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# Reducing the Quantumness of Composite Quantum Systems to Two Classical Compositions

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*Dedicated to Marie-Louise Nykamp*

## Abstract

The *quantumness space* is identified and precisely defined. It is a space which characterizes the difference between states given by classical compositions of systems, and on the other hand, states corresponding to their quantum compositions. Consequently, the quantum composition of systems is *reduced* to two classical compositions. A ranking is also defined for quantumness.

”History is written with the feet ...”

Ex-Chairman Mao,  
of the Long March fame

”Science nowadays is not done  
scientifically, since it is mostly done  
by non-scientists ... ”

Anonymous

## 1. Introduction

As seen in [1-3], there is a natural division of physical systems into Cartesian ones, and on the other hand, non-Cartesian ones, according to the way their state spaces compose.

Namely, given two Cartesian systems  $X$  and  $Y$  with the respective state spaces  $E$  and  $F$ , then the composite system "X and Y" has the state space given by the Cartesian product  $E \times F$ . Classical physical systems are in this sense Cartesian.

On the other hand, quantum systems, for instance, have considerably larger state spaces for their composites. Namely, if  $X$  and  $Y$  are two such systems and their state spaces are the complex Hilbert spaces  $E$  and  $F$ , respectively, then the state space of the composite quantum system "X and Y" is the tensor product  $E \otimes F$ . And indeed, this is a considerably larger space than the Cartesian product  $E \times F$ , since we have the injective mapping, which for convenience we shall consider to be an *embedding*

$$(1.1) \quad E \times F \ni (x, y) \longmapsto x \otimes y \in E \otimes F$$

thus is seen according to the *inclusion*

$$(1.2) \quad E \times F \subset E \otimes F$$

and the difference between the two sets, in this case both complex Hilbert spaces is clearly illustrated already in the finite dimensional case when, if  $m, n$  are the dimensions of  $E$  and  $F$ , respectively, then  $m+n$  is the dimension of  $E \times F$ , while  $E \otimes F$  will have the dimension  $mn$ . Thus in general, the set of *entangled* elements

$$(1.3) \quad (E \otimes F) \setminus (E \times F)$$

is considerably larger than the set  $E \times F$  of non-entangled elements.

An essential difference, therefore, between Cartesian and non-Cartesian physical systems is that in the state spaces of the composites of two

of the latter kind there are states which can be seen as entangled in some sense, namely, those whose state *cannot* be expressed simply in terms of a pair of states, with each state in the pair taken from one of the two component systems. And as is well known in the case of quantum systems, entangled composite states are most important, for instance, in quantum computation, and in general, quantum information technology.

So far, it appears that the only known non-Cartesian physical systems are the quantum ones.

In this regard, in [1-3], the problem was formulated to find physical systems other than the quantum ones and which are non-Cartesian. Needless to say, there may be various applicative advantages in such systems. Among others, they may be used to build computers which - due to the presence of entangled states - could have advantages over usual electronic digital computers.

The main message, see pct. 2) in Remark 1 in section 4, is that for two quantum systems  $X$  and  $Y$ , one has

$$\begin{aligned} X \text{ quantum composed with } Y &= \\ &= ( X \text{ classical composed with } Y ) \text{ classical composed with} \\ &\quad ( \text{quantumness of } X \text{ and } Y ) \end{aligned}$$

or more compactly written

$$(1.4) \quad X \circ_q Y = ( X \circ_c Y ) \circ_c ( \text{quantumness of } X \text{ and } Y )$$

where  $\circ_c$  denotes the classical composition of systems, that is, the Cartesian product of their state spaces, while  $\circ_q$  denotes the quantum composition of systems, thus the tensor product of their state spaces, see [3].

