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Symbolic extensions and continuity properties of the entropy

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Abstract

In this short note we give a new presentation of the entropy theory of symbolic extensions. Then we deduce from the main results of this theory some continuity properties of the entropy regarding the smoothness of the dynamical system. We also prove that generic continuous interval maps have nowhere continuous entropy function.

1 Background

1.1 Monotone operator on a complete lattice and some basic dynamical applications

We first recall some elements of set theory. A (partially) ordered set (L, \leq) is a *complete lattice* if every subset of L has both a supremum and an infimum. A map \mathcal{T} of L into itself is an *increasing operator* if $\mathcal{T}f \leq \mathcal{T}g$ for all $f \leq g$. We recall now an elementary fixed point theorem due to Tarski and Knaster.

Theorem 1 [28] *Let L be a complete lattice and let $\mathcal{T} : L \rightarrow L$ be an increasing operator. Then the set of fixed points of \mathcal{T} in L is also a complete lattice. In particular it is not empty and there is a least fixed point.*

The least fixed point of \mathcal{T} can be obtained in an inductive way as the stationary limit of the transfinite sequence $(\mathcal{T}^\alpha 0)_\alpha$ where 0 is the least element of L and $\mathcal{T}^\alpha 0$ is $\mathcal{T}(\mathcal{T}^{\alpha-1} 0)$ for successor ordinals α and $\mathcal{T}^\alpha 0$ is the least upper bound of $\mathcal{T}^\beta 0$ over $\beta < \alpha$ for limit ordinals α . By the increasing property of \mathcal{T} any fixed point of \mathcal{T} is larger than $\mathcal{T}^\alpha 0$ for all ordinals α . Then the transfinite sequence $(\mathcal{T}^\alpha 0)_\alpha$ is obviously stationary after any ordinal β with cardinal larger than the cardinality of L . Let us denote by $\alpha(\mathcal{T})$ or just α^* the least ordinal α such that $\mathcal{T}^\alpha 0 = \mathcal{T}^{\alpha+1} 0$. This ordinal is called the *order of accumulation* of \mathcal{T} .

We illustrate this formalism with two basic examples. One naive example of increasing operator is given by a nondecreasing map f of the unit interval endowed with the usual order. Then one obtains the least fixed point of f as the stationary limit of the transfinite sequence $(f^\alpha(0))_\alpha$. Clearly the order of accumulation is countable (indeed for any $\alpha < \alpha(f)$ the interval $]f^\alpha(0), f^{\alpha+1}(0)]$ is nonempty and thus contains a rational number). Moreover it is not difficult to construct examples of nondecreasing maps with a given countable ordinal as order of accumulation.

We present now another context where this formalism applies. By a dynamical system we mean a continuous map $T : X \rightarrow X$ on a compact metrizable space X to itself. Recall a point x of X is nonwandering if for every neighborhood U of x there is an integer $n \geq 1$ such that $T^n(U) \cap U$ is nonempty. The set of all nonwandering points is called the nonwandering set of T , and is denoted by $\Omega(T)$. The nonwandering set $\Omega(T)$ of a dynamical system (X, T) differs in general from the nonwandering set of $(\Omega(T), T|_{\Omega(T)})$. The set of forward invariant compact subsets of X ordered with the reverse inclusion \supset is clearly a complete lattice. Then the operator \mathcal{T}_Ω which associates to any forward invariant compact subset K the nonwandering set of $T|_K$ is increasing. It is well known that the least fixed point of \mathcal{T}_Ω is the closure of the set of the recurrent points of (X, T) (Theorem 5.08 of [24]). Moreover by an argument of compactity the order of accumulation of \mathcal{T}_Ω is countable and any countable ordinal can be achieved as the order or accumulation of the operator \mathcal{T}_Ω associated to some dynamical system [16].

1.2 Symbolic extensions

We address the question whether a dynamical system can be encoded with a finite alphabet. More precisely, does a given dynamical system (X, T) admit a *symbolic extension*, that is a subshift (Y, S) of a full shift over a finite alphabet, along with a continuous surjection $\pi : Y \rightarrow X$ such that $\pi \circ S = T \circ \pi$? We are also interested in minimizing the entropy introduced by the code. For example can we choose these extensions to be *principal*, i.e. such that they preserve the entropy of measures? We focus particularly on smooth dynamical systems. In this case, the answer highly depends on the smoothness of the dynamical system :

- there exist \mathcal{C}^1 examples with no symbolic extensions [19],[2],[7];
- there exists \mathcal{C}^r examples with $r > 1$ with no principal symbolic extensions [19],[7]. However \mathcal{C}^r interval maps [13] with $r > 1$ and \mathcal{C}^2 surface diffeomorphisms [6] always admit symbolic extensions (T.Downarowicz and S.Newhouse have conjectured the existence of symbolic extensions for any \mathcal{C}^r map with $r > 1$ on a compact manifold (Conjecture 1.1 of [19]));
- \mathcal{C}^∞ dynamical systems always admit principal symbolic extensions [5].

