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STRICHARTZ ESTIMATES FOR SCHRÖDINGER-LAGUERRE OPERATORS

SALEM BEN SAID

ABSTRACT. In $\mathbb{R}^n \times \mathbb{R}$ we consider the Schrödinger equation

$$i\partial_t u(x, t) + L_k u(x, t) = F(x, t) \tag{\mathcal{E}_1}$$

with given boundary values on \mathbb{R}^n . Here $L_k = \|x\|\Delta_k - \|x\|$ is a differential-difference-multiplication operator on \mathbb{R}^n , where k is a multiplicity function for the Dunkl Laplacian Δ_k . In the $k \equiv 0$ case, L_0 is the classical multi-dimensional Laguerre operator. In this paper we obtain Strichartz estimates for the Schrödinger equation (\mathcal{E}_1). We then prove that Strichartz estimates for the Schrödinger equation

$$i\partial_t v(x, t) + \|x\|\Delta_k v(x, t) = F(x, t) \tag{\mathcal{E}_2}$$

can be obtained from those for (\mathcal{E}_1). The $k \equiv 0$ case is already new.

1. INTRODUCTION AND MAIN RESULTS

Strichartz estimates are a type of a-priori estimates for the solutions of a large class of linear partial differential equations whose common property is that their solutions tend to disperse over time. Originally, such estimates were proved by R. Strichartz [20] in the late 1970's for the wave equation but later researchers extended them to other dispersive equations.

By the mid 1990's Strichartz estimates became a standard tool in the analysis of Schrödinger equations. Such estimates are powerful tools in the study of nonlinear Schrödinger equations. For instance, they proved fruitful for determining whether various Schrödinger equations are well-posed (see e.g. [4, 21, 13, 23, 10, 15, 1] and references therein); thus several attempts have been made to establish Strichartz estimates for perturbed Schrödinger equations.

The primary aim of this paper is to prove Strichartz estimates for a Schrödinger equation associated with a Laguerre type operator coming from Dunkl's theory of differential-difference operators. To illustrate the main results, let us fix notation.

Let R be a reduced root system in \mathbb{R}^n , and let $k : R \rightarrow \mathbb{R}^+$, $\alpha \mapsto k_\alpha$, be a multiplicity function. Let Δ_k be the Dunkl Laplacian operator (see (2.3)). We note that when $k \equiv 0$, Δ_k reduces to the Euclidean Laplacian Δ .

For $1 \leq p \leq \infty$, let $L^p(\mathbb{R}^n, \lambda_k(x) dx)$ be the space of L^p -integrable functions on \mathbb{R}^n with respect to the measure $\lambda_k(x) dx$, where

$$\lambda_k(x) = \|x\|^{-1} \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k_\alpha}.$$

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The density function λ_k has a degree of homogeneity equals to $\gamma_k - 1$, where

$$\gamma_k = \sum_{\alpha \in \mathbb{R}} k_\alpha. \quad (1.1)$$

For a multiplicity function k , we consider the following operator

$$L_k := \|x\| \Delta_k - \|x\|, \quad (1.2)$$

where $\|x\|$ on the right hand side of the formula stands for the multiplication operator by $\|x\|$. This family of operators includes the multi-dimensional Laguerre operator $L_0 = \|x\| \Delta - \|x\|$. There is a rich structure theory associated with L_k , and, in particular, it has a discrete spectrum. We refer to [2, 3] for more details.

The solution u to the k -Laguerre Schrödinger equation

$$\begin{cases} i\partial_t u(x, t) + L_k u(x, t) = F(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x, 0) = f(x), \end{cases} \quad (1.3)$$

is given by

$$u(x, t) = e^{itL_k} f(x) - i \int_0^t e^{i(t-s)L_k} F(x, s) ds. \quad (1.4)$$

The crucial tool in establishing Strichartz estimates for the solution $u(x, t)$ is the integral representation of the semigroup

$$\mathcal{S}_k : \mathbb{C}^+ \ni z \mapsto e^{zL_k}$$

proved in [3]. Here $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \geq 0\}$. We pin down that Strichartz estimates with the specialization $k \equiv 0$ are already new.

As the map $t \mapsto \|e^{itL_k} f\|_{L^p(\mathbb{R}^n, \lambda_k(x) dx)}$ is going to be π -periodic, and thus determined by its values for $t \in I := [-\pi/2, \pi/2]$, we will measure the size of the solution $u(x, t)$ to (1.3) by means of the mixed space-time norm

$$\|u\|_{L^q(I, L^p(\mathbb{R}^n, \lambda_k(x) dx))} := \left(\int_I \left(\int_{\mathbb{R}^n} |u(x, t)|^p \lambda_k(x) dx \right)^{q/p} dt \right)^{1/q}, \quad (1.5)$$

with the obvious modifications if p or $q = \infty$, in which case we use the supremum. To simplify the notation, we shall write $L_t^q L_x^p$ instead of $L^q(I, L^p(\mathbb{R}^n, \lambda_k(x) dx))$, where the lower-indices in $L_t^q L_x^p$ denote the relevant variables.

Definition 1.1. A pair (p, q) of exponents will be called k -admissible if $(\frac{1}{p}, \frac{1}{q})$ lies in the following trapezoid (see Figure 1)

$$\begin{aligned} T_k = & \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \left] \frac{1}{2} \left(\frac{\gamma_k + n - 2}{\gamma_k + n - 1}, \frac{1}{2} \right), \frac{1}{2} \right] \times \left[\frac{1}{2}, 1 \right] \right\} \cup \\ & \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \left] 0, \frac{1}{2} \right] \times \left[0, \frac{1}{2} \left[\mid \frac{1}{q} \geq (\gamma_k + n - 1) \left(\frac{1}{2} - \frac{1}{p} \right) \right] \right\}. \end{aligned} \quad (1.6)$$

Now we state the main results.

