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THE HOMOLOGICAL TORSION OF SL_2 OF THE IMAGINARY QUADRATIC INTEGERS

ALEXANDER D. RAHM

ABSTRACT. The *Bianchi groups* are the groups SL_2 over a ring of integers in an imaginary quadratic number field. We reveal a correspondence between the homological torsion of the Bianchi groups and new geometric invariants, which are effectively computable thanks to their action on hyperbolic space. We expose a novel technique, the *torsion subcomplex reduction*, to obtain these invariants. We use it to explicitly compute the integral group homology of the Bianchi groups.

Furthermore, this correspondence facilitates the computation of the equivariant K -homology of the Bianchi groups. By the Baum/Connes conjecture, which is verified by the Bianchi groups, we obtain the K -theory of their reduced C^* -algebras in terms of isomorphic images of their equivariant K -homology.

1. INTRODUCTION

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by \mathcal{O}_{-m} its ring of integers. The *Bianchi groups* are the groups $SL_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian groups*, which dates back to work of Henri Poincaré [24]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [22]. A wealth of information on the Bianchi groups can be found in the monographs [13, 14, 22]. These groups act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space associated to them. The kernel of this action is the centre $\{\pm 1\}$ of the groups. Thus it is useful to study the quotient of a Bianchi group by its centre, namely $PSL_2(\mathcal{O}_{-m})$, which we also call a Bianchi group. In 1892, Luigi Bianchi [5] computed fundamental domains for this action when $m = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15$ and 19. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of \mathcal{O}_{-m}), so we will call it the *Bianchi fundamental polyhedron*. The computation of the Bianchi fundamental polyhedron has been implemented for all Bianchi groups [25] in the language Pari/GP [1].

The images under $SL_2(\mathcal{O}_{-m})$ of the facets of this polyhedron equip hyperbolic three-space with a cell structure. In order to view clearly the local geometry, we pass to the *refined cell complex*, which we obtain by subdividing this cell structure until the cell stabilisers fix the cells pointwise. We will see how to exploit this cell complex in different ways, in order to see different aspects of the geometry of these groups.

An essential invariant of groups is their homology (defined for instance in [7]). We can compute it for the Bianchi groups using the refined cell complex and the equivariant Leray/Serre spectral

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sequence which starts from the group homology of the stabilisers of representatives of the cells, and converges to the group homology of the Bianchi groups. We will now state the results for simple integer coefficients in the cases $m = 19, 43, 67$ and 163 , which are the non-Euclidean principal ideal domain cases. In contrast to these, the Euclidean principal ideal domain cases are already known from [30]. For some results in class number 2, see [26].

Throughout this article, we will use the “number theorist’s notation” \mathbb{Z}/n for the cyclic group of order n . The virtual cohomological dimension of the Bianchi groups is 2. In degrees strictly above 2, we express their homology in terms of the following Poincaré series at the primes $\ell = 2$ and $\ell = 3$:

$$P_m^\ell(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_\ell} H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell) t^q.$$

These two primes are the only numbers which occur as orders of non-trivial finite elements of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. So it has been shown [27] that the integral homology of these groups is, in all the mentioned degrees, a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$.

Proposition 1. *The integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$, for $m \in \{19, 43, 67, 163\}$, is of isomor-*

$$\text{phism type } H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\beta_1-1} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, & q = 2, \\ \mathbb{Z}^{\beta_1}, & q = 1, \end{cases}$$

where $\frac{m}{\beta_1} \mid \begin{array}{c|ccc} 19 & 43 & 67 & 163 \\ \hline 1 & 2 & 3 & 7 \end{array}$ gives the Betti number β_1 , and is in all higher degrees a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, with the number of copies specified by the Poincaré series










$$P_m^2(t) = \frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)} \text{ and } P_m^3(t) = \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}.$$

We remark that in these four cases, the torsion in the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ is of the same isomorphism type. To understand this, we consider, for a prime ℓ , the subcomplex of the orbit space consisting of the cells with elements of order ℓ in their stabiliser. We call it the ℓ -torsion subcomplex. The following statement on how its homeomorphism type determines the equivariant spectral sequence is proven by the reduction of the torsion subcomplex carried out in [27]. This technique uses lemma 14 to determine the possible type of stabiliser of a vertex v with exactly two adjacent edges which have ℓ -torsion in their stabilisers. Then these two edges, together with v , are replaced by a single edge; and theorem 3 as well as some homological information about the finite groups in question are used to check that the induced morphisms on homology produce the same terms on the second page of the equivariant spectral sequence as before the replacement.

Theorem 2. *The ℓ -primary part of the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ depends in degrees greater than 2 (the virtual cohomological dimension) only on the homeomorphism type of the ℓ -torsion subcomplex.*

This theorem follows directly from the stronger lemmata 30 and 22. Examples for this theorem are given for the prime $\ell = 3$ and thirty-six Bianchi groups in figure 1 (for $\ell = 2$, see [27]). In all the non-Euclidean principal ideal domain cases, the 2-torsion, and respectively 3-torsion subcomplexes are homeomorphic, which explains the results in proposition 1. Underlying theorem 2, there is the following correspondence between the non-trivial cyclic subgroups of the vertex stabilisers and the geodesic lines around which they effect a rotation, and which we shall call *rotation axes*.

Figure 1: Results for the 3-torsion homology, expressed in $P_m^3(t)$

m specifying the Bianchi group	3-torsion subcomplex, homeomorphism type	$P_m^3(t)$
2, 5, 6, 10, 11, 14, 15, 17, 22, 23, 29, 34, 35, 38, 41, 46, 47, 51, 55, 58, 59, 71, 83, 87, 95, 115, 119, 123, 131, 155, 159, 167, 179, 187, 191, 235, 267		$\frac{-2t^3}{t-1}$
26, 42, 143, 195		$2 \left(\frac{-2t^3}{t-1} \right)$
30, 107		$3 \left(\frac{-2t^3}{t-1} \right)$
33		$4 \left(\frac{-2t^3}{t-1} \right)$
1, 7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 571		$\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$
13, 37, 91, 403, 427		$2 \left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)} \right)$
39, 111, 183, 643		$\frac{-2t^3}{t-1} + \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$
259		$\frac{-2t^3}{t-1} + 2 \left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)} \right)$
21		$2 \left(\frac{-2t^3}{t-1} \right) + 2 \left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)} \right)$

Theorem 3. *For any vertex v in hyperbolic space, the action of its stabiliser on the set of rotation axes passing through v , induced by the action of the Bianchi group, is equivalent to the conjugation action of this stabiliser on its non-trivial cyclic subgroups.*

We shall give the proof of theorem 3 in section 2.1.

1.1. Organisation of the paper. We begin with generalities of the action of $SL_2(\mathbb{C})$ on the hyperbolic 3-space. In section 2.1, we describe some particularities which occur when restricting the action to the subgroup $SL_2(\mathcal{O}_{-m})$. This allows us to prove theorem 3. In section 2.2, we describe the refined cellular complex: an $SL_2(\mathcal{O}_{-m})$ -invariant cellular structure on hyperbolic 3-space, in which all cell stabilisation is pointwise; and we elaborate a method to check that this property holds. In section 3, we establish the properties of the action on the refined cell complex that we will need in section 6 in order to prove theorem 2. In section 4, we join some cusps to the refined cellular complex, and retract it equivariantly onto the 2-dimensional, co-compact

Flöge cellular complex. In section 5, we recall a spectral sequence which we can compute with information on this complex, and which converges to the homology of the Bianchi group in question. In sections 5.1 and 5.2, we give a characterisation of the differentials on the first page of the spectral sequence. It makes use of some homological statements on the finite subgroups of the Bianchi groups, which we append in sections 11 and 11.1. We conclude our description of the spectral sequence with the statement that for $q \geq 3$, $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ is a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$. In section 6, we examine the torsion subgraphs and prove a stronger version of theorem 2. In section 7, we establish the results in proposition 1 and in section 6.2 those in figure 1, and the corresponding table in 2-torsion. We further explain in section 8 how we check the correctness of the computed quotient space using the equivariant Euler characteristic. Finally, we give results for the special linear groups in section 9, and for equivariant K -homology and operator K -theory in section 10.

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2. THE ACTION ON HYPERBOLIC SPACE

Consider hyperbolic three-space, for which we will use the upper-half space model \mathcal{H} . As a set,

$$\mathcal{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}.$$

It is diffeomorphic to the symmetric space of $G := \mathrm{SL}_2(\mathbb{C})$, and the natural action of G which we obtain this way on \mathcal{H} can be expressed by the following formula of Poincaré. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, the action of γ on \mathcal{H} is given by $\gamma \cdot (z, \zeta) = (z', \zeta')$, where

$$\zeta' = \frac{|\det \gamma| \zeta}{|cz - d|^2 + \zeta^2 |c|^2},$$

$$z' = \frac{(\bar{d} - cz)(az - b) - \zeta^2 \bar{c}a}{|cz - d|^2 + \zeta^2 |c|^2}.$$

Let us recall Felix Klein's classification of the elements in $\mathrm{SL}_2(\mathbb{C})$, which passes to $\mathrm{PSL}_2(\mathbb{C})$.

Definition 4. An element $\gamma \in \mathrm{SL}_2(\mathbb{C})$, $\gamma \neq \pm 1$, is called loxodromic if its trace is not a real number. Else it is called

$$\begin{cases} \text{parabolic,} & |\mathrm{tr}(\gamma)| = 2, \\ \text{hyperbolic,} & |\mathrm{tr}(\gamma)| > 2 \\ \text{elliptic,} & |\mathrm{tr}(\gamma)| < 2 \end{cases} .$$

We find the geometric meaning of this classification in the following proposition, which is mostly known from Felix Klein's lectures.

Proposition 5 ([13]). Let γ be a non-trivial element of $\mathrm{SL}_2(\mathbb{C})$. Then the following holds:

- γ is parabolic if and only if γ has exactly one fixed point in $\partial\mathcal{H}$.
- γ is elliptic if and only if it has two fixed points in $\partial\mathcal{H}$ and if the points on the geodesic line in \mathcal{H} joining these two points are also left fixed. The action of γ is then a rotation around this line.
- γ is hyperbolic if and only if it has two fixed points in $\partial\mathcal{H}$ and if any circle in $\partial\mathcal{H}$ through these points together with its interior is left invariant. The line in \mathcal{H} joining these two fixed points is then left invariant, but γ has no fixed point in \mathcal{H} .

- γ is loxodromic in all other cases. The action of γ has then two fixed points in $\partial\mathcal{H}$ and no fixed point in \mathcal{H} . The geodesic joining the two fixed points is the only geodesic in \mathcal{H} which is left invariant.

For instance in [28], it is stated that the parabolic elements do not have a fixed point in the interior of \mathcal{H} . So by excluding the parabolic, hyperbolic and loxodromic cases, we obtain the following corollary.

Corollary 6. *Let γ be a non-trivial element of $\mathrm{SL}_2(\mathbb{C})$, admitting a fixed point $v \in \mathcal{H}$. Then γ fixes pointwise a geodesic line through v , and performs a rotation around this line.*

Proposition 5 and corollary 6 again pass to $\mathrm{PSL}_2(\mathbb{C})$, because the center $\{\pm 1\}$ of $\mathrm{SL}_2(\mathbb{C})$ acts trivially on \mathcal{H} .

2.1. The action of the Bianchi groups. Let m be a squarefree positive integer and $\mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic number field with ring of integers \mathcal{O}_{-m} . We restrict the above described action to the Bianchi group $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m}) \subset \mathrm{SL}_2(\mathbb{C})$.

We will work more closely with the geodesic lines of corollary 6, and define them as objects in terms of which to express the following four statements on the geometry of the torsion elements of the Bianchi groups. We will call the matrices 1 and -1 in $\mathrm{SL}_2(\mathbb{C})$ the *trivially acting elements*, because they are the kernel of the action.

Definition 7. *We will call a geodesic line passing through the point $v \in \mathcal{H}$ a Γ_v -axis, if there exists a non-trivially acting element of Γ fixing this line pointwise.*

We will call a group *non-trivially acting* if it admits a non-trivially acting element. Denote by Γ_v the stabiliser in Γ of the cell v .