The main tool which allows to validate or invalidate the existence of (principal) symbolic extensions is the powerful Symbolic Extension Entropy Theorem

[3], due to M.Boyle and T.Downarowicz, which relates the entropy of symbolic extensions with the emergence of entropy at different small scales. This emergence is captured by the convergence of entropy structures which we present in the next subsection. First, we recall some notations and definitions.

We denote by $\mathcal{M}(X, T)$ the set of Borel invariant probability measures of the dynamical system (X, T) endowed with the weak star topology and we denote by *dist* a distance on $\mathcal{M}(X, T)$. Given a symbolic extension $\pi : (Y, S) \rightarrow (X, T)$ we will consider the function $h_{ext}^\pi : \mathcal{M}(X, T) \rightarrow \mathbb{R}^+$:

$$\forall \mu \in \mathcal{M}(X, T), h_{ext}^\pi(\mu) = \sup_{\pi_* \nu = \mu} h(\mu)$$

where h denotes the usual Kolmogorov-Sinai entropy of (X, T) and $\pi_* : \mathcal{M}(Y, S) \rightarrow \mathcal{M}(X, T)$ is the map induced by π on measures. Finally the symbolic extension entropy function is the function $h_{sex} = \inf_{\pi} h_{ext}^\pi$, which is related to the topological entropy of the symbolic extensions (Y, S) of (X, T) by the following variational principle (Theorem 8.1 of [3]) :

$$\sup_{\mu} h_{sex}(\mu) = \inf_S h_{top}(S)$$

1.3 Entropy structure

The *entropy structures* of a dynamical system (X, T) are special nondecreasing sequences $(h_k)_{k \in \mathbb{Z}^+}$ of nonnegative functions which converge pointwise to the Kolmogorov-Sinai entropy function. They recover the usual entropy invariants but also the new ones arising from the entropy theory of symbolic extensions. Two entropy structures $(h_k)_k$ and $(g_k)_k$ satisfy the following property :

$$\forall \gamma > 0 \forall k \in \mathbb{Z}^+ \exists l \in \mathbb{Z}^+, \quad g_l > h_k - \gamma \text{ and } h_l > g_k - \gamma \quad (1)$$

The above relation ensures that the main invariant of the entropy theory of symbolic extensions whose definition involves an entropy structure, namely the operator \mathcal{T}_{sex} introduced in the next section, does not depend on the choice of the entropy structure. Moreover most of the usual approaches to entropy lead to an entropy structure [10], so that entropy structures may be considered as a master entropy invariant unifying the previous theories of entropy. For example if $(P_k)_k$ is a decreasing sequence of partitions whose diameter goes to zero and with small boundaries, i.e. $\mu(\partial P_k) = 0$ for all $\mu \in \mathcal{M}(X, T)$, then the functions $h_k = h(\cdot, P_k)$ define an entropy structure.

We refer to [10] for a complete definition as well as further properties and examples.

2 Symbolic Extension Entropy Theorem revisited

A real function f on a compact metric space E is said to be *upper semicontinuous* if $\limsup_{y \rightarrow x} f(y) \leq f(x)$ for all $x \in E$. When f is a real function defined on E we denote by $\lceil f \rceil$ the smallest upper semicontinuous function larger than or equal to f if f is bounded from above and $\lceil f \rceil$ is the constant function equal to $+\infty$ if not.

Using the framework of operators on lattices presented in the previous section we restate the Symbolic Extension Entropy Theorem. Let (X, T) be a dynamical system. We denote by $\mathcal{S}(X, T)$ the set of nonnegative upper semicontinuous functions defined on $\mathcal{M}(X, T)$ to which we add the function constant equal to $+\infty$. The set $\mathcal{S}(X, T)$ endowed with the usual order is a complete lattice : if F is a family of upper semicontinuous functions then $\inf_f f \in F$ is itself upper semicontinuous and then it is the infimum of F in $\mathcal{S}(X, T)$, and $\lceil \sup_{f \in F} f \rceil$ is clearly the supremum of F in $\mathcal{S}(X, T)$.

Let $\mathcal{H} = (h_k)_k$ be an entropy structure, we define the operator \mathcal{T}_{sex} on $\mathcal{S}(X, T)$ as follows :

$$\begin{aligned} \mathcal{T}_{sex} : \mathcal{S}(X, T) &\rightarrow \mathcal{S}(X, T) \\ f &\mapsto \lim_k \lceil f + h - h_k \rceil \end{aligned}$$

One easily checks from (1) that \mathcal{T}_{sex} does not depend on the choice of the entropy structure $(h_k)_k$. Clearly \mathcal{T}_{sex} is an increasing operator. Moreover, the functions $h - h_k$ being nonnegative, $\mathcal{T}_{sex}f \geq f$ for all $f \in \mathcal{S}(X, T)$. By using the affine structure of the set of invariant probability measures M.Boyle and T.Downarowicz proved that the least fixed point of \mathcal{T}_{sex} coincides with the infimum of the affine fixed points of \mathcal{T}_{sex} . The Symbolic Extension Entropy Theorem can then be restated as follows :