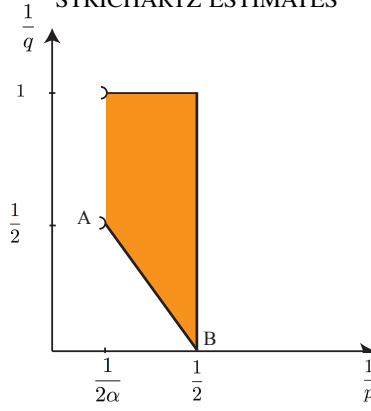


FIGURE 1. The admissible trapezoid T_k (with $\alpha = \frac{\gamma_k + n - 1}{\gamma_k + n - 2}$).

Theorem A (Strichartz estimates). *Suppose k is a non-negative multiplicity function such that*

$$\gamma_k + n - 1 > 0. \quad (1.7)$$

Then, for any k -admissible exponent (p, q) we have:

1. *(The homogeneous Strichartz estimate)*

$$\|e^{itL_k} f\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2}. \quad (1.8)$$

2. *(The dual homogeneous Strichartz estimate)*

$$\left\| \int_I e^{-isL_k} F(x, s) ds \right\|_{L_x^2} \lesssim \|F\|_{L_t^{q'} L_x^{p'}}, \quad (1.9)$$

where p' and q' are the conjugate exponents of p and q .

3. *(The diagonal retarded Strichartz estimate)*

$$\left\| \int_0^t e^{i(t-s)L_k} F(x, s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{q'} L_x^{p'}}. \quad (1.10)$$

Observe that the condition (1.7) is automatically satisfied for $n \geq 2$.

The following is an immediate consequence of the above statement.

Corollary B. *Retain the assumptions of Theorem A. The solution*

$$u(x, t) = e^{itL_k} f(x) - i \int_0^t e^{i(t-s)L_k} F(x, s) ds$$

to the Cauchy problem

$$\begin{cases} i\partial_t u(x, t) + L_k u(x, t) = F(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x, 0) = f(x), \end{cases} \quad (1.11)$$

satisfies

$$\|u\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{q'} L_x^{p'}}, \quad (1.12)$$

for some initial datum $f \in L^2(\mathbb{R}^n, \lambda_k(x) dx)$ and $F \in L^{q'}(I, L^{p'}(\mathbb{R}^n, \lambda_k(x) dx))$.

Write the set of k -admissible pairs (1.6) as $T_k = T_k^{(1)} \cup T_k^{(2)}$ in the obvious way. We will show that for any k -admissible exponents (p, q) and (\tilde{p}, \tilde{q}) in $T_k^{(2)}$ we have the *non-diagonal* Strichartz estimate

$$\|u\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}. \quad (1.13)$$

The proof uses the well know TT^* method (see e.g. [11]).

The second part of this paper deals with the k -Schrödinger equation

$$\begin{cases} i\partial_t v(x, t) + \|x\| \Delta_k v(x, t) = F(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(x, 0) = f(x), \end{cases} \quad (1.14)$$

for real-valued $f \in L^2(\mathbb{R}^n, \lambda_k(x) dx)$. Our aim is to obtain non-diagonal Strichartz estimates for $v(x, t)$ via the non-diagonal Strichartz estimates (1.13). The first step towards this goal is to find a link between the semigroups $e^{it\|x\|\Delta_k}$ and e^{itL_k} . To do so, we searched for the expression of the heat kernel corresponding to the heat operator

$$\partial_t + \|x\| \Delta_k. \quad (1.15)$$

In polar coordinates $x = \|x\|\omega$ and $y = \|y\|\omega'$, we prove that the heat kernel is given by

$$\Gamma_k(x, y; t) = c_k \Gamma\left(\frac{\gamma_k + n - 1}{2}\right) t^{-(\gamma_k + n - 1)} \exp\left(-\frac{\|x\| + \|y\|}{t}\right) V_k\left(\tilde{I}_{\frac{\gamma_k + n - 3}{2}}\left(\frac{\sqrt{2\|x\|\|y\|(1 + \langle \omega, \cdot \rangle)}}{t}\right)\right)(\omega'),$$

where V_k is the so-called Dunkl intertwining operator and $\tilde{I}_\lambda(z) = \left(\frac{z}{2}\right)^{-\lambda} I_\lambda(z)$ is the (normalized) modified Bessel function of the first kind.

This allowed us to obtain the integral representation of the semigroup $e^{it\|x\|\Delta_k}$, which in turn permits us to prove the following:

Theorem C. *For all $1 \leq p, q \leq \infty$ satisfying*

$$\frac{1}{q} = (\gamma_k + n - 1) \left(\frac{1}{2} - \frac{1}{p}\right), \quad (1.16)$$

we have

$$\|e^{itL_k} f\|_{L_t^q L_x^p((0, \pi/2) \times \mathbb{R}^n)} = \|e^{it\|x\|\Delta_k} f\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^n)}. \quad (1.17)$$

As we will see below, it does not matter whether the t -interval in (1.8) is $I = [-\pi/2, \pi/2]$ or $[0, \pi/2]$; the two mixed norms are proportional for real-valued functions f . Thus, as an immediate consequence of the estimate (1.8) and Theorem C, we obtain:

Theorem D. *For any k -admissible exponent (p, q) satisfying the additional condition (1.16), i.e. for any exponent (p, q) such that $(\frac{1}{p}, \frac{1}{q})$ belongs to the borderline AB in Figure 1, we have the homogeneous Strichartz estimate*

$$\|e^{it\|x\|\Delta_k} f\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \|f\|_{L_x^2(\mathbb{R}^n)}. \quad (1.18)$$

Using the homogeneous Strichartz estimate (1.18), and by the same proof as for the non-diagonal estimate (1.13), one obtains:

Corollary E. *For any exponents (p, q) and (\tilde{p}, \tilde{q}) that satisfy the assumptions of Theorem D we have the non-diagonal Strichartz estimate*

$$\|v\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \|f\|_{L_x^2(\mathbb{R}^n)} + \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}(\mathbb{R}^+ \times \mathbb{R}^n)}. \quad (1.19)$$

A very preliminary version of this paper, posted in March 2011 on <http://hal.archives-ouvertes.fr/hal-00578446/fr/>, contained also similar results for the Dunkl harmonic oscillator $H_k = \Delta_k - \|x\|^2$. In [16] (published in 2013) the author considered Strichartz estimates for H_k . In these circumstances, we preferred to drop the harmonic case and to keep only the Laguerre case.