Lemma 8. *For any vertex $v \in \mathcal{H}$, there is a bijection between the Γ_v -axes and the non-trivially acting cyclic subgroups of the stabiliser Γ_v . It is given by associating to a Γ_v -axis the subgroup in Γ of rotations around this axis.*

Proof. First we show that any Γ_v -axis is attributed to some non-trivial cyclic subgroup of Γ_v . Let l be a Γ_v -axis, and let $\dot{\Gamma}_l$ be the set of elements of Γ fixing l pointwise. It is a subset of Γ_v , because $\dot{\Gamma}_l$ fixes l pointwise and thus fixes v . It is a subgroup because the composites and inverses must again fix l pointwise. By the definition of the Γ_v -axes, this subgroup is non-trivially acting. And it is cyclic because by corollary 6, $\dot{\Gamma}_l$ consists only of rotations around l .

Now we show that any non-trivially acting cyclic subgroup of Γ_v is attributed to some Γ_v -axis. Let γ be the generator of a non-trivially acting cyclic subgroup of Γ_v . By corollary 6, there is a geodesic line containing v , around which γ performs a rotation. \square

Proof of theorem 3. Let l be a Γ_v -axis, and $\gamma \in \Gamma_v$. Let $\dot{\Gamma}_l$ be the subgroup of Γ fixing l pointwise. Then $\gamma \cdot l$ is again a Γ_v -axis; and the subgroup of Γ fixing $\gamma \cdot l$ pointwise is $\gamma \dot{\Gamma}_l \gamma^{-1}$. Hence by lemma 8, we can transfer the action to Γ_v -conjugation of the nontrivially acting cyclic subgroups. \square

Lemma 8 and theorem 3 clearly pass from $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$ to $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$.

We will make use of the following list of isomorphy types of finite subgroups in the Bianchi groups, which has been established in [30] and follows directly from the classification in [20].

Lemma 9 (Klein). *The finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$ are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group \mathcal{S}_3 and the alternating group \mathcal{A}_4 .*

The stabilisers of the points inside \mathcal{H} are finite and hence of the above-listed types.

2.2. A cell complex for the Bianchi groups. The Bianchi/Humbert theory [5, 18] gives a fundamental domain for the action of Γ on \mathcal{H} , which we shall call the *Bianchi fundamental polyhedron*. It is a polyhedron in hyperbolic space up to the missing vertex ∞ , and up to a missing vertex for each non-trivial ideal class if \mathcal{O}_{-m} is not a principal ideal domain. We observe the following notion of strictness of the fundamental domain: the interior of the Bianchi fundamental polyhedron contains no two points which are identified by Γ . Swan [32] proves a theorem which implies that the boundary of the Bianchi fundamental polyhedron consists of finitely many cells. Swan further produces a concept for an algorithm to compute the Bianchi fundamental polyhedron. Such an algorithm has been implemented by Cremona [11] for the five cases where \mathcal{O}_{-m} is Euclidean, and by his students Whitley [10] for the non-Euclidean principal ideal domain cases, Bygott [8] for a case of class number 2 and Lingham ([21], used in [9]) for some cases of class number 3; and finally Aranés [2] for arbitrary class numbers. Another algorithm based on this concept has independently been detailed in [27] and implemented in [25] for all Bianchi groups, so we can make explicit use of this cell complex.

We induce a cell structure on \mathcal{H} by the images under Γ of the faces, edges and vertices of the Bianchi fundamental polyhedron.

2.2.1. Pointwise stabilised cells. In order to view clearly the local geometry, we pass to the *refined cell complex*, which we obtain by subdividing this cell structure until the cell stabilisers fix the cells pointwise.

We now give a method for checking effectively if the subdivision is fine enough for the latter property to hold. First we compute which vertices of the Bianchi fundamental polyhedron lie on the same Γ -orbit. This can be deduced from the operation formula and has been also implemented in [25]. Then we perform a check to make sure that no edge of the Bianchi fundamental polyhedron can be sent onto itself reverting its orientation. The latter is only possible when origin and end of the edge are identified by some element of Γ . So to avoid it, we subdivide barycentrically all the edges, origin and end and end of which lie on the same Γ -orbit.

In order to establish an analogous criterion for 2-cells, let us make use of the fact that real hyperbolic space is non-positively curved – it has the CAT(0) property [6].

Lemma 10. *Let σ be a polygon, and G be a group of isometries of an ambient CAT(0) space. Suppose there are at least three vertices of σ which are the unique representative amongst the vertices of σ , of their respective G -orbit. Then the stabiliser of σ in G must fix σ pointwise.*

Proof. Consider an element g of the stabiliser of σ . The isometry g must preserve the set of the vertices of σ up to a permutation. Furthermore, a vertex which is not G -equivalent to any other in this set, must be fixed by g . Under the hypothesis of our lemma, g must hence fix three vertices of σ . As g is a CAT(0) isometry, it must fix pointwise the whole triangle with these three vertices as corners. This triangle is contained in the polygon σ and determines the isometric automorphisms of σ . Thus g must fix σ pointwise. \square

Hence the check on our cell structure consists of making sure that each 2-cell has at least three vertices which are unique as representative of their respective Γ -orbit, amongst the vertices of the 2-cell. Again, we can do this because we already have computed the Γ -equivalences of vertices.

The guarantee that all cells are fixed pointwise, allows us to obtain the stabilisers of the higher dimensional cells simply by intersection of their vertex stabilisers. Even more, in order to check the equivalence of two cells σ and σ' , we only need to intersect the sets of elements of Γ which identify the vertices of σ with the ones of σ' . The following lemma applies to any Γ -cell complex in hyperbolic space, and to all the cells in the refined cell complex.

Lemma 11. *The stabilisers in $PSL_2(\mathcal{O})$ of pointwise-fixed*

- *edges in \mathcal{H} , are cyclic groups of orders one, two or three;*
- *2-cells and 3-cells in \mathcal{H} , are trivial.*

Proof. As $SL_2(\mathbb{C})$ acts as orientation-preserving isometries on hyperbolic three-space, the stabiliser of a pointwise-fixed edge can only perform a rotation, with this edge lying on the rotation axis. This is possible because the edges in \mathcal{H} are geodesic segments. The group of rotations around one given axis must be abelian; and it is easy to see that it cannot be of Klein four-group type. Thus amongst the above-listed types of subgroups of $PSL_2(\mathcal{O})$, which fix points in \mathcal{H} , the only non-trivial ones which can fix edges pointwise, are $\mathbb{Z}/2$ and $\mathbb{Z}/3$.

In a pointwise-fixed 2-cell or 3-cell, we can choose two non-aligned pointwise-fixed edges, a rotation around one of which only fixes the other edge pointwise if it is the trivial rotation. \square

A completely different cell complex for extensions of the Bianchi groups has been obtained by Yasaki [35], who has implemented an algorithm of Gunnells [17] to compute the perfect forms modulo the action of $GL_2(\mathcal{O}_{-m})$, giving the facets of the Voronoï polyhedron arising from a construction of Ash [3].

3. RIGIDITY OF THE ACTION ON THE REFINED CELL COMPLEX

Lemma 12. *Each Γ_v -axis contains two edges of the refined cell complex that are adjacent to v .*

Proof. Let l be a Γ_v -axis for which this is not the case. Then l passes through the interior of a 2- or 3-cell σ adjacent to v . Let γ be a non-trivial element of Γ fixing l . As the Γ -action preserves our cell structure, γ must send σ to another cell of its dimension. Since γ fixes the points in the non-empty intersection of l with the interior of σ , and since the interior of σ intersects trivially with the interior of any other cell, γ must fix σ . Hence γ is in the stabiliser of σ , and trivial by lemma 11. But γ has been chosen non-trivial, contradiction. \square

Lemma 13. *Let e be an edge fixed pointwise by a non-trivially acting element $\gamma \in \Gamma$. Let v be a vertex adjacent to e . Then e lies on a Γ_v -axis.*

Proof. By corollary 6, γ must perform a rotation around an axis passing through v and all the points of e . This is the Γ_v -axis containing e . \square

The above study allows us to deduce from theorem 3 the following lemma, which is useful in order to obtain theorem 2. We write \mathcal{S}_3 for the symmetric group on three letters and \mathcal{A}_4 for the alternating group on four letters.

Lemma 14. *Let v be a non-singular vertex in the refined cell complex. Then the number \mathbf{n} of orbits of edges in the refined cell complex adjacent to v , with stabiliser in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ isomorphic to \mathbb{Z}/ℓ , is given as follows for $\ell = 2$ and $\ell = 3$.*

Isomorphism type of the vertex stabiliser	{1}	$\mathbb{Z}/2$	$\mathbb{Z}/3$	\mathcal{D}_2	\mathcal{S}_3	\mathcal{A}_4
\mathbf{n} for $\ell = 2$	0	2	0	3	2	1
\mathbf{n} for $\ell = 3$	0	0	2	0	1	2.

Proof. Due to lemma 13, any edge with the requested properties must lie on some Γ_v -axis. Thus the cases where $n = 0$ follow directly from lemma 8. It remains to distinguish the following cases.

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{A}_4$.

There is only one conjugacy class of order-2-elements in \mathcal{A}_4 , so by theorem 3, there is just one Γ_v -orbit of Γ_v -axes such that an element of order 2 fixes this axis pointwise. We will see that such an axis is subject to a rotation of angle π around an equivalent Γ_v -axis:

We pick two order-2-elements in Γ_v and denote them by α and β . Denote by l the axis stabilised pointwise by β . The order-2-elements in \mathcal{A}_4 commute, so α sends l to itself. But this cannot be carried out by the identity map, because in that case α would fix two axes and thus also the hyperbolic plane spanned by them, in contradiction to corollary 6. Hence, l is non-trivially rotated onto itself by α , with a rotation axis passing through v . As α preserves our cell structure, it must permute the two edges adjacent to v , lying on l due to lemma 12. So, just one $\mathbb{Z}/2$ -stabilised edge adjacent to v is in the quotient by Γ_v .

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{D}_2$.

There are exactly three conjugacy classes of order-2-elements in \mathcal{D}_2 , so by corollary 3, there are three Γ_v -orbits of Γ_v -axes. We will see that such an axis is subject to rotations of angle π around the other two Γ_v -axes:

We pick two order-2-elements in Γ_v and denote them by α and β . Denote by l the axis stabilised pointwise by β . All elements in \mathcal{D}_2 commute, so α sends l to itself. But this cannot be carried out by the identity map, because in that case α would fix two axes and thus also the hyperbolic plane spanned by them, in contradiction to corollary 6.

Hence, l is non-trivially rotated onto itself by α , with a rotation axis passing through v . As α preserves our cell structure, it must permute the two edges adjacent to v , lying on l due to lemma 12. So, there are exactly three representatives of non-trivially stabilised edges adjacent to v , one on each Γ_v -axis. Their stabilisers are the order-2-subgroups of Γ_v , as we see from lemma 8.

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{S}_3$.

There is only one conjugacy class of order-2-elements in \mathcal{S}_3 , so by theorem 3, there is just one Γ_v -orbit of Γ_v -axes such that an element of order 2 fixes this axis pointwise. Denote its representative by l . Denote by γ the generator of the stabiliser of l . As γ is of order 2, the only elements in $\Gamma_v \cong \mathcal{S}_3$ which commute with γ , are γ itself and 1. So by theorem 3, the only elements of Γ_v sending l to itself are γ and 1. Denote by (a, v) and (v, b) , the two edges which lie on l by lemma 12. Let $\delta \in \Gamma$ be an isometry identifying these two edges. As γ and 1 both fix l pointwise, by the last conclusion δ cannot be an element of Γ_v . Thus δ must send a to v and v to b . Therefore all three vertices a , b and v

are Γ -equivalent, and in particular origin and end of such an edge are identified, which we have excluded for the refined cell complex. Hence δ does not exist, and the two edges lie on different orbits.

- Let Γ_v be a non-trivial cyclic group.

As we see from corollary 6, any vertex with stabiliser a cyclic group lies on a single rotation axis, around which its stabiliser performs rotations. Hence by lemma 12, there are two edges in our cell complex adjacent to v which have the same stabiliser as v ; and any other edge adjacent to v has the trivial stabiliser. It remains to see that these two edges are not identified when passing to the quotient space.