Theorem 2 (Theorem 5.5 of [3]) *The affine fixed points of \mathcal{T}_{sex} are exactly the functions $h_{ext}^\pi - h$, i.e. f is a nonnegative affine upper semicontinuous function on $\mathcal{M}(X, T)$ fixed by \mathcal{T}_{sex} if and only if there exists a symbolic extension π such that $f = h_{ext}^\pi - h$. Moreover $h_{sex} - h$ is the least fixed point of \mathcal{T}_{sex} .*

In [3] the authors considered entropy structures $(h_k)_k$ such that $h_0 = 0$ and $h_{k+1} - h_k$ is upper semicontinuous for all $k \in \mathbb{Z}^+$. Then the Symbolic Extension Entropy Theorem is stated in [3] as follows : a nonnegative affine function f is equal to $h_{ext}^\pi - h$ for some symbolic extension $\pi : (Y, S) \rightarrow (X, T)$ if and only if $f - h_k$ is upper semicontinuous for all $k \in \mathbb{Z}^+$. According to Lemma 2.1.6 of [10] this last condition is equivalent to $\mathcal{T}_{sex}f = f$. Since \mathcal{T}_{sex} does not depend on the choice of the entropy structure $(h_k)_k$ the above Theorem 2 follows at once.

In this setting the transfinite sequence $(u_\alpha)_\alpha$ and the order of accumulation of entropy introduced in [3] are respectively the sequence $(\mathcal{T}_{sex}^\alpha 0)_\alpha$ and the ordinal $\alpha(\mathcal{T}_{sex})$. By an easy argument of compacity the order of accumulation is

a countable ordinal. It was proved in [8] that any countable ordinal is realized as the order of accumulation of entropy of a dynamical system.

As we already mentioned in Section 1.2 it is not known whether dynamical systems of intermediate smoothness (\mathcal{C}^r with $1 < r < +\infty$) always admit symbolic extensions. When T is a \mathcal{C}^1 dynamical system on a compact manifold M of dimension d let us denote by $(\chi_i(x))_{i=1,\dots,d}$ the Lyapunov exponents at a regular point x . The sum of the positive Lyapunov exponents of an invariant measure μ is then given by $\sum_i \chi_i^+(\mu) := \sum_i \int \chi_i^+(x) d\mu(x)$ where we use the notation $a^+ := \max(a, 0)$ for all real numbers a . It is easily checked that $\sum_i \chi_i^+$ defines an affine upper semicontinuous function on $\mathcal{M}(X, T)$ [6]. We conjecture that

Conjecture 1 *Let $T : M \rightarrow M$ be a \mathcal{C}^r map, with $1 < r$, defined on a compact manifold M of dimension d . Then $\frac{d \sum_i \chi_i^+}{r-1}$ is a fixed point of \mathcal{T}_{sex} . In particular it admits a symbolic extension π such that $h_{ext}^\pi - h = \frac{d \sum_i \chi_i^+}{r-1}$.*

It was proved for interval maps [13] and \mathcal{C}^2 surface diffeomorphisms [6]. We have already noted that \mathcal{C}^∞ dynamical systems always admit principal symbolic extensions, i.e. symbolic extensions $\pi : (Y, S) \rightarrow (X, T)$ with $h_{ext}^\pi = h$. Therefore the above conjecture also holds true in this case.

3 Continuity of the entropy

This last section deals with continuity properties of the entropy function. As for symbolic extensions the picture highly depends on the smoothness of the dynamical system :

- there exist (continuous) dynamical systems such that the entropy function h is nowhere continuous;
- there exist \mathcal{C}^1 examples such that the restriction of h to some nonempty compact subset of the set of Borel invariant probability measures is nowhere continuous;
- there exist \mathcal{C}^r examples with $r > 1$ where h is not upper semicontinuous [21][26]. However for \mathcal{C}^r interval maps and \mathcal{C}^2 surface diffeomorphisms h is a difference of upper semicontinuous functions : in particular the set of continuity points of h is a dense \mathcal{G}_δ set;
- h is upper semicontinuous for \mathcal{C}^∞ dynamical systems [23].

As the last item, concerning the special case of \mathcal{C}^∞ maps, is a famous result of S.Newhouse [23] we will not revisit it. We explain in the following how the three first items follow from previous works on entropy structure and symbolic extensions.

3.1 Continuous Examples

The first item follows from the complete characterization of possible entropy functions of (continuous) dynamical systems due to T.Downarowicz and J.Serafin. Recall a *Choquet simplex* K is a compact convex set satisfying the following property. For any $x \in K$ there exists a unique Borel probability measure μ_x on K supported by the extremal set such that $f(x) = \int f(y)d\mu_x(y)$ for all affine continuous functions $f : K \rightarrow \mathbb{R}$. Then a function $g : K \rightarrow \mathbb{R}$ is said to be *harmonic* if $g(x) = \int g(y)d\mu_x(y)$. The *harmonic extension* of a Borel map $f : K \rightarrow \mathbb{R}$ supported on the extremal set of K is the harmonic function $x \mapsto \int_K f(y)d\mu_x(y)$.

