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# Bounds for identifying codes in terms of degree parameters\*

Florent Foucaud<sup>†</sup>

Guillem Perarnau<sup>‡</sup>

May 13, 2011

## Abstract

An identifying code is a subset of vertices of a graph such that each vertex is uniquely determined by its neighbourhood within the identifying code. If  $\gamma^{\text{ID}}(G)$  denotes the minimum size of an identifying code of a graph  $G$ , it was conjectured by F. Foucaud, R. Klasing, A. Kosowski and A. Raspaud that if a connected graph  $G$  has  $n$  vertices and maximum degree  $d$  and admits an identifying code, then  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$ . We use probabilistic tools to show that  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$  holds. For  $d$ -regular graphs and further classes of graphs, this bound is improved to  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^{3/2})}$ . Moreover, if  $G$  is regular and has no vertices sharing the same neighbours, we show that the conjecture holds up to constants, i.e.  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d)}$ . In the second part, we prove that in any graph  $G$  of minimum degree  $\delta$  and girth 5,  $\gamma^{\text{ID}}(G) \leq (1 + o_\delta(1)) \frac{3 \log \delta}{2\delta} n$ . Using the former result, we give sharp estimates for the size of the minimum identifying code of random  $d$ -regular graphs, which is about  $\frac{\log d}{d} n$ .

## 1 Introduction

Given a graph  $G$ , an identifying code  $\mathcal{C}$  is a dominating set such that for any two vertices, their neighbourhoods within  $\mathcal{C}$  are nonempty and distinct. This property can be used to distinguish all vertices of the graph from each other. Identifying codes have found applications to various fields since the introduction of this concept in [18]. These applications include the location of threats in facilities using sensors [23], error-detection schemes [18] and routing [19] in networks, as well as the structural analysis of RNA proteins [17].

In this paper, we address the question of lower and upper bounds on the size of an identifying code, thus extending earlier works on such questions (see e.g. [21, 8, 14, 11, 12]). We focus on degree-related graph parameters such as the minimum and maximum degree, and also study the case of regular graphs. An important part of the paper is devoted to giving the best possible upper bound for the size of an identifying code depending on the order and the maximum degree of the graph, a question raised in [12]. We also give improved bounds for graphs of large girth and study identifying codes in random regular graphs. The main tools used herein are probabilistic.

We begin by giving our notations and defining the concepts used throughout the paper.

As *graphs* and unless specifically mentioned, we understand simple, undirected and finite graphs. The vertex set of a graph  $G$  is denoted by  $V(G)$  and its edge set  $E(G)$ . We also denote its order by  $n = |V(G)|$ . The maximum degree of  $G$  will be denoted by  $d = d(G)$ , its minimum degree, by  $\delta = \delta(G)$ , and its average degree, by  $\bar{d} = \bar{d}(G)$ . We denote by  $u \sim v$ , the adjacency between two vertices  $u$  and  $v$ , and by  $u \not\sim v$ , their non-adjacency. The set of neighbours of some vertex  $v$  is called its *open neighbourhood* and denoted by  $N(v)$ , whereas the set of its neighbours and itself is called its *closed neighbourhood* and denoted by  $N[v]$ . If two distinct vertices  $u, v$  are such that  $N[u] = N[v]$ , they are called *twins*. If  $N(u) = N(v)$  but  $u \not\sim v$ ,  $u$  and  $v$  are called *false twins*. The symmetric difference between two sets  $A$  and  $B$  is denoted by  $A\Delta B$ . We use  $\log(x)$  to denote the natural logarithm of  $x$ . We also make use of the standard asymptotic notations  $o, O, \Theta, \Omega$  and  $\omega$ . Usually the asymptotics are taken either on variables  $d, \delta$  or  $n$ . If we write  $o(1)$  or  $O(1)$ , then by convention the asymptotic is taken only on  $n$ , the number of vertices.

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<sup>†</sup>LaBRI - Université de Bordeaux, 351 cours de la Libération, 33405 Talence cedex, France.

<sup>‡</sup>MA4 - Universitat Politècnica de Catalunya, C/ Jordi Girona 1-3, 08034 Barcelona, Spain.

Given a graph  $G$  and a subset  $\mathcal{C}$  of vertices of  $G$ ,  $\mathcal{C}$  is called a *dominating set* if each vertex of  $V(G) \setminus \mathcal{C}$  has at least one neighbour in  $\mathcal{C}$ . Set  $\mathcal{C}$  is called a *separating set* of  $G$  if for each pair  $u, v$  of vertices of  $G$ ,  $N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C}$  (equivalently,  $(N[u] \Delta N[v]) \cap \mathcal{C} \neq \emptyset$ ). We have the following definition:

**Definition.** *Given a graph  $G$ , a subset of vertices of  $V(G)$  which is both a dominating set and a separating set is called an identifying code of  $G$ .*

First of all it must be stressed that not every graph can have an identifying code. Observe that a graph containing twin vertices does not admit any separating set or identifying code. In fact a graph admits an identifying code if and only if it is *twin-free*, i.e. it has no pair of twins (one can see that if  $G$  is twin-free,  $V(G)$  is an identifying code of  $G$ ). Note that if for three distinct vertices  $u, v, w$  of a twin-free graph  $G$ ,  $N[u] \Delta N[v] = \{w\}$ , then  $w$  belongs to any identifying code of  $G$ . In this case we say that  $w$  is *uv-forced*, or simply *forced*. Observe that any isolated vertex must belong to any identifying code for the reason that it must be dominated. For example, an edgeless graph needs all the vertices in any identifying code. Hence, the bounds of this paper only hold for graphs with few isolated vertices. In order to shorten the statements of our results, we assume that all considered graphs do not have any isolated vertices.