Let $\gamma \in \Gamma$ be an isometry identifying these two edges. Denote these edges by (a, v) and (v, b) . As the only elements of Γ fixing v , fix the whole rotation axis, γ cannot send a to b and fix v . Thus γ must send a to v and v to b . Therefore all three vertices a , b and v are Γ -equivalent, and in particular origin and end of such an edge are identified, which we have excluded for the refined cell complex.

- Let $\ell = 3$ and $\Gamma_v \cong \mathcal{S}_3$.

There is only one cyclic subgroup of order 3 in \mathcal{S}_3 . So by lemma 8, there is just one such Γ_v -axis. Denote this axis by l . Then by theorem 3, l is invariant under the action of the elements of order 2 in Γ_v , because the unique order three subgroup must be preserved under conjugation. But then l must be rotated by the angle π onto itself by the order two elements, which else would fix l and their own axis, hence a hyperbolic plane. So the two edges which lie on l by lemma 12 must be equivalent.

- Let $\ell = 3$ and $\Gamma_v \cong \mathcal{A}_4$.

There are four cyclic subgroups of order 3 in \mathcal{A}_4 . They are all conjugate, so just one Γ_v -axis is in the quotient by Γ_v . Denote its representative by l . Denote by γ the generator of the stabiliser of l . Conjugation with elements of order 2 in \mathcal{A}_4 permutes the four order-3-subgroups. The only elements in $\Gamma_v \cong \mathcal{A}_4$, the conjugation with which fixes the cyclic subgroup generated by γ , are γ itself, γ^2 and 1. So by theorem 3, the only elements of Γ_v sending l to itself are γ , γ^2 and 1. Denote by (a, v) and (v, b) , the two edges which lie on l by lemma 12. Let $\delta \in \Gamma$ be an isometry identifying these two edges. As γ , γ^2 and 1 all fix l pointwise, by the last conclusion δ cannot be an element of Γ_v . Thus δ must send a to v and v to b . Therefore all three vertices a , b and v are Γ -equivalent, and in particular origin and end of such an edge are identified, which we have excluded for the refined cell complex. Hence δ does not exist, and the two edges lie on different orbits.

□

The proof of the following corollary is included in the above proof.

Corollary 15.

- In the case $\Gamma_v \cong \mathcal{D}_2$ of theorem 14, the three stabilisers of edge representatives adjacent to v which are not trivial, are precisely the three order-2-subgroups of Γ_v .
- Consider the cases of theorem 14 where Γ_v is a non-trivial cyclic group. Then the two edges adjacent to v , which have a non-trivial stabiliser, have the same stabiliser as v .

By observing the pre-images of the projection from $SL_2(\mathcal{O}_{-m})$ to $PSL_2(\mathcal{O}_{-m})$, we further obtain the following.

Corollary 16. *Let v be any vertex in the refined cell complex. Then the number n of orbits of edges in the refined cell complex adjacent to v , with stabiliser in $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$ isomorphic to $\mathbb{Z}/2\ell$, is given by the table of theorem 14, if we replace the stabiliser isomorphy types $\{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, \mathcal{D}_2 , \mathcal{S}_3 and \mathcal{A}_4 by their pre-images, which are respectively: $\mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/6$, the 8-elements quaternion group, the 12-elements binary dihedral group and the binary tetrahedral group.*

4. THE FLÖGE CELLULAR COMPLEX

In order to obtain a cell complex with compact quotient space, we proceed in the following way due to Flöge [15]. The boundary of \mathcal{H} is the Riemann sphere $\partial\mathcal{H}$, which, as a topological space, is made up of the complex plane \mathbb{C} compactified with the cusp ∞ . The totally geodesic surfaces in \mathcal{H} are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane. The action of the Bianchi groups extends continuously to the boundary $\partial\mathcal{H}$. The cellular closure of the refined cell complex in $\mathcal{H} \cup \partial\mathcal{H}$ consists of \mathcal{H} and the set of cusps $(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\}) \subset (\mathbb{C} \cup \{\infty\}) \cong \partial\mathcal{H}$. The $\mathrm{SL}_2(\mathcal{O}_{-m})$ -orbit of a cusp $\frac{\lambda}{\mu}$ in $(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\})$ corresponds to the ideal class $[(\lambda, \mu)]$ of \mathcal{O}_{-m} . It is well-known that this does not depend on the choice of the representative $\frac{\lambda}{\mu}$. We extend the refined cell complex to a cell complex \tilde{X} by joining to it, in the case that \mathcal{O}_{-m} is not a principal ideal domain, the $\mathrm{SL}_2(\mathcal{O}_{-m})$ -orbits of the cusps $\frac{\lambda}{\mu}$ for which the ideal (λ, μ) is not principal. At these cusps, we equip \tilde{X} with the ‘‘horoball topology’’ described in [15]. This simply means that the set of cusps, which is discrete in $\partial\mathcal{H}$, is located at the hyperbolic extremities of \tilde{X} : No neighbourhood of a cusp, except the whole \tilde{X} , contains any other cusp.

We retract \tilde{X} in the following, $\mathrm{SL}_2(\mathcal{O}_{-m})$ -equivariant, way. On the Bianchi fundamental polyhedron, the retraction is given by the vertical projection (away from the cusp ∞) onto its facets which are closed in $\mathcal{H} \cup \partial\mathcal{H}$. The latter are the facets which do not touch the cusp ∞ , and are the bottom facets with respect to our vertical direction. The retraction is continued on \mathcal{H} by the group action. It is proven in [16] that this retraction is continuous. We call the retract of \tilde{X} the *Flöge cellular complex* and denote it by X . So in the principal ideal domain cases, X is a retract of the refined cell complex, obtained by contracting the Bianchi fundamental polyhedron onto its cells which do not touch the boundary of \mathcal{H} . In [26], it is checked that the Flöge cellular complex is contractible.

A cell complex constructed by Mendoza [23] that coincides in the principal ideal domain cases with the Flöge cellular complex has been implemented by Vogtmann [33].

5. THE EQUIVARIANT SPECTRAL SEQUENCE TO GROUP HOMOLOGY

Let Γ be a Bianchi group. We will use the Flöge cellular complex X to compute the group homology of Γ with trivial \mathbb{Z} -coefficients, as defined in [7]. We proceed following [7, VII] and [30]. Let us consider the homology $H_*(\Gamma; C_\bullet(X))$ of Γ with coefficients in the cellular chain complex $C_\bullet(X)$ associated to X ; and call it the Γ -equivariant homology of X . As X is contractible, the map $X \rightarrow pt.$ to the point $pt.$ induces an isomorphism

$$H_*(\Gamma; C_\bullet(X)) \rightarrow H_*(\Gamma; C_\bullet(pt.)) \cong H_*(\Gamma; \mathbb{Z}).$$

Denote by X^p the set of p -cells of X , and make use of that the stabiliser Γ_σ in Γ of any p -cell σ of X fixes σ pointwise. Then from

$$C_p(X) = \bigoplus_{\sigma \in X^p} \mathbb{Z} \cong \bigoplus_{\sigma \in \Gamma \backslash X^p} \text{Ind}_{\Gamma_\sigma}^{\Gamma} \mathbb{Z},$$

Shapiro's lemma yields

$$H_q(\Gamma; C_p(X)) \cong \bigoplus_{\sigma \in \Gamma \backslash X^p} H_q(\Gamma_\sigma; \mathbb{Z});$$

and the equivariant Leray/Serre spectral sequence takes the form

$$E_{p,q}^1 = \bigoplus_{\sigma \in \Gamma \backslash X^p} H_q(\Gamma_\sigma; \mathbb{Z}) \implies H_{p+q}(\Gamma; C_\bullet(X)),$$

converging to the Γ -equivariant homology of X , which is, as we have already seen, isomorphic to $H_{p+q}(\Gamma; \mathbb{Z})$ with the trivial action on the coefficients \mathbb{Z} .

We shall also make extensive use of the description given in [30], of the d^1 -differential in this spectral sequence. The technical difference to the cases of trivial class group, treated in [30], is that the stabilisers of the singular points are free abelian groups of rank two. In particular, the Γ -action on our complex X^\bullet is not a *proper action* (in the sense that all stabilisers are finite). As a consequence, the resulting spectral sequence does not degenerate on the E^2 -level like it does in Schwermer and Vogtmann's cases. It is explained in [26] how to handle the non-trivial d^2 -differentials in this spectral sequence.

Let us now describe how to compute explicitly the d^1 -differentials, making use of the knowledge from lemma 9 about the isomorphy types of the stabilisers, and lemma 45 about their inclusions. The bottom row of the E^1 -term, more precisely the chain complex given by the $E_{p,0}^1$ -modules and the $d_{p,0}^1$ -maps, is equivalent to the \mathbb{Z} -chain complex giving the homology of the quotient space of our cell complex by the Γ -action.

From lemma 11, we see that for $q > 0$, the $E_{p,q}^1$ -terms are concentrated in the two columns $p = 0$ and $p = 1$. So for $q > 0$, we only need to compute the differentials

$$\bigoplus_{\sigma \in \Gamma \backslash X^0} H_q(\Gamma_\sigma; \mathbb{Z}) \xleftarrow{d_{1,q}^1} \bigoplus_{\sigma \in \Gamma \backslash X^1} H_q(\Gamma_\sigma; \mathbb{Z}).$$

These differentials arise from the following cell stabiliser inclusions. For any edge in $\Gamma \backslash X^1$, we have, because it is fixed pointwise, an inclusion ι of its stabiliser into the stabiliser of its origin vertex. Choose any matrix g which sends the origin vertex of this edge to its vertex representative in $\Gamma \backslash X^0$. The cell stabiliser inclusion associated to the origin of our edge is the composition of the conjugation by g after the inclusion ι . Up to inner automorphisms of the origin vertex stabiliser, this conjugation map does not depend on the choice of g , because g is determined up to multiplication with elements of the origin vertex stabiliser. The cell stabiliser inclusion associated to the end of an edge is obtained analogously. We see in [7, VII.8] that these cell stabiliser inclusions induce the differential d^1 of the equivariant spectral sequence.

As we see from lemma 11, the inclusions determined by lemma 45 are the only non-trivial inclusions which occur in our $PSL_2(\mathcal{O}_{-m})$ -cell complex. Hence we can decompose the $d_{1,q}^1$ differential in the associated equivariant spectral sequence, for $q > 0$, into a 2-primary and a 3-primary part.

Definition 17. For an abelian group A , the ℓ -primary part is the subgroup consisting of all elements of A of ℓ -power order.

5.1. The 3-primary part. Denote by $(d_{1,q}^1)_{(\ell)}$ the ℓ -primary part of our $d_{1,q}^1$ differential. It suffices to compute $(d_{1,1}^1)_{(3)}$ and $(d_{1,3}^1)_{(3)}$ to get the 3-primary part of our $d_{1,q}^1$ differential, because of the following.

Corollary 18. The 3-primary part $(d_{1,q}^1)_{(3)}$ is of period 4 in q ; and is zero for $q > 0$ even.

Proof. Lemmata 44 and 45. □

An algorithm for the computation of the rank of the differential matrices $(d_{1,1}^1)_{(3)}$ and $(d_{1,3}^1)_{(3)}$ has been given in [27].

5.2. The 2-primary part. As by lemma 11, there are only edges with finite cyclic stabilisers, we see that the $d_{1,q}^1$ differential is zero for $q > 0$ even. Now for q odd, we want to compute its 2-primary part. Any group monomorphism from edge stabilisers of type $\mathbb{Z}/2$ to vertex stabilisers of type $\mathbb{Z}/2$ or \mathcal{S}_3 induces the only possible isomorphism $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ on homology.

By lemma 45, for $q = 1$ the monomorphisms $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ induce zero maps. So, consider the case $q > 1$ odd. By lemma 14, the block in a matrix for $(d_{1,q}^1)_{(2)}$, associated to a vertex orbit of stabiliser type \mathcal{A}_4 , has exactly one non-zero column.