Theorem 3 [12] *Let K be a Choquet simplex and $f : K \rightarrow \mathbb{R}^+$ be a nonnegative function. Then there exists a dynamical system (X, T) and a affine homeomorphism $a : \mathcal{M}(X, T) \rightarrow K$ such that $f \circ a$ is the entropy function of (X, T) if and only if f is a pointwise limit of a nondecreasing sequence of nonnegative harmonic upper semicontinuous functions.*

We only need to construct a Choquet simplex K and an increasing sequence of nonnegative harmonic upper semicontinuous functions $f_k : K \rightarrow \mathbb{R}$ whose limit f is nowhere continuous. Let K be the Choquet simplex $\mathcal{M}([0, 1])$ of the Borel probability measures on $[0, 1]$ and let f_k be the harmonic extension of the characteristic function of the set of Dirac measures $\{\delta_{a_1}, \dots, \delta_{a_k}\}$, where $(a_n)_{n \in \mathbb{Z}^+}$ is a countable enumeration of the rational numbers of $[0, 1]$. The functions f_k are just given by $f_k(\mu) = \sum_{l=1}^k \mu(\{a_l\})$ for all $\mu \in \mathcal{M}([0, 1])$. In particular they are upper semicontinuous. Moreover by the theorem of monotone convergence the limit f is just the harmonic extension of the characteristic function of the Dirac measures at rational numbers. Clearly f is nowhere continuous because any probability measure on $[0, 1]$ is a weak limit of atomic measures supported by either rational or irrational numbers.

In fact this phenomenon is generic among continuous interval maps endowed with the topology of uniform convergence :

Theorem 4 *The entropy function of a generic continuous interval map is nowhere continuous.*

We conjecture the above theorem also holds in higher dimensions. We first begin with the following adaptation of Theorem 3.8 of [9].

Proposition 1 *There exists a residual set \mathcal{S} of continuous interval maps such that the periodic measures of any $f \in \mathcal{S}$ are dense in the set of ergodic measures.*

The proof is based on the following elementary perturbative result :

Lemma 1 *Let μ be an ergodic measure of an interval map f . Then for every C^0 -neighborhood \mathcal{U} of f and every neighborhood \mathcal{V} of μ in the set of probability measures with the weak star topology, there exists $g \in \mathcal{U}$ admitting a periodic point p such that the associated periodic measure γ_p belongs to \mathcal{V} .*

PROOF : Clearly one can assume μ is not periodic for f . By the ergodic theorem $\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}\right)_n$ converges to μ for μ -almost all x . Fix such a point x and let n be an integer such that $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \in \mathcal{V}$. One can also assume x to be recurrent and then choose $f^n x$ close enough to x so that x is a periodic point of period n of $g \in \mathcal{U}$ with $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g^k x} \in \mathcal{V}$ after a small \mathcal{C}^0 -perturbation of f . \square

Given a positive integer n , let \mathcal{K}_n denote the set of continuous interval maps f satisfying the following property :

Each periodic orbit \mathcal{O} of period n is transversal, i.e. on any neighborhood U of $p \in \mathcal{O}$ there exist $x_1 < p < x_2$ in U such that $(f^n(x_1) - x_1)(f^n(x_2) - x_2) < 0$.

Lemma 2 $\mathcal{K} = \bigcap_{n \geq 1} \mathcal{K}_n$ is a residual set in the set of continuous interval maps.

PROOF : It is enough to prove that \mathcal{K}_n is itself residual for each n . For any pair of distinct rational numbers (a, b) let $F_n(a, b) := \{f \in \mathcal{C}([0, 1]), \forall x \in [a, b], f^n(x) \geq x \text{ and } \exists x_0 \in [a, b], f^n(x_0) = x_0\} \cup \{f \in \mathcal{C}([0, 1]), \forall x \in [a, b], f^n(x) \leq x \text{ and } \exists x_0 \in [a, b], f^n(x_0) = x_0\}$. The sets $F_n(a, b)$ are closed and have empty interior because the set of Kupka-Smale maps¹ of \mathcal{C}^1 interval maps is \mathcal{C}^0 -dense in \mathcal{K} [25]. Therefore $\mathcal{K}_n \supset \mathcal{C}([0, 1]) \setminus (\bigcup_{a < b} F_n(a, b))$ is a residual set. \square

We notice that transversal periodic orbits are persistent in the following sense. Assume \mathcal{O} is a transversal periodic orbit of period n , then for any neighborhood U of $p \in \mathcal{O}$ there exists a neighborhood \mathcal{U} of f such that any $g \in \mathcal{U}$ admits a periodic orbit \mathcal{O}' of period n with $\mathcal{O}' \cap U \neq \emptyset$ (this last periodic orbit \mathcal{O}' may be not isolated among g -periodic points of period n , thus \mathcal{O}' is not transversal in general).

Let \mathcal{X} be a subset of $\mathcal{C}([0, 1])$ endowed with the induced topology and let $K(Y)$ be the set of compact subsets of a compact metric space Y endowed with the Hausdorff metric. A map $\phi : \mathcal{X} \rightarrow K(Y)$ is said to be lower semicontinuous (resp. upper semicontinuous) if for any f in \mathcal{X} and for any open set V in Y with $V \cap \phi(f) \neq \emptyset$ (resp. $V \supset \phi(f)$) there exists a neighborhood U of f in \mathcal{X} such that $V \cap \phi(g) \neq \emptyset$ (resp. $V \supset \phi(g)$) for every $g \in U$. The set of continuity points of such (upper or lower semicontinuous) functions is a residual set of \mathcal{X} [18].