For a given graph, the problem of finding a minimum identifying code is known to be NP-hard, even in graphs having small maximum degree and high girth (to be precise, in planar graphs of maximum degree 4 having arbitrarily large girth [2] and planar graphs of maximum degree 3 and girth at least 9 [3]).

The minimum size of an identifying code in a graph  $G$ , denoted  $\gamma^{\text{ID}}(G)$ , is the *identifying code number* of  $G$ . It is known that for a twin-free graph  $G$  on  $n$  vertices having at least one edge, we have:

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

The lower bound is proved in [18] and the upper bound, in [14]. Both bounds are tight and all graphs reaching these two bounds have been classified (see [21] for the lower bound and [11] for the upper bound).

When considering graphs of given maximum degree  $d$ , it was shown in [18] that the lower bound can be improved to  $\gamma^{\text{ID}}(G) \geq \frac{2n}{d+2}$ . This bound is tight and a classification of all graphs reaching it has been proposed in [10]. For any  $d$ , these graphs include some regular graphs and graphs of arbitrarily large girth.

It was conjectured in [12] that the following upper bound holds.

**Conjecture 1** ([12]). *Let  $G$  be a nontrivial connected twin-free graph of maximum degree  $d$ . Then  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$ .*

Graphs of maximum degree  $d$  such that  $\gamma^{\text{ID}}(G) = n - \frac{n}{d}$  are known (e.g. the complete bipartite graph  $K_{d,d}$  and richer classes of graphs described in Section 7). Therefore if Conjecture 1 holds, for any graph  $G$  on  $n$  vertices and of maximum degree  $d$  we would have  $\frac{2}{d+2}n \leq \gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + O(1)$ , with both bounds being tight.

Note that Conjecture 1 holds for graphs of maximum degree 2 (see [15]). It was shown in [11] that  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^5)}$ , and  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$  when  $G$  has no forced vertices (in particular, this is true when  $G$  is regular). It is also known that the conjecture holds (up to constants) if  $G$  is triangle-free: then,  $\gamma^{\text{ID}}(G) \leq n - \frac{n}{3(d+1)}$  [12].

Identifying codes have been previously studied in two models of random graphs, that is the classic random graph model [13] and the model of random geometric graphs [22]. To our knowledge random regular graphs have not been studied in the context of identifying codes.

We summarize our results for regular graphs in Table 1 and compare them to the bound for the dominating set problem (the table contains references for both the bound and its tightness). All bounds are asymptotically tight except for the one of the first entry of the table, which is related to Conjecture 1. We note that identifying codes behave far from dominating sets in general, as shown by the first lines of the table: there are regular graphs having much larger identifying code number than domination number. However, for larger girth and for almost all regular graphs, the bounds for the two problems coincide asymptotically, as shown by the last lines of the table.

In order to prove our results, we use probabilistic techniques. For some results, we use the weighted version of Lovász' Local Lemma to show the existence of an identifying code, together with the Chernoff bound to show that this code is small enough. We also make use of other probabilistic techniques such as the Alteration Method [1] in order to give better bounds in more restricted cases. Finally, we work

	Identifying codes	Dominating sets
in general	$n - \frac{n}{\Theta(d^{3/2})}$ Thm. 10 (Conj. 1: $n - \frac{n}{\Theta(d)}$ )	$\sim \frac{\log d}{d}n$ [1], [24]
no false twins	$n - \frac{n}{61d}$ Thm. 13, Constr. 21	$\sim \frac{\log d}{d}n$ [1], [24]
girth 4	$n - \frac{n}{3d+3}$ [12], Constr. 22	$\sim \frac{\log d}{d}n$ [1], [24]
girth 5	$(1 + o_d(1))\frac{3 \log d}{2d}n$ Thm. 15, Thm. 17	$\sim \frac{\log d}{d}n$ [1], [24]
almost all graphs	$\frac{\log d + \log \log d + O_d(1)}{d}n$ Thm. 16, Thm. 17	$\sim \frac{\log d}{d}n$ [1], [24]

Table 1: Summary of the upper bounds for  $d$ -regular graphs

with the Configuration Model [7] in order to compute the identifying code number of almost all random regular graphs.

The organization of this paper is as follows. In Section 2 we state some preliminary results which will be used throughout the paper. In Section 3, we give new upper bounds on the identifying code number of graphs of maximum degree  $d$ . In Section 4, we improve this result for  $d$ -regular graphs having no false twins. This gives a new family of graphs for which Conjecture 1 holds up to constants. In Section 5, we give a new upper bound for graphs having minimum degree  $\delta$  and girth at least 5. Finally, we extend the latter result to random regular graphs. In Section 6 we give sharp bounds for the identifying code number of almost all  $d$ -regular graphs. A further section is dedicated to various constructions of families of graphs which show the tightness of some of our results (Section 7).

## 2 Preliminary results

We first recall a well-known probabilistic tool: the Lovász Local Lemma. We use its weighted version, a particularization of the general version where each event has an assigned weight.