Lemma 19. Let q be odd. Let v be a vertex representative of stabiliser type \mathcal{D}_2 . Then the block associated to it in a matrix for the $(d_{1,q}^1)_{(2)}$ differential, has exactly three non-zero columns. There is a basis for $(H_q(\mathcal{A}_4; \mathbb{Z}))_{(2)}$ such that these block columns are $(1, 0, \dots, 0)^t$, $(0, \dots, 0, 1)^t$ and $(1, \dots, 1)^t$ (with t the transpose). The latter are linearly independent if and only if $q \geq 3$.

Proof. By lemma 14, there are exactly three Γ -representatives of non-trivially stabilised edges adjacent to v . Furthermore, the stabilisers of these three edges are precisely the three order-2-subgroups of the stabiliser of v . We apply the chain map computation of [30], to each of these three subgroup inclusions, and obtain the claimed three block columns. The length of these block columns is the $\mathbb{Z}/2$ -rank of $H_q(\mathcal{D}_2; \mathbb{Z})$, namely $\frac{q+3}{2}$, so we easily see that these block columns are linearly independent if and only if $q \geq 3$. □

Proposition 20. Let $q \geq 3$ odd. Then $\text{rank}(d_{1,q}^1)_{(2)} = \text{rank}(d_{1,3}^1)_{(2)}$.

Proof. We see from lemma 45 that all inclusions of $\mathbb{Z}/2$ into any vertex stabiliser induce injections on homology in all degrees $q \geq 3$. As the groups $\mathbb{Z}/2$ and \mathcal{S}_3 have their q -th integral homology group $\mathbb{Z}/2$ for all odd q , there is just one possibility for induced injections into it; and hence the matrix block of $(d_{1,q}^1)_{(2)}$ associated to vertex stabilisers of these types is the same for all odd q . Now for vertex stabilisers of type \mathcal{A}_4 , we know from lemma 14 that there is just one 2-torsion edge representative stabiliser inclusion into them. Thus the associated matrix blocks $(1, 0, \dots, 0)^t$ only grow in the number of their zeroes when q grows, but this does not change the rank of $(d_{1,q}^1)_{(2)}$. Finally we see from lemma 19 that associated to vertex representative stabilisers of type \mathcal{D}_2 , there are exactly three matrix sub-blocks, which are linearly independent for all $q \geq 3$. □

Denote by (\tilde{E}, \tilde{d}) the same equivariant spectral sequence, but now with $\mathbb{Z}/2$ -coefficients.

Lemma 21. Let $q \geq 3$ odd. Then the rank of $d_{1,q}^1 \otimes \mathbb{Z}/2$ equals the ranks of the differentials $\tilde{d}_{1,q}^1$ and $\tilde{d}_{1,q+1}^1$.

Proof. Lemma 11 tells us that the edges in our cell complex have cyclic stabilisers, and that only those of type $\mathbb{Z}/2$ can contribute nontrivially to the $\mathbb{Z}/2$ -modules $\tilde{E}_{1,q}^1$, $\tilde{E}_{1,q+1}^1$ and $(E_{1,q}^1)_{(2)}$. Then we see from lemma 44 that $\tilde{E}_{1,q}^1 \cong \tilde{E}_{1,q+1}^1 \cong (E_{1,q}^1)_{(2)}$. Consider matrices for the homomorphisms $d_{1,q}^1 \otimes \mathbb{Z}/2$, $\tilde{d}_{1,q}^1$ and $\tilde{d}_{1,q+1}^1$. Then applying lemma 45 with both the coefficients $M = \mathbb{Z}$ and $M = \mathbb{Z}/2$, we check entry by entry that these matrices are identical. \square

Lemma 22. *Let $q > 2$, $\ell = 2$ or $\ell = 3$, and \mathcal{O}_{-m} any imaginary quadratic ring. Then the ℓ -primary part of the homology $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ is the direct sum over the $(E_{p,q-p}^\infty)_{(\ell)}$ -terms, p running from 0 to q .*

Proof. • The rows of the E^1 -page with q even and $q \geq 2$ do not contain any 3-torsion, so nor does the E^∞ -page. So the assertion follows for $\ell = 3$ knowing that the E^1 -page is concentrated in the first two columns for $q > 0$.
 • Lemma 21 implies that $\tilde{E}_{1,q}^\infty \cong \tilde{E}_{1,q+1}^\infty \cong (E_{1,q}^\infty)_{(2)}$ and $\tilde{E}_{0,q}^\infty \cong \tilde{E}_{0,q+1}^\infty \cong (E_{0,q}^\infty)_{(2)}$. Hence the only possible solution to the dévissage problem is the trivial solution, so we obtain the assertion for $\ell = 2$. \square

Using the well-known fact that the virtual cohomological dimension (abbreviated vcd, see [7] for the definition of this) of the Bianchi groups is 2, we immediately obtain the following statement from this lemma.

Theorem 23. *Let $q > 2$. Then for \mathcal{O}_{-m} any imaginary quadratic ring, $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ is a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$.*

It is not an obvious guess that no greater torsion appears here, because the Bianchi groups admit finite index groups with arbitrarily great primes in their first homology group [31]

6. TORSION SUBCOMPLEX REDUCTION

Let Z be a Γ -cell complex, and let ℓ be a prime number.

Definition 24. *We will call ℓ -torsion subcomplex, the subcomplex of $\Gamma \backslash Z$ consisting of all the cells, the stabiliser in Γ of the pre-images of which contains elements of order ℓ .*

We immediately see that for Z the refined cellular complex, $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$ and ℓ one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because by lemma 11, the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex.

Lemma 25. *Except for the two cases $m = 1$, $\ell = 2$ and $m = 3$, $\ell = 3$, the ℓ -torsion subcomplexes of the Flöge cellular complex and the refined cellular complex coincide for all the Bianchi groups.*

Proof. The only cells which the Flöge cellular complex can admit outside the refined cellular complex, are the cusps associated to non-trivial ideal classes, of stabiliser type \mathbb{Z}^2 . So the first of these ℓ -torsion subcomplexes is always contained in the second. On the other hand, the cells of the Bianchi fundamental polyhedron which are contracted to obtain the Flöge cellular complex, are the 3-cell filling out the interior of the Bianchi fundamental polyhedron and its vertical facets, which all touch the cusp ∞ . As these cells are stabilised pointwise, their stabilisers can only contain ℓ -torsion if the stabiliser of the cusp ∞ contains ℓ -torsion. The latter is only the case for $m = 1$, $\ell = 2$ and $m = 3$, $\ell = 3$. \square

The cells which the 2-torsion and the 3-torsion subgraphs have in common, are precisely the vertices of stabiliser types \mathcal{S}_3 and \mathcal{A}_4 . For example, we find the 2-torsion subgraph drawn in dashed lines (---) and the 3-torsion subgraph drawn in dotted lines (.....) in the fundamental domain for the Flöge cellular complex in figure 3, where the vertices with matching labels are to be identified.

Knowing the types of the cell stabilisers which can appear, (the finite groups of lemma 9 and the cusp stabilisers of type \mathbb{Z}^2), we immediately see that the ℓ -primary part of the $E_{p,q}^1$ -terms of the equivariant spectral sequence converging to the integral homology of the Bianchi groups,

$$(E_{p,q}^1)_{(\ell)} = \bigoplus_{\sigma \in \Gamma \backslash X^p} (\mathbb{H}_q(\Gamma_\sigma; \mathbb{Z}))_{(\ell)},$$

depend for $q \geq 1$ only on the stabilisers of cells in the ℓ -torsion subcomplex. Similarly for the ℓ -primary part of the associated differential $(d_{1,q}^1)_{(\ell)}$: For $q \geq 1$, all cell inclusions which contribute non-trivially to it, can be found on the ℓ -torsion subcomplex.

Now we use the geometric rigidity statements of section 2 to fuse cells in the torsion subcomplexes. Let (a, v) and (v', b) be adjacent edges in the ℓ -torsion subcomplex. This means, they are adjacent to a common vertex orbit $\Gamma \cdot v = \Gamma \cdot v'$.

Definition 26. *If there are exactly two edges adjacent to the vertex v in the ℓ -torsion subcomplex, then we define an edge fusion by replacing the edges (a, v) and (v', b) by the edge (a, b) and forgetting the vertex v .*

Definition 27. *We will call a reduced ℓ -torsion subcomplex, a cell complex obtained from the ℓ -torsion subcomplex of the refined cellular complex by iterating edge fusions as often as this is permitted by definition 26.*

Lemma 28. *In all rows $q \geq 1$, the E^2 -page of the equivariant spectral sequence converging to $\mathbb{H}_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ is invariant under replacing the ℓ -torsion subcomplex by a reduced ℓ -torsion subcomplex for the computation of the ℓ -primary part of the differential $d_{1,q}^1$.*

Proof. A vertex representative v which is removed by an edge fusion must have exactly two orbits of edges of stabiliser type \mathbb{Z}/ℓ adjacent to it. Lemma 14 tells us that then, Γ_v is isomorphic to $\mathbb{Z}/2$ or \mathcal{S}_3 in the case $\ell = 2$, and to $\mathbb{Z}/3$ or \mathcal{A}_4 in the case $\ell = 3$. Now we see from definition 26 and lemma 44 that every edge fusion decreases each by 1 the \mathbb{Z}/ℓ -ranks of the modules $E_{1,q}^1$ and $E_{0,q}^1$ for odd q , because there is a unique isomorphy type of the ℓ -primary part of the homology of the stabilisers for vertices with two adjacent edges in the ℓ -torsion subcomplex:

- For the 2-primary part, $\mathbb{H}_q(\mathcal{S}_3; \mathbb{Z})_{(2)} \cong \mathbb{Z}/2 \cong \mathbb{H}_q(\mathbb{Z}/2; \mathbb{Z})_{(2)}$, for odd q ,
- for the 3-primary part, $\mathbb{H}_q(\mathcal{A}_4; \mathbb{Z})_{(3)} \cong \mathbb{Z}/3 \cong \mathbb{H}_q(\mathbb{Z}/3; \mathbb{Z})_{(3)}$, for odd q ,

and the above ℓ -primary parts are all zero for $q > 0$ even.

Let q be odd. We will show that any edge fusion also decreases by 1 the \mathbb{Z}/ℓ -rank of $(d_{1,q}^1)_{(\ell)}$. Then we can conclude that the E^2 -page is preserved under each edge fusion.

Let \mathcal{G}' be the graph obtained by an edge fusion from an ℓ -torsion subgraph \mathcal{G} . We will show that passing from \mathcal{G}' to \mathcal{G} increases the rank of $(d_{1,q}^1)_{(\ell)}$ by 1. Denote by (a, b) the fused edge. There is a column associated to it in $(d_{1,q}^1)_{(\ell)}$ for \mathcal{G}' of the shape

$$\begin{array}{c|c} & (a, b) \\ \hline a & -1 \\ b & 1 \end{array}$$

in a suitable choice of bases for the ℓ -primary part of the homology groups of the stabilisers. The remaining entries in this column are zeroes.

In the graph \mathcal{G} , we have an additional vertex v ; and we replace our edge by the two edges (a, v) and (v', b) , where v' is on the same orbit as v . By what we have seen in the beginning of this proof, $H_q(\Gamma_v; \mathbb{Z})_{(\ell)} \cong \mathbb{Z}/\ell$. Furthermore, lemma 45 tells us that the inclusions of the stabilisers of (a, v) and (v', b) into Γ_v induce injections on homology. Hence passing to \mathcal{G} , we replace the above matrix column by two columns of the shape

$$\begin{array}{c|cc} & (a, v) & (v', b) \\ \hline v & 1 & -1 \\ a & -1 & 0 \\ b & 0 & 1 \end{array}$$

(again in a suitable choice of bases, and the rest of these columns are zeroes). As the vertex v has exactly two edges adjacent to it in the ℓ -torsion subgraph, the remaining entries in the inserted row associated to the vertex v are zeroes. We further observe that the sum of the two inserted columns equals the replaced column (prolongated by a zero for the row of v). So the differential $(d_{1,q}^1)_{(\ell)}$ for \mathcal{G} has the same rank as the matrix

$$\begin{array}{c|cc} & (a, v) & (a, v) + (v', b) \text{ and remaining columns} \\ \hline v & 1 & 0 \\ a & -1 & \\ \text{other rows} & 0 & (d_{1,q}^1)_{(\ell)} \text{ for } \mathcal{G}' \end{array}$$

Hence the rank of the ℓ -primary part of the differential $d_{1,q}^1$ has increased exactly by 1. \square

The geometrical meaning of a reduced torsion subgraph is the following.