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2. BACKGROUND

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in \mathbb{R}^n as well as its bilinear extension to $\mathbb{C}^n \times \mathbb{C}^n$.

Let $G \subset O(n)$ be a finite reflection group on \mathbb{R}^n . Let R be the corresponding root system. We will assume that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$.

The norm $\|\cdot\|$ which is induced on \mathbb{R}^n by $\langle \cdot, \cdot \rangle$ extends to a G -invariant norm on \mathbb{C}^n , which will also be denoted by $\|\cdot\|$.

For $\alpha \in R$, let r_α denote the corresponding reflection, i.e.

$$r_\alpha(x) := x - \langle \alpha, x \rangle \alpha, \quad x \in \mathbb{R}^n.$$

The subgroup $G \subset O(n)$ is generated by the reflections $\{r_\alpha \mid \alpha \in R\}$. We refer to [12] for more details on the theory of finite reflection groups.

Definition 2.1. *A function $k : R \rightarrow \mathbb{C}$ is a multiplicity function if k is G -invariant.*

Setting $k_\alpha := k(\alpha)$ for $\alpha \in R$, we have $k_{h\alpha} = k_\alpha$ for all $h \in G$. Henceforth, K^+ denotes the set of multiplicity functions $k = (k_\alpha)_{\alpha \in R}$ such that $k_\alpha \geq 0$ for all $\alpha \in R$.

Dunkl's discovery in 1989 of the operators that now bear his name is one of the most important developments in the theory of special functions associated with root systems [6].

Definition 2.2. *Let $k \in K^+$ and $\xi \in \mathbb{C}^n$. The Dunkl operator $T_\xi := T_\xi(k)$ is defined on $C^1(\mathbb{R}^n)$ by*

$$T_\xi f(x) := \partial_\xi f(x) + \frac{1}{2} \sum_{\alpha \in R} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle},$$

where ∂_ξ denotes the directional derivative corresponding to ξ .

It is shown in [7] that for any $k \in K^+$, there is a unique linear isomorphism V_k (Dunkl's intertwining operator) on polynomials that intertwines the action of the Dunkl operators and the ordinary partial derivatives. In [8] this result had been extended to a larger

algebra of real analytic functions. Few years later, Rösler [19] proved that for each $x \in \mathbb{R}^n$ there exists a unique positive probability measure μ_x^k on \mathbb{R}^n such that

$$V_k f(x) = \int_{\mathbb{R}^n} f(\xi) d\mu_x^k(\xi). \quad (2.1)$$

The measures μ_x^k are compactly supported and $\text{supp } \mu_x^k \subset \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq \|x\|\}$.

Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. The Dunkl Laplacian operator is defined as

$$\Delta_k := \sum_{j=1}^n T_{\xi_j}^2. \quad (2.2)$$

By the normalization $\langle \alpha, \alpha \rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha \in R} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right\}, \quad (2.3)$$

where Δ and ∇ are the usual Laplacian and gradient operators, respectively. Observe that for $k \equiv 0$, the Dunkl Laplace operator Δ_k reduces to the Euclidean Laplacian Δ .

Definition 2.3. *The space of k -spherical harmonics of degree m is defined by*

$$\mathcal{H}_k^m(\mathbb{R}^n) = \text{Ker } \Delta_k \cap \mathcal{P}_m(\mathbb{R}^n),$$

where $\mathcal{P}_m(\mathbb{R}^n)$ is the space of homogeneous polynomials of degree m on \mathbb{R}^n .

Let $L^p(\mathbb{R}^n, \lambda_k(x) dx)$ be the space of L^p -integrable functions on \mathbb{R}^n with respect to the measure $\lambda_k(x) dx$ where

$$\lambda_k(x) := \|x\|^{-1} \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k_\alpha}.$$

Clearly $\lambda_k(x)$ is G -invariant and homogeneous of degree $\gamma_k - 1$, where

$$\gamma_k := \sum_{\alpha \in R} k_\alpha. \quad (2.4)$$

The following normalization constant

$$c_k := \left\{ \int_{\mathbb{R}^n} \exp(-\|x\|) \lambda_k(x) dx \right\}^{-1} \quad (2.5)$$

is needed for later use. For $k \in K^+$ such that $\langle k \rangle + n - 1 > 0$, the integral in (2.5) converges and we have

$$c_k = \Gamma(\gamma_k + n - 1)^{-1} d_k, \quad (2.6)$$

where

$$d_k := \left\{ \int_{S^{n-1}} \prod_{\alpha \in R} |\langle \alpha, \omega \rangle|^{k_\alpha} d\sigma(\omega) \right\}^{-1}. \quad (2.7)$$

The notation $X \lesssim Y$ will be used to indicate that $X \leq CY$ with a positive constant C independent of significant quantities.

3. A LAGUERRE SEMIGROUP

Consider the operator

$$L_k = \|x\|\Delta_k - \|x\|,$$

where $\|x\|$ on the right hand side of the formula stands for the multiplication operator by $\|x\|$. The operator L_k belongs to a large family of operators introduced and studied by the author, T. Kobayashi and B. Ørsted [3]. It is proved in [loc. cit.] that L_k extends to a self-adjoint operator on $L^2(\mathbb{R}^n, \lambda_k(x) dx)$, when $k \in K^+$ such that $\gamma_k + n - 1 > 0$. Moreover, there is no continuous spectrum of L_k and all the discrete spectra are positive.

For $z \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$, let

$$\mathcal{S}_k(z) := \exp(zL_k)$$

be the so-called k -Laguerre semigroup [2, 3]. In the $k \equiv 0$ case, $\mathcal{S}_0(z)$ is the Laguerre semigroup studied by T. Kobayashi and G. Mano [14]. In the following theorem we resume the main properties of $\mathcal{S}_k(z)$.

Theorem 3.1. (see [3, Theorem 3.39]) *Suppose $k \in K^+$ satisfies the condition $\gamma_k + n - 1 > 0$. Then,*

1. *The map*

$$\mathbb{C}^+ \times L^2(\mathbb{R}^n, \lambda_k(x) dx) \longrightarrow L^2(\mathbb{R}^n, \lambda_k(x) dx), \quad (z, f) \longmapsto \mathcal{S}_k(z)f$$

is continuous.

