Study of a $3D$ Ginzburg-Landau functional with a discontinuous pinning term

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Abstract

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a $3D$-Ginzburg-Landau type energy with a discontinuous pinning term among $H^1(\Omega, \mathbb{C})$-maps subject to a boundary Dirichlet condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}_+^*$ takes a constant value $b \in (0, 1)$ in ω, an inner strictly convex subdomain of Ω, and 1 outside ω. We prove energy estimates with various error terms depending on assumptions on Ω, ω and g. In some special cases, we identify the vorticity lines via the concentration of the energy.

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1 Introduction

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a $3D$-Ginzburg-Landau type energy with a discontinuous pinning term among $H^1(\Omega, \mathbb{C})$-maps subject to a boundary Dirichlet condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}_+^*$ takes a constant value $b \in (0, 1)$ in

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ω, an inner strictly convex subdomain of Ω, and 1 outside ω. The strict convexity of ω is not necessary but it allows to make a simpler description of the technics used in this article.

Our Ginzburg-Landau type energy is

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}(a(x)^2 - |u(x)|^2)^2 \right\} \mathrm{d}x. \tag{1}$$

In (1), $u \in H^1_g := \{u \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g\}$.

Following [8], let U_ε be **the** unique minimizer of E_ε in H^1_1. If $v \in H^1(\Omega, \mathbb{C})$ and $|v| \equiv 1$ on $\partial\Omega$, then [8]

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v), \text{ where } F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

Consequently the study of minimizers of E_ε in H^1_g is related to the study of minimizers of F_ε in H^1_g.

Our technics are directly inspired from whose initially developed by Sandier in [12] (whose purpose was to give, in some special situations, a simple proof of the $3D$ analysis of the Ginzburg-Landau equation, by Lin and Riviere [9]), and by their adaptations in [2].

We prove energy estimates with various error terms depending on our assumptions on Ω and g. In some special cases, we identify the vorticity lines via the concentration of the energy. At the end of this section, we will present a strategy which could lead to the localization of the vortex lines.

The results we present are a first step towards a more precise description of the vorticity defects and of the asymptotic of minimizers.

Before stating our own results, we start by recalling the asymptotic expansion of the energy in the standard Ginzburg-Landau model in $3D$.

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, if we let

$$E^0_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right\},$$

then we have

$$\inf_{H^1_g} E^0_\varepsilon = C(g)|\ln\varepsilon| + o(|\ln\varepsilon|). \tag{2}$$

Moreover, $\dfrac{C(g)}{\pi}$ is given by the length of a minimal connection connecting the singularities of g (in the spirit of Brezis, Coron, Lieb [5]). (See [9], [10], [12] and [2]).

For special g's and for a convex domain Ω, (2) was obtained by Lin and Rivière [9] (see also [10]) and Sandier [12]. The case of a general data $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and a simply connected Ω is due to Bourgain, Brezis and Mironescu [2].

The above articles are our main references in this work. One of our main results is the analog of (2) for the minimization of F_ε (Theorem 1). This result is first proved when g is in a dense set $\mathcal{H} \subset H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and then extended by density. The upper bound is obtained directly using the technics developed in [12] and [2]. The lower bound needs an adaptation in the argument of Sandier [12]. The main ingredient used to obtain a lower bound in Sandier [12] is the existence of a "structure function" adapted to the singularities of g. In the spirit of [12], we prove, under suitable assumptions on Ω, ω and g, the existence of structure functions adapted to our situation. We presente below constructions (in the spirit of Sandier) of structure functions under restrictive hypotheses on the geometries of Ω, ω and on the singularities of g (see Corollaries 1, 2 and Proposition 7).

In our situation, when g admits a finite number of singularities, the constant $\dfrac{C(g)}{\pi}$ is the length of a minimal connection between the singularities of g. This minimal connection is computed with respect to a metric d_{a^2} depending only on a (see (9)). (This generalizes the case of the standard potential $(1 - |u|^2)^2$, where the distance is the euclidean one.)

When g has a finite number of singularities, one may prove a concentration of the energy along the vorticity lines (See Theorems 2 and 3). As in [9] and [12], we obtain, after normalization, that

the energy of minimizer is uniform along the vorticity lines (See Theorem 2). These vorticity lines are identified: they are geodesic segments associated to d_{a^2}.

The goal of this work is to explain how the vorticity lines are modified under the effect of a pinning term. Although from the theorems below we have an idea on the form of the vorticity lines, in order to have a complete description of the defects, we need an η-ellipticity results in the spirit of [1] for the minimizers of F_ε. Namely: fix $r > 0$ then for small ε and v a minimizer of F_ε

$$\textit{if, in a ball } B(x,r), \textit{ the quantity } \frac{F_\varepsilon(v, B(x,r))}{|\ln \varepsilon|} \textit{ is small, then } |v(x)| \simeq 1.$$

It seems that an η-ellipticity result cannot be obtained by the standard method, which relies on a monoticity formula obtained from a Pohozaev identity. The oscillating behavior of U_ε yields impossible the direct application of monotonicity formulae. When U_ε does not oscillate, it is possible to derive η-ellipticity (see *e.g.* [11]). In our case, η-ellipticity would require a uniform control of the Lipschitz norm of U_ε; this does not hold in our situation.

This paper is divided as follows:

- We first present the fundamental properties of the special solution U_ε (Section 2).
- In Section 3, we define and describe the main geometrical objects in the study. Once this done, we state the main results.
- The proofs of the main results are sketched Section 4. In particular we explain how we may use [12] and [2] and we underline the required adaptations.
- The heart of the argument is based on energetic estimates. In Section 5 we proof upper bounds according to various assumptions and in Section 7 we obtain lower bounds. Section 6 is dedicated to the key tool used Section 7.
- Section 8 is devoted to the last argument in the proof of Theorem 1.

2 Description of the special solution U_ε

Let $\overline{\omega} \subset \Omega \subset \mathbb{R}^3$ be two smooth bounded open sets s.t. Ω is convex and ω is strictly convex. For $b \in (0,1)$ we define

$$\begin{array}{rccl} a: & \mathbb{R}^3 & \to & \{b,1\} \\ & x & \mapsto & \begin{cases} b & \text{if } x \in \omega \\ 1 & \text{otherwise} \end{cases} \end{array}.$$

We denote E_ε the Ginzburg-Landau functional with a as pinning term, namely

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}(a(x)^2 - |u(x)|^2)^2 \right\} \mathrm{d}x.$$

For $\varepsilon > 0$, we denote (see [8]) U_ε **the** unique global minimizer of E_ε in

$$H^1_1 := \{u \in \ H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u \equiv 1\}.$$

In the following, we will denote also $U_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ the extension by 1 of the unique global minimizer of E_ε in H^1_1.

Proposition 1. *The following assertions are true*

1. $U_\varepsilon : \mathbb{R}^3 \to [b,1]$ *(from [8]),*
2. $-\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2} U_\varepsilon (a^2 - U_\varepsilon^2)$ *in* Ω,
3. $E_\varepsilon(U_\varepsilon) \underset{\varepsilon\to 0}{\sim} \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega (a^2 - U_\varepsilon^2)^2 \underset{\varepsilon\to 0}{\sim} \dfrac{1}{\varepsilon}$ *(same argument as in [8]),*

4. There are $C, \gamma > 0$ s.t. for $x \in \Omega$ we have (same proof as in [7])

$$|U_\varepsilon(x) - a(x)| \leq C\mathrm{e}^{-\gamma \mathrm{dist}(x,\partial\omega)/\varepsilon}, \tag{3}$$

5. If $v \in \mathcal{J} := \{v \in H^1(\Omega, \mathbb{C}) \,|\, |\mathrm{tr}_{\partial\Omega} v| = 1\}$ then $E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v)$ (same proof as [8]) with

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}, \tag{4}$$

6. If v minimises F_ε in $H^1_g := \{v \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} v = g\}$ then $|v| \leq 1$ in Ω (same proof as [8]).

3 Minimal connections, geodesic links and main results

In this section we define the main geometrical objects which appear in the description of the vorticity lines.

3.1 Length of a minimal connection of a map $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, following [2], one may associate to g a continuous linear form

$$T_g : (\mathrm{Lip}(\partial\Omega, \mathbb{R}), \|\cdot\|_{\mathrm{Lip}}) \to \mathbb{R}.$$

Here $\|\varphi\|_{\mathrm{Lip}} = \|\varphi\|_{L^\infty} + \sup\limits_{\substack{x,y\in\partial\Omega \\ x\neq y}} \dfrac{|\varphi(x) - \varphi(y)|}{|x-y|}$ with $|x-y| = d_{\mathrm{eucl}}(x,y)$ is the euclidean distance in $\mathbb{R}^3$ between x and y.

The map T_g is defined by the following way: let $\varphi \in \mathrm{Lip}(\partial\Omega, \mathbb{R})$ and $g \in H^{1/2}(\partial\Omega, \mathbb{R})$;

- fix $u \in H^1_g$ and consider $H = 2(\partial_2 u \times \partial_3 u\,,\, \partial_3 u \times \partial_1 u\,,\, \partial_1 u \times \partial_2 u)$;
- fix $\phi \in \mathrm{Lip}(\Omega, \mathbb{R})$ s.t. $\phi = \varphi$ on $\partial\Omega$;

then

$$\begin{array}{rccc} T_g : & \mathrm{Lip}(\partial\Omega, \mathbb{R}) & \to & \mathbb{R} \\ & \varphi & \mapsto & \displaystyle\int_\Omega H \cdot \nabla\phi \end{array}$$

is independent of the choice of u and ϕ.

Notation 1. Here "$\times$" stands for the "vectorial product" in $\mathbb{C}$: $(w_1 + \imath w_2) \times (z_1 + \imath z_2) = w_1 z_2 - w_2 z_1$, $w_1, w_2, z_1, z_2 \in \mathbb{R}$.

Following [2], we denote, for $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and d an equivalent distance with d_{eucl} on $\partial\Omega$,

$$L(g,d) := \sup\{T_g(\varphi) \,|\, |\varphi|_d \leq 1\} = \max\{T_g(\varphi) \,|\, |\varphi|_d \leq 1\} \tag{5}$$

with

$$|\varphi|_d := \sup_{\substack{x\neq y \\ x,y\in\partial\Omega}} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)}.$$

Note that $L(g,d)$ is finite, since $T_g : (\mathrm{Lip}(\partial\Omega, \mathbb{R}), \|\cdot\|_{\mathrm{Lip}}) \to \mathbb{R}$ is continuous and d, d_{eucl} are equivalent on $\partial\Omega$.

In the spirit of [9],[12] and [2] we deal with prepared boundary conditions g's. In this article we use the dense subset $\mathcal{H} \subset H^{1/2}(\partial\Omega, \mathbb{S}^1)$

$$\mathcal{H} = \left\{ g \in \bigcap_{1\leq p<2} W^{1,p}(\partial\Omega, \mathbb{S}^1) \;\middle|\; \begin{array}{c} g \text{ is smooth outside a finite set } \mathcal{C}, \\ \forall M \in \mathcal{C} \text{ we have for } x \text{ close to } M: \\ |\nabla g(x)| \leq C/|x - M|, \\ \exists\, R_M \in \mathcal{O}(3) \text{ s.t. } \left| g(x) - R_M\left(\frac{x-M}{|x-M|}\right)\right| \leq C|x-M|. \end{array} \right\}.$$

Here we considered $\mathbb{S}^1 \simeq \{0\} \times \mathbb{S}^1 \subset \mathbb{S}^2$.