Remark 29. From the proof of lemma 14, we see that for $\mathrm{PSL}_2(\mathcal{O}_{-m})$ and the refined cell complex, any pair of fused edges has pre-images that lie on the same rotation axis. On the other hand, the quotient of any axis for rotations of order ℓ is a chain of fusionable adjacent edges in the ℓ -torsion subgraph. Hence a reduced ℓ -torsion subgraph contains one edge for every $\mathrm{PSL}_2(\mathcal{O}_{-m})$ -representative of axes for rotations of order ℓ .

6.1. Classifying the reduced torsion subcomplexes.

Given an ℓ -torsion subgraph for $\mathrm{PSL}_2(\mathcal{O}_{-m})$, the only difference that can occur between two of its reductions, is the following. If there is a loop in the graph, then this loop will become a single edge with identical origin and end vertex. But this vertex can be chosen arbitrarily from the vertices which are originally on the loop. However, the topology of the reduced graph does not depend on this choice of vertex. So as a topological space, it is well defined to speak of *the* reduced ℓ -torsion subgraph. Now we are ready to give the proof of the lemma that, together with lemma 22, implies theorem 2 because the 2-cells are trivially stabilised.

Lemma 30. *The ℓ -primary part of the terms $E_{p,q}^2$ of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ depends in all rows $q \geq 1$ only on the homeomorphism type of the ℓ -torsion subcomplex.*

Proof. By lemma 25, for each of the occurring primes $\ell = 2$ and $\ell = 3$, there is only one case where the ℓ -torsion subgraph of the Flöge cellular complex does not coincide with the ℓ -torsion subgraph of the refined cellular complex. In this case, the second subgraph is not closed, as it

has edges reaching out to the cusp ∞ . So it cannot be homeomorphic to the ℓ -torsion subgraph \mathcal{G} for any other m , because cells are fixed pointwise and hence all vertices of edges of \mathcal{G} not reaching out to cusps are contained in \mathcal{G} . So we need only consider the cases where we can identify the ℓ -torsion subgraph of the Flöge cellular complex and the refined cellular complex.

By lemma 28, we can pass from the ℓ -torsion subgraph to a reduced ℓ -torsion subgraph. Then on the connected components which contain no loops, only vertices with one or three edges adjacent to them remain: we have eliminated all vertices with two adjacent edges by edge fusions; and vertices with no adjacent edges cannot be in the ℓ -torsion subgraph due to lemma 14. By the latter theorem, there is a unique stabiliser type of vertices with one adjacent edge in the ℓ -torsion subgraph. Furthermore, the group \mathcal{D}_2 is the unique stabiliser type of vertices with three adjacent edges in the 2-torsion subgraph; and in the 3-torsion subgraph, there is no vertex with three adjacent edges at all. We can recognise vertices with one, respectively three adjacent edges as end points respectively bifurcation points in the reduced ℓ -torsion subgraph considered as a topological space Y . The end points and bifurcation points are preserved by homeomorphisms. So, a reduced ℓ -torsion subgraph can be reconstructed from the homeomorphism type of Y , as well as the associated stabiliser types. By lemma 19, it is sufficient to know that the bifurcation points come from vertices with stabiliser type \mathcal{D}_2 in order to establish the matrix block in the differential matrix. And for the vertices with just one adjacent edge, it suffices to use lemma 45 to get the associated matrix blocks. Computing the rank of this differential matrix, gives us the ℓ -primary part of the terms $E_{p,q}^2$, in all cases $q \geq 1$. So it only remains to see that the above arguments still work when there are loops in the ℓ -torsion subgraph. We classify the types of loops by the number n of bifurcation points they contain. This number can be read off from the homeomorphism type of Y . A loop with $n = 0$ consists of one edge with identified origin and end point in a reduced ℓ -torsion subgraph. The stabiliser of this point is of type $\mathbb{Z}/2$ or \mathcal{S}_3 in the case $\ell = 2$ and of type $\mathbb{Z}/3$ or \mathcal{A}_4 in the case $\ell = 3$. As stated in the proof of theorem 28, the ℓ -primary part of the homology groups is the same for both possible stabiliser types, and by lemma 45 the inclusions of \mathbb{Z}/ℓ into them induce always injections, so we do not need to know which is precisely this stabiliser type to reobtain the original contribution to the E^2 -page. Finally, if $n > 0$, then such a loop contains no more edges with two adjacent vertices in the reduced ℓ -torsion subgraph, and the arguments of the situation without loops work. \square

For the discussion in the rest of this section, we leave the two special cases $m = 1, \ell = 2$ and $m = 3, \ell = 3$ excluded, and treat them in subsection 7.2.

Observation 31. Consider the case $\ell = 3$. By lemma 14, there can be no bifurcation point in the 3-torsion subgraph. Hence, every connected component of a reduced 3-torsion subgraph consists of a single edge,

- either with two vertices of stabiliser type \mathcal{S}_3 ,
- or with identified origin and end point of stabiliser type $\mathbb{Z}/3$ or \mathcal{A}_4 , so this connected component is a loop.

In the second case, the contribution to the E^2 -page does not depend on the occurring stabiliser type, as we see from lemma 30.

Notation 32. Let ℓ be a prime number. Consider the Poincaré series in the dimensions over the field with ℓ elements, of the homology with \mathbb{Z}/ℓ -coefficients of $PSL_2(\mathcal{O}_{-m})$,

$$P_m^\ell(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_\ell} H_q(PSL_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell) t^q.$$

This Poincaré series depends only on the ℓ -primary part of the E^2 -page because we have cut off the degrees smaller than or equal to the virtual cohomological dimension of Γ , namely 2.

We observe that we can decompose the 3-torsion Poincaré series $P_m^3(t)$ as a sum over the series obtained from the connected components of the 3-torsion subgraph, because there can be no interference in the following sense.

Proposition 33. *The matrix for the ℓ -primary part of $d_{1,q}^1$ can be decomposed as a direct sum of the blocks associated to the connected components of the ℓ -torsion subgraph.*

Proof. As there is no adjacency between different connected components, all entries off these blocks are zero. \square

Observation 34. Hence it suffices to compute the 3-torsion Poincaré series $P^{3,1}(t)$ and $P^{3,2}(t)$ associated to the first and the second homeomorphism type appearing in observation 31, and for any Bianchi group $PSL_2(\mathcal{O}_{-m})$ count the numbers n_1 of connected components of first type, n_2 of connected components of second type. Then the 3-torsion Poincaré series associated to this Bianchi group equals

$$P_m^3(t) = n_1 P^{3,1}(t) + n_2 P^{3,2}(t).$$

As $P^{3,1}(t)$ and $P^{3,2}(t)$ are linearly independent, the reduced 3-torsion subgraph can be easily computed from the 3-torsion Poincaré series.

For the connected components in the 2-torsion subgraph, a priori infinitely many homeomorphism types may occur. But we still have, by proposition 33, a direct sum decomposition of the 2-primary part of the E^2 -page, for q greater than the virtual cohomological dimension of the Bianchi group.

6.2. Computations of the homological torsion. The following lemma is useful in order to transform the Poincaré series of notation 32 into fractions of finite polynomials in t .











Lemma 35. *The equation $\sum_{k=0}^{\infty} (ak + b)t^{ik+j} = \frac{t^j(b+t^i(a-b))}{(1-t^i)^2}$ holds for all $a, b, i, j \in \mathbb{N}$.*





Proof.

$$\begin{aligned} & (1-t^i)^2 \sum_{k=0}^{\infty} (ak+b)t^{ik+j} \\ &= \sum_{k=0}^{\infty} (ak+b)t^{ik+j} - 2 \sum_{k=0}^{\infty} (ak+b)t^{i(k+1)+j} + \sum_{k=0}^{\infty} (ak+b)t^{i(k+2)+j} \\ &= \sum_{k=0}^{\infty} (ak+b)t^{ik+j} - 2 \sum_{k=1}^{\infty} (a(k-1)+b)t^{ik+j} + \sum_{k=2}^{\infty} (a(k-2)+b)t^{ik+j} \end{aligned}$$

Now the multiplicity in this term of $\sum_{k=2}^{\infty} t^{ik+j}$ is $ak+b-2ak+2a-2b+ak-2a+b=0$, so the above term equals $bt^j + (a+b)t^{i+j} - 2bt^{j+i} = t^j(b-bt^i+at^i)$. \square

Figure 2: The results for the Poincaré series $P_m^2(t)$ in 2-torsion

m	reduced 2-torsion subgraph	$P_m^2(t)$
7, 15, 35, 39, 87, 91, 95, 115, 151, 155, 159, 191, 403		$\frac{-2t^3}{t-1}$
46		$2 \left(\frac{-2t^3}{t-1} \right)$
235, 427		$3 \left(\frac{-2t^3}{t-1} \right)$
3, 11, 19, 43, 67, 139, 163		$\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}$
51, 123, 187, 267		$2 \left(\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)} \right)$
6, 22		$\frac{-2t^3}{t-1} + \frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}$
5, 10, 13, 29, 58		$\frac{-t^3(3t-5)}{(t-1)^2}$
37		$\frac{-t^3(3t-5)}{(t-1)^2} + 2 \left(\frac{-2t^3}{t-1} \right)$
2		$\frac{-2t^3(t^3-t^2-2)}{(t-1)^2(t^2+t+1)}$
34		$2 \left(\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)} \right) + 2 \left(\frac{-2t^3}{t-1} \right)$

Observation 36. In figure 2, we see that in the cases of class numbers 1 and 2, and in the cases where the absolute value of the discriminant is inferior to 200, there are just four homeomorphism types of connected components in the 2-torsion subgraph, , ,  and . Their associated 2-torsion Poincaré series are linearly independent. As a result, within our observed range of values for m , we are able to decompose a 2-torsion Poincaré series in order to reobtain the homeomorphy type.

7. THE NON-EUCLIDEAN PRINCIPAL IDEAL DOMAIN CASES

We now give the results in the cases $m = 19, 43, 67$ and 163 , which are the non-Euclidean principal ideal domain cases. The Euclidean principal ideal domain cases are already known from [30]. We observe that in these four cases, the torsion in the integral homology of $PSL_2(\mathcal{O}_{-m})$ is of the same isomorphism type. This comes from the fact that their 2-torsion and 3-torsion subgraphs are homeomorphic (see figure 3 below). Lemma 30 then explains this isomorphy.

Proposition 37. *For $m \in \{19, 43, 67, 163\}$, the E^2 -page of the equivariant spectral sequence is concentrated in the columns $p = 0$, $p = 1$ and $p = 2$, given as follows.*

$q = 12n + 14$	$(\mathbb{Z}/2)^{4n+6}$		
$q = 12n + 13$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 12$	$(\mathbb{Z}/2)^{4n+4}$		
$q = 12n + 11$	$(\mathbb{Z}/2)^{4n+5} \oplus \mathbb{Z}/3$		
$q = 12n + 10$	$(\mathbb{Z}/2)^{4n+2}$		
$q = 12n + 9$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 8$	$(\mathbb{Z}/2)^{4n+4}$		
$q = 12n + 7$	$(\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3$		
$q = 12n + 6$	$(\mathbb{Z}/2)^{4n+2}$		
$q = 12n + 5$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 4$	$(\mathbb{Z}/2)^{4n}$		
$q = 12n + 3$	$(\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3$		
$q = 2$	$(\mathbb{Z}/2)^2$		
$q = 1$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	
$q = 0$	\mathbb{Z}	\mathbb{Z}^{β_1}	\mathbb{Z}^{β_2} ,
	$p = 0$	$p = 1$	$p = 2$

where $\frac{m}{\beta_1} \mid \frac{19}{1} \frac{43}{2} \frac{67}{3} \frac{163}{7}$ gives the Betti number β_1 , and $\beta_2 = \beta_1 - 1$.

The Betti numbers are related by the vanishing of the naive Euler characteristic [33], $\beta_0 + \beta_1 + \beta_2 = 0$.