It follows that the quantum composition  $X \circ_q Y$  is *reduced* to two classical compositions, namely, first

$$X \circ_c Y$$

the result of which is then further composed classically with

*quantumness of X and Y*

Needless to say, by the above reduction the quantum aspects are not eliminated completely, since they remain in the

*quantumness of X and Y*

However, the interest in such reduction is in the consequent relegation of whatever the quantum aspects may ever be as such to a specific well circumscribed place, namely, as a mere component in two successive classical compositions.

## 2. Constructing Tensor Products

Let us start with a more general, and thus simpler setup in order to better highlight what is going on. Let  $E$  and  $F$  be two abelian groups. Then their tensor product  $E \otimes F$  is constructed in the following *five steps*, [1-3].

### Step 1 :

Let  $G$  be the *free monoid* generated by the elements of the usual Cartesian product  $E \times F$ . In other words, the elements of  $G$  are all the finite sequences

$$(2.1) \quad (a_1, b_1), (a_2, b_2), (a_3, b_3), \dots, (a_n, b_n)$$

where  $n \geq 1$  and  $a_1, a_2, a_3, \dots, a_n \in E, b_1, b_2, b_3, \dots, b_n \in F$ . We also include the *empty sequence*, which thus corresponds to  $n = 0$ .

### Step 2 :

We recall that the *monoidal composition* of these sequences is done

simply by their concatenation. Furthermore, in order to simplify the notation, the comas between pairs of brackets in (2.1) will be omitted. It will be convenient to denote the resulting *monoid* by

$$(2.2) \quad (G, \diamond)$$

Clearly,  $(G, \diamond)$  is a *noncommutative* monoid, whenever at least one of the groups  $E$  or  $F$  has more than one single element. Further, we have the *injective* mapping

$$(2.3) \quad E \times F \ni (a, b) \longmapsto (a, b) \in G$$

which in fact is the *embedding*

$$(2.4) \quad E \times F \subseteq G$$

and this embedding is *strict*, whenever  $m + n < mn$ , where  $E$  has at least  $m$  elements, while  $F$  has at least  $n$  elements.

### Step 3 :

We define an *equivalence relation*  $\approx$  on  $G$  as follows. Given two elements

$$g = (a_1, b_1)(a_2, b_2)(a_3, b_3) \dots (a_n, b_n),$$

$$h = (c_1, d_1)(c_2, d_2)(c_3, d_3) \dots (c_m, d_m) \in G$$

they are equivalent, if and only if any of the following conditions holds :

$$(2.5) \quad g = h$$

or one of the elements  $g$  or  $h$  can be obtained from the other by a finite number of applications of any of the following operations :

$$(2.6) \quad \text{a permutation of pairs } (a, b) \text{ in } g$$

$$(2.7) \quad \text{a permutation of pairs } (c, d) \text{ in } h$$

(2.8) replacement of a pair  $((a' + a''), b)$  in  $g$  with the pair of pairs  $(a', b)(a'', b)$ , or vice-versa

(2.9) replacement of a pair  $(a, (b' + b''))$  in  $g$  with the pair of pairs  $(a, b')(a, b'')$ , or vice-versa

(2.10) replacement of a pair  $((c' + c''), d)$  in  $h$  with the pair of pairs  $(c', d)(c'', d)$ , or vice-versa

(2.11) replacement of a pair  $(c, (d' + d''))$  in  $h$  with the pair of pairs  $(c, d')(c, d'')$ , or vice-versa

where  $+$  is the group operation in the respective abelian groups  $E$  and  $F$ .

It follows easily that  $\approx$  is an equivalence relation which is *compatible* with the monoid  $(G, \diamond)$ .

**Step 4 :**

Finally, one defines the tensor product as the *quotient space*

$$(2.12) \quad E \otimes F = G / \approx$$

and in view of (2.3), (2.4), obtains the *injective* mapping

$$(2.13) \quad E \times F \ni (a, b) \longmapsto a \otimes b \in E \otimes F$$

where  $a \otimes b$  denotes the *coset*, or in other words, the *equivalence class* of  $(a, b) \in G$ , see (2.4), with respect to the equivalence relation  $\approx$  on  $G$ .