One may define $\deg(u, M)$, the topological degree of u with respect to M: if $R_M \in \mathcal{O}(3)^+$ then $\deg(u, M) = 1$ otherwise $\deg(u, M) = -1$.

In order to justify the term of "degree", assume that in a neighborhood of $M \in \mathcal{C}$, $\partial\Omega$ is flat. Then, for $r > 0$ sufficiently small, $C = \partial B(M, r) \cap \partial\Omega$ is a circle centered in M. This circle has a natural orientation induced by $B(M, r) \cap \Omega$. Thus $g_{|C} \in C^\infty(C, \mathbb{S}^1)$ admits a well defined topological degree (see *e.g.* [4]), and this degree does not depend on small r.

We consider

$$P = \{M \in \mathcal{C} \,|\, \deg(u, M) = 1\} \text{ and } N = \{M \in \mathcal{C} \,|\, \deg(u, M) = -1\}.$$

One may also consider for $g \in \mathcal{H}$ the degree of g with respect to ∂U for U a non empty smooth open set of $\partial\Omega$ s.t. ∂U does not contain any singularities of g. This degree is defined as

$$\deg(g, \partial U) = \text{Card}(\{p \in P \,|\, p \in U\}) - \text{Card}(\{n \in N \,|\, n \in U\}). \tag{6}$$

From [2], we have the following

Proposition 2. *Let $g, h \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, then we have*

1. *$T_{gh} = T_g + T_h$ and $T_{\overline{g}} = -T_g$ (Lemma 9),*
2. *$|(T_g - T_h)(\varphi)| \leq C|g - h|_{H^{1/2}}(|g|_{H^{1/2}} + |h|_{H^{1/2}})|\varphi|_{d_{\text{eucl}}}$, $\varphi \in \text{Lip}(\partial\Omega, \mathbb{R})$ (Lemma 9),*
3. *$\mathcal{H}$ is dense in $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ (Lemma B.1),*
4. *if $u \in \mathcal{H}$, then $\text{Card}(P) = \text{Card}(N)$ and $T_g = 2\pi \sum_{p \in P} \delta_p - 2\pi \sum_{n \in N} \delta_n$ (Lemma 2),*
5. *if $u \in \mathcal{H}$, then $L(g, d) = \min_{\sigma \in S_k} \sum_i d(p_i, n_{\sigma(i)})$ where d is a distance equivalent with d_{eucl} on $\partial\Omega$ (Theorem 1).*

3.2 Minimal connections, minimal length, geodesic links

In the last assertion of Proposition 2, we used the notion of length of a minimal connection. Namely, consider d a distance on $\mathcal{C} = P \cup N$, $P, N \subset \mathbb{R}^3$ two sets of k distinct points s.t. $P \cap N = \emptyset$, $P = \{p_1, ..., p_k\}$ and $N = \{n_1, ..., n_k\}$.

We denote by $L(\mathcal{C}, d)$ the length of a minimal connection of $\mathcal{C}$ in $(\mathcal{C}, d)$, *i.e.*,

$$L(\mathcal{C}, d) = \min_{\sigma \in S_k} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)}). \tag{7}$$

In [5] (Lemma 4.2), the authors proved that

$$L(\mathcal{C}, d) = \max \left\{ \sum_{i=1}^{k} \{\varphi(p_i) - \varphi(n_i)\} \,|\, \varphi : \mathcal{C} \to \mathbb{R},\, |\varphi|_d^{\mathcal{C}} \leq 1 \right\} \tag{8}$$

with

$$|\varphi|_d^{\mathcal{C}} = \sup_{\substack{x \neq y \\ x, y \in \mathcal{C}}} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)}.$$

A permutation σ s.t. $\sum_i d(p_i, n_{\sigma(i)}) = L(P \cup N, d)$ is called a minimal connection of $(P \cup N, d)$.

In the following we will consider a special form of distance d on $\partial\Omega$: the geodesic distance in $\overline{\Omega}$ equipped with a metric we will describe below.

Let us first introduce some notations. Let $f : \mathbb{R}^3 \to [b^2, 1]$ be a Borel function and let $\Gamma \subset \overline{\Omega}$ be Lipschitz curve. We denote by $\text{long}_f(\Gamma)$ the length of Γ in the metric fh (here h is the euclidean metric in $\mathbb{R}^3$), *i.e.*,

$$\text{long}_f(\Gamma) := \int_0^1 f(\gamma(s))\, |\gamma'(s)| \text{d}s,\ \gamma : [0, 1] \to \Gamma \text{ is a admissible parametrization of } \Gamma.$$

In this paper, when we consider a curve (or arc) Γ, it will be implicitly that it is a Lipschitz one.

We define d_f as the geodesic distance in fh (h is the euclidean metric in $\mathbb{R}^3$).

Thus, for $x, y \in \mathbb{R}^3$, $x \neq y$, we have

$$d_f(x,y) = \inf_{\substack{\Gamma \text{ Lipschitz arc} \\ \text{with endpoints } x,\, y}} \operatorname{long}_f(\Gamma). \tag{9}$$

In the special case $f = a^2$, one may easily prove the following proposition

Proposition 3. *Let $x, y \in \mathbb{R}^3$, $x \neq y$. The following assertions are true*

1. *In* (9) *the* infimum *is attained.*

   We denote by Γ_0 a minimal curve in (9).

2. *If $x, y \in \overline{\Omega}$ then a geodesic Γ_0 is included in $\overline{\Omega}$.*

3. *A geodesic $\Gamma_0 = \cup_{i=1}^{l} S_i$ is a union of at most three line segments.*

4. *These line segments are such that*

   a. *if $x, y \in \omega$ then $l = 1$,*
   b. *if $l = 2$ then $S_1 \cap S_2 \subset \partial\omega$,*
   c. *if $l = 3$ then $x, y \in \mathbb{R}^3 \setminus \overline{\omega}$ and S_2 is a chord of ω,*
   d. *if $[x,y] \cap \overline{\omega} = \{z\}$ then $l \in \{2, 3\}$.*

In the case $d = d_{a^2}$ and $\mathcal{C} = P \cup N \subset \partial\Omega$, we say that $\cup_i \Gamma_i$ is a geodesic link when σ is a minimal connexion in $(\mathcal{C}, d_{a^2})$ and Γ_i is a geodesic joining p_i to $n_{\sigma(i)}$. In Figure 1, we have represented a geodesic link for $k = 2$ and a certain $b \in (0,1)$.

3.3 The main results

Theorem 1. *Let $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. Then we have*

$$\inf_{v \in H^1_g} F_\varepsilon(v) = \pi L(g, d_{a^2}) |\ln \varepsilon| + o(|\ln \varepsilon|).$$

Theorem 2. *Let $g \in \mathcal{H}$ be s.t. $(\mathcal{C} = P \cup N, d_{a^2})$ admits a unique geodesic link which is denoted $\cup_i \Gamma_i$.*

Let v_ε be a minimizer of F_ε in H^1_g. Then the normalized energy density

$$\mu_\varepsilon = \frac{\frac{U_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{U_\varepsilon^4}{4\varepsilon^2}(1 - |v_\varepsilon|^2)^2}{|\ln \varepsilon|}\, \mathscr{H}^3 \text{ weakly converges in } \Omega \text{ in the sense of the measure to } \pi a^2 \mathscr{H}^1_{|\cup_i \Gamma_i}.$$

Here $\mathscr{H}^3$ is the 3-dimensional Hausdorff measure and $\mathscr{H}^1_{|\cup_i \Gamma_i}$ is the one dimensional Hausdorff mesure on $\cup_i \Gamma_i$.

In other words

$$\forall \phi \in C^0(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \text{ we have } \int_\Omega \phi \, \mathrm{d}\mu_\varepsilon \to \pi \int_{\cup_i \Gamma_i} \phi a^2 \, \mathrm{d}\mathscr{H}^1.$$

Note that this result gives an (uniform) energy concentration property of the minimizers along the geodesic link. Namely, for all compact K s.t. $K \cap \cup_i \Gamma_i = \emptyset$, we have $F_\varepsilon(v_\varepsilon, K) = o(|\ln \varepsilon|)$.

In order to obtain a more precise statement we assume that $\Omega = B(0,1)$ and $\omega = B(0, r_0)$, $r_0 \in (0,1)$, $g \in \mathcal{H}$ is s.t. $\mathcal{C} = \{p, n\}$ with $p = -n$. Under these hypotheses we have

Theorem 3. *The following estimate holds*

$$\inf_{H^1_g} F_\varepsilon = \pi d_{a^2}(p,n) |\ln \varepsilon| + \mathcal{O}(1).$$

Moreover, for all $\eta > 0$, there is $C_\eta > 0$ s.t. denoting $V_\eta = \{x \in \Omega \,|\, \operatorname{dist}(x, [p,n]) \geq \eta\}$ and v_ε a minimizer of F_ε in H^1_g, we have

$$F_\varepsilon(v_\varepsilon, V_\eta) \leq C_\eta.$$

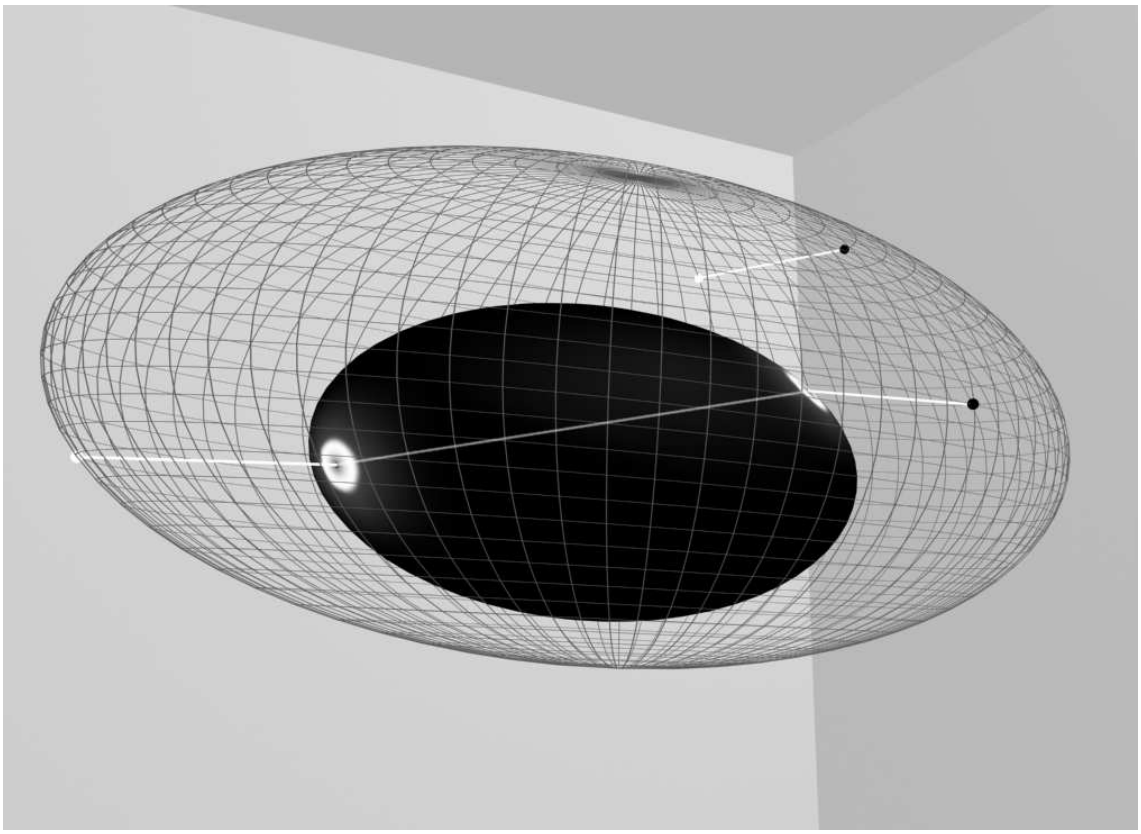

Figure 1: Illustration of a geodesic link with $k = 2$: the boundary of Ω is in wire, the one of ω is black filled, the positive points are white, the negative ones are black and a geodesic link is represented in white. The shaded off on the two penetration points gives indications about the $3D$-geometry of the geodesic link and of the inclusion. (Courtesy of Alexandre Marotta)

4 Outline of the proofs

The proofs of the above theorems strongly rely on the technics developed in [12]. The proofs of theorems 1, 3 consist essentially into two parts devoted to obtaining respectively lower and upper bounds.