**Lemma 2** (Weighted Local Lemma [20]). *Let  $\mathcal{E} = \{E_1, \dots, E_M\}$  be a set of (typically “bad”) events such that each  $E_i$  is mutually independent of  $\mathcal{E} \setminus (\mathcal{D}_i \cup \{E_i\})$  where  $\mathcal{D}_i \subseteq \mathcal{E}$ . Suppose that there exist some integer weights  $t_1, \dots, t_M \geq 1$  and a real  $p \leq \frac{1}{4}$  such that for each  $1 \leq i \leq M$ :*

- $\Pr(E_i) \leq p^{t_i}$ , and
- $\sum_{E_j \in \mathcal{D}_i} (2p)^{t_j} \leq \frac{t_i}{2}$

Then  $\Pr(\bigcap_{i=1}^M \overline{E_i}) \geq \prod_{i=1}^M (1 - (2p)^{t_i}) > 0$ .

Note that in the previous Lemma, since  $p \leq \frac{1}{4}$ , we have

$$\Pr(\bigcap_{i=1}^M \overline{E_i}) \geq \exp \left\{ -2 \log 2 \sum_{i=1}^M (2p)^{t_i} \right\}. \quad (1)$$

We also use the following version of the well-known Chernoff bound, which is a reformulation of Theorem A.1.13 in [1].

**Theorem 3** (Chernoff bound). *Let  $X$  be a random variable of  $n$  independent trials of probability  $p$ , and let  $a > 0$  be a real number. Then  $\Pr(X - np \leq -a) \leq e^{-\frac{a^2}{2np}}$ .*

The following observation gives an equivalent condition for a set to be an identifying code, and follows from the fact that for two vertices  $u, v$  at distance at least 3 from each other,  $N[u] \Delta N[v] = N[u] \cup N[v]$ .

**Observation 4.** *For a graph  $G$  and a set  $\mathcal{C} \subseteq V(G)$ , if  $\mathcal{C}$  is dominating and  $N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C}$  for each pair of vertices  $u, v$  at distance at most two from each other, then  $N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C}$  for each pair of vertices of the graph.*

The next observation is immediate, but it is worth mentioning here.

**Observation 5.** *Let  $G$  be a twin-free graph and  $\mathcal{C}$ , an identifying code of  $G$ . Any set  $\mathcal{C}'$  such that  $\mathcal{C} \subseteq \mathcal{C}'$  is also an identifying code of  $G$ .*

The following lemma was first proved in [4], and a proof can be found in [11] (as [4] is not accessible).

**Lemma 6** ([4]). *If  $G$  is a finite twin-free graph without isolated vertices, then for every vertex  $u$  of  $G$ , there is a vertex  $v \in N[u]$  such that  $V(G) \setminus \{v\}$  is an identifying code of  $G$ .*

We obtain the following lower bound as a corollary of Lemma 6, which is crucial in some steps of our proofs.

**Proposition 7.** *Let  $G$  be a twin-free graph on  $n$  vertices and of maximum degree  $d$ . Then there are at least  $\frac{n}{d+1}$  non-forced vertices in  $G$ .*

*Proof.* Observe that a vertex  $v$  of  $G$  is not forced only if  $V(G) \setminus \{v\}$  is an identifying code of  $G$ . Hence, by Lemma 6, the set  $S$  of non-forced vertices is a dominating set of  $G$ . Hence  $|S| \geq \frac{n}{d+1}$ .  $\square$

Note that Proposition 7 is tight. Indeed, consider the graph  $A_k$  on  $2k$  vertices defined in [11] as follows:  $V(A_k) = \{x_1, \dots, x_{2k}\}$  and  $E(A_k) = \{x_i x_j, |i - j| \leq k - 1\}$ .  $A_k$  can be seen as the  $(k - 1)$ -th power of the path  $P_{2k}$ . In the graph  $A_k$  with an additional universal vertex  $x$  (i.e.  $x$  is adjacent to all vertices of  $A_k$ ), one can check that all vertices but  $x$  are forced. This graph has  $n = 2k + 1$  vertices, maximum degree  $2k$  and exactly  $1 = \frac{n}{d+1}$  non-forced vertex. Taking all forced vertices gives a minimum identifying code of this graph.

However, note that since for a fixed value of  $d$ , we know only one such graph, it is not enough to give a counterexample to Conjecture 1. Indeed in this case  $n - \frac{n}{d+1} = n - \frac{n}{d} + O(1)$ . So we ask the following question:

**Question 8.** *For a fixed value of  $d$  and arbitrarily large values of  $n$ , do there exist graphs of maximum degree  $d$  on  $n$  vertices having exactly  $\frac{n}{d+1}$  non-forced vertices?*

Answering this question in positive would provide counterexamples to Conjecture 1. Note that for the similar question where we replace  $d + 1$  by  $d$ , the answer is positive by Construction 20 of Section 7. For any  $d$ , this construction provides arbitrarily large graphs having  $\frac{n}{d}$  non-forced vertices.

Finally, the next proposition shows an upper bound on the number of false twins in a graph.