2. *The operator norm $\|\mathcal{S}_k(z)\|_{\text{op}}$ is $\exp(-(\gamma_k + n - 1)\operatorname{Re} z)$.*

3. *If $\operatorname{Re} z > 0$, then $\mathcal{S}_k(z)$ is a Hilbert Schmidt operator.*

4. *If $\operatorname{Re} z = 0$, then $\mathcal{S}_k(z)$ is a unitary operator.*

In view of the above theorem, it follows from the Schwartz kernel theorem that the operator $\mathcal{S}_k(z)$ can be expressed by means of a distribution kernel $\Lambda_k(x, y; z)$. If we adopt Gelfand's notation on generalized functions, we may write the operator $\mathcal{S}_k(z)$ on $L^2(\mathbb{R}^n, \lambda_k(x) dx)$ as an 'integral transform' against the measure $\lambda_k(x) dx$:

$$\mathcal{S}_k(z)f(x) = c_k \int_{\mathbb{R}^n} f(y)\Lambda_k(x, y; z)\lambda_k(y) dy, \quad (3.1)$$

where c_k is as in (2.6). In [3, Theorem 4.23], with T. Kobayashi and B. Ørsted, we were able to express the kernel $\Lambda_k(x, y; z)$ in terms of elementary functions. Below we will recall the expression of $\Lambda_k(x, y; z)$. To do so, let us introduce the following continuous function of $u \in [-1, 1]$ with parameters $r, s > 0$ and $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$:

$$h_k(r, s; z; u) := \Gamma\left(\frac{\gamma_k + n - 1}{2}\right) \frac{\exp\left(- (r + s) \coth(z)\right)}{\sinh(z)^{\gamma_k + n - 1}} \tilde{I}_{\frac{\gamma_k + n - 3}{2}}\left(\frac{\sqrt{2}(rs)^{1/2}}{\sinh(z)}(1 + u)^{1/2}\right), \quad (3.2)$$

where $\tilde{I}_\lambda(w) := \left(\frac{w}{2}\right)^{-\lambda} I_\lambda(w)$ is the (normalized) modified Bessel function of the first kind. Using polar coordinates $x = r\omega$ and $y = s\eta$, the kernel Λ_k is given by

$$\Lambda_k(x, y; z) = V_k^\eta h_k(r, s; z; \langle \omega, \cdot \rangle)(\eta), \quad (3.3)$$

where V_k is the Dunkl intertwining operator (see (2.1)), and the upper-script in V_k^η denotes the relevant variable. Note that if $k \equiv 0$, then $\Lambda_0(x, y; z) = h_0(r, s; z; \langle \omega, \eta \rangle)$.

Fact 3.2. From (3.2), we have for all $t \in \mathbb{R} \setminus \pi\mathbb{Z}$,

1. $\Lambda_k(x, y; -it) = \overline{\Lambda_k(x, y; it)}$.
2. $\Lambda_k(x, y; i(t + \pi)) = e^{-i\pi(\gamma_k+n-1)} \Lambda_k(x, y; it)$.

In view of Fact 3.2.1, the function $t \mapsto \|e^{itL_k} f\|_{L_x^p}$ (whenever it makes sense) is π -periodic, and thus determined by its values for $t \in [-\pi/2, \pi/2]$. For this reason, we will consider the mixed space-time norm (1.5) to measure the size of the solution $u(x, t)$ to the Cauchy problem (1.3).

The following estimates for the growth of the kernel $\Lambda_k(x, y; z)$ play a significant role in the next section.

Proposition 3.3. (see [3, Proposition 4.26]) *The kernel $\Lambda_k(x, y; z)$ satisfies:*

1. For $\operatorname{Re} z > 0$, there exists a constant $c > 0$ depending on z such that

$$|\Lambda_k(x, y; z)| \leq |\sinh z|^{-(\gamma_k+n-1)} \exp(-c(\|x\| + \|y\|)). \quad (3.4)$$

2. For $z = r + it$, with $r \geq 0$ and $t \in \mathbb{R} \setminus \pi\mathbb{Z}$, we have

$$|\Lambda_k(x, y; z)| \leq |\sin t|^{-(\gamma_k+n-1)}. \quad (3.5)$$

Fact 3.4. *As an immediate consequence of Proposition 3.3 is that e^{zL_k} is bounded on the space $L^1(\mathbb{R}^n, \lambda_k(x) dx)$ for $\operatorname{Re} z > 0$. Moreover, by Theorem 3.1, we knew that e^{zL_k} is bounded on $L^2(\mathbb{R}^n, \lambda_k(x) dx)$ for $\operatorname{Re} z \geq 0$. Then, by complex interpolation, we deduce that $e^{zL_k} f$ belongs to $L^p(\mathbb{R}^n, \lambda_k(x) dx)$ for all $1 \leq p \leq 2$ whenever $f \in L^p(\mathbb{R}^n, \lambda_k(x) dx)$ and $\operatorname{Re} z > 0$. Duality will then takes care of the range $2 \leq p \leq \infty$.*

4. A LAGUERRE-SCHRÖDINGER EQUATION

We will prove the Strichartz estimates listed in Theorem A. The first step towards those decay estimates is the following statement:

Lemma 4.1. *Let $z = r + it$ with $r \geq 0$ and $t \in \mathbb{R} \setminus \pi\mathbb{Z}$. For all $2 \leq p \leq \infty$ we have*

$$\|e^{zL_k} f\|_{L_x^p} \leq |\sin t|^{2(\gamma_k+n-1)(\frac{1}{p}-\frac{1}{2})} \|f\|_{L_x^{p'}}, \quad (4.1)$$

where p' is the conjugate exponent of p .

Proof. On the one hand, by the uniform estimate (3.5), we have

$$\|e^{zL_k} f\|_{L_x^\infty} \leq |\sin t|^{-(\gamma_k+n-1)} \|f\|_{L_x^1}, \quad \forall z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}.$$

On the other hand, by Theorem 3.1.2, we have

$$\|e^{zL_k} f\|_{L_x^2} \leq e^{-r(\gamma_k+n-1)} \|f\|_{L_x^2}.$$

Now, for all $2 \leq p \leq \infty$, Riesz-Thorin's interpolation implies

$$\|e^{zL_k} f\|_{L_x^p} \leq |\sin t|^{2(\gamma_k+n-1)(\frac{1}{p}-\frac{1}{2})} \|f\|_{L_x^{p'}}.$$

□

The following technical lemma is needed for later use.