The upper bound is obtained by the construction of a test function. The test function was obtained by Sandier in [12] in the situation where there is a geodesic link in $(\mathcal{C}, d_{a^2})$ which is a union of line segments. In this special case, one may obtain (see Section 5.1):

$$\inf_{v \in H^1_g} F_\varepsilon(v) \leq \pi L(g, d_{a^2}) |\ln \varepsilon| + \mathcal{O}(1). \tag{10}$$

For the general case, when the geodesic links are not unions of line segments, in [2], Bourgain, Brezis and Mironescu adapted the construction of Sandier. In our case this leads to the bound:

$$\inf_{v \in H^1_g} F_\varepsilon(v) \leq \pi L(g, d_{a^2}) |\ln \varepsilon| + o(|\ln \varepsilon|). \tag{11}$$

(See Section 5.2.)

The lower bounds are obtained as in [12]. The key ingredient is the construction of a "structure function" $\xi : \mathbb{R}^3 \to \mathbb{R}$ (see Section 6 for a precise definition). Due to the fact that for $M \in \mathbb{R}^3$, $x \to \psi_M(x) = d_{a^2}(x, M)$ is not C^1 (its gradient is not continuous on $\partial\omega$ since $|\nabla \psi_M| = a^2$ in $\mathbb{R}^3 \setminus \partial\omega$), we cannot obtain ξ with exactly the same properties as in [12] (see Corollary 1). The consequence of this lack of smoothness for the distance function implies that our best lower bound is

$$\inf_{v \in H^1_g} F_\varepsilon(v) \geq \pi L(g, d_{a^2}) |\ln \varepsilon| - o(|\ln \varepsilon|). \tag{12}$$

However, under strong symmetry hypotheses, namely, $\Omega = B(0,1), \omega = B(0,r_0)$ and $\mathcal{C} = \{p, n = -p\}$, the structure function ξ enjoys additional properties (see Proposition 7). In this symmetric case, one may obtain the sharper bound

$$\inf_{v \in H^1_g} F_\varepsilon(v) \geq \pi L(g, d_{a^2}) |\ln \varepsilon| - \mathcal{O}(1). \tag{13}$$

The estimate on $\inf_{H^1_g} F_\varepsilon$ in Theorem 1 (resp. Theorem 3) is a direct consequence of (11), (12) (resp. (10) and (13)) and of the density of $\mathcal{H}$ in $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ (see Section 8) .

Theorem 2 is proved along the main lines in [12].

Roughly speaking, under the hypotheses of Theorem 2, for all $x \in \Omega$, there is $\rho_x > 0$ s.t. for $K = \overline{B(x, \rho_x)}$, one may consider a structure function ξ adapted to $\mathcal{C}$ which is constant in K (see Section 6.2). Arguing as in [12], if K does not intersect the geodesic link, then we obtain that in K, a minimizer of F_ε has its energy of order $o(|\ln \varepsilon|)$ (see (27)). Thus μ, the weak limit of μ_ε (which exists up to subsequence), is supported in $\overline{\Omega} \setminus K$. Therefore, one may prove that the support of μ is included in the geodesic link.

Otherwise, if x is on the geodesic link, as explain in [12], then we obtain for v_ε a minimizer and ρ sufficiently small that

$$\limsup \frac{F_\varepsilon(v_\varepsilon, K)}{|\ln \varepsilon|} \leq \pi \mathrm{long}_{a^2}(K \cap \cup_i \Gamma_i).$$

Theorem 2 is obtained by comparing μ to $\pi a^2 \mathscr{H}^1_{|\cup_i \Gamma_i}$.

5 First step in the proofs of theorems : an upper bounds for $\inf_{H^1_g} F_\varepsilon$, $g \in \mathcal{H}$

5.1 The case where $\mathcal{C}$ admits a geodesic link in $(\mathbb{R}^3, d_{a^2})$ which is a union of lines

Assume that there is $\Gamma = \cup \Gamma_i$, a geodesic link of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$ s.t. Γ_i is a line segment for all i. One may assume that the minimal connection associated to Γ is the identity.

In this situation, we may mimic the construction of the test function made in Section 1 of Sandier [12].

The test function is a fixed (independent of ε) $\mathbb{S}^1$-valued function outside $\mathcal{V}_\eta$, an η-tubular neighborhood of Γ (η small and independent of ε).

Inside each tubular neighborhood $\mathcal{V}_{\eta,i}$ of a geodesic $p_i \Gamma_i n_i$, the test function takes the form (in the basis $\{p_i, (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\}$ where $n_i = (0, 0, |p_i - n_i|)$)

$$v_\varepsilon(x,y,z) = \begin{cases} \alpha \dfrac{(x,y)}{|(x,y)|} \text{ if } \eta < z < |p_i - n_i| - \eta \text{ and } \varepsilon < |(x,y)| < \eta \\ \alpha \dfrac{(x,y)}{\varepsilon} \text{ if } \eta < z < |p_i - n_i| - \eta \text{ and } |(x,y)| < \varepsilon \\ \dfrac{1}{2} \displaystyle\int_{\tilde{\mathcal{V}}_{\eta,i}} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq 2\eta\pi |\ln \varepsilon| + C(g, \eta) \\ \text{with } \tilde{\mathcal{V}}_{\eta,i} = \mathcal{V}_{\eta,i} \cap \{0 < z < \eta, |p_i - n_i| - \eta < z < |p_i - n_i|\} \end{cases} . \tag{14}$$

Here $\alpha \in \mathbb{S}^1$ is a fixed constant.

From the strict convexity of ω, for all line $D \subset \mathbb{R}^3$, we have

$$\mathrm{long}(D \cap \{x \in \mathbb{R}^3 \,|\, \mathrm{dist}(x, \partial\omega) \leq \sqrt{\varepsilon}\} \leq C\sqrt{\varepsilon} \text{ with } C > 0 \text{ is independent of } \varepsilon.$$

Thus one may obtain from Proposition 1 Assertion *4.* that (10) holds.

5.2 The general case

One may adapt the above construction to the more general situation where the geodesic links are not unions of line segments.

For the standard Ginzburg-Landau energy, this has been done in [2]; there, Ω is not supposed convex. Roughly speaking, the argument there consists in replacing in Sandier's proof, line segments by curves.

Their construction begins with the modification of Ω (flattening $\partial\Omega$ close to the singularities) and, for $\eta > 0$, by the construction of an approximate (smooth) geodesic link $\Gamma_\eta \subset \Omega$ s.t. $\mathrm{long}_{a^2}(\Gamma_\eta) \leq \mathrm{long}_{a^2}(\Gamma) + \eta$. Here Γ is a geodesic link.

In order to be applicable to our situation, this construction requires the additional property $\mathscr{H}^1(\Gamma_\eta \cap \{\mathrm{dist}(x, \partial\omega) < \sqrt{\varepsilon}\}) \sim \sqrt{\varepsilon}$; we can clearly find Γ_η satisfying this property.

By adapting the construction of v_ε in (14), one may construct a test function v_ε^η having Γ_η as set of zeroes and satisfying, for each $\eta > 0$, the estimate

$$\frac{\inf_{v \in H^1_g} F_\varepsilon(v)}{|\ln \varepsilon|} \leq \frac{F_\varepsilon(v_\varepsilon^\eta)}{|\ln \varepsilon|} + o_\varepsilon(1) = \pi \mathrm{long}_{a^2}(\Gamma_\eta) + o_\varepsilon(1) \leq \pi \mathrm{long}_{a^2}(\Gamma) + \eta + o_\varepsilon(1).$$

In order to obtain this estimate we rely on the formula of v_ε^η, Proposition 1 Assertion *4.* and the assumption $\mathscr{H}^1(\Gamma_\eta \cap \{\mathrm{dist}(x, \partial\omega) < \sqrt{\varepsilon}\}) \sim \sqrt{\varepsilon}$.

Consequently we deduce that (11) holds.

6 The structure functions

For $g \in \mathcal{H}$, we will construct a suitable structure function adapted to the singularities of g.

Roughly speaking, a structure function ξ is a smooth map which almost maximizes (5). More qualitative properties of ξ will be describe in Corollaries 1, 2 and Proposition 7.

We present below three constructions of structure functions, corresponding to three different settings.

Throughout this section, we fix $\mathcal{C} = P \cup N$, $\mathrm{Card}(P) = \mathrm{Card}(N) = k \in \mathbb{N}^*$, $P \cap N = \emptyset$.

Let $\delta_0 = 10^{-2} \cdot \mathrm{dist}(\partial\omega, \partial\Omega)$. For $0 < \delta < \delta_0$, we define $\omega_\delta := \omega + B(0, \delta)$, $\alpha_0 = a^2$ and

$$\begin{array}{rccl} \alpha_\delta : & \mathbb{R}^3 & \to & \{1, b^2\} \\ & x & \mapsto & \begin{cases} b^2 & \text{if } x \in \omega_\delta \\ 1 & \text{otherwise} \end{cases} . \end{array}$$

For $x, y \in \mathbb{R}^3$ and $0 \leq \delta < \delta' \leq \delta_0$, we have

$$d_{\alpha_{\delta'}}(x, y) \leq d_{\alpha_\delta}(x, y) \leq d_{\alpha_{\delta'}}(x, y) + \mathcal{O}(\delta' - \delta). \tag{15}$$

The first inequality is a direct consequence of $\alpha_{\delta'} \leq \alpha_\delta$. We prove the second inequality. Consider $x, y \in \mathbb{R}^3$ s.t. $d_{\alpha_{\delta'}}(x, y) < d_{\alpha_\delta}(x, y)$. We obtain that if Γ is a geodesic joining x and y in $(\mathbb{R}^3, d_{\alpha_{\delta'}})$, then we have $\Gamma \cap \partial\omega_{\delta'} \neq \emptyset$.

Note that by Proposition 3, we have $\mathrm{Card}(\Gamma \cap \partial\omega_{\delta'}) \in \{1, 2\}$.

Assume that $\Gamma \cap \partial\omega_{\delta'} = \{x', y'\}$ with $d_{\alpha_{\delta'}}(x, x') < d_{\alpha_{\delta'}}(y, x')$. The situation where $\Gamma \cap \partial\omega_{\delta'} = \{z\}$ is similar.

