Stability analysis of discontinous quantum control systems with dipole and polarizability coupling

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Abstract

Closed quantum systems under the influence of a laser field, whose interaction is modeled by a Schrödinger equation with a coupling control operator containing both a linear (dipole) and a quadratic (polarizability) term are analyzed. Discontinuous feedbacks, obtained by a Lyapunov trajectory tracking procedure, have been recently proposed to control these type of systems. The purpose of this paper is to study the asymptotic stability by considering the solutions in the Filippov sense. The analysis is developed by applying a variant of LaSalle invariance principle for differential inclusions. Numerical simulations are included to illustrate the efficiency of the discontinuous control.

Key words: Quantum systems, Stabilization, Lyapunov Control function, Tracking, Differential inclusions

1 Introduction

Control of quantum systems using laser fields has been subject to significant developments in the last two decades ([1,4,13,15,27] etc.). The increasing interest on this domain is motivated by the effects of the technique: we can create or break chemical bonds, each time with finesse far beyond the usual macroscopic means (temperature, pression, etc.).

Since the first successful laboratory experiments obtained at the beginning of the 90s [1,13] many applications of this method have been developed: designing logical gates in future quantum computers, investigations of imaging by nuclear magnetic resonance - NMR, study of protein dynamics, molecular detection, molecular orientation and alignment, construction of ultra-short laser etc..

From the beginning, the complexity of chemical phenomena that arise during the interaction laser-quantum system has required the introduction of theoretical methods as an important step to experimental phase. This type of analysis can reveal the set of objectives that can be achieved, and the nature of the laser pulse that can be used. In this context, we consider the time dependent Schrödinger equation, that models the evolution of a quantum system:

$$i\frac{d}{dt}\Psi(t) = H(t)\Psi(t) \tag{1}$$

where $H(t)$ is an Hermitian operator, called the Hamiltonian and Ψ a complex function called the wavefunction. When the system is controlled by selecting a convenient laser intensity $\epsilon(t)$, the interaction between the laser and the system is described by an operator μ_1, also called dipole coupling [19]. Thus, we recover a billinear form of the Schrödinger equation, formally written:

$$i\frac{d}{dt}\Psi(t) = (H_0 + \epsilon(t)\mu_1)\Psi(t). \tag{2}$$

In this case $H(t) = H_0 + \epsilon(t)\mu_1$, where H_0 is the internal Hamiltonian operator, that characterizes the system when the laser is shut down ($\epsilon(t) = 0$). In the limit of small laser intensities the first order term $\epsilon(t)\mu_1$ may be enough to adequately describe the interaction, however, situations exist where the dipole coupling does not have enough influence on the system to reach the control goal; the goal may become accessible only after taking into account terms of higher order in the expansion of $H(t)$, for example a polarizability term $\epsilon^2(t)\mu_2$ (see e.g. [7,8] and related works).

In the following, we focus on the case where a second order them is added in the expansion of the Hamiltonian:

$$H(t) = H_0 + \epsilon(t)\mu_1 + \epsilon^2(t)\mu_2. \tag{3}$$

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For numerical reasons a finite dimensional setting is considered. The operators will be restrained to a linear space spanned by a N dimensional set D. This set can contain for example the first N eigenvalues of the infinite dimensional internal hamiltonian H_0. For simplicity we conserve the same notations, i.e. we denote by H_0, μ_1 and μ_2, $N \times N$ Hermitian matrices with complex coefficients and by Ψ a N dimensional complex vector.
One important problem is to determine efficient laser fields to control quantum systems whose Hamiltonian are defined by (3). For this purpose an analysis of the controllability has to be pursued, i.e. ask if any admissible quantum state can be attained with some admissible laser field. This can be studied via the general accessibility criteria [3,23] based on Lie brackets; more specific results can be found in [26]. A detailed presentation has been made in [6].

Even if positive results of controllability for systems, with Hamiltonian defined by (3), have been obtained, finding efficient numerical algorithms to determine the control field remains a very difficult task. A solution is to present the problem as a minimization of a cost functional, that describes the goal to be achieved, and eventually some other constraints. This approach leaded to procedures such as stochastic iterative approaches (e.g., genetic algorithms) [16], iterative critical point methods (monotonic algorithms) [17,24,28], trajectory tracking or local control procedures ([2,5,9,12,18,20,22] etc.). One advantage of this class of methods is that we obtain explicit control fields. Another one is that few propagations in time are required to approach the solution of the time-dependent Schrödinger equation (TDSE). This is an important aspect when larger systems are considered.