**Step 5 :**

Since the equivalence  $\approx$  is compatible with the monoid structure of  $(G, \diamond)$ , and in view of (2.8), (2-9), it follows that the tensor product  $E \otimes F$  obtains an *abelian group* structure.

It is useful to note the fact that in the above steps 1 and 2, there is absolutely no need for any structure on the sets  $E$  and  $F$ , and thus they can be arbitrary nonvoid sets.

Furthermore, in step 3 above, the only place the structure on the sets  $E$  and  $F$  appears is in (2.8), (2.9). And the way this structure is involved allows for *wide ranging* generalizations, far beyond any algebra, [2-6].

### 3. Universal Property of Tensor Products

For convenience, we recall here certain main features of the usual tensor product of vector spaces, and relate them to certain properties of Cartesian products.

Let  $\mathbb{K}$  be a field and  $E, F, G$  vector spaces over  $\mathbb{K}$ .

#### 3.1. Cartesian Product of Vector Spaces

Then  $E \times F$  is the vector space over  $\mathbb{K}$  where the operations are given by

$$\lambda(x, y) + \mu(u, v) = (\lambda x + \mu u, \lambda y + \mu v)$$

for any  $x, y \in E$ ,  $u, v \in F$ ,  $\lambda, \mu \in \mathbb{K}$ .

#### 3.2. Linear Mappings

Let  $\mathcal{L}(E, F)$  be the set of all mappings

$$f : E \longrightarrow F$$

such that

$$f(\lambda x + \mu u) = \lambda f(x) + \mu f(u)$$

for  $u, v \in E$ ,  $\lambda, \mu \in \mathbb{K}$ .



### 3.3. Bilinear Mappings

Let  $\mathcal{L}(E, F; G)$  be the set of all mappings

$$g : E \times F \longrightarrow G$$

such that for  $x \in E$  fixed, the mapping  $F \ni y \mapsto g(x, y) \in G$  is linear in  $y$ , and similarly, for  $y \in F$  fixed, the mapping  $E \ni x \mapsto g(x, y) \in G$  is linear in  $x \in E$ .

It is easy to see that

$$\mathcal{L}(E, F; G) = \mathcal{L}(E, \mathcal{L}(F, G))$$

### 3.4. Tensor Products

The aim of the tensor product  $E \otimes F$  is to establish a close connection between the *bilinear* mappings in  $\mathcal{L}(E, F; G)$  and the *linear* mappings in  $\mathcal{L}(E \otimes F, G)$ .

Namely, the *tensor product*  $E \otimes F$  is :

(3.4.1) a vector space over  $\mathbb{K}$ , together with

(3.4.2) a bilinear mapping  $t : E \times F \longrightarrow E \otimes F$ , such that we have the following :

#### UNIVERSALITY PROPERTY

$\forall$   $V$  vector space over  $\mathbb{K}$ ,  $g \in \mathcal{L}(E, F; V)$  bilinear mapping :

$\exists !$   $h \in \mathcal{L}(E \otimes F, V)$  linear mapping :

$$h \circ t = g$$

or in other words :

(3.4.3) the diagram commutes

$$\begin{array}{ccc}
 E \times F & \xrightarrow{t} & E \otimes F \\
 & \searrow g & \swarrow \exists! h \\
 & & V
 \end{array}$$

and

(3.4.4) the tensor product  $E \otimes F$  is *unique* up to vector space isomorphism.

