + 2r_{m-1} = mq\}$ . We decompose  $B_m$  as follows:  $B_m = D_m \cup E_m$ , with  $D_m = B_m \cap \{0, \frac{q}{2}, q\}^{m-1}$  and  $E_m = B_m \setminus D_m$ , so that

$$\begin{aligned} \varphi[I(f_n)^m] &= \sum_{(r_1, \dots, r_{m-1}) \in D_m} (\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n \\ &+ \sum_{(r_1, \dots, r_{m-1}) \in E_m} (\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n. \end{aligned} \quad (4.9)$$

By the forthcoming Lemma 5.1, we have  $\|f_n \frown^{q/2} f_n - f_n\| \rightarrow 0$  and  $\|f_n \frown^r f_n\| \rightarrow 0$  for  $r \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$ . The conclusion is then obtained by observing that the first sum in (4.9) converges to  $\varphi[Z(\lambda)^m]$  by Proposition 2.4 and the forthcoming Lemma 5.2, whereas the second sum converges to zero by the forthcoming Lemma 5.4.  $\square$

### 4.2 Proof of Proposition 1.4

Assume that  $F = I(f)$ , where  $f$  is a mirror symmetric element of  $L^2(\mathbb{R}_+^q)$  for some even  $q \geq 4$ , and also that  $\varphi[F^2] = \|f\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda > 0$ . If  $F$  had the same law of  $Z(\lambda)$ , then  $\varphi(F^4) - 2\varphi(F^3) = 2\lambda^2 - \lambda$ , and the forthcoming Lemma 5.1 would imply that  $\|f \frown^{q/2} f - f\|_{L^2(\mathbb{R}_+^q)} = 0$  and  $\|f \frown^r f\|_{L^2(\mathbb{R}_+^{2q-2r})} = 0$  for all  $r \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$ . As shown in [5, Proof of Corollary 1.7], the relation  $\|f \frown^{q-1} f\|_{L^2(\mathbb{R}_+^2)} = 0$  implies that necessarily  $f = 0$ , and therefore that  $F = 0$ . Since  $\varphi[F^2] = \lambda > 0$  we have achieved a contradiction, and the proof is concluded.  $\square$

## 5 Ancillary lemmas

This section collects some technical results that are used in the proof of Theorem 1.3.

**Lemma 5.1** *Let  $q \geq 2$  be an even integer, and consider a sequence  $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$  of mirror symmetric functions such that  $\|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda > 0$  for every  $n$ . As  $n \rightarrow \infty$ , one has that*

$$\varphi[I(f_n)^4] - 2\varphi[I(f_n)^3] \rightarrow 2\lambda^2 - \lambda$$

*if and only if  $\|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \rightarrow 0$  and  $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \rightarrow 0$  for all  $r \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$ .*

*Proof.* The product formula yields

$$I(f_n)^2 - I(f_n) = \lambda + I(f_n \overset{0}{\frown} f_n) + I(f_n \overset{q/2}{\frown} f_n - f_n) + \sum_{\substack{1 \leq r \leq q-1 \\ r \neq q/2}} I(f_n \overset{r}{\frown} f_n).$$

Using the isometry property and the fact that multiple Wigner integrals of different orders are orthogonal in  $L^2(\mathcal{A}, \varphi)$ , we deduce that

$$\begin{aligned} & \varphi[(I(f_n)^2 - I(f_n))^2] \\ &= \lambda^2 + \|f_n \overset{0}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q})}^2 + \|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)}^2 + \sum_{\substack{1 \leq r \leq q-1 \\ r \neq q/2}} \|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})}^2 \\ &= 2\lambda^2 + \|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)}^2 + \sum_{\substack{1 \leq r \leq q-1 \\ r \neq q/2}} \|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})}^2, \end{aligned}$$

and the desired conclusion follows because  $\varphi[I(f_n)^2] = \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda$ .  $\square$

**Lemma 5.2** *Let  $m \geq 2$  be an integer, let  $q \geq 2$  be an even integer, and recall the notation adopted in (4.9). Assume  $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$  is a sequence of mirror symmetric functions satisfying  $\|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda > 0$  for every  $n$ . If  $\|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \rightarrow 0$  as  $n \rightarrow \infty$ , then*

$$\sum_{(r_1, \dots, r_{m-1}) \in D_m} (\dots ((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \dots f_n) \overset{r_{m-1}}{\frown} f_n \rightarrow \varphi[Z(\lambda)^m] = \sum_{j=1}^m \lambda^j R_{m,j}, \quad (5.10)$$

as  $n \rightarrow \infty$ .