Lyapunov trajectory tracking techniques have been applied for systems with Hamiltonian (3) in order to determine the control ϵ. A first positive result has been obtained by adapting the analysis presented in [14,18], that deals with bilinear quantum systems $H_0 + \epsilon(t)\mu_1$. The success of the feedback control depends on whether there exists (non-zero) direct coupling, through μ_1, between the target state and **all** other eigenstates. When the same property holds for Hamiltonian $H(t) = H_0 + \epsilon(t)\mu_1 + \epsilon^2(t)\mu_2$ the same type of feedback formulas hold. When some of the (direct) coupling is realized through μ_2 instead of μ_1, the previous feedback formulas do not hold any more and two alternatives have been proposed (see [6] for more details): discontinuous feedback and time varying feedback.

Only approximative asymptotic stability results have been proved for this last two situations. This paper focuses on the case of discontinuous feedback obtained for quantum systems with Hamiltonian defined by (3). The goal is to prove stability results, and especially asymptotic stability considering the solutions of the quantum system (1), with Hamiltonian H given by (3), in the Filippov sense.

The balance of the paper is as follows: in Section 2 we introduce the main notations, the Lyapunov tracking procedure followed by the construction of the discontinuous feedback. Then, we study the existence of solutions in the Filippov sense. In Section 3 we prove a first stability result followed by an asymptotic stability result. The last two sections are dedicated to numerical simulations and conclusions.

2 Lyapunov trajectory tracking

2.1 Lyapunov function

We consider equation (1), with Hamiltonian H(t) given by (3), that describes the evolution of a N-level quantum system submitted to an external action:

$$i\frac{d}{dt}\Psi(t) = (H_0 + \epsilon(t)\mu_1 + \epsilon^2(t)\mu_2)\Psi(t). \tag{4}$$

The wave function $\Psi = (\Psi_j)_{j=1}^N$ is a vector in $\mathbb{C}^N$, verifying $\sum_{j=1}^N |\Psi_j|^2 = 1$, i.e. Ψ belongs to the unit sphere $\mathcal{S}^N(0,1)$ of $\mathbb{C}^N$. The function Ψ represents a complete physical description of the state of the quantum system at every instant t.

Recall that two wave functions Ψ_1 and Ψ_2 that differ by a phase $\theta(t) \in \mathbb{R}$, i.e. $\Psi_1 = \exp(i\theta(t))\Psi_2$, describe the same physical state. To take into account the property we add a fictitious control ω (see also [18]). Hence we will replace the evolution equation (4) by:

$$i\frac{d}{dt}\Psi(t) = (H_0 + \epsilon(t)\mu_1 + \epsilon^2\mu_2 + \omega(t))\Psi(t), \tag{5}$$

where $\omega \in \mathbb{R}$ is a new control. We can choose it arbitrarily without changing the physical quantities attached to Ψ. We assume in the sequel that the state space is $\mathcal{S}^N(0,1)$ and the dynamics given by (5) admits two independent controls ϵ and ω.

In order to obtain an explicit formula for the laser field $\epsilon(t)$, we apply a Lyapunov trajectory tracking technique. The method consists in introducing a time varying function $V(\Psi(t))$:

$$V(\Psi(t)) = \langle \Psi - \phi | \Psi - \phi \rangle = \|\Psi - \phi\|^2, \tag{6}$$

with Ψ a smooth solution of (5) and ϕ an eigenvector of H_0 associated to the eigenvalue λ.