Therefore we have the *injective* mapping

$$\mathcal{L}(E, F; V) \ni g \longmapsto h \in \mathcal{L}(E \otimes F, V) \quad \text{with} \quad h \circ t = g$$

The converse mapping

$$\mathcal{L}(E \otimes F, V) \ni h \longmapsto g = h \circ t \in \mathcal{L}(E, F; V)$$

obviously exists. Thus we have the *bijective* mapping

$$\mathcal{L}(E \otimes F, V) \ni h \longmapsto g = h \circ t \in \mathcal{L}(E, F; V)$$

More specifically, we have the bilinear mapping

$$(3.4.5) \quad E \times F \ni (x, y) \longmapsto t(x, y) = x \otimes y \in E \otimes F$$

and the linear mapping

$$(3.4.6) \quad E \otimes F \ni \sum_i x_i \otimes y_i \longmapsto h(\sum_i x_i \otimes y_i) = \sum_i g(x_i, y_i) \in V$$

In particular, if  $V = E \times F$  and  $g = id_{E \times F}$ , then we have the *surjective* linear mapping

$$(3.4.7) \quad h : E \otimes F \ni \sum_i x_i \otimes y_i \longmapsto \sum_i (x_i, y_i) \in E \times F$$

### 3.5. Lack of Interest in $\mathcal{L}(E \times F, G)$

Let  $f \in \mathcal{L}(E \times F, G)$  and  $(x, y) \in E \times F$ , then  $(x, y) = (x, 0) + (0, y)$ , hence

$$f(x, y) = f((x, 0) + (0, y)) = f(x, 0) + f(0, y)$$

thus  $f(x, y)$  depends on  $x$  and  $y$  in a *particular* manner, that is, separately on  $x$ , and separately on  $y$ .

### 3.6. Universality Property of Cartesian Products

Let  $A, B$  be two nonvoid sets. Their cartesian product is :

$$(3.6.1) \quad \text{a set } A \times B, \text{ together with}$$

$$(3.6.2) \quad \text{two projection mappings } p_A : A \times A \longrightarrow A, \\ p_B : A \times B \longrightarrow B, \text{ such that we have the following :}$$

#### UNIVERSALITY PROPERTY

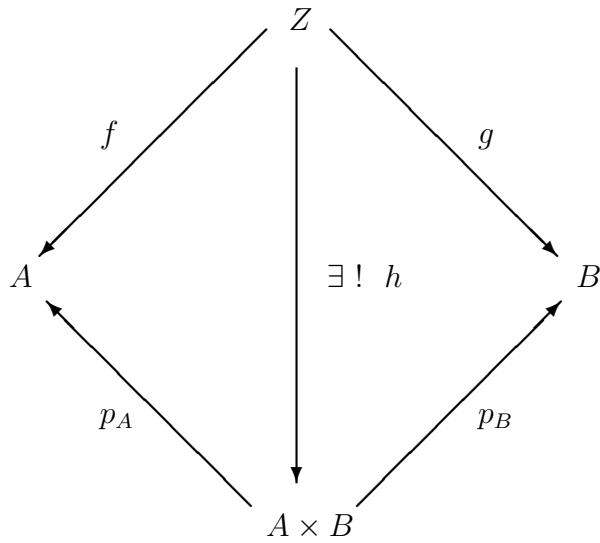
$$\forall \quad Z \text{ nonvoid set, } f : Z \longrightarrow A, \quad g : Z \longrightarrow B :$$

$$\exists ! \quad h : Z \longrightarrow A \times B :$$

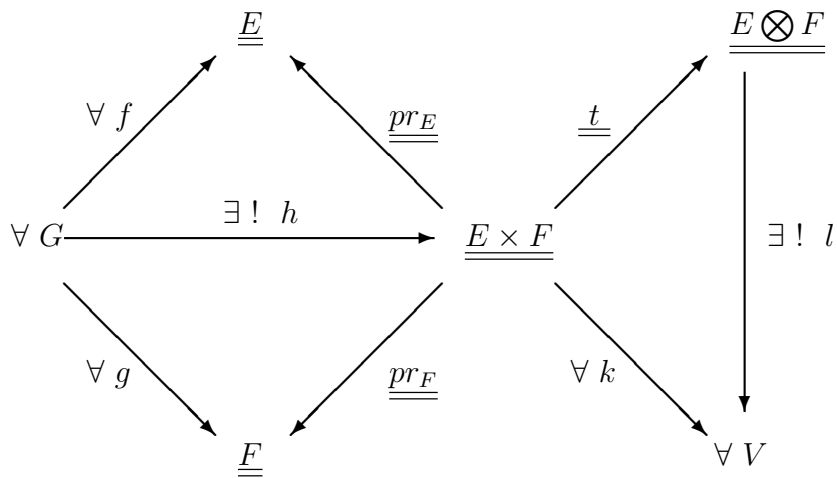
$$f = p_A \circ h, \quad g = p_B \circ h$$