**Proposition 9.** *Let  $G$  be a graph on  $n$  vertices having maximum degree  $d$  and no isolated vertices, then  $G$  has at most  $\frac{n(d-1)}{2}$  pairs of false twins.*

*Proof.* Let us build a graph  $H$  on  $V(G)$ , where two vertices  $u, v$  are adjacent in  $H$  if they are false twins in  $G$ . Note that since a vertex can have at most  $d - 1$  false twins,  $H$  has maximum degree  $d - 1$ . Therefore it has at most  $\frac{n(d-1)}{2}$  edges and the claim follows.  $\square$

Note that the bound of Lemma 9 is tight since in a complete bipartite graph  $K_{d,d}$ ,  $n = 2d$  and there are exactly  $2\binom{d}{2} = \frac{n(d-1)}{2}$  pairs of false twins.

### 3 Upper bounds for the general case

In this section, we improve the known upper bounds of [11] on the identifying code number by using the Weighted Local Lemma, stated in Lemma 2.

**Theorem 10.** *There exists an integer  $d_0$  such that for each twin-free graph  $G$  on  $n$  vertices having maximum degree  $d \geq d_0$ , the following holds:*

$$\gamma^{ID}(G) \leq \begin{cases} n - \frac{nf(G)}{9d^{3/2}} & \text{if } f(G) \geq 13.5d^{-1/2} \\ n - \frac{nf(G)^2}{122d} & \text{if } f(G) < 13.5d^{-1/2} \end{cases}$$

where  $f(G)$  denotes the proportion of non-forced vertices of  $G$ .

*Proof.* Let  $F$  be the set of forced vertices of  $G$ , and  $V' = V(G) \setminus F$ . Note that  $|V'| = nf(G)$ . By the definition of a forced vertex, any identifying code must contain all vertices of  $F$ .

In this proof, we first build a set  $S$  in a random manner by choosing vertices from  $V'$ . Then we exhibit some “bad” configurations - if none of those occur, the set  $F \cup (V(G) \setminus S)$  is an identifying code of  $G$ . Using the Weighted Local Lemma, we compute a lower bound on the (non-zero) probability that this holds. Finally, we use the Chernoff bound to show that with non-zero probability, the size of  $S$  is also large enough for our purposes. This shows that such a “good” large set  $S$  exists, and it can be used to build an identifying code that has a sufficiently small size.

Let  $p = p(d)$  be a probability which will be determined later. We build the set  $S \subseteq V'$  such that each vertex of  $V'$  independently belongs to  $S$  with probability  $p$ . Therefore the random variable  $|S|$  follows a binomial distribution  $\text{Bin}(nf(G), p)$  and has expected value  $\mathbb{E}(|S|) = pnf(G)$ .

Let us now define the set  $\mathcal{E}$  of “bad” events. These are of four types. An illustration of these events is given in Figure 1.

- **Type A:** for each vertex  $u \in V'$ , let  $A_u$  be the event that  $N[u] \subseteq S$ .
- **Type B:** for each pair  $\{u, v\}$  of adjacent vertices of  $V(G)$ , let  $B_{u,v}$  be the event that  $(N[u] \Delta N[v]) \subseteq S$ .
- **Type C:** for each pair  $\{u, v\} \in V'$  of vertices at distance two from each other such that  $u$  and  $v$  are not false twins, let  $C_{u,v}$  be the event that  $(N[u] \Delta N[v]) \subseteq S$ .
- **Type D:** for each pair  $\{u, v\} \in V'$  of false twins, let  $D_{u,v}$  be the event that  $(N[u] \Delta N[v]) = \{u, v\} \subseteq S$ .

All the events of type  $B_{u,v}$  where  $|N[u] \Delta N[v]| = 1$  do not happen because  $F$  belongs to the code. Observe that the events  $C_{u,v}$  and  $D_{u,v}$  are just defined over the pairs of vertices in  $V'$  because if either  $u$  or  $v$  belongs to  $F$ , the event does not happen.

If no event of type  $A$  occurs,  $V(G) \setminus S$  is a dominating set of  $G$ . If no event of type  $B$  occurs, all pairs of adjacent vertices are separated by  $V(G) \setminus S$ . If no event of type  $C$  or  $D$  occurs, all pairs of vertices at distance 2 from each other are separated. Thus by Observation 4,  $V(G) \setminus S$  is also a separating set of  $G$ , and therefore it is an identifying code of  $G$ .

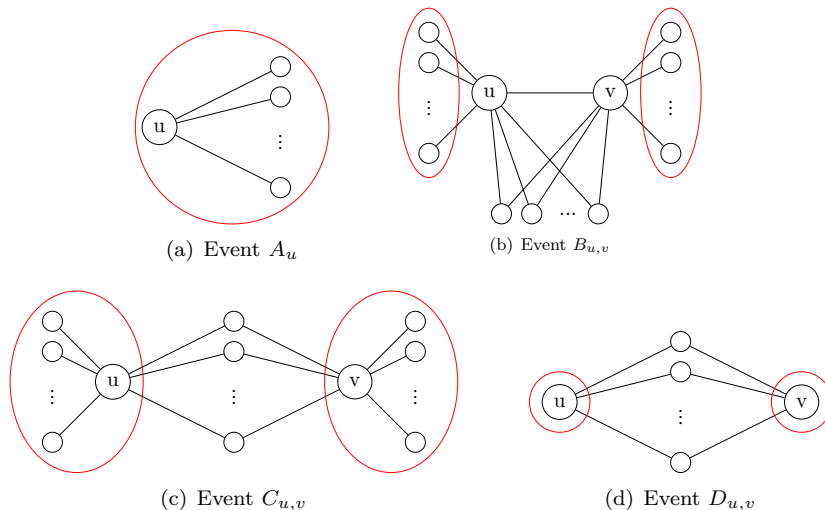


Figure 1: The “bad” events. The circled vertices belong to set  $S$ .