The function V is nonnegative for all $t > 0$ and all $\Psi \in \mathcal{S}^N(0,1)$ and vanishes when $\Psi = \phi$. We search for feedback controls such that V is a Lyapunov function. To do that we compute formally the derivative of V along the trajectories of (5):

$$\frac{dV}{dt} = 2\epsilon Im(\langle\mu_1\Psi(t)|\phi\rangle) + 2\epsilon^2 Im(\langle\mu_2\Psi(t)|\phi\rangle) + 2(\omega+\lambda)Im(\langle\Psi(t) \mid \phi\rangle), \quad (7)$$

where Im denotes the imaginary part. For convenience we denote: $I_1 = Im(\langle\mu_1\Psi(t)|\phi\rangle)$ and $I_2 = Im(\langle\mu_2\Psi(t)|\phi\rangle)$.
Then note that if, for example, one takes

$$\begin{cases} \epsilon(I_1, I_2) = -kI_1/(1+kI_2) \\ \omega = -\lambda - cIm(\langle\Psi(t)|\phi\rangle), \end{cases} \quad (8)$$

with k and c strictly positive parameters, one gets

$$dV/dt = -2k(I_1/(1+kI_2))^2 - 2c(Im(\langle\Psi(t) \mid \phi\rangle))^2 \leq 0,$$

and thus V is nonincreasing.
However, even if the feedback is chosen such that is V monotonically decreasing, this does not automatically imply that the minimum value will be reached. A convergence analysis is required.

2.2 Discontinuous feedback

The theoretical result (see Theorem 2.1) in [11] shows that tracking to ϕ works well when all eigenstates of H_0, $\phi_2, ...\phi_N$, other than ϕ are coupled to ϕ by μ_1, *i.e.* $\langle\phi_j, \mu_1\phi\rangle \neq 0, j = 2, \ldots, N$. For the important case when some of the coupling are realized by μ_2 instead of μ_1 formulas (8) are ineffective. Discontinuous and time varying feedback have been proposed to stabilize the system (see [6]).

The introduction of discontinuous feedback laws is motivated by the formula of the derivative of V with respect to time. We remark that dV/dt reads as the sum of $2(\omega+\lambda)Im(\langle\Psi(t) \mid \phi)$ and a function $U(\epsilon)$:

$$\frac{dV}{dt} = 2U(\epsilon) + 2(\omega+\lambda)Im(\langle\Psi(t) \mid \phi). \quad (9)$$

Here $U(\epsilon) = \epsilon^2 I_2 + \epsilon I_1$ is a second order function of ϵ, with coefficients depending on I_1 and I_2. Consequently, the condition $dV/dt \leqslant 0$ depends on the sign of a second order function. Thus, the idea is to divide the space defined by I_1 and I_2 into disjoint regions. To each region we assign different formulas for the control $\epsilon(I_1, I_2)$ such that $U(\epsilon) \leqslant 0$ in any point (I_1, I_2).

To this goal we consider the regions (see Fig. 1):

$$\begin{aligned} A &= \{\Psi|\ I_2(\Psi) < -\sqrt{|I_1|}\}, \\ B &= \{\Psi||I_2(\Psi) > \sqrt{|I_1|}\}, \\ C &= \{\Psi| - \sqrt{|I_1|} \leqslant I_2(\Psi) \leqslant \sqrt{|I_1|}\} \end{aligned} \quad (10)$$

and we define the control as follows:

$$\epsilon(I_1(\Psi), I_2(\Psi)) = \begin{cases} k_1 I_2, \text{ in A} \\ 0, \quad \text{in B} \\ -k_2 I_1/(1+k_2 I_2), \text{in C} \end{cases} \quad (11)$$

$$\omega = -\lambda - cIm(\langle\Psi(t) \mid \phi\rangle).$$

with $k_1, k_2, c > 0$.

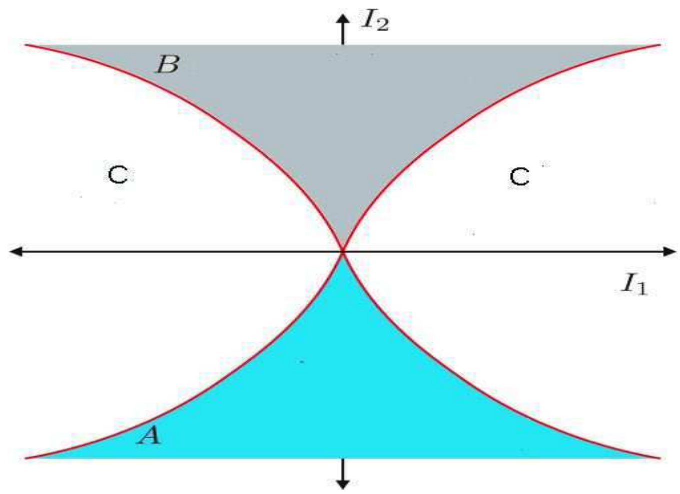

Fig. 1. Schematic view of the regions A, B,C. We consider arbitrary units for I_1 and I_2.