Consider $x'' = \Pi_{\overline{\omega_\delta}}(x')$ and $y'' = \Pi_{\overline{\omega_\delta}}(y')$. Here $\Pi_{\overline{\omega_\delta}}$ stands for the orthogonal projection on $\overline{\omega_\delta}$. By the definition of x'' and y'' we have $d_{\mathrm{eucl}}(x', x'') = d_{\mathrm{eucl}}(y', y'') = \delta' - \delta$. By Proposition 3, we deduce that $d_{\alpha_{\delta'}}(x'', y'') = d_{\alpha_\delta}(x'', y'')$.

Since $x', y' \in \Gamma$, we have

$$\begin{array}{rcl} d_{\alpha_{\delta'}}(x, y) & = & d_{\alpha_{\delta'}}(x, x') + d_{\alpha_{\delta'}}(x', y') + d_{\alpha_{\delta'}}(y', y) \\ & \geq & d_{\alpha_{\delta'}}(x, x') + d_{\alpha_{\delta'}}(x'', y'') + d_{\alpha_{\delta'}}(y', y) - 2b^2|\delta' - \delta| \\ & \geq & d_{\alpha_\delta}(x, x') + d_{\alpha_\delta}(x'', y'') + d_{\alpha_\delta}(y', y) - 2b^2|\delta' - \delta| \\ & \geq & d_{\alpha_\delta}(x, x') + d_{\alpha_\delta}(x', y') + d_{\alpha_\delta}(y', y) - 2(1 + b^2)|\delta' - \delta| \\ & \geq & d_{\alpha_\delta}(x, y) - 2(1 + b^2)|\delta' - \delta| \end{array}$$

Consequently, (15) holds.

Thus, for $\mathcal{C} = P \cup N$ as defined above, we obtain that

$$L(\mathcal{C}, d_{\alpha_\delta}) = L(\mathcal{C}, d_{\alpha_{\delta'}}) + \mathcal{O}(\delta' - \delta). \tag{16}$$

6.1 Second step in the proof of Theorem 1: construction of a structure function

We have the following proposition

Proposition 4. *For $\eta > 0$ there is $\delta_\eta > 0$ s.t. for $\delta_\eta > \delta > 0$ there are $C_{\eta,\delta} > 0$, $E_{\eta,\delta} \subset \mathbb{R}$ and $\xi_{\eta,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ s.t.*

1. *$|\nabla \xi_{\eta,\delta}| \leq \alpha_\delta$ in $\mathbb{R}^3$*
2. *$\sum_{i \in \mathbb{N}_k} \{\xi_{\eta,\delta}(p_i) - \xi_{\eta,\delta}(n_i)\} \geq L(\mathcal{C}, d_{\alpha_\delta}) - \eta$*
3. *$\mathscr{H}^1(E_{\eta,\delta}) \leq \eta$ and for all $t \in \mathbb{R} \setminus E_{\eta,\delta}$, $\{\xi_{\eta,\delta} = t\}$ is a closed two dimensional surface with its second fundamental form which is bounded by $C_{\eta,\delta}$.*

Proof. We construct $\xi_{\eta,\delta}$ in five steps.

Let $\eta > 0$ and $0 < \delta < \delta' < \delta_0$. We denote $\alpha = \alpha_\delta$ and $\alpha' = \alpha_{\delta'}$. Assume that $P = \{p_1, ..., p_k\}$ and $N = \{n_1, ..., n_k\}$ are s.t. $\sigma = \mathrm{Id}$ is a minimal connection in $(\mathcal{C}, d_{\alpha'})$.

Step 1: There is $\xi_0 : \mathcal{C} \to \mathbb{R}$ s.t. ξ_0 is 1-Lipschitz in $(\mathcal{C}, d_{\alpha'})$ and $\xi_0(p_i) - \xi_0(n_i) = d_{\alpha'}(p_i, n_i)$

This step is a direct consequence of Lemma 4.2 in [5] (see also Lemma 2.2 in [12] or Lemma 2 in [3]).

Step 2: We extend ξ_0 to $\mathbb{R}^3$: there is some $\xi_1 \in \mathrm{Lip}(\mathbb{R}^3, \mathbb{R})$ s.t. $|\nabla \xi_1| = \alpha'$ and $\xi_{1|\mathcal{C}} \equiv \xi_0$

Although the argument is the same as in [12], for the convenience of the reader, we recall the construction.

Consider

$$\xi_1(x) = \max_i \{\xi_0(p_i) - d_{\alpha'}(x, p_i)\}, \ x \in \mathbb{R}^3.$$

Then we have

- $\xi_{1|\mathcal{C}} \equiv \xi_0$: let $M \in \mathcal{C}$ and i be s.t. $M \in \{p_i, n_i\}$ and $j \neq i$, it is clear that

$$\xi_0(p_i) - d_{\alpha'}(M, p_i) - \xi_0(p_j) + d_{\alpha'}(M, p_j) = \begin{cases} \xi_0(p_i) - \xi_0(p_j) + d_{\alpha'}(p_i, p_j) \geq 0 & \text{if } M = p_i \\ \xi_0(n_i) - \xi_0(p_j) + d_{\alpha'}(n_i, p_j) \geq 0 & \text{if } M = n_i \end{cases}.$$

- $|\nabla \xi_1| = \alpha'$: for all i we have

$$|\nabla [\xi_0(p_i) - d_{\alpha'}(x, p_i)]| = |\nabla d_{\alpha'}(x, p_i)| = \alpha' \text{ in } L^\infty(\mathbb{R}^3).$$

Step 3: We construct a smooth approximation: $\xi_2 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is s.t. $|\nabla \xi_2| \leq \lambda \alpha$ ($\lambda < 1$) and

$$\sum_{i \in \mathbb{N}_k} \{\xi_2(p_i) - \xi_2(n_i)\} \geq L(\mathcal{C}, d_\alpha) - \eta/2 \tag{17}$$

Let $\delta > \beta > 0$ and let $(\rho_t)_{\delta > t > 0}$ be a classical mollifier, namely $\rho_t(x) = t^{-3}\rho(x/t)$ with $\rho \in C^\infty(\mathbb{R}^3, [0, 1])$, $\mathrm{Supp}\, \rho \subset B(0, 1)$ and $\int_{\mathbb{R}^3} \rho = 1$.

Consider

$$\xi_2(x) := (1 - \beta)\xi_1 * \rho_t(x).$$

Condition (17) is clearly satisfied when t and β are small. On the other hand, the point estimate $|\nabla \xi_2(x)| \leq (1 - \beta)\|\nabla \xi_1\|_{L^\infty(B(x,t))}$ implies that $|\nabla \xi_2| \leq \lambda \alpha$ for appropriate $\lambda < 1$, provided t is sufficiently small.

Step 4: Let $\tilde{\Omega}$ be a smooth and bounded open neighborhood of $\overline{\Omega}$. We approximate ξ_2 by $\xi_{\eta,\delta}$ s.t. we have $\xi_{\eta,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ and

$$\|\xi_{\eta,\delta} - \xi_2\|_{L^\infty(\tilde{\Omega})} \leq \eta/(4k),$$

$$|\nabla \xi_{\eta,\delta}| \leq \alpha,$$

$$\xi_{\eta,\delta} \text{ is a Morse function,}$$

$$\exists R = R(\eta,\delta) > 0 \text{ s.t. in } \mathbb{R}^3 \setminus B(0,R),\ \xi_{\eta,\delta} = |x|/2$$

Clearly $\xi_{\eta,\delta}$ satisfies *1.* et *2.* of Proposition 4.

Step 5: We follow [12]. We construct $E_{\eta,\delta}$

Let $\{x_1, ..., x_l\}$ be the set of the critical points of $\xi_{\eta,\delta}$. Then there is $C = C(\eta,\delta) > 0$ s.t.:

$$\inf_{B(0,R)\setminus \cup_i B(x_i,\rho)} |\nabla \xi_{\eta,\delta}| \geq \frac{\rho}{C} \text{ since the critical points are not degenerate}$$

and

$$\mathscr{H}^1\left[\xi_{\eta,\delta}(\cup_i B(x_i,\rho))\right] \leq C\rho^2.$$

We consider $\rho > 0$ s.t. $C\rho^2 \leq \eta$ and set $E_{\eta,\delta} = \xi_{\eta,\delta}(\cup_i B(x_i,\rho))$.

For $t \notin E_{\eta,\delta}$, we have

- if $x \in \{\xi_{\eta,\delta} = t\} \setminus B(0,R)$, then the second fundamental form of $\{\xi_{\eta,\delta} = t\}$ in x is bounded,
- if $x \in \{\xi_{\eta,\delta} = t\} \cap B(0,R)$, then the second form is bounded by $C_{\eta,\delta} = \dfrac{C \sup_{B(0,R)} |D^2 \xi_{\eta,\delta}|}{\inf_{B(0,R)\setminus \cup_i B(x_i,\rho)} |\nabla \xi_{\eta,\delta}|}$.

We find that the second fundamental form is globally bounded. □

Our next result provides a sharper estimate on the gradient of structure functions.

Corollary 1. *For all $\eta > 0$, there is $C_\eta > 0$, $E_\eta \subset \mathbb{R}$, $\xi_\eta \in C^\infty(\mathbb{R}^3, \mathbb{R})$ and $\varepsilon_\eta > 0$ s.t. for $0 < \varepsilon < \varepsilon_\eta$,*

1. *$|\nabla \xi_\eta| \leq \min(a^2, U_\varepsilon^2 + \varepsilon^4)$ in $\mathbb{R}^3$,*
2. *$\sum_{i \in \mathbb{N}_k} \{\xi_\eta(p_i) - \xi_\eta(n_i)\} \geq L(\mathcal{C}, d_{a^2}) - \eta$,*
3. *$\mathscr{H}^1(E_\eta) \leq \eta$ and for all $t \in \mathbb{R} \setminus E_\eta$, $\{\xi_\eta = t\}$ is a closed hypersurface whose second fundamental form is bounded by C_η.*

Proof. Let $\eta > 0$ and fix $0 < \delta < \delta_\eta$ (δ_η given by Proposition 4) s.t.

$$L(\mathcal{C}, d_{\alpha_\delta}) + \frac{\eta}{2} \geq L(\mathcal{C}, d_{a^2}).$$

Consider $\varepsilon_\eta > 0$ s.t. for $0 < \varepsilon < \varepsilon_\eta$ we have

$$C\mathrm{e}^{-\gamma\delta/\varepsilon} < \varepsilon^4 \,(C \text{ and } \gamma \text{ are given by (3)}).$$

We take $\xi_\eta = \xi_{\eta/2,\delta}$ obtained from Proposition 4.

Clearly, ξ_η satisfies *2.* and *3.* with $E_\eta = E_{\eta/2,\delta}$ and $C_\eta = C_{\eta/2,\delta}$.

It is direct to obtain that

$$|\nabla \xi_\eta| - U_\varepsilon^2 \leq \alpha_\delta - U_\varepsilon^2 \leq \begin{cases} b^2 - U_\varepsilon^2 \leq 0 & \text{if } \operatorname{dist}(x,\omega) < \delta \\ \varepsilon^4 & \text{otherwise} \end{cases}.$$

It follows that ξ_η satisfies *1* since $\alpha_\delta \leq a^2$. □

6.2 Second step in the proof of Theorem 2: construction of a structure function

6.2.1 Definition and properties of a special pseudometric

Let $f : \mathbb{R}^3 \to [b^2, 1]$ be a Borel function and let $K \subset \mathbb{R}^3$ be a smooth compact set. We define

$$d_f^K(x,y) = \min\left\{d_f(x,y), d_f(x,K) + d_f(y,K)\right\}.$$

Here $d_f(x,K) = \min_{y \in K} d_f(x,y)$.