We define the weight  $t_i$  of each event  $E_i \in \mathcal{E}$  as the number of vertices participating in it (i.e. the number of vertices that belong to set  $S$  when  $E_i$  holds, as defined previously). Note that since  $G$  has maximum degree  $d$  and no isolated vertices, for each vertex  $u$ , we have  $2 \leq |N[u]| \leq d + 1$ . An event of type  $B$  has weight at least 2, otherwise there would be a unique vertex belonging to this event. This vertex would be  $uv$ -forced and would belong to  $F$ , and therefore not to  $S$ . Thus, denoting  $t_T^{\min}$  and  $t_T^{\max}$  the minimum and maximum weights of an event of type  $T$  ( $T \in \{A, B, C, D\}$ ) we obtain the following bounds:

$$\begin{aligned}
t_A^{min} &= 2 \leq t_i \leq d+1 = t_A^{max} && \text{if } E_i \text{ is of type } A \\
t_B^{min} &= 2 \leq t_i \leq 2d-2 = t_B^{max} && \text{if } E_i \text{ is of type } B \\
t_C^{min} &= 3 \leq t_i \leq 2d = t_C^{max} && \text{if } E_i \text{ is of type } C \\
t_D^{min} &= 2 = t_i = 2 = t_D^{max} && \text{if } E_i \text{ is of type } D
\end{aligned}$$

Some vertex  $x$  can intersect at most  $d+1$  events  $A_u$  since  $u \in N[x]$ . Vertex  $x$  can intersect at most  $d(d-1)$  events  $B_{u,v}$ : supposing  $u$  is the vertex adjacent to  $x$ , there are  $d$  ways to choose  $u$ , and  $d-1$  ways to choose  $v$  among  $N(u) \setminus x$ . Similarly  $x$  can intersect at most  $d^2(d-1)$  events  $C_{u,v}$ :  $d(d-1)$  possibilities if  $x = u$  or  $x = v$  and  $d(d-1)^2$  if  $u$  or  $v$  is a neighbour of  $x$ . Finally,  $x$  can intersect at most  $d-1$  events  $D_{u,v}$  since  $x$  can have at most  $d-1$  false twins. For each type  $T$  of events ( $T \in \{A, B, C, D\}$ ), let us define  $int(v, T)$  to be the number of events of type  $T$  containing a given vertex  $v$  of  $G$ . Hence:

$$int(v, A) \leq d+1 \quad int(v, B) \leq d(d-1) \quad int(v, C) \leq d^2(d-1) \quad int(v, D) \leq d-1$$

Let us call  $E_{IC}$  the event that no event of  $\mathcal{E}$  occurs. Using the Weighted Local Lemma, we want to show that  $\Pr(E_{IC}) > 0$ . For two events  $E_i$  and  $E_j$  of  $\mathcal{E}$ , we note  $i \sim j$  if  $E_i$  and  $E_j$  are not mutually independent. In order to apply the Weighted Local Lemma (Lemma 2), the following conditions must hold for each event  $E_i \in \mathcal{E}$ :

$$\sum_{i \sim j} (2p)^{t_j} \leq \frac{t_i}{2} \quad (2)$$

The latter conditions are implied by:

$$t_T^{max} int(A)(2p)^{t_A^{min}} + t_T^{max} int(B)(2p)^{t_B^{min}} + t_T^{max} int(C)(2p)^{t_C^{min}} + t_T^{max} int(D)(2p)^{t_D^{min}} \leq \frac{t_T^{min}}{2}$$

where  $T \in \{A, B, C, D\}$  and where  $int(T)$  denotes the maximum value of  $int(v, T)$  over all vertices  $v$  of  $G$ .

This leads to the following inequalities:

$$(d+1)^2(2p)^2 + (d+1)d(d-1)(2p)^2 + (d+1)d^2(d-1)(2p)^3 + (d+1)(d-1)(2p)^2 \leq 1 \quad (3)$$

$$2(d+1)(d-1)(2p)^2 + 2d(d-1)^2(2p)^2 + 2(d-1)^2d^2(2p)^3 + 2(d-1)^2(2p)^2 \leq 1 \quad (4)$$

$$2d(d+1)(2p)^2 + 2d^2(d-1)(2p)^2 + 2d^3(d-1)(2p)^3 + 2d(d-1)(2p)^2 \leq \frac{3}{2} \quad (5)$$

$$2(d+1)(2p)^2 + 2d(d-1)(2p)^2 + 2d^2(d-1)(2p)^3 + 2(d-1)(2p)^2 \leq 1 \quad (6)$$

Intuitively, we need to set  $p = O(d^{-3/2})$  in order to solve these inequalities. In this case, for some sufficiently large  $d$ , observe that Inequality (4) implies Inequalities (3), (5) and (6). Moreover, then, Inequality (4) is implied by the following one:

$$9d^3p^2 \leq 1 \quad (7)$$

Hence, we fix  $p = \frac{1}{k(d)d^{3/2}}$ . This implies  $k(d) \geq 3$ . Moreover, since we are dealing with large values of  $d$ , we have  $p \leq \frac{1}{4}$ . Under this condition the Weighted Local Lemma can be applied.