Remark 2.1 *Under some restrictions for k_1 and k_2 that will be introduced later on, the condition $U(\epsilon) \leqslant 0$ is fulfilled on the region A and C. On region B we have $U(\epsilon) = 0$.*

Remark 2.2 *In order to guarantee $1+k_2I_2 > 0$ in equation (11), one notes that $|I_2| \leq |\langle\mu_2\Psi(t)|\phi\rangle| \leq \|\mu_2\|$; therefore $1+k_2I_2 > 0$ as soon as $k_2 < \frac{1}{\|\mu_2\|}$. From now on, unless otherwise specified, this condition will be supposed satisfied.*

We replace the feedback (11) into equation (5) and we obtain a discontinuous right side equation:

$$i\frac{d}{dt}\Psi(t) = \begin{cases} \Big(H_0 + k_1I_2\mu_1 + (k_1I_2)^2\mu_2 - \lambda - cIm(\langle\Psi(t)|\phi\rangle)\Big)\Psi, & \text{in A} \\ \Big(H_0 - \lambda - cIm(\langle\Psi(t)|\phi\rangle\Big)\Psi, & \text{in B} \\ \Big(H_0 - k_2\frac{I_1}{1+k_2I_2}\mu_1 + (k_2\frac{I_1}{1+k_2I_2})^2\mu_2 - \lambda - cIm(\langle\Psi(t)|\phi\rangle)\Big)\Psi, & \text{in C} \end{cases} \quad (12)$$

2.3 Existence of solutions in Filippov sense

Before discussing the stability of the system defined by (12) we need to study the existence of solutions. One idea is to consider solutions in the Filippov sense.

Definition 2.1 *Let us consider the equation*

$$\dot{x}(t) = f(x(t)) \tag{13}$$

with piecewise discontinuous function $f : D \to \mathbb{R}^d$, where d is the dimension of the space and $D \subseteq \mathbb{R}^d$ a compact set. We recall that a solution in the Filippov sense of (13) *is a locally absolutely continuous map such that:*

$$\dot{x} \in \mathcal{F}(x(t)) \tag{14}$$

with

$$\mathcal{F} := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{conv}(f(x+\delta\mathcal{B}) \setminus N)) \tag{15}$$

where μ is the Lebesgue measure, S is a arbitrary set of measure zero, $\overline{conv}(\mathcal{A})$ is the smallest closed convex set containing $\mathcal{A}$, $\mathcal{B}$ is the unit ball of $\mathbb{R}^d$ and f is a discontinuous function.

In our case a solution in the Filippov sense, Ψ of (12) is a locally absolutely continuous map such that:

$$\frac{d}{dt}\Psi \in \mathcal{F}(\Psi(t)) \tag{16}$$

with $\mathcal{F}$ defined by (15) and f given by

$$f(\Psi) = \begin{cases} -i\Big(H_0 + k_1 I_2 \mu_1 + (k_1 I_2)^2 \mu_2 - \\ \qquad \lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi, & \text{in A} \\ -i\Big(H_0 - \lambda - cIm(\langle \Psi(t)|\phi\rangle\Big)\Psi, & \text{in B} \\ -i\Big(H_0 - k_2 \frac{I_1}{1+k_2 I_2}\mu_1 + (k_2 \frac{I_1}{1+k_2 I_2})^2 \mu_2 \\ \qquad -\lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi, & \text{in C} \end{cases} \tag{17}$$

The way $\mathcal{F}(\Psi)$ is defined implies that it is a nonempty, bounded, closed, convex set. In the same time it is an upper semicontinuous function of Ψ. Thus, we are in the hypothesis of Theorem 1(page 70) in [10]) and the existence of an absolutely continuous function solution of (16) is assured.