PROOF OF PROPOSITION 1 : Let X_g be the closure of the periodic measures of $g \in \mathcal{K}$ in the set $\mathcal{M}([0, 1])$ of probability measures of $[0, 1]$ endowed with the weak star topology. Transversal periodic points are persistent so that the map $g \mapsto X_g$ from \mathcal{K} to $K(\mathcal{M}([0, 1]))$ is lower semicontinuous. Since \mathcal{K} is a residual set it follows that the set \mathcal{S} of continuity points of $g \mapsto X_g$ is also a residual

¹A \mathcal{C}^1 interval map f is said Kupka-Smale when every periodic point p of f is hyperbolic, i.e. $|(f^P)'(p)| \neq 1$ where P is the period of p .

set of $\mathcal{C}([0, 1])$. Let $f \in \mathcal{S}$ be such a point of continuity and let μ be an ergodic f -invariant measure. By the previous lemma μ is approximated by a periodic measure of a continuous map arbitrarily close to f . After another small \mathcal{C}^0 perturbation one can assume this last map in \mathcal{K} . By continuity of $g \mapsto X_g$ at f the measure μ belongs then to the closure of the periodic measures of f . \square

Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous interval map. We recall that a family $J = (J_1, \dots, J_p)$ of closed disjoint intervals is called a p -horseshoe of f if $J_k \subset f(J_i)$ for all i, k . Let us denote $H_J := \bigcap_{n \in \mathbb{Z}} f^n J$ and $(\{1, \dots, p\}^{\mathbb{Z}^+}, \sigma)$ the one sided shift with p symbols. The map $\pi : (H_J, f) \rightarrow (\{1, \dots, p\}^{\mathbb{Z}^+}, \sigma)$ defined by $(\pi(x))_k = q$ iff $f^k(x) \in J_q$ is a semi-conjugacy. In particular $h_{top}(f) \geq \log p$. In fact horseshoes characterize entropy of continuous interval maps [22] : if f is a continuous interval map with entropy $h_{top}(f) > 0$ then for all $h < h_{top}(f)$ there exists a p -horseshoe for f^N with entropy $\log p/N > h$.

We show now as in [19] that horseshoes accumulate on periodic orbits for generic maps. For a given continuous interval map $f : [0, 1] \rightarrow [0, 1]$ and a given positive integer n we say that a set S of periodic points of f satisfies the property (P_n) when :

For each periodic point p in S there exists a horseshoe with entropy $\log 3$ such that any invariant measure supported on this horseshoe lies in the open $\frac{1}{n}$ -neighborhood of γ_p .

We observe that if S satisfies the property (P_n) then the closure of S satisfies the property (P_{n-1}) . Let \mathcal{R}_n denote the set of continuous interval maps f such that the set of all periodic points satisfies the above property (P_n) .

Proposition 2 $\mathcal{R} = \bigcap_{n \geq 1} \mathcal{R}_n$ is residual in the set of continuous interval maps.

PROOF : For any interval map $f \in \mathcal{K}$ and for any integer m we denote by $Per_m(f)$ the set of periodic points of period less than m . This set is a closed subset of the interval with empty interior because all the periodic points of f are transversal. Therefore, if for all positive integers l and m we let $Per_{l,m}(f) = \{p \in Per_m(f), \text{ s.t. } [p, p + \frac{1}{l} \cap Per_m(f) = \{p\} \text{ or }]p - \frac{1}{l}, p] \cap Per_m(f) = \{p\}\}$, the union $\bigcup_{l \in \mathbb{Z}^+} Per_{l,m}(f)$ is a dense subset of $Per_m(f)$. Let $\mathcal{R}_n(l, m)$ be the subset of \mathcal{K} such that the property (P_n) holds for $Per_{l,m}(f)$. Then we have the following inclusions $\mathcal{R}_n \subset \bigcap_{l,m} \mathcal{R}_n(l, m) \subset \mathcal{R}_{n-1}$ so that $\mathcal{R} = \bigcap_{l,m,n} \mathcal{R}_n(l, m)$. Therefore we only need to prove $\mathcal{R}_n(l, m)$ is an open dense set in \mathcal{K} because \mathcal{K} is residual by Lemma 2.