Lemma 4.2. *Let $h \in L^q(I, L^2(\mathbb{R}^n, \lambda_k(x) dx))$ with $1 \leq q \leq \infty$. For all $r \geq 0$, the function*

$$F_r(x, t, s) := e^{(r+it)L_k} h(x, t) \overline{e^{(r+is)L_k} h(x, s)}$$

belongs to the space $L^1(\mathbb{R}^n \times I \times I, \lambda_k(x) dx dt ds)$.

Proof. Since the function $x \mapsto h(x, t)$ belongs to $L^2(\mathbb{R}^n, \lambda_k(x) dx)$, it follows from Theorem 3.1 (or Lemma 4.1 with $p = 2$) that $\|e^{(r+it)L_k} h(\cdot, t)\|_{L_x^2} \leq \|h(\cdot, t)\|_{L_x^2}$. Hence, by Cauchy-Schwartz inequality we have

$$\int_{\mathbb{R}^n} |F_r(x, t, s)| \lambda_k(x) dx \leq \|h(\cdot, t)\|_{L_x^2} \|h(\cdot, s)\|_{L_x^2}.$$

Now, applying Hölder's inequality we obtain

$$\begin{aligned} \int_I \int_I \int_{\mathbb{R}^n} |F_r(x, t, s)| \lambda_k(x) dx dt ds &\leq \left(\int_I \|h(\cdot, t)\|_{L_x^2}^2 dt \right) \\ &\lesssim \|h\|_{L_t^q L_x^2}^2 \end{aligned}$$

whenever $1 \leq q \leq \infty$. □

Proof of Theorem A.2. The proof is mimicked from [17]. Let us start with a function $h \in L_t^q(L^{p'} \cap L^2)_x$. Then

$$\left\| \int_I e^{(r+it)L_k} h(\cdot, t) dt \right\|_{L_x^2}^2 = \int_I \int_I \left\{ \int_{\mathbb{R}^n} e^{(r+it)L_k} h(x, t) \overline{e^{(r+is)L_k} h(x, s)} \lambda_k(x) dx \right\} dt ds, \quad (4.2)$$

where the interchange of the order of integration is accomplished by Lemma 4.2. Using the integral representation (3.1) of the semigroup $\mathcal{S}_k(z) = e^{zL_k}$ together with the fact that

$$c_k \int_{\mathbb{R}^n} \Lambda_k(x, y; z_1) \Lambda_k(x, y'; z_2) \lambda_k(x) dx = \Lambda_k(y, y'; z_1 + z_2) \quad (4.3)$$

(see [3, p. 1297]), we can rewrite the integral within brackets on the right hand side of (4.2) as

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{(r+it)L_k} h(x, t) \overline{e^{(r+is)L_k} h(x, s)} \lambda_k(x) dx \\ &= c_k^2 \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \Lambda_k(x, y, r+it) h(y, t) \lambda_k(y) dy \right\} \\ &\quad \left\{ \int_{\mathbb{R}^n} \overline{\Lambda_k(x, y', r+is)} \overline{h(y', s)} \lambda_k(y') dy' \right\} \lambda_k(x) dx \\ &= c_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(y, t) \overline{h(y', s)} \Lambda_k(y, y', 2r+i(t-s)) \lambda_k(y') \lambda_k(y) dy dy' \\ &= \int_{\mathbb{R}^n} \overline{h(y', s)} e^{(2r+i(t-s))L_k} h(y', t) \lambda_k(y') dy'. \end{aligned}$$

Thus, for all $p, q \geq 1$, the identity (4.2) becomes

$$\begin{aligned} \left\| \int_I e^{(r+it)L_k} h(\cdot, t) dt \right\|_{L_x^2}^2 &= \int_I \int_{\mathbb{R}^n} \left\{ \int_I e^{(2r+i(t-s))L_k} h(y', t) dt \right\} \overline{h(y', s)} \lambda_k(y') dy' ds \\ &\lesssim \left\| \int_I e^{(2r+i(t-s))L_k} h(\cdot, t) dt \right\|_{L_s^q L_x^p} \|h\|_{L_s^{q'} L_x^{p'}}. \end{aligned}$$

We claim that for every k -admissible pair (p, q) (see Definition 1.1) we have

$$\left\| \int_I e^{(2r+i(t-s))L_k} h(\cdot, t) dt \right\|_{L_s^q L_x^p} \lesssim \|h\|_{L_s^{q'} L_x^{p'}}. \quad (4.4)$$

Indeed, by Minkowski's inequality [9] and Lemma 4.1, we have

$$\begin{aligned} \left\| \int_I e^{(2r+i(t-s))L_k} h(\cdot, t) dt \right\|_{L_x^p} &\leq \int_I \left\| e^{(2r+i(t-s))L_k} h(\cdot, t) \right\|_{L_x^p} dt \\ &\leq \int_I \frac{\|h(\cdot, t)\|_{L_x^{p'}}}{|\sin(t-s)|^{2(\gamma_k+n-1)(\frac{1}{2}-\frac{1}{p})}} dt, \end{aligned}$$

for all $2 \leq p \leq \infty$. Let $\xi(t) := |\sin(t)|^{-2(\gamma_k+n-1)(\frac{1}{2}-\frac{1}{p})}$. Thus we may rewrite the last integral placed on the right-hand side as the Euclidean convolution of the function $t \mapsto \|h(\cdot, t)\|_{L_x^{p'}}$ with $\xi(t)$. Hence, using [18, Lemma 2] (see also the proof of Proposition 2 in [18]), the estimate

$$\left\| \int_I e^{(2r+i(t-s))L_k} h(\cdot, t) dt \right\|_{L_s^q L_x^p} \lesssim \|h\|_{L_s^{q'} L_x^{p'}}$$

holds true for every k -admissible pair (p, q) . This finishes the proof of the claim (4.4).