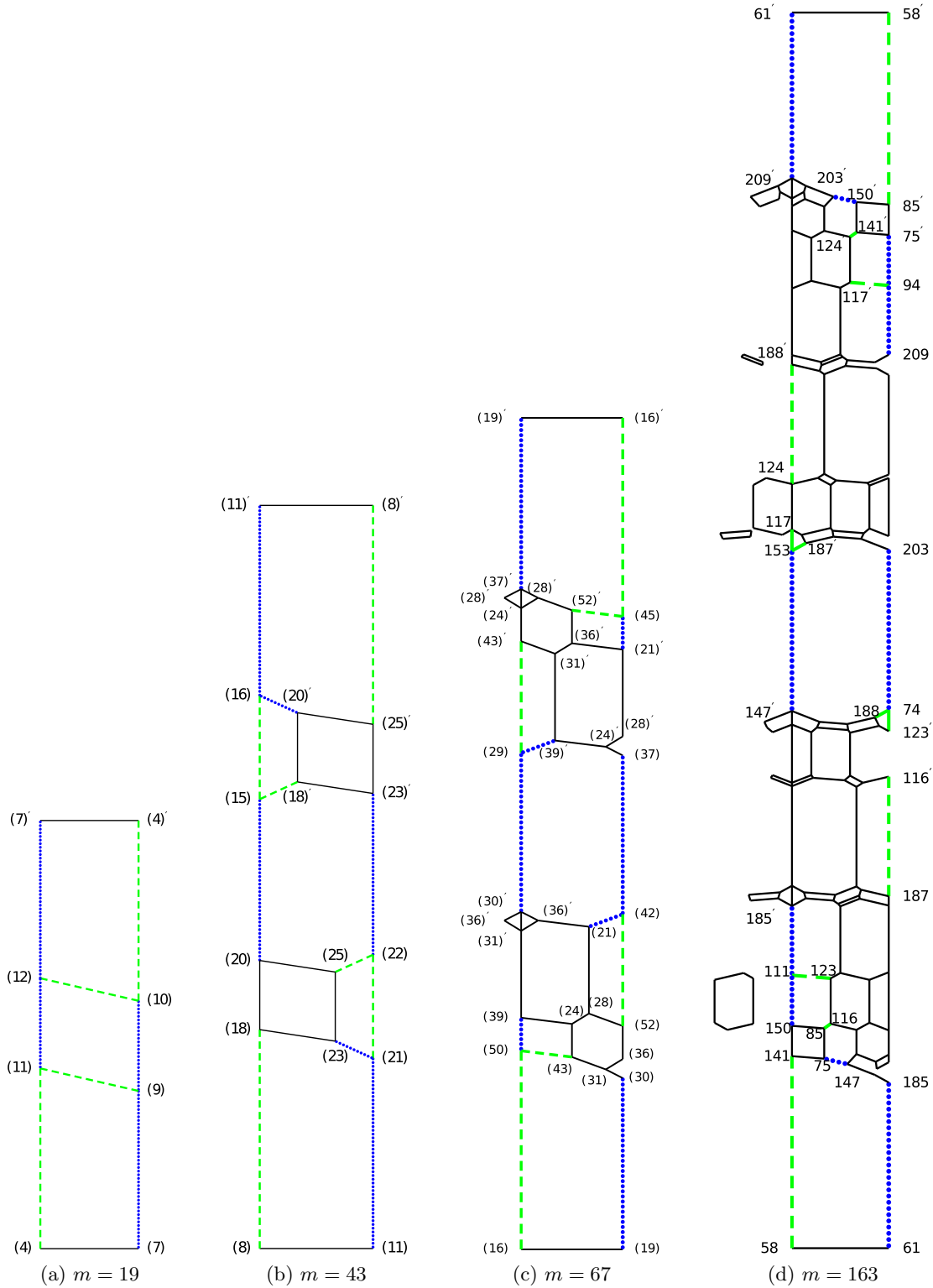
We observe that the $E_{0,1}^2$ -term vanishes completely. This term is the target of the only d^2 -arrow which can for arbitrary m be non-zero, namely $d_{2,0}^2$. Hence our spectral sequence degenerates at the E^2 -level. The only ambiguity in the dévissage concerns $H_2(PSL_2(\mathcal{O}_{-m}); \mathbb{Z})$; the above E^∞ -page says that its 2-primary part is either $(\mathbb{Z}/2)^3$ or $\mathbb{Z}/4 \oplus \mathbb{Z}/2$. Computing

$$\dim_{\mathbb{F}_2} H_q(PSL_2(\mathcal{O}_{-m}); \mathbb{Z}/2) = \begin{cases} 4k + 5, & q = 6k + 8, \\ 4k + 3, & q = 6k + 7, \\ 4k + 5, & q = 6k + 6, \\ 4k + 3, & q = 6k + 5, \\ 4k + 1, & q = 6k + 4, \\ 4k + 3, & q = 6k + 3, \\ \beta_2 + 2, & q = 2, \\ \beta_1, & q = 1, \end{cases}$$

and comparing with the help of the Universal Coefficient Theorem, we can exclude the first possibility. We obtain proposition 1.

7.1. Intermediary results. We now give the intermediary results we have used to compute the above E^2 -pages, in the case $m = 67$. Our fundamental domain for the Flöge cellular complex (which coincides with Mendoza's spine in the principal ideal domain cases) is drawn in figure 4c.

Figure 3: Fundamental domains for $\mathrm{PSL}_2(\mathcal{O}_{-m})$ in the Flöge cellular complex, in the non-Euclidean principal ideal domain cases. The vertices with matching labels are identified by $\mathrm{PSL}_2(\mathcal{O}_{-m})$, and then the 2-torsion subcomplex is obtained from the dashed edges and the 3-torsion subcomplex from the dotted edges.



We denote by (k) the vertex number k in the output files of the program [25], and by $\Gamma_{(k)}$ its stabiliser. We do the same for the edges, which we denote by their origin and end point. We will write $(k)'$ for the other vertices on the same Γ -orbit as (k) . We will use the following notations:

$$\begin{aligned}
A &:= \pm \begin{pmatrix} -\omega & 16 \\ 1 & 1+\omega \end{pmatrix}, & B &:= \pm \begin{pmatrix} -\omega & 8 \\ 2 & 1+\omega \end{pmatrix}, \\
C &:= \pm \begin{pmatrix} 5-\omega & 10+2\omega \\ 2+\omega & -6+\omega \end{pmatrix}, & D &:= \pm \begin{pmatrix} 13 & 10+10\omega \\ \omega & -13 \end{pmatrix}, \\
F &:= \pm \begin{pmatrix} -\omega & 4 \\ 4 & 1+\omega \end{pmatrix}, & G &:= \pm \begin{pmatrix} -2-\omega & 4-\omega \\ 3 & 2+\omega \end{pmatrix}, \\
H &:= \pm \begin{pmatrix} -1-\omega & 15-\omega \\ 1 & 1+\omega \end{pmatrix}, & J &:= \pm \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \\
L &:= \pm \begin{pmatrix} -1-\omega & 7-\omega \\ 2 & 2+\omega \end{pmatrix}, & N &:= \pm \begin{pmatrix} 9-\omega & 14+6\omega \\ 2+\omega & -10+\omega \end{pmatrix}, \\
S &:= \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.
\end{aligned}$$

We observe the following stabilisers of the vertex representatives.

$$\begin{aligned}
\Gamma_{(36)} = \Gamma_{(24)} &= \Gamma_{(31)} = \Gamma_{(28)} &= 1 \\
\Gamma_{(43)} &= \langle JF \mid (JF)^2 = 1 \rangle &\cong \mathbb{Z}/2 \\
\Gamma_{(16)} &= \langle J \mid J^2 = 1 \rangle &\cong \mathbb{Z}/2 \\
\Gamma_{(52)} &= \langle G \mid G^2 = 1 \rangle &\cong \mathbb{Z}/2 \\
\Gamma_{(42)} &= \langle L, G \mid L^3 = G^2 = (LG)^3 = 1 \rangle &\cong \mathcal{A}_4 \\
\Gamma_{(29)} &= \langle B, N \mid B^3 = N^3 = (BN)^2 = 1 \rangle &\cong \mathcal{A}_4 \\
\Gamma_{(45)} &= \langle D, H \mid D^2 = H^2 = (DH)^3 = 1 \rangle &\cong \mathcal{S}_3 \\
\Gamma_{(50)} &= \langle J, F \mid J^2 = F^3 = (JF)^2 = 1 \rangle &\cong \mathcal{S}_3 \\
\Gamma_{(37)} &= \langle L \mid L^3 = 1 \rangle &\cong \mathbb{Z}/3 \\
\Gamma_{(21)} &= \langle C \mid C^3 = 1 \rangle &\cong \mathbb{Z}/3 \\
\Gamma_{(39)} &= \langle F \mid F^3 = 1 \rangle &\cong \mathbb{Z}/3 \\
\Gamma_{(19)} = \Gamma_{(30)} &= \langle S \mid S^3 = 1 \rangle &\cong \mathbb{Z}/3.
\end{aligned}$$

The following edge representatives have stabiliser type $\mathbb{Z}/2$:

$$\begin{aligned}
\Gamma_{(42),(52)} &= \langle G \mid G^2 = 1 \rangle \\
\Gamma_{(18),(45)} &= \langle H \mid H^2 = 1 \rangle \\
\Gamma_{(45),(51)} &= \langle D \mid D^2 = 1 \rangle \\
\Gamma_{(43),(50)} &= \langle JF \mid (JF)^2 = 1 \rangle \\
\Gamma_{(16),(50)} &= \langle J \mid J^2 = 1 \rangle \\
\Gamma_{(29),(44)} &= \langle BN \mid (BN)^2 = 1 \rangle.
\end{aligned}$$

In order to compare these with the vertex representative stabilisers, we note that there are respectively two elements of Γ sending the vertex (18) to (16), (44) to (43) and (51) to (52).

The following edge representatives have stabiliser type $\mathbb{Z}/3$:

$$\begin{aligned}
\Gamma_{(19),(30)} &= \langle S \mid S^3 = 1 \rangle \\
\Gamma_{(21),(42)} &= \langle C \mid C^3 = 1 \rangle \\
\Gamma_{(37),(42)} &= \langle L \mid L^3 = 1 \rangle \\
\Gamma_{(26),(45)} &= \langle DH \mid (DH)^3 = 1 \rangle \\
\Gamma_{(20),(23)} &= \langle A \mid A^3 = 1 \rangle \\
\Gamma_{(29),(54)} &= \langle N \mid N^3 = 1 \rangle \\
\Gamma_{(29),(47)} &= \langle B \mid B^3 = 1 \rangle \\
\Gamma_{(39),(50)} &= \langle F \mid F^3 = 1 \rangle
\end{aligned}$$

In order to determine the homomorphisms into the vertex stabilisers, we give a matrix for each of the following vertex identifications. The whole coset of matrices performing this vertex identification is obtained by multiplying the edge stabiliser from the right onto this matrix.

The matrix	sends vertex number to vertex number	
$\pm \begin{pmatrix} 7 & 5+5\omega \\ \omega & -12 \end{pmatrix}$	(26)	(21)
$\pm \begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}$	(20)	(19)
$\pm \begin{pmatrix} 8-2\omega & 30+7\omega \\ 3+\omega & -12+2\omega \end{pmatrix}$	(23)	(37)
$\pm \begin{pmatrix} 5 & 3+3\omega \\ \omega & -10 \end{pmatrix}$	(54)	(39)
$\pm \begin{pmatrix} -3-\omega & 6-\omega \\ 5 & 1+2\omega \end{pmatrix}$	(47)	(30).

The remaining seventeen edge orbits have trivial stabiliser. There are fifteen orbits of 2-cells. The above cardinalities sum up to the equivariant Euler characteristic

$$\chi_{\Gamma}(X) = 4 + \frac{3}{2} + \frac{2}{12} + \frac{2}{6} + \frac{5}{3} - 17 - \frac{6}{2} - \frac{8}{3} + 15 = 0,$$

whence there is a check of our calculations in view of proposition 41.

The d^1 differentials in the equivariant spectral sequence.