3 Stability analysis

3.1 A first stability result

Before we can analyze the stability, it is necessary to define several notions:

Definition 3.1 *A solution $\overline{\Psi}$ of the differential inclusion* (16) *is stable if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that if $|\Psi(0) - \overline{\Psi}| < \delta$ then*

$$|\Psi(t) - \overline{\Psi}| < \delta, \quad \text{for every } \ t \geqslant 0. \tag{18}$$

Definition 3.2 *A solution $\overline{\Psi}$ of the differential inclusion* (16) *is asymptotically stable if is stable and δ can be chosen such that if $|\Psi(0) - \overline{\Psi}| < \delta$ then*

$$\lim_{t\to\infty} (\Psi(t) - \overline{\Psi}) = 0. \tag{19}$$

We can give now a first stability result:

Theorem 3.1 *Consider* (12) *with $\Psi \in \mathcal{S}^N(0,1)$ a solution in the Filippov sense and an eigenstate $\phi \in \mathcal{S}^N(0,1)$ of H_0 associated to the eigenvalue λ. Take the constants $k_1 > 1$, $k_2 < \frac{1}{\|\mu_2\|}$ and $c > 0$. The solution $\Psi = \phi$ of the inclusion* (16) *is stable.*

Proof 3.1 *Up to a shift on ω and H_0, we can assume that $\lambda = 0$. Since the function $V(\Psi)$ is C^1 with respect to Ψ, we can define the upper derivate by:*

$$\dot{V}^* = \left(\frac{dV}{dt}\right)^* = \sup_{y\in F(\Psi)} (\nabla V \cdot y). \tag{20}$$

For almost all t the derivative $\dot{\Psi}$ exists and satisfies the differential inclusion (16)*. For these t there exists:*

$$\dot{V} = \frac{d}{dt} V(\Psi(t)) = \nabla V \cdot \dot{\Psi}. \tag{21}$$

Theorem 1, page 153 in [10] says that if $\dot{V}^* \leqslant 0$ then ϕ is a stable point. In order to verify the fulfillment of this condition it is sufficient to make sure that $dV/dt = \partial V \cdot f \leqslant 0$ only on the domains of continuity of the function f defined by (17)*. In this domains we have $\mathcal{F}(\Psi) = f(\Psi)$. On the discontinuity points of the function f the set $\mathcal{F}$ is defined as the closure of a convex set. This operation does not increase the upper boundary of the expression $\nabla V \cdot f$ (see [10] for more details).*
On the interior of the region $B = \{\Psi || I_2(\Psi) > \sqrt{|I_1|}\}$ the control $\epsilon(t)$ is zero, this implies

$$\frac{dV}{dt} = -2c(Im(\langle \Psi(t)|\phi\rangle))^2 \leqslant 0. \tag{22}$$

We have the same property on the interior of the region $C = \{\Psi| - \sqrt{|I_1|} \leqslant I_2(\Psi) \leqslant \sqrt{|I_1|}\}$, since the control $\epsilon(t)$, is chosen such that:

$$\begin{aligned} \frac{dV}{dt} &= 2\frac{-k_2 I_1}{1+kI_2} I_1 + 2\frac{k_2^2 I_1^2}{(1+kI_2)^2} I_2 - 2c(Im(\langle \Psi(t)|\phi\rangle))^2 \\ &= -2k_2 \frac{I_1^2}{(1+k_2 I_2)^2} - 2c(Im(\langle \Psi(t)|\phi\rangle))^2 \leqslant 0, \end{aligned} \tag{23}$$

same conclusion on the interior of the region $A = \{\Psi |\ I_2(\Psi) < -\sqrt{|I_1|}\}$ *since by hypothesis* $k_1 > 1$. *The condition* $\dot{V}^* \leqslant 0$ *is fulfilled, thus we can apply here Theorem 1, page 153 in [10], and the conclusion follows.*

3.2 Asymptotic stability analysis

In the following we prove an asymptotic stability result for the system defined by (12) around the target ϕ. We apply a LaSalle type result for differential inclusions introduced in [21].