Let us first prove that this set is open in \mathcal{K} . Since horseshoes are persistent under small \mathcal{C}^0 perturbations it is enough to show that $f \mapsto Per_{l,m}(f)$ is upper semicontinuous from \mathcal{K} to $K([0, 1])$. For any $f \in \mathcal{K}$ there exists a finite subset $\{p_1 < p_2 < \dots < p_N\}$ of $Per_m(f)$ containing $Per_{l,m}(f)$ such that either $[p_i - p_{i+1} < \frac{1}{l}]$ or $[p_i - p_{i+1} \geq \frac{1}{l} \text{ and }]p_i, p_{i+1} \cap Per_m(f) \neq \emptyset]$ for

$i = 0, \dots, N$ with $p_0 = -1$ and $p_{N+1} = 2$. In the second case we remark that p_i and p_{i+1} belong to $Per_{l,m}(f) \cup \{p_0, p_{N+1}\}$. Let $I = \{i, |p_i - p_{i+1}| < \frac{1}{l}\}$ and let $r > 0$ be such that $2r + |p_i - p_{i+1}| < \frac{1}{l}$ for all $i \in I$. The transversal periodic points p_1, \dots, p_N are persistent so that any $g \in \mathcal{K}$ close enough to f admits periodic points q_1, \dots, q_N of period less than m with $q_i \in B(p_i, r)$ for $i = 1, \dots, N$. We can also assume that $[p_i + r, p_{i+1} - r] \cap Per_m(g) = \emptyset$ for all $i \notin I$. Then it is easily seen that $Per_{l,m}(g)$ is contained in the open r -neighborhood of $Per_{l,m}(f)$.

To prove the density it is enough to approximate any \mathcal{C}^1 Kupka-Smale maps by a map in $\mathcal{R}_n(l, m)$. To do it one just blows up the finitely many periodic points of period m in small intervals where we "put" an interval map of entropy $\log 3$ (a 3-tent map for example). We refer to [20] for the technical details of the construction. In doing so we may introduce other periodic points of period n but they are all $\frac{1}{n}$ -close to a horseshoe with entropy $\log 3$ when the inserted intervals are small enough. □

Theorem 4 follows immediately from Proposition 1 and Proposition 2. Indeed ergodic measures of any $f \in \mathcal{S} \cap \mathcal{R}$ are approximated by periodic measures (with zero entropy) and by ergodic measures with $\log 3$ entropy (which correspond to the measures of maximal entropy of the horseshoes closed to the periodic points). Therefore any invariant measure is approximated by a convex combination of ergodic measures with either zero entropy or with $\log 3$ entropy.

Remark 1 *By using the robustness of horseshoes it is easily seen that generic continuous maps on a compact manifold have infinite topological entropy (see Section 5 of [1]). In particular they do not admit symbolic extensions.*

3.2 \mathcal{C}^1 examples

In [19], examples of \mathcal{C}^1 surface volume preserving diffeomorphisms (M, T) without symbolic extensions are built by accumulating horseshoes on periodic points. These examples ² satisfy more precisely the following property (see in [19] the proof of Theorem 1.3 after Lemma 5.1) :

There are a nonempty compact subset \mathcal{E} of $\mathcal{M}(M, T)$ and a positive real number $\rho_0 > 0$ such that :

- *the periodic measures are dense in \mathcal{E} ;*
- $\forall \mu \in \mathcal{E}$,

$$\limsup_{\nu \in \mathcal{E}, \nu \rightarrow \mu} h(\nu) > \rho_0$$

In particular the entropy function restricted to \mathcal{E} is nowhere continuous. Indeed any invariant measure in \mathcal{E} with positive entropy is a limit of periodic measures in \mathcal{E} according to the first item. Then, if $\mu \in \mathcal{E}$ is an invariant measure

²They are generic in the complementary set of the Anosov diffeomorphisms in the set of \mathcal{C}^1 volume preserving diffeomorphisms on a surface M .

with zero entropy, the second item implies that μ is a limit of invariant measures in \mathcal{E} with entropy larger than $\rho_0 > 0$.

3.3 Entropy as a difference of upper semicontinuous functions

A way to estimate how a nonnegative function h defined on a compact metric space fails to be upper semicontinuous consists in wondering whether it is a difference of nonnegative upper semicontinuous functions and in determining the "smallest" writing as a difference of nonnegative upper semicontinuous functions : what is the smallest upper semicontinuous function $f \geq 0$ such that $f + h$ is upper semicontinuous, which can be written as $f = \inf\{g \geq 0 : g \text{ and } g + h \text{ is upper semicontinuous}\}$? To this end we consider the operator \mathcal{T}_{usc} :

$$\begin{aligned} \mathcal{T}_{usc} : \mathcal{S}(X, T) &\rightarrow \mathcal{S}(X, T) \\ f &\mapsto \left[\mu \mapsto \limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) \right] \end{aligned}$$

As \mathcal{T}_{sex} the operator \mathcal{T}_{usc} is increasing and satisfies $\mathcal{T}_{usc}f \geq f$ for all $f \in \mathcal{S}(X, T)$.

Lemma 3 $h = g - f$ with $f, g \in \mathcal{S}(X, T)$ if and only if f is a fixed point of \mathcal{T}_{usc} .

PROOF : Assume $h + f$ is upper semicontinuous. We have

$$\limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) \leq \max \left(\limsup_{\nu \rightarrow \mu} f(\nu), \limsup_{\nu \rightarrow \mu} (h(\nu) - h(\mu) + f(\nu)) \right)$$

and then by upper semicontinuity of f and $h + f$:

$$\limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) \leq f(\mu)$$

Conversely assume $\mathcal{T}_{usc}(f) = f$. This implies that

$$\begin{aligned} \limsup_{\nu \rightarrow \mu} (h(\nu) - h(\mu) + f(\nu)) &\leq \limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) \\ &\leq f(\mu) \end{aligned}$$

and thus $f + h$ is upper semicontinuous. \square

We denote by h_{usc} the smallest upper semicontinuous function larger than h such that $h_{usc} - h$ is itself upper semicontinuous. In the previous setting $h_{usc} - h$ is the smallest fixed point, $\mathcal{T}_{usc}^{\alpha(\mathcal{T}_{usc})}0$, of the monotone operator \mathcal{T}_{usc} .