Then d_f^K is a pseudometric in $\mathbb{R}^3$. If, in addition $K \cap \mathcal{C} = \emptyset$, then d_f^K is a distance in $\mathcal{C}$. Therefore the minimal connection of $\mathcal{C}$ and the length of a minimal connection $L(\mathcal{C}, d_f^K)$ with respect to d_f^K make sense.

Clearly, if $x, y \in \mathbb{R}^3$, then we have $d_f^K(x,y) = 0 \Leftrightarrow x = y$ or $x, y \in K$. One may easily prove that

$$d_f^K(x,y) \leq d_f(x,y) \leq d_f^K(x,y) + \operatorname{diam}(K).$$

We are interested in the special case $K = \overline{B(x_0, r)}$ for some $x_0 \in \Omega$ and $f = \alpha_\delta$ with $\delta \in [0, \delta_0]$. Note that we have a similar estimate to (16), namely for $0 \leq \delta < \delta' < \delta_0$

$$L(\mathcal{C}, d_{\alpha_\delta}^K) = L(\mathcal{C}, d_{\alpha_{\delta'}}^K) + \mathcal{O}(|\delta' - \delta|). \tag{18}$$

Definition 1. For $y \notin K$ and $x \in \mathbb{R}^3$, we say that

- Γ is a K-curve joining x, y if Γ is a finite union of curves included in $\mathbb{R}^3 \setminus K$ s.t. their endpoints are either x or y or an element of ∂K,
- Γ is a minimal K-curve joining x, y if $\Gamma = \cup_i \Gamma_i$ is a K-curve joining x, y, where the Γ_i's are disjoint curves and $\sum_i \operatorname{long}_{a^2}(\Gamma_i) = d_{a^2}^K(x,y)$.

We next sum up the main properties of $d_{a^2}^K$.

Proposition 5. *Let $x_0 \in \mathbb{R}^3$, $r > 0$ and $K = \overline{B(x_0, r)}$. Then:*

1. *If $y \notin K$ then for all $x \in \mathbb{R}^3$ there is a minimal K-curve joining x, y. Moreover, a minimal K-curve is the union of at most two geodesics in $(\mathbb{R}^3, d_{a^2})$.*
2. *For $x_0, x, y \in \mathbb{R}^3$, $x \neq y$ and $x_0 \neq x, y$, we have:*
   - *i. If $x_0 \in \mathbb{R}^3 \setminus \partial\omega$ and x_0 is on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y) - 2a^2(x_0)r$,*
   - *ii. If $x_0 \in \partial\omega$ and x_0 is on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y) - (1+b^2)r$,*
   - *iii. If x_0 is not on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y)$.*

Proof. We prove the first assertion. There are two cases to consider: $x \in K$ and $x \notin K$.

If $x \in K$ and $y \notin K$, then we have the existence of a unique point $y_0 \in K$ which minimizes $d_{a^2}(y,z)$ among the points $z \in K$. Clearly considering Γ a geodesic in $(\mathbb{R}^3, d_{a^2})$ joining y with y_0, by definition of y_0, $\Gamma \cap K = \emptyset$. Thus Γ is a minimal K-curve according to the definition given above.

If $x, y \notin K$, then we consider Γ a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$ and, for $z \in \{x, y\}$, let Γ_z be a minimal curve in $(\mathbb{R}^3, d_{a^2})$ joining z with K.

If $\operatorname{long}_{a^2}(\Gamma_x) + \operatorname{long}_{a^2}(\Gamma_y) < \operatorname{long}_{a^2}(\Gamma \setminus K) \leq d_{a^2}(x,y)$, then one may consider $\Gamma_x \cup \Gamma_z$ as a minimal K-curve. Indeed, in this situation, $d_{a^2}^K(x,y) < d_{a^2}(x,y)$ which implies that a minimizing sequence of K-curves $\tilde{\Gamma}_n$ satisfies for large n that $\tilde{\Gamma}_n$ contains curves with an endpoint on ∂K. More precisely, by definition, there are Γ_x^n, Γ_z^n two connected components of $\tilde{\Gamma}_n$ s.t. for $z \in \{x, y\}$, Γ_z^n has z and z_n' for endpoints with $z_n' \in \partial K$. Therefore

$$\operatorname{long}_{a^2}(\Gamma_x) + \operatorname{long}_{a^2}(\Gamma_y) \leq \operatorname{long}_{a^2}(\tilde{\Gamma}_n),$$

and thus $\Gamma_x \cup \Gamma_z$ is a minimal K-curve.

Otherwise, $\operatorname{long}_{a^2}(\Gamma_x) + \operatorname{long}_{a^2}(\Gamma_y) \geq d_{a^2}(x,y)$. Consequently, denoting Γ a geodesic in $(\mathbb{R}^3, d_{a^2})$ joining x with y, $\Gamma \setminus K$ is a K-curve and has a minimal length.

It remains to prove that Γ, a minimal K-curve, is a union of at most two geodesics in $(\mathbb{R}^3, d_{a^2})$. If Γ is connected, then, by the definition of a K-curve, $\Gamma \cap K = \emptyset$. Thus Γ is a geodesic joining x, y.

Otherwise, assume that Γ is not connected. By the definition of a K-curve and by the minimality of Γ, for $z \in \{x, y\}$, there are $z' \in \partial K$ and Γ_z a connected component of Γ s.t. z, z' are the endpoints of Γ_z. Thus, by minimality of Γ, Γ_z is a geodesic joining z, z' and $\Gamma = \Gamma_x \cup \Gamma_y$.

Now we prove the second assertion. First, we assume that $x_0 \notin \partial\omega$ and that x_0 is on a geodesic curve joining x, y in $(\mathbb{R}^3, d_{a^2})$.

Consider $r_{x_0,x,y} = 10^{-2}\min\{|x-x_0|, |y-x_0|, \mathrm{dist}(x_0, \partial\omega)\}$. Then, for $r < r_{x_0,x,y}$, considering the K-curve $\Gamma \setminus K$ where Γ a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$ and containing x_0, we obtain that

$$d^K_{a^2}(x,y) \leq d_{a^2}(x,y) - 2a^2(x_0)r. \tag{19}$$

This comes from the fact that $\Gamma \cap K$ is a diameter of K and that this diameter is contained in the same connected component of $\mathbb{R}^3 \setminus \partial\omega$ as x_0. To obtain the reverse estimate, it suffices to consider Γ, a minimal K-curve joining x, y. From (19), we know that Γ as exactly two connected components: Γ_x, Γ_y with Γ_z has z, z' for endpoints with $z \in \{x, y\}$ and $z' \in \partial K$. Thus it suffices to complete Γ by the line segments $[x_0, x']$ and $[x_0, y']$ to obtain the reverse inequality. (Note that in this situation, $[x', y']$ is a diameter of K)

If $x_0 \in \partial\omega$, then the argument is similar taking $0 < r_{x_0,x,y} < 10^{-2}\min\{|x-x_0|, |y-x_0|\}$ sufficiently small s.t.:

- $B(x_0, r_{x_0,x,y}) \setminus \partial\omega$ has exactly two connected components,
- For all geodesic Γ joining x, y in $(\mathbb{R}^3, d_{a^2})$, if $x_0 \in \Gamma$ then $(\Gamma \cap K) \setminus \partial\omega$ has exactly two connected components: one in ω and the other in $\mathbb{R}^3 \setminus \overline{\omega}$.

Note that from Proposition 3, Assertion *4.d.*, $r_{x_0,x,y}$ is well defined.

Now we prove the last assertion arguing by contradiction. Assume that there is $r_n \downarrow 0$ s.t. denoting $K_n = \overline{B(x_0, r_n)}$, we have $d^{K_n}_{a^2}(x,y) < d_{a^2}(x,y)$. Consequently there are $x_n, y_n \in \partial K_n$ and $\Gamma_n = \Gamma^n_x \cup \Gamma^n_y$ where Γ^n_z is a geodesic joining z and z_n in $(\mathbb{R}^3, d_{a^2})$, $z \in \{x, y\}$. Consequently, for $z \in \{x, y\}$, one may complete Γ^n_z by the line segment $[z'_n, x_0]$ whose length in $(\mathbb{R}^3, d_{a^2})$ is at most r_n. We denote $\tilde{\Gamma}^n_z$ this curve. Clearly $d_{a^2}(z, x_0) \leq \mathrm{long}_{a^2}(\tilde{\Gamma}^n_z) \leq \mathrm{long}_{a^2}(\Gamma^n_z) + r_n$.

It suffices to claim that in a metric space (X, d) which admits geodesic curves we have for x_0, x, y three distinct points in X

$$x_0 \text{ is on a geodesic joining } x, y \Longleftrightarrow d(x,y) = d(x,x_0) + d(x_0,y).$$

Since x_0 is not on a geodesic curve joining x, y in $(\mathbb{R}^3, d_{a^2})$, there is $\eta > 0$ s.t. $d_{a^2}(x,y) + \eta < d_{a^2}(x,x_0) + d_{a^2}(x_0,y)$ and thus

$$\mathrm{long}_{a^2}(\Gamma^n_x) + \mathrm{long}_{a^2}(\Gamma^n_y) = d^{K_n}_{a^2}(x,y) < d_{a^2}(x,y) \leq \mathrm{long}_{a^2}(\Gamma^n_x) + \mathrm{long}_{a^2}(\Gamma^n_y) + 2r_n - \eta.$$

Clearly we obtain a contradiction for n sufficiently large s.t. $r_n < \eta/2$. □

Let $x_0 \in \Omega$ and $\mathcal{C} \subset \partial\Omega$ as above. If for all minimal connexion σ of $\mathcal{C}$ and for $i \in \{1, ..., k\}$, we have that x_0 is not on a geodesic joining $p_i, n_{\sigma(i)}$ in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,\mathcal{C}} > 0$ s.t. for all $r < r_{x_0,\mathcal{C}}$, we have for $K = \overline{B(x_0, r)}$

$$L(\mathcal{C}, d^K_{a^2}) = L(\mathcal{C}, d_{a^2}). \tag{20}$$

6.2.2 Construction of a structure function

Proposition 6. *Let $K = \overline{B(x_0, r)}$ be s.t. $\overline{B(x_0, 2r)} \subset \mathbb{R}^3 \setminus \mathcal{C}$ and $\eta > 0$. Then there is $\delta_{\eta,K} > 0$ s.t. for $0 < \delta < \delta_{\eta,K}$ there are $C_{\eta,K,\delta}$, $E_{\eta,K,\delta} \subset \mathbb{R}$ and $\xi_{\eta,K,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying*

1. *$|\nabla\xi_{\eta,K,\delta}| \leq \alpha_\delta$ in $\mathbb{R}^3$ and $\xi_{\eta,K,\delta}$ is constant in K,*
2. *$\sum_{i\in\mathbb{N}_k}\{\xi_{\eta,K,\delta}(p_i) - \xi_{\eta,K,\delta}(n_i)\} \geq L(\mathcal{C}, d^K_{\alpha_\delta}) - \eta$,*
3. *$\mathscr{H}^1(E_{\eta,K,\delta}) \leq \eta$ and for $t \in \mathbb{R} \setminus E_{\eta,K,\delta}$, $\{\xi_{\eta,K,\delta} = t\}$ is a closed hypersurface whose second fundamental form is bounded by $C_{\eta,K,\delta}$.*

Proof. The main point is that we require that $\xi_{\eta,K,\delta}$ is constant in K. All the other requirements are satisfied by the map $\xi_{\eta,\delta}$ constructed in Proposition 4.