or in other words :

$$(3.6.3) \quad \text{the diagram commutes}$$



### 3.7. Cartesian and Tensor Products seen together



## 4. The Quantumness of Composite Quantum Systems

Let us see what is the difference between the Cartesian product  $E \times F$  and the tensor product  $E \otimes F$  in (1.1) or (3.4.7), since it is precisely this difference which stands for the assumed *quantumness* that dis-

tinguishes between the classical composition of the respective systems leading to the Cartesian product  $E \times F$ , and on the other hand, the quantum composition which gives the tensor product  $E \otimes F$ . For convenience, we shall assume here that both  $E$  and  $F$  are vector spaces on a given field  $\mathbb{K}$ , a case which obviously contains in particular the situation of quantum interest, when  $E$  and  $F$  are complex Hilbert spaces.

Clearly,  $E \otimes F$  is in general considerably larger than  $E \times F$ . And since in view of (1.1), the latter is a vector subspace of the former, it follows that we have infinitely many direct sum representations

$$(4.1) \quad E \otimes F = (E \times F) \oplus G$$

for suitable vector subspaces  $G$  in  $E \otimes F$ .

An important fact here is that, in view of (3.4.7),  $G$  can be chosen *canonically*, as follows

$$(4.2) \quad \begin{aligned} G &= \ker h = \\ &= \{ \sum_{1 \leq i \leq n} x_i \otimes y_i \mid \sum_{1 \leq i \leq n} (x_i, y_i) = 0 = (0, 0) \in E \times F \} = \\ &= \{ \sum_{1 \leq i \leq n} x_i \otimes y_i \mid \sum_{1 \leq i \leq n} x_i = 0 \in E, \sum_{1 \leq i \leq n} y_i = 0 \in F \} \end{aligned}$$

and it will be useful in the sequel to introduce the notation in

**Definition 1.**

We call

$$(4.3) \quad (E \otimes F)_0 = \{ \sum_{1 \leq i \leq n} x_i \otimes y_i \mid \sum_{1 \leq i \leq n} x_i = 0 \in E, \sum_{1 \leq i \leq n} y_i = 0 \in F \}$$

the *quantumness space* of the quantum-like composition  $E \otimes F$ . □

Thus in view of (4.1), (3.4.7), we obtain

$$(4.4) \quad E \otimes F = (E \times F) \oplus (E \otimes F)_0$$

which, except for a vector space isomorphism, obviously gives

$$(4.5) \quad E \otimes F = (E \times F) \times (E \otimes F)_0$$

Consequently, (4.4) results in

**Theorem 1.**

Given two vector spaces  $E$  and  $F$  on a field  $\mathbb{K}$ , then for every  $\sum_{1 \leq i \leq n} x_i \otimes y_i \in E \otimes F$ , there exists a *unique* representation

$$(4.6) \quad \sum_{1 \leq i \leq n} x_i \otimes y_i = x \otimes y + \sum_{1 \leq j \leq m} u_j \otimes v_j$$

where  $x_i, x, u_j \in E, y_i, y, v_j \in F$ , and

$$(4.7) \quad \sum_{1 \leq j \leq m} u_j = 0 \in E, \quad \sum_{1 \leq j \leq m} v_j = 0 \in F$$