The 2-primary part $(d_{1,1}^1)_{(2)}: (\mathbb{Z}/2)^5 \leftarrow (\mathbb{Z}/2)^6$ can be expressed by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \text{ of rank 5.}$$

For $q > 1$ odd, $(d_{1,q}^1)_{(2)}$ has preimage $(\mathbb{Z}/2)^6$ and full rank 6, and can be expressed by the following matrices with $2k$ zero rows to be inserted. For the differential $(d_{1,6k+5}^1)_{(2)}$, with target space $(\mathbb{Z}/2)^{7+2k+2}$, this is the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

And for the differentials $(d_{1,6k+3}^1)_{(2)}$ and $(d_{1,6k+7}^1)_{(2)}$, with target space $(\mathbb{Z}/2)^{7+2k}$, we obtain their matrix by removing the two zero rows from the above matrix.

On the 3-primary part, for $q \equiv 1 \pmod{4}$, we express $(d_{1,q}^1)_{(3)}: (\mathbb{Z}/3)^7 \leftarrow (\mathbb{Z}/3)^8$ by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \text{ of rank 7;}$$

and for $q \equiv 3 \pmod{4}$, $(d_{1,q}^1)_{(3)}: (\mathbb{Z}/3)^9 \leftarrow (\mathbb{Z}/3)^8$ can be expressed by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ of rank 8.}$$

So in both cases the highest possible rank occurs. We deduce the E^2 -page of proposition 37.

7.2. Gaussian and Eisenstein integers. In [27], the computation for the cases of the Gaussian integers $\mathcal{O}_{-1} = \mathbb{Z}[\sqrt{-1}]$ and the Eisenstein integers $\mathcal{O}_{-3} = \mathbb{Z}[\omega]$, with $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3} = e^{\frac{2\pi i}{3}}$, has been redone by hand, following step by step the description in [30]. This enables us to clean up some typographical impacts of the publication process (presumably the recomposition) to the results in the editor's version of the latter paper. Some of the implied corrections have already been suggested by Berkove [4].

For the Gaussian integers, the integral homology of $PSL_2(\mathcal{O}_{-1})$ is a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$, with the number of copies specified by

$$H_q(PSL_2(\mathcal{O}_{-1}); \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3, & q = 2, \\ (\mathbb{Z}/2)^2, & q = 1 \end{cases}$$

and the Poincaré series $P_1^2(t) = \frac{-2t^3(2t^3-t^2-3)}{(t-1)^2(t^2+t+1)}$ and $P_1^3(t) = \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$.

For the Eisenstein integers,

$$H_q(PSL_2(\mathcal{O}_{-3}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3, & q = 1 \end{cases}$$

and $H_q(PSL_2(\mathcal{O}_{-3}); \mathbb{Z})$ is for $q \geq 3$ a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ with the number of copies specified by the Poincaré series $P_3^2(t) = \frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}$ and $P_3^3(t) = \frac{-t^3(t^2+2)}{(t-1)(t^2+1)}$.

8. EQUIVARIANT EULER CHARACTERISTIC

We will use the Euler characteristic to check the geometry of the quotient $\Gamma \backslash X$. Recall the following definitions and proposition.

Definition 38 (Euler characteristic). *Suppose Γ' is a torsion-free group. Then we define its Euler characteristic as*

$$\chi(\Gamma') = \sum_i (-1)^i \dim H_i(\Gamma'; \mathbb{Q}).$$

Suppose further that Γ' is a torsion-free subgroup of finite index in a group Γ . Then we define the Euler characteristic of Γ as

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

The latter formula is well-defined because of [7, IX.6.3].

Definition 39 (Equivariant Euler characteristic). *Suppose X is a Γ -complex such that*

- every isotropy group Γ_σ is of finite homological type;
- X has only finitely many cells mod Γ .

Then we define the Γ -equivariant Euler characteristic of X as

$$\chi_\Gamma(X) := \sum_{\sigma} (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

where σ runs over the orbit representatives of cells of X .

Proposition 40 ([7, IX.7.3 e']). *Suppose X is a Γ -complex such that $\chi_\Gamma(X)$ is defined. If Γ is virtually torsion-free, then Γ is of finite homological type and $\chi(\Gamma) = \chi_\Gamma(X)$.*

Let now Γ be $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Then the above proposition applies to X taken to be Flöge's (or still, Mendoza's) Γ -equivariant deformation retract of \mathcal{H} .

Using $\chi(\Gamma_\sigma) = \frac{1}{\mathrm{card}(\Gamma_\sigma)}$ for Γ_σ finite, the fact that the singular points have stabiliser \mathbb{Z}^2 , and the torsion-free Euler characteristic

$$\chi(\mathbb{Z}^2) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}}(H_i \mathbb{Z}^2) = 1 - 2 + 1 = 0,$$

we get the formula

$$\chi(\Gamma) = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

where σ runs over the orbit representatives of cells of X with finite stabilisers.

Proposition 41. *The Euler characteristic $\chi(\Gamma)$ vanishes for the Bianchi groups.*

This is a well-known fact, see for instance [26] for a proof. We obtain a “mass formula”

$$0 = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

which allows us to check the topology of the computed quotient space. For example, in the case $m = 427$, the mass formula takes the expression

$$184 + \frac{12}{2} + \frac{18}{3} + \frac{4}{6} - \left(441 + \frac{16}{2} + \frac{20}{3} \right) + 259 = 0,$$

which comes from 184 trivially stabilised vertices, 12 vertices with stabiliser of order two, 18 vertices with stabiliser of order three, 4 vertices with stabiliser of type \mathcal{S}_3 , 441 trivially stabilised edges, 16 edges with stabiliser of order two, 20 edges with stabiliser of order three, and 259 two-cells in the quotient cell complex. Tables with the expression in other cases, including all cases of class number 2, are given in [27]. A further check which has been carried out on the results of [25] is the vanishing of the naive Euler characteristic, that is proven in [33].

9. RESULTS FOR THE SPECIAL LINEAR GROUPS

In order to keep the ℓ -torsion subcomplex low-dimensional, it is important to divide the arithmetic group by the subgroup generated by all the elements of order ℓ which are in the kernel of the action. Else, when ℓ occurs as the order of an element in the kernel, the ℓ -torsion subcomplex is the whole quotient complex. For instance, for $SL_2(\mathcal{O}_{-m})$, where \mathcal{O}_{-m} is a non-Euclidean principal ideal domain, we remark that for each 2-cycle in the quotient complex (which corresponds to a generator of $H_2(SL_2(\mathcal{O}_{-m}); \mathbb{Q})$), we have a constant summand $\mathbb{Z}/2$ to its integral homology in all degrees $q > 1$.

Proposition 42. *The integral homology of $SL_2(\mathcal{O}_{-m})$, for $m \in \{19, 43, 67, 163\}$, is given as*

$$H_q(SL_2(\mathcal{O}_{-m}); \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{\beta_2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3, & q = 6 + 4n, \\ (\mathbb{Z}/2)^{\beta_2}, & q = 5 + 4n, \\ (\mathbb{Z}/2)^{\beta_2} \oplus \mathbb{Z}/2, & q = 4 + 4n, \\ (\mathbb{Z}/2)^{\beta_2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3, & q = 3 + 4n, \\ \mathbb{Z}^{\beta_2} \oplus (\mathbb{Z}/2)^{\beta_2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3, & q = 2, \\ \mathbb{Z}^{\beta_2+1}, & q = 1, \end{cases}$$

where the Betti number β_2 is given in proposition 37.

The results of this proposition have been computed with HAP [12] from the cell complex information computed with [25].

10. K -THEORY

With the above information about the action of the Bianchi groups, we can, analogously to the way of [29], compute the Bredon homology of the Bianchi groups, from which we can deduce their equivariant K -homology. The results of the computations [27] are the following.

Theorem 43. *Let β_2 be the Betti number specified in proposition 37. For \mathcal{O}_{-m} principal, the equivariant K -homology of $\Gamma := PSL_2(\mathcal{O}_{-m})$ is isomorphic to*

	$m = 1$	$m = 2$	$m = 3$	$m = 7$	$m = 11$	$m \in \{19, 43, 67, 163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	\mathbb{Z}^6	$\mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}^3	$\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{\beta_2} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$
$K_1^\Gamma(\underline{E}\Gamma)$	\mathbb{Z}	\mathbb{Z}^3	0	\mathbb{Z}^3	\mathbb{Z}^3	$\mathbb{Z} \oplus \mathbb{Z}^{\beta_2+1}$.

The remainder of the equivariant K -homology of Γ is given by Bott 2-periodicity. By the Baum/Connes conjecture, which holds for the Bianchi groups [19], we obtain the K -theory of the reduced C^* -algebras of the Bianchi groups as isomorphic images.

11. APPENDIX: THE MAPS INDUCED ON COHOMOLOGY BY FINITE SUBGROUP INCLUSIONS

Concerning the possible finite stabiliser groups of vertices in hyperbolic space, we can establish their integral homology by elementary means. This has been done in [30], and we obtain their homology with \mathbb{Z}/ℓ -coefficients by the universal coefficient theorem. Then it only remains to correct a minor topographical error for \mathcal{A}_4 in order to obtain the following lemma.

Lemma 44 (Schwermer/Vogtmann). *The homology with trivial \mathbb{Z} - respectively \mathbb{Z}/ℓ -coefficients, for $\ell = 2$ or 3 , of the finite subgroups of $PSL_2(\mathcal{O}_{-m})$ listed in lemma 9 is*

$$\begin{aligned}
H_q(\mathbb{Z}/n; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/n, & q \text{ odd}, \\ 0, & q \text{ even}, q > 0; \end{cases} & H_q(\mathbb{Z}/n; \mathbb{Z}/n) &\cong \mathbb{Z}/n, \text{ for } n, q \in \mathbb{N} \cup \{0\}; \\
H_q(\mathcal{D}_2; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^{\frac{q+3}{2}}, & q \text{ odd}, \\ (\mathbb{Z}/2)^{\frac{q}{2}}, & q \text{ even}, q > 0; \end{cases} & H_q(\mathcal{D}_2; \mathbb{Z}/2) &\cong (\mathbb{Z}/2)^{q+1} & H_q(\mathcal{D}_2; \mathbb{Z}/3) &= 0, q \geq 1; \\
H_q(\mathcal{S}_3; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/2, & q \equiv 1 \pmod{4}, \\ 0, & q \equiv 2 \pmod{4}, \\ \mathbb{Z}/6, & q \equiv 3 \pmod{4}, \\ 0, & q \equiv 0 \pmod{4}, q > 0; \end{cases} & H_q(\mathcal{S}_3; \mathbb{Z}/3) &\cong \begin{cases} \mathbb{Z}/3, & q = 0, \\ 0, & q \equiv 1 \pmod{4}, \\ 0, & q \equiv 2 \pmod{4}, \\ \mathbb{Z}/3, & q \equiv 3 \pmod{4}, \\ \mathbb{Z}/3, & q \equiv 0 \pmod{4}, q > 0; \end{cases} & H_q(\mathcal{S}_3; \mathbb{Z}/2) &\cong \mathbb{Z}/2, q \in \mathbb{N} \cup \{0\}; \\
H_q(\mathcal{A}_4; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3, & q = 6k + 1, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2, & q = 6k + 2, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6, & q = 6k + 3, \\ (\mathbb{Z}/2)^k, & q = 6k + 4, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6, & q = 6k + 5, \\ (\mathbb{Z}/2)^{k+1}, & q = 6k + 6. \end{cases} & H_q(\mathcal{A}_4; \mathbb{Z}/2) &\cong \begin{cases} \mathbb{Z}/2, & q = 0, \\ (\mathbb{Z}/2)^{2k}, & q = 6k + 1, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k + 2, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k + 3, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k + 4, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k + 5, \\ (\mathbb{Z}/2)^{2k+3}, & q = 6k + 6. \end{cases} & H_q(\mathcal{A}_4; \mathbb{Z}/3) &\cong \mathbb{Z}/3, q \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

With the Lyndon/Hochschild/Serre spectral sequence, we can further establish the following.

Lemma 45 (Schwermer/Vogtmann [30]). *Let M be \mathbb{Z} or $\mathbb{Z}/2$. Consider group homology with trivial M -coefficients. Then the following holds.*

- Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$ induces an injection on homology.
- An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$ induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.
- Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$ induces an injection on homology in all degrees.
- An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$ induces injections on homology in all degrees.
- An inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on H_1 .