Let  $M_T$  be the number of events of type  $T$ , where  $T \in \{A, B, C, D\}$ . We have:

$$\Pr(E_{IC}) \geq \prod_{i=1}^{M_A} (1 - (2p)^{t_A^{min}}) \prod_{i=1}^{M_B} (1 - (2p)^{t_B^{min}}) \prod_{i=1}^{M_C} (1 - (2p)^{t_C^{min}}) \prod_{i=1}^{M_D} (1 - (2p)^{t_D^{min}})$$

Note that  $M_A \leq nf(G)$  since by definition there is one event of type  $A$  for each vertex of  $V'$ . Moreover,  $M_B \leq \frac{nd}{2}$  since there is at most one event  $B$  for each edge in  $G$ . We also have that  $M_C$  is at most the number of pairs of vertices in  $V'$  at distance 2 from each other. This is also at most the number of paths of length 2 with both endpoints in  $V'$ , which is upper-bounded by  $\frac{nf(G)d(d-1)}{2}$ . Finally,  $M_D$  is at most the number of pairs of false twins in  $V'$ , i.e.  $nf(G)\frac{d-1}{2}$  by Proposition 9. Hence, we have

$$\Pr(E_{IC}) \geq (1 - (2p)^2)^{nf(G)} (1 - (2p)^2)^{\frac{nd}{2}} (1 - (2p)^3)^{\frac{nf(G)d(d-1)}{2}} (1 - (2p)^2)^{\frac{nf(G)(d-1)}{2}}$$

Using Lemma 2 (more precisely, we use Equation (1)) and the fact that  $p = \frac{1}{k(d)d^{3/2}}$

$$\begin{aligned} \Pr(E_{\text{IC}}) &\geq \exp \left\{ -2 \log 2 \left( f(G)(2p)^2 + \frac{d}{2}(2p)^2 + \frac{f(G)d(d-1)}{2}(2p)^3 + \frac{f(G)(d-1)}{2}(2p)^2 \right) n \right\} \\ &\geq \exp \left\{ -\frac{4 \log 2}{k(d)^2 d^2} \left( \frac{2f(G)}{d} + 1 + \frac{2f(G)}{k(d)d^{1/2}} + f(G) \right) n \right\} \end{aligned}$$

Since  $f(G) \leq 1$ , one can check that for sufficiently large  $d$ :

$$\Pr(E_{\text{IC}}) \geq \exp \left\{ -\frac{9 \log 2}{k(d)^2 d^2} n \right\}$$

The Weighted Local Lemma shows that  $S$  has the desired properties with probability  $\Pr(E_{\text{IC}}) > 0$ , implying that such a set exists. Note that we have no guarantee on the size of  $S$ . In fact, if  $S = \emptyset$  then  $V(G) \setminus S = V(G)$  is always an identifying code. Therefore we need to estimate the probability that  $|S|$  is far below its expected size. In order to do this, we use the Chernoff bound of Theorem 3 by putting  $a = \frac{nf(G)}{c(d)}$  where  $c(d)$  can be an arbitrary positive function of  $d$ . Let  $E_{\text{BIG}}$  be the event that  $|S| - np > -\frac{nf(G)}{c(d)}$ . We obtain:

$$\begin{aligned} \Pr(\overline{E_{\text{BIG}}}) &\leq \exp \left\{ -\frac{\left( \frac{nf(G)}{c(d)} \right)^2}{2pnf(G)} \right\} \\ &= \exp \left\{ -\frac{k(d)f(G)d^{3/2}}{2c(d)^2} n \right\} \end{aligned}$$

Now we have

$$\begin{aligned} \Pr(E_{\text{IC}} \text{ and } E_{\text{BIG}}) &= 1 - \Pr(\overline{E_{\text{IC}}} \text{ or } \overline{E_{\text{BIG}}}) \\ &\geq 1 - \Pr(\overline{E_{\text{IC}}}) - \Pr(\overline{E_{\text{BIG}}}) \\ &= 1 - (1 - \Pr(E_{\text{IC}})) - \Pr(\overline{E_{\text{BIG}}}) \\ &= \Pr(E_{\text{IC}}) - \Pr(\overline{E_{\text{BIG}}}) \\ &\geq \exp \left\{ -\frac{9 \log 2}{k(d)^2 d^2} n \right\} - \exp \left\{ -\frac{k(d)f(G)d^{3/2}}{2c(d)^2} n \right\} \end{aligned}$$

Thus,  $\Pr(E_{\text{IC}} \text{ and } E_{\text{BIG}}) > 0$  if  $c(d) < \frac{k(d)^{3/2} d^{7/4} f(G)^{1/2}}{\sqrt{18 \log 2}}$ . We arbitrarily set  $c(d) = \frac{k(d)^{3/2} d^{7/4} f(G)^{1/2}}{\sqrt{18}}$  in order to simplify the calculations.

Now we have to check that  $E_{\text{BIG}}$  implies that  $S$  is still large enough.

$$\begin{aligned} |S| &\geq \mathbb{E}(|S|) - \frac{nf(G)}{c(d)} \\ &= \frac{nf(G)}{k(d)d^{3/2}} - \frac{nf(G)}{c(d)} \\ &= \left( \frac{1}{k(d)} - \frac{\sqrt{18}}{k(d)^{3/2} d^{1/4} f(G)^{1/2}} \right) \frac{nf(G)}{d^{3/2}} \end{aligned} \tag{8}$$

Let us now distinguish between two cases depending on the behaviour of the function  $f(G)$ .