Now the desired estimate (1.9) follows from the density of $L_t^{q'}(L^{p'} \cap L^2)_x$ in $L_t^{q'} L_x^{p'}$ for $1 < p \leq \infty$. \square

Proof of Theorem A.1. By an abstract result due to Ginibre and Velo [11, Lemma 2.1] the estimates (1.8) and (1.9) are equivalent. However, for the sake of simplicity we will give the details. We shall prove the estimate (1.8) for the semigroup $e^{(r+it)L_k}$ with $r \geq 0$ (not just for $r = 0$). Let $h \in L_t^{q'} L_x^{p'}$. By invoking $(e^{zL_k})^* = e^{\bar{z}L_k}$ (see [3, p. 1297]) and by using Theorem A.2, we have

$$\begin{aligned} \left| \int_I \langle e^{(r+it)L_k} f, h(\cdot, t) \rangle_{L_x^2} dt \right| &= \left| \int_I \langle f, e^{(r-it)L_k} h(\cdot, t) \rangle_{L_x^2} dt \right| \\ &\leq \|f\|_{L_x^2} \left\| \int_I e^{(r-it)L_k} h(\cdot, t) dt \right\|_{L_x^2} \\ &\lesssim \|f\|_{L_x^2} \|h\|_{L_s^{q'} L_x^{p'}}. \end{aligned}$$

\square

Proof of Theorem A.3. Below, the asterisk denotes the Euclidean convolution product. By means Minkowski's inequality and the estimates (4.1) and (4.4), we have

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)L_k} F(\cdot, s) \, ds \right\|_{L_t^q L_x^p} &\lesssim \left\| \int_0^t \|e^{i(t-s)L_k} F(\cdot, s)\|_{L_x^p} \, ds \right\|_{L_t^q} \\
&\lesssim \left\| \int_I \|e^{i(t-s)L_k} F(\cdot, s)\|_{L_x^p} \, ds \right\|_{L_t^q} \\
&\lesssim \left\| \|F\|_{L_x^{p'}} * |\sin t|^{-2(\gamma_k+n-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \right\|_{L_t^q} \\
&\lesssim \|F\|_{L_t^{q'} L_x^{p'}}.
\end{aligned}$$

□

Proof of Theorem B. Via Duhamel's principle,

$$u(x, t) = e^{itL_k} f(x) - i \int_0^t e^{i(t-s)L_k} F(x, s) \, ds \quad (4.5)$$

is the unique solution to the Cauchy problem (1.3). Now, by using Theorem A.1 and Theorem A.3, we get

$$\begin{aligned}
\|u(x, t)\|_{L_t^q L_x^p} &\lesssim \|f\|_{L_x^2} + \left\| \int_0^t e^{i(t-s)L_k} F(x, s) \, ds \right\|_{L_t^q L_x^p} \\
&\lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{q'} L_x^{p'}}
\end{aligned} \quad (4.6)$$

whenever (p, q) is a k -admissible pair. □

We shall prove the non-diagonal version (1.13) of the estimate (1.12). The proof uses the Christ-Kiselev lemma which is very useful in establishing non-diagonal retarded Strichartz estimates (see e.g. [5], [21, Lemma 2.4]). The concept that motivates this lemma is that if an operator is bounded from one space to another, then any appropriate 'localization' of that operator must also be bounded.

Let

$$T_k^{(2)} = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \left] 0, \frac{1}{2} \right[\times \left[0, \frac{1}{2} \left[\left| \frac{1}{q} \geq (\gamma_k + n - 1) \left(\frac{1}{2} - \frac{1}{p} \right) \right. \right. \right\} \subset T_k$$

(see Definition 1.1).

Proposition 4.3. *For any k -admissible exponents (p, q) and (\tilde{q}, \tilde{p}) in $T_k^{(2)}$, the following non-diagonal Strichartz estimate*

$$\|u\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}$$

holds true.

Proof. The proof is standard and uses the so-called TT^* method. Consider the operator

$$Tf(x, t) = e^{itL_k} f(x)$$

and its formal adjoint

$$T^*F(x) = \int_I e^{-isL_k} F(x, s) \, ds.$$

Composing the two estimates (1.8) and (1.9) we deduce the $L_t^{\tilde{q}'} L_x^{\tilde{p}'}$ \rightarrow $L_t^q L_x^p$ boundedness of

$$TT^* F(x, t) = \int_I e^{i(t-s)L_k} F(x, s) ds$$

(we refer the reader to [11, Corollary 2.1] for an abstract argument). That is

$$\left\| \int_I e^{i(t-s)L_k} F(x, s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}. \quad (4.7)$$

However, the non-diagonal untruncated estimate (4.7) still does not apply to (4.5). At a fixed t , the time integral on the left hand side of (4.7) is over I , while the corresponding integral on the right hand side of (4.5) only goes up to t . Thus, using Christ-Kiselev lemma, one converts the non-diagonal untruncated estimate (4.7) to the non-diagonal truncate estimate

$$\left\| \int_0^t e^{i(t-s)L_k} F(x, s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}} \quad (4.8)$$

as long as $\tilde{q}' < q$. The hypothesis that $\tilde{q}' < q$ is unfortunately necessary. \square

5. THE HEAT EQUATION

Recall from Definition 2.3 the space $\mathcal{H}_k^m(\mathbb{R}^n)$ of k -spherical harmonics of degree m . Let $\{Y_{m,j}, j = 1, \dots, d(m,n)\}$ be a real-coefficient orthonormal basis of $\mathcal{H}_k^m(\mathbb{R}^n)$ in the Hilbert space $L^2(S^{n-1}, d_k \lambda_k(\omega) d\sigma(\omega))$. Here S^{n-1} is the unit sphere in \mathbb{R}^n , $d\sigma$ is the Lebesgue surface measure, $\lambda_k(\omega) = \prod_{\alpha \in R} |\langle \alpha, \omega \rangle|^{k_\alpha}$, and d_k is as in (2.7).

Let k be a non-negative multiplicity function such that $\gamma_k + n - 1 > 0$. For $x, y \in \mathbb{R}^n$, let

$$\mathcal{J}_k(x, y) := e^{i\pi \left(\frac{\gamma_k + n - 1}{2}\right)} \Lambda_k\left(x, y; i\frac{\pi}{2}\right),$$

where Λ_k is as in (3.3). The kernel \mathcal{J}_k appeared in [3] as the integral kernel of a generalized Fourier transform.