Theorem 3.2 *Consider* (12) *with* $\Psi \in \mathcal{S}^N(0,1)$ *a solution in the Filippov sense and an eigenstate* $\phi \in \mathcal{S}^N(0,1)$ *of* H_0 *associated to the eigenvalue* λ. *Take the feedback* (11) *with* $k_1 > 1$, $k_2 < \frac{1}{\|\mu_2\|}$ *and* $c > 0$. *Under the hypothesis:*

(1) $\lambda_j \neq \lambda_l$ *for* $j \neq l$,
(2) for any $j = 2, .., N$: $\langle \mu_1 \phi_j | \phi \rangle \neq 0$ *or* $\langle \mu_2 \phi_j | \phi \rangle \neq 0$, *where* $\phi_1, \ldots, \phi_N$ *is an orthogonal system of eigenvectors of* H_0 *corresponding to the eigenvalues* $(\lambda_i)_{i=1,\ldots,N}$,

the ω *limit set of* $\Psi(t)$ *reduces to* $\pm\phi$.

Proof 3.2 *Up to a shift in* ω *and* H_0, *we may assume that* $\lambda = 0$. *Since we consider the solutions of the system* (12) *in the Filippov sense, the stability analysis will be made for the system defined by* (16).

Theorem 2.11 in [21] says that the trajectories of the system (16) *converge to the largest weekly invariant set contained in* $E = \{\Psi \in \mathcal{S}^N(0,1) | 0 \in \dot{\overline{V}}\}$, *where* $\dot{\overline{V}}(\Psi) = \{\nabla V(\Psi) \cdot u, u \in \mathcal{F}(\Psi)\}$, *with* V *defined by* (6).

Let us first compute the differential inclusion $\mathcal{F}$ *defined by* (15), *associated to the discontinous function* f *defined by* (12). *If* $I_2(\Psi) > \sqrt{|I_1|}$ *the function* f *is continuous then:*

$$\mathcal{F} = -i(H_0 - \lambda I)\Psi. \tag{24}$$

If $-\sqrt{|I_1|} < I_2(\Psi) < \sqrt{|I_1|}$ *the function* f *is continuous thus:*

$$\mathcal{F} = -i\Big(H_0 - k_2 \frac{I_1}{1 + k_2 I_2}\mu_1 + (k_2 \frac{I_1}{1 + k_2 I_2})^2 \mu_2 - \lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi. \tag{25}$$

If $I_2(\Psi) < -\sqrt{|I_1|}$ *the function* f *has the same property and:*

$$\mathcal{F} = -i\Big(H_0 + k_1 I_2 \mu_1 + (k_1 I_2)^2 \mu_2 - \lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi. \tag{26}$$

On the contrary on the set $\{\Psi | I_2(\Psi) = \sqrt{|I_1(\Psi)|}\}$ *the function* f *is discontinuous, hence*

$$\mathcal{F}(\Psi) = \begin{cases} [b, c_1], & \text{if b} \leqslant \text{c}_1 \\ [c_1, b], & \text{if c}_1 \leqslant \text{b} \end{cases} \tag{27}$$

Same property on the set $\{\Psi | I_2(\Psi) = -\sqrt{|I_1(\Psi)|}\}$:

$$\mathcal{F}(\Psi) = \begin{cases} [a, c_1], & \text{if a} \leqslant \text{c}_1 \\ [c_1, a], & \text{if c}_1 \leqslant \text{a}, \end{cases} \tag{28}$$

where we have denoted:

$$\begin{aligned} a &= -i\Big(H_0 + k_1 I_2 \mu_1 + (k_1 I_2)^2 \mu_2 - \lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi, \\ b &= -i\Big(H_0 - \lambda - cIm(\langle \Psi(t)|\phi\rangle\Big)\Psi \\ c_1 &= -i\Big(H_0 - k_2 \frac{I_1}{1 + k_2 I_2}\mu_1 + (k_2 \frac{I_1}{1 + k_2 I_2})^2 \mu_2 - \\ &\quad \lambda - cIm(\langle \Psi(t)|\phi\rangle)\Big)\Psi. \end{aligned} \tag{29}$$