Lemma 4 For all $f \in \mathcal{S}(X, T)$

$$\mathcal{T}_{sex} f \geq \mathcal{T}_{usc} f$$

In particular the fixed points of \mathcal{T}_{sex} are fixed points of \mathcal{T}_{usc} and $h_{sex} \geq h_{usc}$.

PROOF : Any dynamical system admits an entropy structure $(h_k)_k$ with upper semicontinuous functions $(h_k)_k$ (see for instance the remark following Theorem 7.0.1 of [13]). Consider such an entropy structure $(h_k)_k$ of (X, T) . Let $f \in \mathcal{S}(X, T)$ and $\mu \in \mathcal{M}(X, T)$. For all $k \in \mathbb{Z}^+$ we have by upper semicontinuity of h_k

$$\begin{aligned} \limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) &\leq \limsup_{\nu \rightarrow \mu} \left((h - h_k)(\nu) + (h_k(\nu) - h(\mu))^+ + f(\nu) \right) \\ &\leq \limsup_{\nu \rightarrow \mu} \left((h - h_k)(\nu) + f(\nu) \right) \end{aligned}$$

and finally by taking the limit in k we get for all $\mu \in \mathcal{M}(X, T)$

$$\limsup_{\nu \rightarrow \mu} \left((h(\nu) - h(\mu))^+ + f(\nu) \right) \leq \mathcal{T}_{sex} f(\mu)$$

The inclusion of the sets of fixed points follows then immediately from the inequality $\mathcal{T}_{sex} f \geq f$ which holds for all $f \in \mathcal{S}(X, T)$. \square

From this last lemma we deduce easily the following corollary :

Corollary 1 Let (X, T) be a dynamical system admitting a symbolic extension. Then the entropy function h is a difference of nonnegative upper semicontinuous functions. In particular it is the case for \mathcal{C}^r interval maps with $r > 1$ and \mathcal{C}^2 surface diffeomorphisms; more precisely we have then that $h + \frac{d \sum_i X_i^+}{r-1}$ is upper semicontinuous.

On the other hand there exist dynamical systems with continuous entropy function h but without symbolic extension (Example 2.20 of [3]). In particular $h_{sex} \neq h_{usc}$ and there exist fixed points of \mathcal{T}_{usc} which are not fixed points of \mathcal{T}_{sex} . However recall that when (X, T) admits symbolic extensions then h_{sex} and thus h_{usc} coincide with h on a dense \mathcal{G}_δ set of $\mathcal{M}(X, T)$ (Proposition 3.1 of [3]). When the entropy is a difference of nonnegative upper semicontinuous function then h_{usc} coincides also with h on a residual set for similar reasons.

To compare with the Realization Theorem of Downarowicz and Serafin (Theorem 3) observe that any difference of nonnegative upper semicontinuous functions is a nondecreasing limit of upper semicontinuous functions. Finally let us mention that a real function ϕ defined on $\mathcal{M}(X, T)$ can be written as a difference of nonnegative upper semicontinuous functions if and only if there exists a sequence of continuous functions $(\phi_n)_n$ with $\phi(\mu) = \sum_n \phi_n(\mu)$ and $\sum_n \phi_n(\mu) \leq C$ for all $\mu \in \mathcal{M}(X, T)$ and for some constant C [14].

As a function of the first Baire class the set of continuity points of any difference of upper semicontinuous functions is a dense \mathcal{G}_δ set. In particular the

examples of T.Downarowicz and S.Newhouse discussed in Section 3.2 have no symbolic extensions by Corollary 1 (consider the restriction of h to \mathcal{E}). This gives an alternative proof of Proposition 4.4 of [19]. We recall that for a large class of dynamical systems the continuity points of the entropy function are measures of zero entropy. Indeed periodic measures are often dense in the set of invariant probability measures (which implies also that generic invariant measures have zero entropy, see Proposition 5.4 of [9]). For example it is the case of dynamical systems satisfying the so-called *specification* property (uniformly hyperbolic dynamical systems [27], continuous piecewise monotone maps [15],...) and it is also the case of \mathcal{C}^1 generic diffeomorphisms [9]. Continuity points are also zero entropy measures for $\mathcal{C}^{1+\alpha}$ surface diffeomorphisms because in this case the invariant measures with positive entropy are approximated by periodic measures [17].

3.4 \mathcal{C}^r examples and related questions

In all the known smooth examples with large symbolic extension entropy [2],[19],[7], we accumulate horseshoes on periodic measures at infinitely many scales. The same process gives large lower bounds on h_{usc} according to the following lemma. It follows that these examples are relevant from the point of view of the continuity properties of the entropy independently of the question of symbolic extensions.