**Case 1.**  $f(G) = \omega(d^{-1/2})$ .

Plugging this into Equation (8) together with the fact that  $k(d) \geq 3$  (see Equation (7)) and  $d$  is sufficiently large, we obtain:

$$|S| \geq \left( 1 - \frac{\sqrt{18}}{k(d)^{1/2} d^{1/4} f(G)^{1/2}} \right) \frac{nf(G)}{3d^{3/2}} = (1 + o_d(1)) \frac{nf(G)}{3d^{3/2}} \geq \frac{nf(G)}{9d^{3/2}}$$



**Case 2.**  $f(G) = O(d^{-1/2})$ .

Since  $|S|$  must be positive, from Equation (8) we need  $k(d)^{3/2}d^{1/4}f(G)^{1/2} > \sqrt{18}k(d)$ , which leads to  $k(d) = \frac{c_0}{f(G)\sqrt{d}}$  for  $c_0 > 18$ .<sup>1</sup> The size of  $S$  is optimized when  $c_0 = \frac{81}{2}$ .

The only constraints on  $k(d)$  are the one of the previous paragraph and the condition  $k(d) \geq 3$  from Equation (7). Hence we can choose  $k(d) = \max\{3, \frac{81}{2\sqrt{d}f(G)}\}$ . Expressing it as a piecewise function, we obtain:

$$k(d) = \begin{cases} 3 & \text{if } f(G) \geq 13.5d^{-1/2} \\ \frac{81}{2\sqrt{d}f(G)} & \text{if } f(G) < 13.5d^{-1/2} \end{cases}$$

Note that for large enough  $d$ ,  $k(d) \geq 3$ .

Now we can see that:

$$|S| \geq \left( \frac{1}{k(d)} - \frac{1}{c(d)} \right) \frac{nf(G)}{d^{3/2}} \geq \begin{cases} \frac{f(G)}{9d^{3/2}}n & \text{if } f(G) \geq 13.5d^{-1/2} \\ \frac{f(G)^2}{122d}n & \text{if } f(G) < 13.5d^{-1/2} \end{cases}$$

Finally, considering Case 1 and Case 2, we obtain an identifying code  $\mathcal{C} = V(G) \setminus S$  such that:

$$|\mathcal{C}| \leq \begin{cases} n - \frac{nf(G)}{9d^{3/2}} & \text{if } f(G) \geq 13.5d^{-1/2} \\ n - \frac{nf(G)^2}{122d} & \text{if } f(G) < 13.5d^{-1/2} \end{cases}$$

□

Note that for regular graphs,  $f(G) = 1$  because a forced vertex implies the existence of two vertices with distinct degrees. We obtain the following result:

**Corollary 11** (Graphs with constant proportion of non-forced vertices). *There exists an integer  $d_0$  such that for each twin-free graph  $G$  on  $n$  vertices having maximum degree  $d \geq d_0$  and  $f(G) = \frac{1}{\alpha}$  for some constant  $\alpha \geq 1$ ,  $\gamma^{ID}(G) \leq n - \frac{n}{9\alpha d^{3/2}}$ . In particular if  $G$  is  $d$ -regular,  $\gamma^{ID}(G) \leq n - \frac{n}{9d^{3/2}}$ .*

By Proposition 7 we know that  $f(G) \geq \frac{1}{d+1}$ . Thus we obtain the following general result.

**Corollary 12** (General case). *There exists an integer  $d_0$  such that for each twin-free graph  $G$  on  $n$  vertices having maximum degree  $d \geq d_0$ ,  $\gamma^{ID}(G) \leq n - \frac{n}{122d(d+1)^2} = n - \frac{n}{\Theta(d^3)}$ .*

## 4 A tight upper bound for regular graphs without false twins

In this section we show that Conjecture 1 asymptotically holds for the class of regular graphs without false twins. Moreover, as shown by Construction 21 of Section 7, for any  $d$  there are arbitrarily large graphs from this class that need  $n - \frac{n}{d}$  vertices in any identifying code. Therefore this result is asymptotically tight for this class. Moreover this class of graphs is interesting as it is still very large, and is a natural extension of the class of twin-free graphs.

We also remark that the technique used in this section with non-regular graphs, would give a weaker bound than the ones of the previous section.

**Theorem 13.** *There exists an integer  $d_0$  such that for each  $d \geq d_0$  and for each twin-free  $d$ -regular graph  $G$  on  $n$  vertices having no false twins,  $\gamma^{ID}(G) \leq n - \frac{n}{61d}$ .*

*Proof.* The proof for this theorem is based on the same ideas as the one of Theorem 10. Observe that since  $G$  is regular, there are no forced vertices in  $G$ . Thus, it is not necessary to add them beforehand as we did in the proof of Theorem 10. Let  $\mathcal{E} = \{E_1, \dots, E_M\}$  be the set of “bad events”. Since  $G$  has no false twins, there appear no events of type  $D$ . In this proof, in order to be more accurate, we split the events of type  $B$  and  $C$  according to their weight. Hence, our types of events are:

- **Type A:** for each vertex  $u$ , let  $A_u$  be the event that  $N[u] \subseteq S$ .
- **Type B<sup>j</sup>** ( $2 \leq j \leq 2d - 2$ ): for each pair  $\{u, v\}$  of adjacent vertices, let  $B_{u,v}^j$  be the event that  $|(N[u] \Delta N[v])| = j$  and  $(N[u] \Delta N[v]) \subseteq S$ .