If we take in consideration the computations made in the proof of Theorem 3.1 *we have:*

$$\dot{\overline{V}} = \begin{cases} -k_1 I_2 (I_1 + k_1 I_2^2) - 2c(Im(\langle \Psi(t)|\phi\rangle))^2, \\ \qquad\qquad\qquad \text{if I}_2 < -\sqrt{|\text{I}_1|} \\ -2k_2 \frac{I_1^2}{(1+k_2 I_2)^2} - 2c(Im(\langle \Psi(t)|\phi\rangle))^2, \\ \qquad\qquad\qquad \text{if } -\sqrt{|\text{I}_1|} < \text{I}_2 < \sqrt{|\text{I}_1|} \\ -2c(Im(\langle \Psi(t)|\phi\rangle))^2, \qquad \text{if I}_2 > \sqrt{|\text{I}_1|} \end{cases} \tag{30}$$

On the contrary on the discontinuity set $\{\Psi | I_2(\Psi) = \sqrt{|I_1(\Psi)|}\}$ *we use relation* (27) *and we obtain:*

$$\dot{\overline{V}} = \begin{cases} [f, g], & \text{if f} \leqslant \text{g} \\ [g, f], & \text{if g} \leqslant \text{f} \end{cases} \tag{31}$$

In the same way, on the set $\{\Psi | I_2(\Psi) = -\sqrt{|I_1(\Psi)|}\}$, *considering relation* (28) *and we have:*

$$\dot{\overline{V}} = \begin{cases} [g, h], & \text{if g} \leqslant \text{h} \\ [h, g], & \text{if h} \leqslant \text{g}, \end{cases} \tag{32}$$

where:

$$f = -k_1 I_2(I_1 + k_1 I_2^2) - 2c(Im(\langle\Psi(t)|\phi\rangle))^2,$$
$$g = -2c(Im(\langle\Psi(t)|\phi\rangle))^2$$
$$h = -2k_2\frac{I_1^2}{(1+k_2I_2)^2} - 2c(Im(\langle\Psi(t)|\phi\rangle))^2. \quad (33)$$

Since $(I_1 + k_1 I_2^2) > 0$, it follows that the limit set is characterized by:

$$I_1 = 0,\ I_2 = 0,\ Im(\langle\Psi(t)|\phi\rangle) = 0, \quad (34)$$

and therefore $\epsilon = 0$.
This implies that the set E consists in fact of trajectories of the uncontrolled system:

$$i\frac{d}{dt}\Psi = H_0\Psi. \quad (35)$$

with solutions of the form:

$$\Psi = \sum_{j=1}^{N} b_j e^{-i\lambda_j t}\phi_j. \quad (36)$$

We substitute (36) *in* (34) *and we obtain:*

$$Im(\langle\Psi(t)|\phi\rangle) = Im(b_1)\langle\phi,\phi\rangle + \sum_{j=2}^{N} Im(b_j\langle\phi_j,\phi\rangle e^{-i\lambda_j t}) = 0. \quad (37)$$

$$I_1(\Psi) = Im(b_1)\langle\mu_1\phi,\phi\rangle + \sum_{j\in J_1} Im(b_j\langle\mu_1\phi_j,\phi\rangle e^{-i\lambda_j t}) = 0. \quad (38)$$

$$I_2(\Psi) = Im(b_1)\langle\mu_2\phi,\phi\rangle + \sum_{j\in J_2} Im(b_j\langle\mu_2\phi_j,\phi\rangle e^{-i\lambda_j t}) = 0. \quad (39)$$

Without loss of generality we take $\phi = \phi_1$. From equation (34) *and* (37)*, together with $\langle\phi_j,\phi\rangle = 0$ for all $j = 2,...,N$ we obtain $Im(b_1) = 0$. Since along the trajectories Ψ in $\Omega(\Psi)$, $I_1(\Psi) \equiv 0$, we have $\sum_{j\in J_1} Im(b_j\langle\mu_1\phi_j,\phi\rangle e^{-i\lambda_j t}) = \sum_{j\in J_1} B_j^{'}\sin(\lambda_j t + \theta_j) = 0$. The functions $\sin(\lambda_j t+\theta_j)$ are linearly independent as λ_j are all different, hence the sum can only vanish if all coefficients $B_j^{'}$ vanish. Observe now that $B_j^{'} = 0, j \in J_1$ if and only if $b_j = 0, j \in J_1$. Using $Im(b_1) = 0$ we have:*

$$I_2(\Psi) = \sum_{j\in J_2} Im(b_j\langle\mu_2\phi_j,\phi\rangle e^{-i\lambda_j t}) = \sum_{j\in J_2} B_j\sin(\lambda_j t + \theta_j). \quad (40)$$