Lemma 5 *Let (X, T) be a dynamical system and let μ be a T -invariant probability measure. We assume that for all $k \in \mathbb{Z}^+$, there exist periodic points $(p_{(i_1, \dots, i_{2k+1})})_{(i_1, \dots, i_{2k+1}) \in \mathbb{Z}^{+2k+1}}$ and invariant probability measures $(\mu_{(i_1, \dots, i_{2k})})_{(i_1, \dots, i_{2k}) \in \mathbb{Z}^{+2k}}$ (we put $\mathbb{Z}^{+0} = \{\emptyset\}$ and $\mu_\emptyset = \mu$) such that :*

- for all $(i_1, \dots, i_{2k}) \in \mathbb{Z}^{+2k}$, the periodic measures $\gamma_{p_{(i_1, \dots, i_{2k+1})}}$ are converging to $\mu_{(i_1, \dots, i_{2k})}$ when i_{2k+1} goes to $+\infty$;
- for all $(i_1, \dots, i_{2k+1}) \in \mathbb{Z}^{+2k+1}$, the measures $\mu_{(i_1, \dots, i_{2k+2})}$ are converging to $\gamma_{p_{(i_1, \dots, i_{2k+1})}}$ when i_{2k+2} goes to $+\infty$;
- the limits $\lim_{i_1 \rightarrow +\infty} \left(\lim_{i_2 \rightarrow +\infty} \dots \left(\lim_{i_{2k+2} \rightarrow +\infty} h(\mu_{(i_1, \dots, i_{2k+2})}) \right) \dots \right)$ exist;

Then for all $n \in \mathbb{Z}^+$:

$$\mathcal{T}_{usc}^n 0(\mu) \geq \sum_{l=1}^n \limsup_{i_1 \rightarrow +\infty} \left(\lim_{i_2 \rightarrow +\infty} \dots \left(\lim_{i_{2l} \rightarrow +\infty} h(\mu_{(i_1, \dots, i_{2l})}) \right) \dots \right)$$

PROOF : The proof is done by induction on n . Assume the lemma is true for $n - 1$ and let us prove it for n . By the induction hypothesis we have for every μ_{i_1, i_2}

$$\mathcal{T}_{usc}^{n-1} 0(\mu_{i_1, i_2}) \geq \sum_{l=2}^n \lim_{i_3 \rightarrow +\infty} \left(\lim_{i_4 \rightarrow +\infty} \dots \left(\lim_{i_{2l} \rightarrow +\infty} h(\mu_{(i_1, \dots, i_{2l})}) \right) \dots \right)$$

Then

$$\begin{aligned} \mathcal{T}_{usc}^n 0(\gamma_{p_{i_1}}) &\geq \limsup_{i_2} (h(\mu_{i_1, i_2}) + \mathcal{T}_{usc}^{n-1} 0(\mu_{i_1, i_2})) \\ &\geq \lim_{i_2} h(\mu_{i_1, i_2}) + \sum_{l=2}^n \lim_{i_2} \left(\lim_{i_3} \left(\lim_{i_4} \dots \left(\lim_{i_{2l}} h(\mu_{(i_1, \dots, i_{2l})}) \right) \dots \right) \right) \end{aligned}$$

Then by upper semicontinuity we conclude that

$$\begin{aligned} \mathcal{T}_{usc}^n 0(\mu) &\geq \limsup_{i_1} \mathcal{T}_{usc}^n 0(\gamma_{p_{i_1}}) \\ &\geq \sum_{l=1}^n \lim_{i_1} \left(\lim_{i_2} \dots \left(\lim_{i_{2l}} h(\mu_{(i_1, \dots, i_{2l})}) \right) \dots \right) \end{aligned}$$

□

We finally ask three questions :

Question 1 *We mentioned examples of (topological) dynamical systems with $h_{sex} = +\infty$ and $h_{usc} = h$. Do there exist examples of C^r dynamical systems with $1 \leq r < +\infty$ with $h_{sex} \neq h_{usc}$? with $h_{sex} = +\infty$ and $h_{usc} = h$?*

Question 2 *The entropy function of a (topological) dynamical system may be continuous and nonzero. What about C^r dynamical systems with $1 \leq r \leq +\infty$? If T is a C^r dynamical system with $1 \leq r \leq +\infty$, must every continuity point of the entropy function have zero entropy?*

Question 3 *We have seen that the existence of symbolic extensions is stronger than the property of the entropy function to be a difference of nonnegative upper semicontinuous functions. Regarding Conjecture 1 one can ask directly : Is the entropy function of any C^r dynamical system with $r > 1$ a difference of nonnegative upper semicontinuous functions? Does it characterize the entropy function of C^r systems with $r > 1$, i.e. given a Choquet simplex K and a nonnegative function $f : K \rightarrow \mathbb{R}^+$ which is a difference of nonnegative upper semicontinuous functions, does there exist a C^r dynamical system (M, T) with $r > 1$ on a compact manifold M and an affine homeomorphism $a : \mathcal{M}(M, T) \rightarrow K$ such that $f \circ a$ is the entropy function of (M, T) ?*

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