Furthermore, we have  $\sum_{1 \leq j \leq m} u_j \otimes v_j \in (E \otimes F)_0$ , if and only if

$$(4.8) \quad \sum_{1 \leq j \leq m} u_{\alpha(j)} \otimes v_{\beta(j)} \in (E \otimes F)_0$$

where  $\alpha$  and  $\beta$  are arbitrary permutations of  $\{1, 2, 3, \dots, m\}$ . □

Now equivalently, (4.5) yields

**Theorem 2.**

Given two vector spaces  $E$  and  $F$  on a field  $\mathbb{K}$ , then the quantum composition  $E \otimes F$  is the Cartesian composition of the Cartesian composition  $E \times F$  with the quantumness space  $(E \otimes F)_0$ . □

Regarding the *unique* representation (4.6), we have

**Theorem 3.**

Given two vector spaces  $E$  and  $F$  on a field  $\mathbb{K}$ , then for every  $\sum_{1 \leq i \leq n} x_i \otimes y_i \in E \otimes F$ , the *unique* representation (4.6) is given by the decomposition

$$(4.9) \quad \sum_{1 \leq i \leq n} x_i \otimes y_i = \left( \sum_{1 \leq i \leq n} x_i \right) \otimes \left( \sum_{1 \leq i \leq n} y_i \right) + \left( \sum_{1 \leq i \leq n} x_i \otimes y_i - \left( \sum_{1 \leq i \leq n} x_i \right) \otimes \left( \sum_{1 \leq i \leq n} y_i \right) \right)$$

where the *classical* composition part is

$$(4.10) \quad \left( \sum_{1 \leq i \leq n} x_i \right) \otimes \left( \sum_{1 \leq i \leq n} y_i \right) \in E \times F \subseteq E \otimes F$$

while the *quantumness* part is

$$(4.11) \quad \sum_{1 \leq i \leq n} x_i \otimes y_i - \left( \sum_{1 \leq i \leq n} x_i \right) \otimes \left( \sum_{1 \leq i \leq n} y_i \right) \in (E \otimes F)_0$$

**Remark 1.**

1) It follows that  $(E \otimes F)_0$  measures the *quantumness* of the states in  $E \otimes F$  which are *not* classical, that is, which do not belong to  $E \times F$ , and thus are in  $(E \otimes F) \setminus (E \times F)$ .

2) Furthermore, each quantum-like state  $\sum_{1 \leq i \leq n} x_i \otimes y_i \in E \otimes F$  is uniquely represented as the classical composition of a classical composition  $x \otimes y$  and of a component  $\sum_{1 \leq j \leq m} u_j \otimes v_j$  in the corresponding quantumness space.

Symbolically, for two quantum systems  $X$  and  $Y$ , one may indicate the above as

$$\begin{aligned} X \text{ quantum composed with } Y &= \\ &= ( X \text{ classical composed with } Y ) \text{ classical composed with} \\ &\quad ( \text{quantumness of } X \text{ and } Y ) \end{aligned}$$

Further related comments can be found at the end of section 1.

3) In case the dimensions of  $E$  and  $F$  are finite, then (4.4) obviously gives

$$(4.12) \quad \dim(E \otimes F)_0 = \dim E \dim F - \dim E - \dim F$$

**Remark 2.**

The above extend trivially to any finite number of quantum compositions.

**5. Examples**

Clearly we have

$$(5.1) \quad u \otimes v \in (E \otimes F)_0 \iff u = 0 \in E, \quad v = 0 \in F$$

Also

$$(5.2) \quad u_1 \otimes v_1 + u_2 \otimes v_2 \in (E \otimes F)_0 \iff u_2 = -u_1, \quad v_2 = -v_1$$

in which case

$$(5.3) \quad u_1 \otimes v_1 + u_2 \otimes v_2 = u_1 \otimes v_1 + (-u_1) \otimes (-v_1) = 2(u_1 \otimes v_1)$$

thus

$$(5.4) \quad u_1 \otimes v_1 + u_2 \otimes v_2 \in (E \otimes F)_0 \iff u_1 \otimes v_1 \in (E \otimes F)_0 \iff \\ \iff u_1 = u_2 = 0 \in E, \quad v_1 = v_2 = 0 \in F$$