Schwermer and Vogtmann prove this for $M = \mathbb{Z}$. We will make use of the following statements to prove it for $M = \mathbb{Z}/2$. As the only automorphism of $\mathbb{Z}/2$ is the identity, $\mathbb{Z}/2$ -coefficients are always trivial coefficients.

Lemma 46. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into \mathcal{D}_2 .*

Proof. Consider the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$ -coefficients of the trivial extension $1 \rightarrow \mathbb{Z}/2 \rightarrow \mathcal{D}_2 \rightarrow \mathbb{Z}/2 \rightarrow 1$. It takes the form

$$E_{p,q}^2 = H_p(\mathbb{Z}/2; H_q(\mathbb{Z}/2; \mathbb{Z}/2)) \Rightarrow H_{p+q}(\mathcal{D}_2; \mathbb{Z}/2).$$

As $H_q(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $q \in \mathbb{N} \cup \{0\}$, we obtain $E_{p,q}^2 = \mathbb{Z}/2$ for all $p, q \in \mathbb{N} \cup \{0\}$. As we know from lemma 44 that $H_q(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$, all the differentials must be zero and $E^2 = E^\infty$. Hence we obtain the claimed injections on homology. \square

Lemma 47. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into \mathcal{S}_3 .*

Proof. Consider the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$ -coefficients of the non-trivial extension $1 \rightarrow \mathbb{Z}/3 \rightarrow \mathcal{S}_3 \rightarrow \mathbb{Z}/2 \rightarrow 1$. It takes the form

$$E_{p,q}^2 = H_p(\mathbb{Z}/2; H_q(\mathbb{Z}/3; \mathbb{Z}/2)) \Rightarrow H_{p+q}(\mathcal{S}_3; \mathbb{Z}/2).$$

As $H_q(\mathbb{Z}/3; \mathbb{Z}/2) = 0$ for $q > 0$, the E^2 -page is concentrated in the row $q = 0$ and equals the E^∞ -page. Thus we have isomorphisms $H_p(\mathbb{Z}/2; H_0(\mathbb{Z}/3; \mathbb{Z}/2)) \cong H_p(\mathcal{S}_3; \mathbb{Z}/2)$, from which we obtain the claimed morphisms on homology. \square

Let t be a generator of $\mathbb{Z}/3$. Let F be the periodic resolution of \mathbb{Z} over $\mathbb{Z}[\mathbb{Z}/3]$ given by

$$\cdots \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{t^2+t+1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{\text{augmentation}} \mathbb{Z}.$$

Lemma 48. *Let A be an abelian group consisting only of elements of order 2 and the neutral element. Then $F \otimes_{\mathbb{Z}[\mathbb{Z}/3]} A$ is exact in degrees greater than zero, regardless of the $\mathbb{Z}[\mathbb{Z}/3]$ -module structure attributed to A .*

Proof. We will show the two equations

$$\text{image}(t^2 + t + 1) = \ker(t - 1) \text{ and } \ker(t^2 + t + 1) = \text{image}(t - 1).$$

As $t^3 = 1$, the equation $(t^2 + t + 1)(t - 1) = 0$ holds, and yields the inclusions

$$\text{image}(t^2 + t + 1) \subset \ker(t - 1) \text{ and } \ker(t^2 + t + 1) \supset \text{image}(t - 1).$$

Now we want to show that $\text{image}(t^2 + t + 1) \supset \ker(t - 1)$. Let $v \in \ker(t - 1)$. Then $(t - 1) \cdot v = 0$, or equivalently, $t \cdot v = v$. We apply this three times to obtain $(t^2 + t + 1) \cdot v = 3v$. As $2v = 0$ in A , we have $(t^2 + t + 1) \cdot v = v$ and hence the claimed inclusion.

It remains to show that $\ker(t^2 + t + 1) \subset \text{image}(t - 1)$. Let $v \in \ker(t^2 + t + 1)$. We will see that $t \cdot v$ is a preimage of v for the multiplication by $t - 1$. Namely, $(t - 1) \cdot (t \cdot v) = t^2 \cdot v - t \cdot v = -2t \cdot v - v$ because $t^2 \cdot v + t \cdot v + v = 0$. Using that multiplication by 2 is zero on A , we obtain the image v and hence the last inclusion. \square

Lemma 49. *The E^2 -page of the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$ -coefficients for the extension $1 \rightarrow \mathcal{D}_2 \rightarrow \mathcal{A}_4 \rightarrow \mathbb{Z}/3 \rightarrow 1$ is concentrated in the column $p = 0$.*

Proof. The E^2 -page of the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$ -coefficients is given by $E_{p,q}^2 = H_p(\mathbb{Z}/3; H_q(\mathcal{D}_2; \mathbb{Z}/2))$.

The action of $\mathbb{Z}/3$ on $H_q(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$ is determined by the non-trivial conjugation action of $\mathbb{Z}/3$ on \mathcal{D}_2 . But applying lemma 48, we obtain $H_p(\mathbb{Z}/3; (\mathbb{Z}/2)^{q+1}) = 0$ for $p > 0$ and any action of $\mathbb{Z}/3$ on $(\mathbb{Z}/2)^{q+1}$. \square

We can now do the last remaining step to prove lemma 45 in the case $M = \mathbb{Z}/2$.

Corollary 50. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into \mathcal{A}_4 .*

Proof.

- For an inclusion $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$, this follows from the fact that $H_p(\mathbb{Z}/3; \mathbb{Z}/2) = 0$ for all $p > 0$.
- For an inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$, factorise by an inclusion $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$. Lemma 49 gives an isomorphism $H_q(\mathcal{A}_4; \mathbb{Z}/2) \cong H_0(\mathbb{Z}/3; H_q(\mathcal{D}_2; \mathbb{Z}/2))$ for $q \in \mathbb{N} \cup \{0\}$. From lemma 46 and the low terms of our resolution for \mathcal{A}_4 , we deduce that the induced map on homology is injective whenever $H_q(\mathcal{A}_4; \mathbb{Z}/2)$ is non-zero. From lemma 44, we see that this is the case for all $q \in \mathbb{N} \cup \{0\}$ except for $q = 1$, where we obtain the zero map.

\square

11.1. Appendix: The low terms of a free resolution for the alternating group on 4 objects. We will use Wall's Lemma to construct a free resolution for \mathcal{A}_4 , and compute its three differentials of lowest degrees explicitly. This resolution will help us determine the maps induced on homology by inclusions into vertex stabilisers of type \mathcal{A}_4 .

So let us recall Wall's lemma. Given a group extension $1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$, and free resolutions B for K , and C for H , we construct a free resolution for G in terms of the following double chain complex. For any $s \in \mathbb{N} \cup \{0\}$, let C_s be free on α_s generators. We define D_s as the direct sum of α_s copies of $\mathbb{Z}[G] \otimes_K B$. Then we have an augmentation of D_s onto the direct sum of α_s copies of $\mathbb{Z}[H]$, which we will identify with C_s , and write $\varepsilon_s : D_s \rightarrow C_s$. If $A_{r,s}$ is the submodule of D_s which is the direct sum of α_s copies of $\mathbb{Z}[G] \otimes_K B_r$, then $A_{r,s}$ is a free G -module, and D_s is the direct sum of the $A_{r,s}$.

Lemma 51 (C.T.C. Wall [34]). *There exist G -maps $d_{r,s}^k : A_{r,s} \rightarrow A_{r+k-1,s-k}$ for $k \geq 1$, $s \geq k$ such that*

- $\varepsilon_{s-1} \circ d_{0,s}^1 = d_s^C \circ \varepsilon_s : A_{0,s} \rightarrow C_{s-1}$ where d^C denotes the differential in C ,
- $\sum_{i=0}^k d^{k-i} \circ d^i = 0$, for each k , where $d_{r,s}^k$ is interpreted as zero if $r = k = 0$, or if $s < k$.

Finally, we let A denote the direct sum of the $A_{r,s}$ graded by $\dim A_{r,s} = r + s$. And let $d := \sum_k d^k$.

Theorem 52 (C.T.C. Wall [34]). *(A, d) is acyclic, and so yields a free resolution for G .*

Let t be a generator of \mathbb{Z}/n . Let $F(n)$ be the periodic resolution of \mathbb{Z} over $\mathbb{Z}[\mathbb{Z}/n]$ given by

$$\dots \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t^n + \dots + t + 1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{\text{augmentation}} \mathbb{Z}.$$

We consider the group extension $1 \rightarrow \mathcal{D}_2 \rightarrow \mathcal{A}_4 \rightarrow \mathbb{Z}/3 \rightarrow 1$, the resolution $F(3)$ for $\mathbb{Z}/3$ and the resolution $F(2) \otimes F(2)$ for \mathcal{D}_2 . Then, $A_{r,s} = \mathbb{Z}[\mathcal{A}_4] \otimes_{\mathbb{Z}[\mathcal{D}_2]} (\mathbb{Z}[\mathcal{D}_2])^r \cong (\mathbb{Z}[\mathcal{A}_4])^r$.

Let us use the cycle notation for the elements in the alternating group on four letters. Then in low degrees, the differential of $F(2) \otimes F(2)$ becomes

$$\begin{aligned} d_{1,s}^0 &= ((12)(34) - 1, (14)(23) - 1), \\ d_{2,s}^0 &= \begin{pmatrix} (12)(34) + 1 & 1 - (14)(23) & 0 \\ 0 & (12)(34) - 1 & (14)(23) + 1 \end{pmatrix}, \\ d_{3,s}^0 &= \begin{pmatrix} (12)(34) - 1 & (14)(23) - 1 & 0 & 0 \\ 0 & -(12)(34) - 1 & (14)(23) + 1 & 0 \\ 0 & 0 & (12)(34) - 1 & (14)(23) - 1 \end{pmatrix}, \end{aligned}$$

for all $s \in \mathbb{N}$. At the same time, we can set $d_{0,2k}^1 = ((132) + (123) + 1)$ and

$d_{0,2k+1}^1 = ((123) - 1)$ for all $k \in \mathbb{N}$, which satisfies the first condition in Wall's Lemma. Further, we set

$$\begin{aligned} d_{1,1}^1 &= \begin{pmatrix} 1 & (142) \\ -(123) & (134) + 1 \end{pmatrix}, \\ d_{1,2}^1 &= \begin{pmatrix} -1 - (123) + (134) - (124) & (234) - (123) \\ (142) - (132) & (142) - 1 + (124) + (143) \end{pmatrix}, \end{aligned}$$

$$d_{2,1}^1 = \begin{pmatrix} -1 & 0 & -(123) \\ 0 & (142) - 1 & (243) \\ (134) & 0 & -(123) - 1 \end{pmatrix}.$$

We sum up, and obtain the low degree terms of a free resolution for \mathcal{A}_4 :

$$\cdots \longrightarrow (\mathbb{Z}[\mathcal{A}_4])^{10} \xrightarrow{d_3} (\mathbb{Z}[\mathcal{A}_4])^6 \xrightarrow{d_2} (\mathbb{Z}[\mathcal{A}_4])^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{A}_4] \rightarrow 0,$$

where $d_1 = (d_{0,1}^1, d_{1,0}^0) = ((123) - 1, (12)(34) - 1, (14)(23) - 1)$,

$$d_2 = \begin{pmatrix} d_{0,2}^1 & d_{1,1}^0 & 0 \\ 0 & d_{1,1}^1 & d_{2,0}^0 \end{pmatrix} \\ = \begin{pmatrix} (132) + (123) + 1 & (12)(34) - 1 & (14)(23) - 1 & 0 & 0 & 0 \\ 0 & 1 & (142) & (12)(34) - 1 & (14)(23) - 1 & 0 \\ 0 & -(123) & (134) + 1 & 0 & (12)(34) - 1 & (14)(23) - 1 \end{pmatrix},$$

$$\text{and we assemble analogously } d_3 = \begin{pmatrix} d_{0,3}^1 & d_{1,2}^0 & 0 & 0 \\ 0 & d_{1,2}^1 & d_{2,1}^0 & 0 \\ 0 & d_{1,2}^2 & d_{2,1}^1 & d_{3,0}^0 \end{pmatrix}.$$

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