According to [3, Theorem 4.20], the kernel \mathcal{J}_k admits the following series expansion

$$\mathcal{J}_k(x, y) = \Gamma(\gamma_k + n - 1) \sum_{m=0}^{\infty} \sum_{j=1}^{d(m,n)} \mathcal{J}_k^{(m)}(\|x\| \|y\|) Y_{m,j}(\omega) Y_{m,j}(\omega'), \quad (5.1)$$

where

$$\mathcal{J}_k^{(m)}(u) = e^{-im\pi} u^{-(\gamma_k + n)/2 + 1} J_{\lambda_m}(2\sqrt{u}).$$

Here $\lambda_m = 2m + \gamma_k + n - 2$ and J_ν is the Bessel function of the first kind.

For $t > 0$ and $x, y \in \mathbb{R}^n$, define

$$\Gamma_k(x, y; t) := c_k^2 \int_{\mathbb{R}^n} e^{-t\|\xi\|} \mathcal{J}_k(x, \xi) \mathcal{J}_k(y, \xi) \lambda_k(\xi) d\xi, \quad (5.2)$$

where the constant c_k is given by (2.6). Indeed Γ_k is a heat kernel so that the function

$$\vartheta(x, t) = \int_{\mathbb{R}^n} f(y) \Gamma_k(x, y; t) \lambda_k(y) dy$$

solves the following heat equation

$$\begin{cases} \partial_t \vartheta(x, t) = \|x\| \Delta_k \vartheta(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ \vartheta(x, 0) = f(x). \end{cases} \quad (5.3)$$

This is due to the fact that

$$\|x\| \Delta_k^x \mathcal{J}_k(x, \xi) = -\|\xi\| \mathcal{J}_k(x, \xi), \quad (5.4)$$

where the superscript in Δ_k^x indicates the relevant variable (see [3, Theorem 5.7]).

Theorem 5.1. *In polar coordinates $x = \|x\|\omega$ and $y = \|y\|\omega'$, the heat kernel Γ_k is given by*

$$\Gamma_k(x, y; t) = c_k \Gamma\left(\frac{\gamma_k + n - 1}{2}\right) t^{-(\gamma_k + n - 1)} \exp\left(-\frac{\|x\| + \|y\|}{t}\right) V_k\left(\tilde{I}_{\frac{\gamma_k + n - 3}{2}}\left(\frac{\sqrt{2\|x\|\|y\|(1 + \langle \omega, \cdot \rangle)}}{t}\right)\right)(\omega'),$$

where $\tilde{I}_\lambda(z) = \left(\frac{z}{2}\right)^{-\lambda} I_\lambda(z)$ is the (normalized) modified Bessel function of the first kind.

Proof. In view of the series expansion (5.1) of \mathcal{J}_k and by the orthogonality of the k -spherical harmonics, we have

$$\begin{aligned} & \int_{S^{n-1}} \mathcal{J}_k(x, \|\xi\|\eta) \mathcal{J}_k(y, \|\xi\|\eta) \lambda_k(\eta) d\sigma(\eta) \\ &= d_k^{-1} \Gamma(\gamma_k + n - 1)^2 \\ & \quad \sum_{m=0}^{\infty} \sum_{j=1}^{d(m,n)} \mathcal{J}_k^{(m)}(\|x\| \|\xi\|) \mathcal{J}_k^{(m)}(\|y\| \|\xi\|) Y_{m,j}(\omega) Y_{m,j}(\omega') \\ &= d_k^{-1} \Gamma(\gamma_k + n - 1)^2 (\|x\| \|y\|)^{-(\gamma_k + n)/2 + 1} \|\xi\|^{-\gamma_k - n + 2} \\ & \quad \sum_{m=0}^{\infty} \sum_{j=1}^{d(m,n)} Y_{m,j}(\omega) Y_{m,j}(\omega') J_{\lambda_m}(2\sqrt{\|x\| \|\xi\|}) J_{\lambda_m}(2\sqrt{\|y\| \|\xi\|}) \\ &= d_k c_k^{-2} (\|x\| \|y\|)^{-(\gamma_k + n)/2 + 1} \|\xi\|^{-\gamma_k - n + 2} \\ & \quad \sum_{m=0}^{\infty} \sum_{j=1}^{d(m,n)} Y_{m,j}(\omega) Y_{m,j}(\omega') J_{\lambda_m}(2\sqrt{\|x\| \|\xi\|}) J_{\lambda_m}(2\sqrt{\|y\| \|\xi\|}). \end{aligned}$$

Invoking the integral (5.2) leads to

$$\Gamma_k(x, y; t) = d_k (\|x\| \|y\|)^{-(\gamma_k + n)/2 + 1} \sum_{m=0}^{\infty} \sum_{j=1}^{d(m,n)} Y_{m,j}(\omega) Y_{m,j}(\omega') \psi_m(t, \|x\|, \|y\|), \quad (5.5)$$

where

$$\psi_m(t, \|x\|, \|y\|) = \int_0^\infty e^{-tu} J_{\lambda_m}(2\sqrt{\|x\|u}) J_{\lambda_m}(2\sqrt{\|y\|u}) du.$$

The integral formula (see [22, Exercise 68])

$$\int_0^\infty e^{-p^2 u^2} J_\lambda(\alpha u) J_\lambda(\beta u) u \, du = \frac{1}{2p^2} \exp\left(-\frac{\alpha^2 + \beta^2}{4p^2}\right) I_\lambda\left(\frac{\alpha\beta}{2p^2}\right)$$

implies

$$\psi_m(t, \|x\|, \|y\|) = \frac{1}{t} \exp\left(-\frac{\|x\| + \|y\|}{t}\right) I_{\lambda_m}\left(2\frac{\sqrt{\|x\|\|y\|}}{t}\right).$$

Then the identity (5.5) becomes

$$\begin{aligned} \Gamma_k(x, y; t) &= d_k t^{-(\gamma_k+n-1)} \exp\left(-\frac{\|x\| + \|y\|}{t}\right) \\ &\quad \sum_{m=0}^\infty \sum_{j=1}^{d(m,n)} \left(\frac{\sqrt{\|x\|\|y\|}}{t}\right)^{2m} \tilde{I}_{\lambda_m}\left(2\frac{\sqrt{\|x\|\|y\|}}{t}\right) Y_{m,j}(\omega) Y_{m,j}(\omega'). \end{aligned}$$

Now, the statement follows from the addition formula (4.41) in [3]. \square

We close this section by observing that Γ_k extends naturally to complex time arguments $z \in \mathbb{C}^*$.