For $\delta < r/2$, let $K_1 = \overline{B(x_0, r+2\delta)}$ and $K_2 = \overline{B(x_0, r+\delta)}$. We denote $\alpha = \alpha_\delta$ et $\alpha' = \alpha_{2\delta}$.

Step 1: As in the proof of Proposition 4, there is a function $\xi_0 : \mathcal{C} \to \mathbb{R}$, 1-Lipschitz function with

respect to $d^{K_1}_{\alpha'}$ and s.t. $\xi_0(p_i) - \xi_0(n_i) = d^{K_1}_{\alpha'}(p_i, n_i)$.

Step 2: We extend ξ_0 to a map $\xi_1 : \mathbb{R}^3 \to \mathbb{R}$, 1-Lipschitz and constant in K_1

For example, we may take

$$\xi_1(x) = \max_i \xi_0(p_i) - d^{K_1}_{\alpha'}(x, p_i).$$

As in the proof of Proposition 4, $\xi_{1|\mathcal{C}} = \xi_0$ and $|\nabla \xi_1| \leq \alpha'$. Moreover, ξ_1 is constant in K_1. Indeed, for all $x \in K_1$, we have $\xi_0(x) = \max_i \xi_0(p_i) - d^{K_1}_{\alpha'}(x, p_i) = \max_i \xi_0(p_i) - d_{\alpha'}(p_i, K_1)$.

Step 3: We approximate ξ_1 by $\xi_2 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying $|\nabla \xi_2| \leq \lambda\alpha$ ($\lambda < 1$), $\sum_{i \in \mathbb{N}_k} \{\xi_2(p_i) - \xi_2(n_i)\} \geq L(\mathcal{C}, d^K_\alpha) - \eta/2$ (for δ sufficiently small), and s.t. ξ_2 is constant in K_2

The approximation ξ_2 is obtain (as in Proposition 4) by regularization using a mollifier and noting that

$$L(\mathcal{C}, d^K_\alpha) \geq L(\mathcal{C}, d^K_{\alpha'}) \geq L(\mathcal{C}, d^{K_2}_{\alpha'}) \geq L(\mathcal{C}, d^{K_1}_{\alpha'}) \geq L(\mathcal{C}, d^K_{\alpha'}) - \mathcal{O}(\delta) = L(\mathcal{C}, d^K_\alpha) - \mathcal{O}(\delta).$$

Step 4: Let $\tilde{\Omega}$ be a neighborhood of $\overline{\Omega}$. We approximate ξ_2 by ξ_3 where $\xi_3 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\|\xi_3 - \xi_2\|_{L^\infty(\tilde{\Omega})} < \eta^2\delta^2,$$

$$|\nabla \xi_3| \leq \frac{1+\lambda}{2}\alpha,$$

$$\xi_3 \text{ is a Morse function,}$$

$$\exists R > 0 \text{ s.t. in } \mathbb{R}^3 \setminus B(0,R),\ \xi_3 = |x|/2.$$

Step 5: We modify ξ_3 in order to have $\xi_{\eta,K,\delta} \equiv C_0$ in K

By construction, there is $C_0 \in \xi_3(K)$ s.t. $\|\xi_3 - C_0\|_{L^\infty(K_2)} < \eta^2\delta^2$. Noting that $\text{dist}(\partial K_2, K) = \delta$, one may construct $\xi_{\eta,K,\delta} \in C^\infty(\mathbb{R}^3)$ s.t.

$$\begin{cases} \xi_{\eta,K,\delta} = \xi_3 \text{ in } \mathbb{R}^3 \setminus K_2,\ \xi_{\eta,K,\delta} \equiv C_0 \text{ in } K, \\ \|\xi_{\eta,K,\delta} - C_0\|_{L^\infty(K_2)} < \eta^2\delta^2 \text{ and } |\nabla \xi_{\eta,K,\delta}| \leq b^2 \text{ in } K_2. \end{cases}$$

Clearly $\xi_{\eta,K,\delta}$ satisfies *1.* and *2.* in Proposition 6.

Step 6: We construct $E_{\eta,K,\delta}$

For $\rho > 0$, we consider $E^1_{\eta,K,\delta} = \xi_{\eta,K,\delta}(\cup_i B(x_i, \rho))$ where $\{x_1, ..., x_l\}$ is the set of the critical points of $\xi_{\eta,K,\delta}$ in $B(0,R) \setminus K_2$.

For the same reasons as in Proposition 4, we have $\mathscr{H}^1(E^1_{\eta,K,\delta}) \leq C\rho$.

We also define $E^2_{\eta,K,\delta} = \xi_{\eta,K,\delta}(K_2)$. By construction, we have $\mathscr{H}^1(E^2_{\eta,K,\delta}) \leq 2\eta^2\delta^2$.

Thus it suffices to consider δ, ρ s.t. $C\rho + 2\eta^2\delta^2 \leq \eta$ and to set $E_{\eta,K,\delta} = E^1_{\eta,K,\delta} \cup E^2_{\eta,K,\delta}$. □

In the spirit of Corollary 1, we have

Corollary 2. *Let $x_0 \in \mathbb{R}^3$ be s.t. x_0 does not belong any minimal link of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$. There is $r_{x_0} > 0$ s.t. denoting $K = \overline{B(x_0, r_{x_0})}$, for $\eta > 0$ there are $\xi_{\eta,K} \in C^\infty(\mathbb{R}^3, \mathbb{R})$, $E_{\eta,K} \subset \mathbb{R}$, $C_{\eta,K} > 0$ and $\varepsilon_{\eta,K} > 0$ s.t. for $0 < \varepsilon < \varepsilon_{\eta,K}$,*

1. $|\nabla \xi_{\eta,K}| \leq \min(a^2, U^2_\varepsilon + \varepsilon^4)$ *in* $\mathbb{R}^3$
2. $\sum_{i \in \mathbb{N}_k} \{\xi_{\eta,K}(p_i) - \xi_{\eta,K}(n_i)\} \geq L(\mathcal{C}, d_{a^2}) - \eta$
3. $\mathscr{H}^1(E_{\eta,K}) \leq \eta$ *and for all* $t \in \mathbb{R} \setminus E_{\eta,K}$, $\{\xi_{\eta,K} = t\}$ *is a closed hypersurface whose second fundamental form is bounded by* $C_{\eta,K}$.

Proof. Assume that $\sigma = \mathrm{Id}$ is a minimal connexion in $(\mathcal{C}, d_{a^2})$.

Let $r_{x_0,\mathcal{C}} > 0$ (given by Proposition 5, Assertion *2.iii*) be s.t. for $K = \overline{B(x_0, r_{x_0,\mathcal{C}}/2)}$, we have for all $i \in \{1, ..., k\}$: $d^K_{a^2}(p_i, n_i) = d_{a^2}(p_i, n_i)$. Consequently, $L(\mathcal{C}, d_{a^2}) = L(\mathcal{C}, d^K_{a^2})$.

Now we apply Proposition 6: there is $\delta_{\eta/2,K} > 0$ s.t. for $0 < \delta < \delta_{\eta/2,K}$, there are $C_{\eta/2,K,\delta} > 0$, $E_{\eta/2,K,\delta} \subset \mathbb{R}$ and $\xi_{\eta/2,K,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying the conclusions of Proposition 6.

From (20), one may fix $0 < \delta < \delta_{\eta/2,K}$ s.t.

$$L(\mathcal{C}, d_{a^2}) - L(\mathcal{C}, d^K_{\alpha_\delta}) < \eta/2.$$

Consequently, considering $\varepsilon_{\eta,K} > 0$ s.t. for $0 < \varepsilon < \varepsilon_{\eta,K}$, we have $C\mathrm{e}^{-\gamma\delta/\varepsilon} < \varepsilon^4$ (C and γ are given by (3)).

We obtain the result taking $C_{\eta,K} = C_{\eta/2,K,\delta}$, $E_{\eta,K} = E_{\eta/2,K,\delta}$ and $\xi_{\eta,K} = \xi_{\eta/2,K,\delta}$.

□

6.3 Second step in the proof of Theorem 3: a structure function in presence of symmetries

In this section we assume that $\Omega = B(0,1)$ and that $\omega = B(0, r_0)$, with $r_0 \in (0,1)$.

Consider $\mathcal{C} = \{(1,0,0), (-1,0,0)\} = \{p, n\}$, $p = (1,0,0)$. It is clear that in this situation, the line segment $[p,n]$ is the unique geodesic between p and n in $(\mathbb{R}^3, d_{a^2})$.

The main result of this section is

Proposition 7. *Let $M \in \Omega \setminus [p,n]$. Then there is $\mathcal{V}$, an open neighbourhood of M s.t. for $\varepsilon > 0$, there is $\xi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ a Lipschitz function s.t.*

1. *$\xi_\varepsilon(p) - \xi_\varepsilon(n) = d_{U^2_\varepsilon}(p,n)$,*
2. *$|\nabla \xi_\varepsilon| \leq U^2_\varepsilon$,*
3. *$\xi_\varepsilon \equiv 0$ in $\mathcal{V}$,*
4. *$\forall\, t \in \xi_\varepsilon(\mathbb{R}^3) \setminus \{0, \xi_\varepsilon(p), \xi_\varepsilon(n)\}$, $\{\xi_\varepsilon = t\}$ is a sphere whose radius is at least 1.*

Using the spherical symmetry of Ω, ω and the minimality of U_ε, one may easily prove the following proposition.

Proposition 8. *The unique minimizer U_ε of E_ε in H^1_1, is radially symmetric and non decreasing.*

Proposition 7 is a particular case of the following lemma.

Lemma 1. *[The dumbbell lemma]*

Let $U : \mathbb{R}^3 \to [b,1]$ be a radially symmetric and non decreasing Borel function. Fix $p, n \in \mathbb{S}^2$, $p = -n$ and let $M \in \Omega \setminus [p,n]$.

Then there are $\xi : \mathbb{R}^3 \to \mathbb{R}$ and B^+, B^- two distinct open balls, B^+, B^- are exteriorly tangent and independent of U s.t.

1. *$\xi(p) - \xi(n) = d_{U^2}(p,n)$,*
2. *$|\nabla \xi| \leq U^2$,*
3. *$\xi \equiv 0$ in $\mathcal{V} := \mathbb{R}^3 \setminus (B^+ \cup B^-)$,*
4. *$M \in T$ with T which is the common tangent plan of B^+ and B^-,*
5. *B^+ is centered in $2p$, B^- is centered in $2n$,*
6. *denoting $\tilde{B}^+$ (resp. $\tilde{B}^-$) the ball centered in $2p$ (resp. $2n$) with radius 1, ξ is locally constant in $\tilde{B}^+ \cup \tilde{B}^-$,*
7. *$\forall t \in \xi(\mathbb{R}^3) \setminus \{0, \xi(p), \xi(n)\}$, $\{\xi = t\}$ is a sphere centered in $2p$ or $2n$.*

Using the symmetry of the situation, the function ξ is represented in the Figure 2.