Let now  $E = F = \mathbb{C}^2$ , with  $\mathbb{K} = \mathbb{C}$ . We consider the non-normalized Bell state



$$(5.5) \quad z = |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle \in E \otimes F$$

where  $|a\rangle, |b\rangle \in \mathbb{C}^2$  is an orthonormal basis in  $\mathbb{C}^2$ , and compute its unique representation (4.6) as follows. Assume that

$$(5.6) \quad z = x \otimes y + \sum_{1 \leq j \leq m} u_j \otimes v_j$$

then applying to it the linear mapping  $h$  in (3.4.7), it follows that

$$(5.7) \quad x + \sum_{1 \leq j \leq m} u_j = a + b, \quad y + \sum_{1 \leq j \leq m} v_j = a + b$$

while at the same time, see (4.7), we have

$$(5.8) \quad \sum_{1 \leq j \leq m} u_j = 0, \quad \sum_{1 \leq j \leq m} v_j = 0$$

Therefore

$$(5.9) \quad x = a + b, \quad y = a + b$$

and it only remains to determine the component of quantumness

$$(5.10) \quad \sum_{1 \leq j \leq m} u_j \otimes v_j = ?$$

which in view of (5.6), (5.9) and (5.10), results as follows

$$(5.11) \quad \begin{aligned} \sum_{1 \leq j \leq m} u_j \otimes v_j &= \\ &= |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle - |a+b\rangle \otimes |a+b\rangle \end{aligned}$$

In conclusion, the decomposition of the Bell state according to the *unique* representation (4.6) is given by

$$(5.12) \quad \begin{aligned} |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle &= \\ &= ( |a+b\rangle \otimes |a+b\rangle ) + \\ &+ ( |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle - |a+b\rangle \otimes |a+b\rangle ) \end{aligned}$$

with the *classical* composition part

$$(5.13) \quad |a + b\rangle \otimes |a + b\rangle \in E \times F = \mathbb{C}^2 \times \mathbb{C}^2$$

and the *quantumness* part

$$(5.14) \quad |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle - \\ - |a + b\rangle \otimes |a + b\rangle \in (E \otimes F)_0 = (\mathbb{C}^2 \otimes \mathbb{C}^2)_0$$

## 6. Rank of Quantumness

In view of the above, we are led to

### Definition 2.

Given  $z \in (E \otimes F)_0$ , we define

$$(6.1) \quad \text{rank } z$$

to be smallest  $m \in \mathbb{N}$  for which we have

$$(6.2) \quad z = \sum_{1 \leq j \leq m} u_j \otimes v_j$$

with

$$(6.3) \quad \sum_{1 \leq j \leq m} u_j = 0 \in E, \quad \sum_{1 \leq j \leq m} v_j = 0 \in F$$

□

We note that in view of (5.1) - (5.4), we have

$$(6.4) \quad \text{rank}(0 \otimes 0) = 1$$

$$(6.5) \quad \text{rank}(x \otimes y + (-x) \otimes (-y)) = 1$$

while in view of (5.12) - (5.14), we have

$$(6.6) \quad \text{rank} ( |a\rangle \otimes |a\rangle + |b\rangle \otimes |b\rangle - |a+b\rangle \otimes |a+b\rangle ) = 3$$

In view of the unique representation (4.6), one can extend Definition 2, as follows

**Definition 3.**

Given  $z \in (E \otimes F)$ , we define

$$(6.7) \quad \text{rank } z = \text{rank } w$$

where one has the unique (4.6) representation

$$(6.8) \quad z = x \otimes y + w$$

with  $w \in (E \otimes F)_0$ .

□

In view of the above, we obtain

**Theorem 4.**

For all  $z \in E \otimes F$ , we have

$$(6.9) \quad \text{rank } z \neq 2$$

## References

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