Since $I_2(\Psi) = 0$ following the same arguments as above $b_j = 0$ for $j = 2,...,N$. Together with equality (36) *this leaves only $\Psi = b_1 e^{-i\lambda t}\phi = b_1\phi$ (we assumed $\lambda = 0$). Since $Im(b_1) = 0$ the only case remained is $\Psi = \pm\phi$. This concludes the proof of Theorem 3.2.*

4 Numerical simulations

We consider next the five-dimensional system (see [25]) defined by:

$$H_0 = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 1.4 & 0 \\ 0 & 0 & 0 & 0 & 2.15 \end{pmatrix},$$
$$\mu_1 = \begin{pmatrix} 0&0&1&1&1 \\ 0&0&1&1&1 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \end{pmatrix}, \mu_2 = \begin{pmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}. \quad (41)$$

In Fig. 2 simulations describe the evolution of the population of the trajectory $\Psi = (\Psi_1, \Psi_2, ..., \Psi_5)$, for the initial state $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$. We take $k_1 = 1.1, k_2 = c = 0.8$. We remark that the discontinuous laser field (11) is efficient to reach the first eigenstate $\phi = (1,0,0,0,0,0)$ of energy $\lambda = 1$, at the final time T. Note that here $\|\mu_2\| = 1$.
The Fig. 3 describes the evolution of the Lyapunov function defined by (6).

5 Conclusions

We study in this paper the control of Schrödinger equation. The particularity of the problem is that the interaction between the system and the laser is not described just by a first order term $\epsilon(t)\mu_1$, but also by a second order, polarizability term $\epsilon^2(t)\mu_2$. In a previous work discontinuous feedback with memory terms were introduced in order to exploit the polarizability coupling. The present paper studies this discontinuous case and focuses on obtaining an asymptotic stability result. Related numerical simulations are also presented.

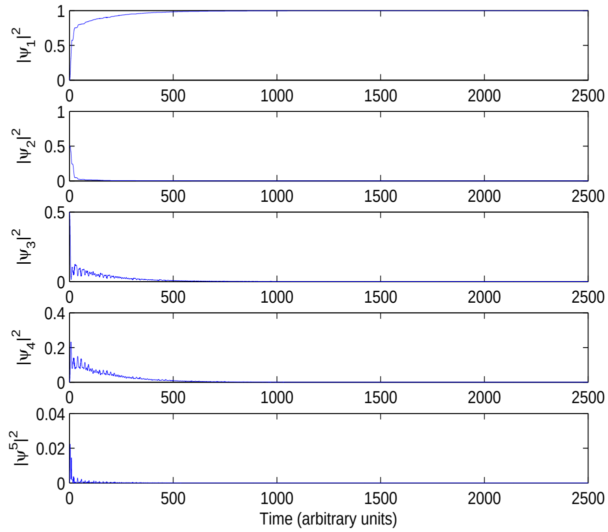

Fig. 2. The population of the system (41) with trajectory $\Psi = (\Psi_1, \Psi_2, ..., \Psi_5)$; initial condition: $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$; the feedback is defined by (11), with $k_1 = 1.1, k_2 = c = 0.8$.

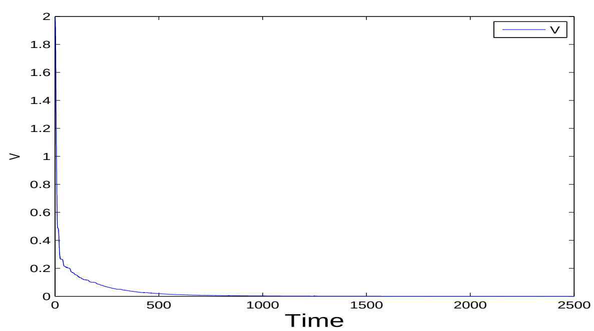

Fig. 3. Evolution of the Lyapunov function $V(\Psi)$ defined by (6); initial condition: $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$; system defined by (41) with feedback (11). We take $k_1 = 1.1, k_2 = c = 0.8$.

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