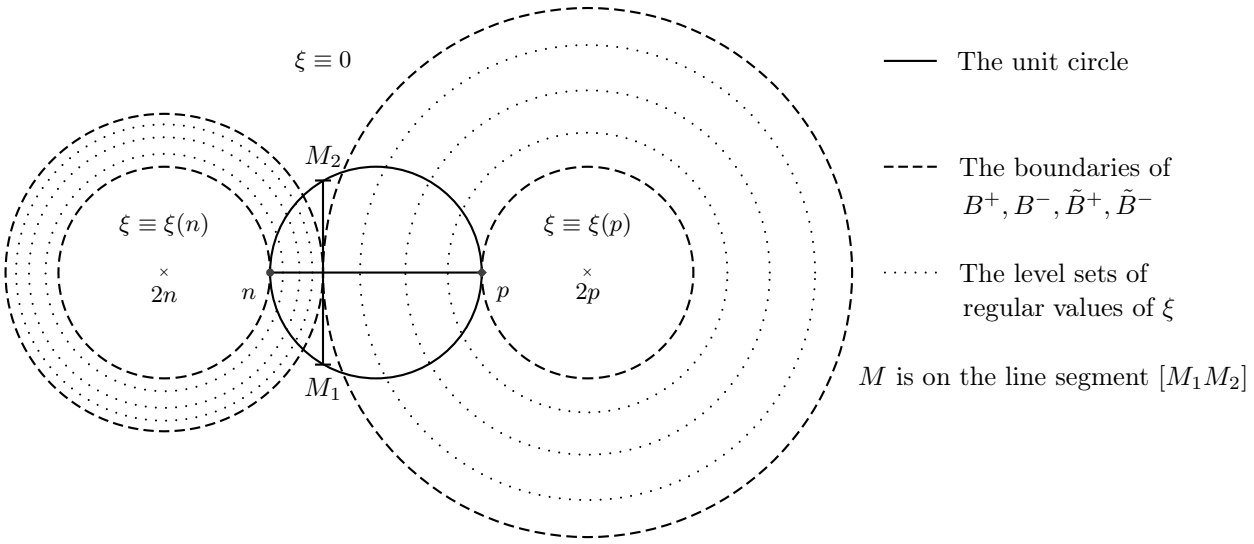

Figure 2: The geometry of the level sets of ξ (intersected with the plane defined by p, n, M)

Proof. Let $p, n \in \partial\Omega$, $p = -n$ and $\{0, (e_1, e_2, e_3)\}$ an orthonormal and direct coordinate system of $\mathbb{R}^3$ s.t. $p = (1, 0, 0)$ et $n = (-1, 0, 0)$. Let $M(x_0, y_0, z_0) \in \Omega \setminus [p, n]$.

Step 1: We construct $\xi_0 : [-1, 1] \to \mathbb{R}$ s.t. $\xi_0(1) - \xi_0(-1) = d_{U^2}(p, n)$, ${\xi_0}'(s) = U^2(s, 0, 0)$ and $\xi_0(x_0) = 0$

It suffices to consider

$$\xi_0(s) = \int_{x_0}^{s} U^2(t, 0, 0)\, \mathrm{d}t.$$

Step 2: We construct $\xi : \mathbb{R}^3 \to \mathbb{R}$

We denote

$$\Sigma_r^+ = \partial B((2, 0, 0), r) \text{ for } r \in (1, 2 - x_0)$$

and

$$\Sigma_r^- = \partial B((-2, 0, 0), r) \text{ for } r \in (1, 2 + x_0).$$

We define $\xi : \mathbb{R}^3 \to \mathbb{R}$ by its level sets:

$$\xi = \begin{cases} \xi_0(2 - r) & \text{on } \Sigma_r^+,\ r \in (1, 2 - x_0) \\ \xi_0(r - 2) & \text{on } \Sigma_r^-,\ r \in (1, 2 + x_0) \\ \xi_0(-1) & \text{in } \overline{B((-2, 0, 0), 1)} \\ \xi_0(1) & \text{in } \overline{B((2, 0, 0), 1)} \\ 0 & \text{otherwise} \end{cases}.$$

Step 3: ξ satisfies the properties of Lemma 1

Assertion *1.* is easily satisfied since $\xi(p) = \xi_0(1)$, $\xi(n) = \xi_0(-1)$ and $\xi_0(1) - \xi_0(-1) = d_{U^2}(p, n)$.

We take $B^+ = B((2, 0, 0), 2 - x_0)$ et $B^- = B((-2, 0, 0), 2 + x_0)$.

Clearly Assertions *3., 4., 5., 6.* and *7.* hold.

We check *2.*. Since ξ is locally constant in

$$V := [\mathbb{R}^3 \setminus (B^+ \cup B^-)] \cup \overline{\tilde{B}^+ \cup \tilde{B}^-},$$

it suffices to prove that $|\nabla\xi| \leq U^2$ in $\mathbb{R}^3 \setminus V$.

The key argument is the fact that for $Q, Q' \in \mathbb{R}^3$, $Q \neq Q'$ and $0 < r < |Q - Q'|$ we have $\mathrm{dist}(Q, \partial B(Q', r)) = |Q - Q'| - r = |Q - Q_0|$ where $[Q, Q'] \cap \partial B(Q', r) = \{Q_0\}$. This is obvious if we draw a picture and may be easily justified. Indeed, if Q_0 is a minimal point, then line segment $[Q, Q_0]$ is orthogonal to $\partial B(Q', r)$. Only two points on $\partial B(Q', r)$ satisfy this condition and one of them is clearly not minimal.

Consequently, taking $Q = 0$ and $Q' \in \{2p, 2n\}$ we have that

$$\min_{Q_0 \in \Sigma_r^{\pm}} |Q_0| = |(\pm(2-r), 0, 0)|.$$

Note that U is radially symmetric and non decreasing. Since in each connected components of

$$(B^+ \cup B^-) \setminus \overline{\tilde{B^+} \cup \tilde{B^-}},$$

ξ admits a spherical symmetry, we have

$$\begin{aligned} |\nabla \xi(x)| &= \begin{cases} |{\xi_0}'(2-r)| = U^2(2-r,0,0) = \min_{\Sigma_r^+} U^2 & \text{if } x \in \Sigma_r^+ \\ |{\xi_0}'(r-2)| = U^2(r-2,0,0) = \min_{\Sigma_r^-} U^2 & \text{if } x \in \Sigma_r^- \end{cases} \\ &\leq U^2(x). \end{aligned}$$

□

7 Lower bound for $F_\varepsilon(v_\varepsilon)$ when $g \in \mathcal{H}$: the argument of Sandier

In the computation of a sharp lower bound for $F_\varepsilon(v_\varepsilon)$, one of the main ingredients is Proposition 3.5 in [12]. For the convenience of the reader, we recall this result.

Proposition 9. *Let $\tilde{\Sigma}$ be a closed and oriented hypersurface in $\mathbb{R}^3$ whose second fundamental form is bounded by K. We denote by $d(\cdot,\cdot)$ the Euclidean distance restricted to $\tilde{\Sigma}$.*

Consider $\Sigma \subset \tilde{\Sigma}$, a bounded open set and $v : \Sigma \to \mathbb{C}$ s.t. there is $0 < \beta < 1$ satisfying

$$\operatorname{dist}(x, \partial\Sigma) < \beta \Rightarrow |v(x)| \geq 1/2.$$

Then we have the existence of $C > 0$ depending only on K and $\deg(v, \partial\Sigma)$ s.t.

$$\frac{1}{2}\int_\Sigma \left\{ |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \right\} \geq \pi |\deg(v, \partial\Sigma)| \ln \frac{\beta}{\varepsilon} - C.$$

This section is devoted to the proof the following propostion.

Proposition 10. *Let $g \in \mathcal{H}$ and $\mathcal{C} = P \cup N$ the set of its singularity.*

1) We have

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon)}{|\ln \varepsilon|} \geq \pi L(g, d_{a^2}). \tag{21}$$

2) We denote $< \Gamma >$ the union of all minimal links of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$ and for $\mu > 0$, $K_\mu := \{x \in \Omega \,|\, \operatorname{dist}(x, < \Gamma >) \geq \mu\}$. Then we have

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu)}{|\ln \varepsilon|} \geq \pi L(g, d_{a^2}). \tag{22}$$

3) Moreover, if we are in the symmetric case of Section 6.3, then there is $C_\mu > 0$ s.t.

$$F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu) \geq \pi d_{a^2}(p,n) |\ln \varepsilon| - C_\mu. \tag{23}$$

Theorem 1 for $g \in \mathcal{H}$, as well as Theorems 2, 3, are straightforward consequences of Proposition 10 combined with the upper bounds (10), (11).

We prove in detail (21), and we will sketch the proofs of (22),(23) which are, as explain in [12], obtained exactly in the same way as (21).

We prove that for all $\tilde{\eta} := \eta(8k^2 + 3k + 1) > 0$, the following holds

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon)}{|\ln \varepsilon|} \geq \pi L(g, d_{a^2}) - \tilde{\eta}. \tag{24}$$

Let $\eta > 0$, $\varepsilon_n \downarrow 0$, let $(v_n)_n \subset H^1_g$ be a sequence of minimizers of F_{ε_n} in H^1_g and let ξ_η, C_η, E_η be given by Corollary 1 (for n sufficiently large).

Let $0 < \rho < \eta$ and set

$$\Omega_\rho := \{x \in \mathbb{R}^3 \,|\, \mathrm{dist}(x, \Omega) < \rho \text{ et } \mathrm{dist}(x, \mathcal{C}) > \rho\}.$$

One may assume that ρ is sufficiently small s.t. in $\Omega_\rho \setminus \Omega$, $\Pi_{\partial\Omega}$, the orthogonal projection on $\partial\Omega$, is well defined and smooth.

Then we extend v_n (we use the same notation for the extension) by letting

$$v_n : \Omega_\rho \to \mathbb{R}^2,\ x \mapsto \begin{cases} v_n(x) & \text{if } x \in \Omega \\ g(\Pi_{\partial\Omega}(x)) & \text{if } x \in \Omega_\rho \setminus \Omega \end{cases}.$$

Since $g \in \mathcal{H}$ and $v_{n|\Omega_\rho \setminus \Omega}$ does not depend on n and takes its values in $\mathbb{S}^1$, we obtain the existence of $C(\rho)$ depending only on ρ, Ω, g s.t.

$$F_{\varepsilon_n}(v_n, \Omega) \geq F_{\varepsilon_n}(v_n, \Omega_\rho) - C(\rho)$$

If we define $F = F_{\eta,\rho} := E_\eta \cup [\xi_\eta(\mathcal{C}) - 2\rho, \xi_\eta(\mathcal{C}) + 2\rho]$, then we have

$$\mathscr{H}^1(F) \leq 8k\rho + \eta \leq (8k+1)\eta.$$

If $t \in \mathbb{R} \setminus F$, we denote by $\tilde{\Sigma}_t = \{\xi_\eta = t\}$. We construct for almost all $t \in \mathbb{R} \setminus F$ a closed submanifold $\Sigma_t \subset \tilde{\Sigma}_t$.

Note that for $t \in \mathbb{R} \setminus F$, we have $\mathrm{dist}(t, \xi_\eta(\mathcal{C})) \geq 2\rho$. Consequently, for $t \in \mathbb{R} \setminus F$, we obtain that $\tilde{\Sigma}_t \cap \{\Omega + B(0,\rho)\} = \tilde{\Sigma}_t \cap \Omega_\rho$.

Since $t \in \mathbb{R} \setminus F$ is not a critical value of ξ_η, the connected components W's of $\tilde{\Sigma}_t = \partial\{\xi_\eta \geq t\} = \{\xi_\eta = t\}$ have no boundary. If such W intersects Ω_ρ, then we distinguish two cases:

a) $W \cap \partial\Omega_\rho = \emptyset$

b) $W \cap \partial\Omega_\rho \neq \emptyset$.