*Proof.* Assume that  $f_n \overset{q/2}{\frown} f_n \approx f_n$  (given two sequences  $\{a_n\}$  and  $\{b_n\}$  with values in some normed vector space, we write  $a_n \approx b_n$  to indicate that  $a_n - b_n \rightarrow 0$  with respect to the associated norm), and consider  $(r_1, \dots, r_{m-1}) \in D_m$ . Using the identities  $f_n \overset{0}{\frown} f_n = f_n \otimes f_n$ ,  $f_n \overset{q}{\frown} f_n = \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda$  and  $f_n \overset{q/2}{\frown} f_n \approx f_n$ , it is evident that

$$(\dots ((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \dots f_n) \overset{r_{m-1}}{\frown} f_n \rightarrow \lambda^j,$$

where  $j$  equals the number of the entries of  $(r_1, \dots, r_{m-1})$  that are equal to  $q$ . It follows that, for every  $m \geq 2$ , there exists a polynomial  $w_m(\lambda)$  (independent of  $q$ ) such that, for every sequence  $\{f_n\}$  as in the statement,

$$\sum_{(r_1, \dots, r_{m-1}) \in D_m} (\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n \rightarrow w_m(\lambda).$$

Now consider the case  $q = 2$  and  $f_n = f = \sum_{i=1}^p e_i \otimes e_i$ , where  $p \geq 1$  and  $\{e_i : i = 1, \dots, p\}$  is an orthonormal system in  $L^2(\mathbb{R}_+^q)$ . The following three facts are in order: (a)  $I(\sum_{i=1}^p e_i \otimes e_i)$  has the same law as  $Z(p)$  (see Remark 3.2), (b)  $\|f\|_{L^2(\mathbb{R}_+^2)}^2 = p$ , and (c)  $f \frown^1 f = f$ . Since  $E_m = \emptyset$  for  $q = 2$ , the previous discussion (combined with (4.9) and Proposition 2.4) yields that, for every  $m \geq 2$ ,  $w_m(p) = \varphi[Z(p)^m] = \sum_{j=1}^m p^j R_{m,j}$ , for every  $p = 1, 2, \dots$ . Since two polynomials coinciding on a countable set are necessarily equal, we deduce that  $w_m(\lambda) = \varphi[Z(\lambda)^m]$  for every  $\lambda > 0$ .  $\square$

**Remark 5.3** By inspection of the arguments used in the proof of Lemma 5.2, one deduces that  $R_m = |D_m|$ , for every  $m \geq 2$ .

**Lemma 5.4** *Let  $m \geq 2$  be an integer, let  $q \geq 2$  be an even integer, and recall the notation adopted in (4.9). Assume  $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$  is a sequence of mirror symmetric functions satisfying  $\|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda > 0$  for every  $n$ . If  $(r_1, \dots, r_{m-1}) \in E_m$  and if  $\|f_n \frown^r f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \rightarrow 0$  for all  $r \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$ , then*

$$(\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots f_n) \frown^{r_{m-1}} f_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*Proof.* Let  $j \in \{1, \dots, m-1\}$  be the smallest integer such that  $r_j \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$ . Then

$$\begin{aligned} & |(\dots ((f_n \frown^{r_1} f_n) \frown^{r_2} f_n) \dots \frown^{r_{m-1}} f_n)| \\ &= |(\dots ((f_n \frown^{r_1} \dots \frown^{r_{j-1}} f_n) \frown^{r_j} f_n) \frown^{r_{j+1}} \dots \frown^{r_{m-1}} f_n)| \\ &\approx_{(*)} C |(\dots ((f_n \otimes \dots \otimes f_n) \frown^{r_j} f_n) \frown^{r_{j+1}} \dots \frown^{r_{m-1}} f_n)| \quad (\text{using } f_n \frown^{q/2} f_n \approx f_n \text{ and } f_n \frown^q f_n = \lambda) \\ &\leq_{(**)} C \|(f_n \otimes \dots \otimes f_n) \otimes (f_n \otimes_{r_j} f_n)\| \times \|f_n\|_{L^2(\mathbb{R}_+^q)}^{m-j-1} \quad (\text{by Cauchy-Schwarz}) \\ &= C \|f_n \frown^{r_j} f_n\|_{L^2(\mathbb{R}_+^{2q-2r_j})} \quad (\text{since } \|f_n\|_{L^2(\mathbb{R}_+^q)}^2 = \lambda) \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

In the previous computations, we have adopted the following conventions: (a)  $C$  is a finite constant independent of  $n$  that may change from line to line, (b) the tensor products in  $(*)$  and  $(**)$  have an unspecified order which depends on  $(r_1, \dots, r_{k-1})$ , and (c) the first norm in  $(**)$  refers to an appropriate  $L^2(\mathbb{R}_+^s)$  space, for some  $s$  depending of the order of the tensor product.  $\square$

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