6. A SCHRÖDINGER EQUATION

In this section we turn our attention to the following Cauchy problem

$$\begin{cases} i\partial_t v(x, t) + \|x\| \Delta_k v(x, t) = F(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(x, 0) = f(x). \end{cases} \quad (6.1)$$

Formally, the free Schrödinger equation

$$\begin{cases} i\partial_t v_f(x, t) + \|x\| \Delta_k v_f(x, t) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v_f(x, 0) = f(x) \end{cases} \quad (6.2)$$

is obtained by the transformation $t \mapsto it$ of the heat equation (5.3) to ‘imaginary time’. In other words, for all $f \in L^2(\mathbb{R}^n, \lambda_k(x) \, dx)$, the function

$$v_f(x, t) = e^{it\|x\|\Delta_k} f(x) = \int_{\mathbb{R}^n} f(y) \Gamma_k(x, y; it) \lambda_k(y) \, dy$$

solves the problem (6.2).

Next we will measure the $L_t^q L_x^p$ -size of $v_f(x, t)$ by means of the $L_t^q L_x^p$ -size of the solution $u(x, t)$ to the Cauchy problem (1.11) with $F \equiv 0$. To do so, we will prove a relation between the semigroups $e^{it\|x\|\Delta_k}$ and e^{itL_k} .

Recall from Fact 3.2 that the integral kernel $\Lambda_k(x, y; it)$ of the semigroup $\mathcal{S}_k(it) = e^{itL_k}$ satisfies

$$\Lambda_k(x, y; -it) = \overline{\Lambda_k(x, y; it)}, \quad \Lambda_k(x, y; i(t + \pi)) = e^{-i\pi(\gamma_k+n-1)} \Lambda_k(x, y; it).$$

Thus, for a real-valued function f , the map $t \mapsto \|e^{itL_k} f\|_{L_x^p(\mathbb{R}^n)}$ is even and π -periodic, and therefore determined by its values for $0 \leq t \leq \pi/2$. Then, the two mixed norms $\|e^{itL_k} f\|_{L_t^q L_x^p(I \times \mathbb{R}^n)}$ and $\|e^{itL_k} f\|_{L_t^q L_x^p((0, \pi/2) \times \mathbb{R}^n)}$ are proportional for real-valued functions f .

Proposition 6.1. *For all $f \in L^2(\mathbb{R}^n, \lambda_k(x) dx)$ and for all $s > 0$, we have*

$$e^{i \arctan(s) L_k} f(x) = (1 + s^2)^{\left(\frac{\gamma_k + n - 1}{2}\right)} \exp(-is\|x\|) e^{is\|x\|\Delta_k} f((1 + s^2)x).$$

Proof. The proof uses the change of variable $s = \tan(t)$ with $t \in (0, \pi/2)$. \square

Now we are ready to prove Theorem C.

Proof of Theorem C. Assuming $p, q < \infty$. Proposition 6.1 gives

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \left\{ \int_{\mathbb{R}^n} |e^{itL_k} f(x)|^p \lambda_k(x) dx \right\}^{q/p} dt \\ &= \int_0^\infty \left\{ \int_{\mathbb{R}^n} |e^{is\|x\|\Delta_k} f(x)|^p \lambda_k(x) dx \right\}^{q/p} (1 + s^2)^{q\left\{\left(\frac{\gamma_k + n - 1}{2}\right) - \frac{1}{q} - \frac{1}{p}(\gamma_k + n - 1)\right\}} ds. \end{aligned}$$

If the pair (p, q) satisfies the condition

$$(\gamma_k + n - 1) \frac{1}{p} + \frac{1}{q} = \left(\frac{\gamma_k + n - 1}{2} \right), \quad (6.3)$$

then we have

$$\int_0^{\frac{\pi}{2}} \left\{ \int_{\mathbb{R}^n} |e^{itL_k} f(x)|^p \lambda_k(x) dx \right\}^{q/p} dt = \int_0^\infty \left\{ \int_{\mathbb{R}^n} |e^{is\|x\|\Delta_k} f(x)|^p \lambda_k(x) dx \right\}^{q/p} ds.$$

The cases when p or q is infinite are similar. \square

Proof of Theorem D. The statement follows from the estimate (1.8) and Theorem C. \square

Proof of Corollary E. Let (p, q) and (\tilde{p}, \tilde{q}) be two k -admissible pairs satisfying the assumptions of Theorem D. The abstract Lemma 2.1 in [11] applied to the estimate (1.18) yields

$$\left\| \int_{\mathbb{R}^+} e^{-is\|x\|\Delta_k} F(x, s) ds \right\|_{L_x^2(\mathbb{R}^n)} \lesssim \|F\|_{L_t^q L_x^{\tilde{p}'}(\mathbb{R}^+ \times \mathbb{R}^n)}. \quad (6.4)$$

Following the same argument as in Proposition 4.3, we obtain

$$\left\| \int_{\mathbb{R}^+} e^{i(t-s)\|x\|\Delta_k} F(x, s) ds \right\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}(\mathbb{R}^+ \times \mathbb{R}^n)}. \quad (6.5)$$

Now, by Christ-Kiselev's lemma we get the non-diagonal truncated estimate

$$\left\| \int_0^t e^{i(t-s)\|x\|\Delta_k} F(x, s) ds \right\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}(\mathbb{R}^+ \times \mathbb{R}^n)}$$

(by the assumptions on (p, q) and (\tilde{p}, \tilde{q}) , the exponents q and \tilde{q} satisfy already the condition $\tilde{q}' < q$). This finishes the proof of Corollary E. \square

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