Denote by W_a, resp. W_b, the set of the connected components satisfying a), resp. b).

If $W_b = \emptyset$, then we define $\Sigma_t = \tilde{\Sigma}_t \cap \Omega_\rho = \{\xi_\eta = t\} \cap \Omega_\rho$.

Thus it remains to construct Σ_t when $W_b \neq \emptyset$. Consider

$$f : \begin{array}{ccc} \Omega + B(0,\rho) & \to & \mathbb{R}^2 \\ x & \mapsto & (\xi_\eta(x), \mathrm{dist}[x, \partial(\Omega + B(0,\rho))]) \end{array}.$$

Using the Constant Rank Theorem (see Theorem 4.3.2, page 91 in [6]), the set $f^{-1}(\{t\} \times [r, \infty))$ $(r \in (0, \rho/2))$ is a manifold with boundary when

- t is a regular value of ξ_η,
- (t, r) is a regular value of f.

Thus, using Sard's Lemma, for almost all $t \in \mathbb{R} \setminus F$ s.t. $W_b \neq \emptyset$, there is $r = r(t) \in (0, \rho/2)$ s.t. $\Sigma_t = f^{-1}(\{t\} \times [r, \infty]) \subset \tilde{\Sigma}_t$ is a closed submanifold with boundary. Moreover, we have $\partial\Sigma_t \subset \partial\{\Omega + B(0, \rho - r)\} \cap \Omega_\rho$.

We denote by G the set

$$G := \{t \in \mathbb{R} \setminus F \,|\, W_b = \emptyset \text{ or } W_b \neq \emptyset \text{ and } \Sigma_t = f^{-1}(\{t\} \times [r, \infty)) \text{ with } r \in (0, \rho/2)\}.$$

For $t \in G$ we have

$$\mathrm{dist}(\partial\Sigma_t, \Omega) \geq \rho/2. \tag{25}$$

Let $x \in \Sigma_t$ be s.t. $\mathrm{dist}(x, \partial\Sigma_t) < \rho/2$. Using (25), we have $x \in \Omega_\rho \setminus \Omega$ and therefore $|v_n(x)| = 1$.

Finally, we are in position to apply Proposition 9:

$$\frac{1}{2}\int_{\Sigma_t} \left\{ |\nabla v_n|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_n|^2)^2 \right\} \geq \pi |\deg(v_n, \partial\Sigma_t)| \ln \frac{\rho}{\varepsilon_n} - C(\deg(v_n, \partial\Sigma_t)). \tag{26}$$

For $M \in \mathcal{C}$ and for $t \in G$ we denote $M^t \in \partial(\Omega + B(\rho - r(t))$ s.t. $\Pi_{\partial\Omega}(M^t) = M$. Here we set $r(t) = 0$ when $W_b = \emptyset$, *i.e.*, when $\Sigma_t = \tilde{\Sigma}_t \cap \Omega_\rho$. It is clear that M^t is uniquely defined.

Since $d(n,t) = \deg(v_n, \partial\Sigma_t) = \mathrm{Card}(\{p_i^t \in \{\xi_\eta \geq t\}\}) - \mathrm{Card}(\{n_i^t \in \{\xi_\eta \geq t\}\})$ takes at most $2k$ values, one may assume that $C(\deg(v_n, \partial\Sigma_t))$ is uniformly bounded in n and t. Note that $d(n,t)$ is defined for almost all t.

The key argument in this proof is the way to pass from lower bounds on hypersurfaces to a lower bound in Ω. We have the following lemma.

Lemma 2. *The following lower bound holds*

$$\int_{\mathbb{R}} d(n,t)\,\mathrm{d}t \geq L(g, d_{a^2}) - \eta(2k+1).$$

Proof. Let $m = \inf_{\Omega_\rho} \xi_\eta$, then we have

$$\begin{aligned}
\int_{\mathbb{R}} d(n,t) &= \int_{\mathbb{R}} |\{p_i^t \in \{\xi_\eta \geq t\}\}| - |\{n_i^t \in \{\xi_\eta \geq t\}\}| \\
&\geq \int_{\mathbb{R}} |\{p_i \in \{\xi_\eta \geq t+\rho\}\}| - |\{n_i \in \{\xi_\eta \geq t-\rho\}\}| \\
&\geq \sum_{i=1}^{k} \int_m^\infty \{\mathbb{1}_{\xi_\eta(p_i) > t+\rho} - \mathbb{1}_{\xi_\eta(n_i) > t-\rho}\} \\
&\geq \sum_{i=1}^{k} \{\xi_\eta(p_i) - \xi_\eta(n_i)\} - 2k\rho \geq L(g, d_{a^2}) - \eta(2k+1).
\end{aligned}$$

□

With the help of Lemma 2, we have

$$\begin{aligned}
F_{\varepsilon_n}(v_n, \Omega_\rho) \geq (\text{Corollary 1}) &\geq \frac{1}{2}\int_\Omega (|\nabla\xi_\eta| - \varepsilon_n^4)\left[|\nabla v_n|^2 + \frac{U_{\varepsilon_n}^2}{2\varepsilon_n^2}(1-|v_n|^2)^2\right] \\
&\geq (11) \geq \frac{1}{2}\int_{\mathbb{R}\setminus F}\frac{1}{2}\int_{\Sigma_t}\left\{|\nabla v_n|^2 + \frac{b^2}{2\varepsilon_n^2}(1-|v_n|^2)^2\right\}\mathrm{d}t - C_0 \\
&\geq (26) \geq \pi(\ln\frac{\rho}{\varepsilon_n} - C)\int_{\mathbb{R}\setminus F}|d(n,t)| - C_0 \\
&\geq (\text{Lemma 2}) \geq \pi(\ln\frac{\rho}{\varepsilon_n} - C)\left[L(g, d_{a^2}) - \eta(2k+1) - k\mathscr{H}^1(F)\right] - C_0 \\
&\geq \pi|\ln\varepsilon_n|\left[L(g, d_{a^2}) - \eta(8k^2+3k+1)\right] - \tilde{C}.
\end{aligned}$$

It follows that

$$\liminf_n \frac{F_{\varepsilon_n}(v_n, \Omega)}{|\ln\varepsilon_n|} \geq \pi L(g, d_{a^2}) - \eta(8k^2+3k+1),\ \forall\,\eta > 0.$$

Proposition 10.1) is obtained by letting $\eta \to 0$ in the above estimate.

We now briefly sketch the arguments leading to (22) and (23). The fundamental ingredient is a lower bound for $F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu)$. Without loss of generality, by compactness of K_μ, we may only consider the situation $K = \overline{B(x, r_x)}$ for some x which does not belong to a geodesic link between the singularities of g; here, $r_x > 0$ is some small number.

In order to prove (22), we use Corollary 2.

Following the same lines of proof of lower bound as in Proposition 10.1), we find that

$$\liminf_n \frac{F_{\varepsilon_n}(v_n, \Omega \setminus K)}{|\ln\varepsilon_n|} \geq \pi L(g, d_{a^2}).$$

Combining this lower bound with (11), we obtain

$$F_{\varepsilon_n}(v_n, K) = o(|\ln\varepsilon_n|). \tag{27}$$

In the symmetric case, using Proposition 7, we obtain the existence of r_x s.t., with $K = \overline{B(x, r_x)}$, we have

$$F_{\varepsilon_n}(v_n, \Omega \setminus K) \geq \pi d_{a^2}(p, n)|\ln \varepsilon| - C_K.$$

Consequently from the upper bound (10), we deduce

$$F_{\varepsilon_n}(v_n, K) \leq C'_K.$$

8 Extension by density of Theorem 1

From (22) and (11), we obtain that Theorem 1 holds for $g \in \mathcal{H}$. This section is devoted to the extension of Theorem 1 to the general case $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$.

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, we denote

$$f_{\varepsilon,g} = \min_{v \in H^1_g} F_\varepsilon(v).$$

Using exactly the same argument as in [2], we have

Proposition 11. *1. Let $\delta \in (0,1)$ Then there is $C(\delta) > 0$ s.t. for $g_1, g_2 \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, we have ((5.1),(5.2) in [2])*

$$(1-\delta) f_{\varepsilon,g_1} - C(\delta) f_{\varepsilon,g_2} \leq f_{\varepsilon,g_1 g_2} \leq (1+\delta) f_{\varepsilon,g_1} + C(\delta) f_{\varepsilon,g_2}. \tag{28}$$

2. There is $C > 0$ depending only on Ω s.t. for $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ we have ((5.4) in [2])

$$f_{\varepsilon,g} \leq C|g|^2_{H^{1/2}(\partial\Omega)}(1 + |\ln \varepsilon|). \tag{29}$$

3. If $(g_n)_n \subset \mathcal{H}$ is s.t. $g_n \to g$ in $H^{1/2}(\partial\Omega)$ then Lemma 17 in [2] applied with $u_n = g_n/g$ and $v = g$ yields

$$\left| \frac{g_n}{g} \right|_{H^{1/2}(\partial\Omega)} \to 0. \tag{30}$$

4. There is $C > 0$ depending only on Ω and on a s.t. for $g_1, g_2 \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ we have ((2.6) in [2])

$$|L(g_1, d_{a^2}) - L(g_2, d_{a^2})| \leq C|g_1 - g_2|_{H^{1/2}(\partial\Omega)} \left(|g_1|_{H^{1/2}(\partial\Omega)} + |g_2|_{H^{1/2}(\partial\Omega)} \right). \tag{31}$$

Using this proposition, Theorem 1 is proved as follows.

Let $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. By Proposition 2 (the third assertion), there is $(g_n)_n \subset \mathcal{H}$ s.t. $g_n \to g$ in $H^{1/2}(\partial\Omega)$.

Let $\varepsilon \in (0,1)$ and $\delta > 0$. Then, by (28), we have

$$(1-\delta)\frac{f_{\varepsilon,g_n}}{|\ln \varepsilon|} - C(\delta)\frac{f_{\varepsilon,g/g_n}}{|\ln \varepsilon|} \leq \frac{f_{\varepsilon,g}}{|\ln \varepsilon|} \leq (1+\delta)\frac{f_{\varepsilon,g_n}}{|\ln \varepsilon|} + C(\delta)\frac{f_{\varepsilon,g/g_n}}{|\ln \varepsilon|}.$$

From (29) and the fact that Theorem 1 holds for g_n, we have

$$\begin{aligned} (1-\delta)\pi L(g_n, d_{a^2}) - C'(\delta)|g/g_n|_{H^{1/2}} &\leq \liminf_\varepsilon \frac{f_{\varepsilon,g}}{|\ln \varepsilon|} \\ &\leq \limsup_\varepsilon \frac{f_{\varepsilon,g}}{|\ln \varepsilon|} \\ &\leq (1-\delta)\pi L(g_n, d_{a^2}) + C'(\delta)|g/g_n|_{H^{1/2}}. \end{aligned} \tag{32}$$

Using (31), we obtain that $L(g_n, d_{a^2}) \to L(g, d_{a^2})$. If, in (32), we first let $n \to \infty$, we use (30) and we next let $\delta \to 0$, we obtain that

$$\lim_\varepsilon \frac{f_{\varepsilon,g}}{|\ln \varepsilon|} = \pi L(g, d_{a^2}).$$

The proof of Theorem 1 is complete.

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