<sup>1</sup>Note that if  $f(G) = \omega(d^{-1/2})$ , we obtain  $k(d) = o_d(1)$ . In that case the conditions for applying the Weighted Local Lemma (see Equation (7)) are not satisfied, so this approach cannot be used in Case 1.

- **Type  $C^j$**  ( $4 \leq j \leq 2d$ ): for each pair  $\{u, v\} \in V(G)$  of vertices at distance two from each other, let  $C_{u,v}^j$  be the event that  $|(N[u]\Delta N[v])| = j$  and  $(N[u]\Delta N[v]) \subseteq S$ .

Observe that since  $G$  is regular, the events  $B^j$  and  $C^j$  are just defined for even  $j$ . For  $j \geq 2$  and for  $T \in \{A, B^j, C^j\}$ , let  $t_T$  be the weight of an event of type  $T$  (for an event  $E_i \in \mathcal{E}$  of type  $T$ ,  $t_i = t_T$ ). We have the following:

$$t_A = d + 1 \quad t_{B^j} = j \quad t_{C^j} = j$$

As in the proof of Theorem 10, we define  $\text{int}(v, T)$  to be the number of events of type  $T$  containing a given vertex  $v$  of  $G$ . By the same arguments as in the proof of Theorem 10, we have:

$$\text{int}(v, A) = d + 1 \quad \sum_{j=2}^{2d-2} \text{int}(v, B^j) \leq d(d-1) \quad \sum_{j=4}^{2d} \text{int}(v, C^j) \leq d^2(d-1) \quad (9)$$

The conditions for the different types of events given by Equation (2) is now implied by the following set of equations (where  $E_i \in \mathcal{E}$ ):

$$\sum_{v \in E_i} \text{int}(v, A)(2p)^{t_A} + \sum_{j=2}^{2d-2} \sum_{v \in E_i} \text{int}(v, B^j)(2p)^{t_{B^j}} + \sum_{j=4}^{2d} \sum_{v \in E_i} \text{int}(v, C^j)(2p)^{t_{C^j}} \leq \frac{t_i}{2}$$

Using the bounds of (9) and noting that for any  $j$ ,  $(2p)^{t_{B^j}} \leq (2p)^2$  and  $(2p)^{t_{C^j}} \leq (2p)^4$ , these equations are implied by:

$$(d+1)(2p)^{d+1} + d(d-1)(2p)^2 + d^2(d-1)(2p)^4 \leq \frac{1}{2} \quad (10)$$

Note that in order to satisfy the equation, we need  $p = O\left(\frac{1}{d}\right)$ . Then for sufficiently large  $d$ , it is implied by:

$$9d^2p^2 \leq 1$$

Taking  $p = \frac{1}{kd}$  for some  $k \geq 3$ , the previous equation is satisfied.

As in the proof of Theorem 10,  $E_{\text{IC}}$  denotes the event that  $V(G) \setminus S$  is a valid identifying code (i.e. no “bad” event occurs), and  $E_{\text{BIG}}$  is the event that the size of  $S$  is not too small, i.e.  $|S| - pn > -\frac{n}{cd}$  for some constant  $c$ .

The Weighted Local Lemma and Equation (1) give the following lower bound:

$$\Pr(E_{\text{IC}}) \geq \exp \left\{ -2 \log 2 \left( (2p)^{d+1} + \frac{d}{2}(2p)^2 + \frac{d(d-1)}{2}(2p)^4 \right) n \right\} \geq \exp \left\{ -\frac{9 \log 2}{2k^2d} n \right\}$$

Whereas the Chernoff bound gives:

$$\Pr(\overline{E_{\text{BIG}}}) \leq \exp \left\{ -\frac{k}{2c^2d} n \right\}$$

Now, using the same arguments as in the proof of Theorem 10, this leads to  $c < \frac{k^{3/2}}{3\sqrt{\log 2}}$ . For example we assume  $c = \frac{k^{3/2}}{3}$ . The size of the code is optimized when  $k = \frac{81}{4}$ . Hence we obtain  $\gamma^{\text{ID}}(G) \leq |V(G) \setminus S| \leq n - \left( \frac{n}{kd} - \frac{n}{cd} \right) \leq n - \frac{n}{61d}$ .  $\square$

## 5 Upper bounds for graphs with girth at least 5

This section is devoted to the study of graphs that have girth at least 5. We will use these results in Section 6, which deals with random regular graphs.

Despite being different than our previous proofs, the ones of this section have also a probabilistic flavour. One can check that for graphs of girth 5, applying the Local Lemma does not lead to a satisfying result. However, by using the Alteration method, a better bound can be given.

We start by defining an auxiliary notion that will be used in this section. A subset  $D \subseteq V(G)$  is called a *2-dominating set* if for each vertex  $v$  of  $V(G) \setminus D$ ,  $|N(v) \cap D| \geq 2$  [9]. The next lemma shows that we can use a 2-dominating set to construct an identifying code.

**Lemma 14.** *Let  $G$  be a twin-free graph on  $n$  vertices having girth at least 5. Let  $D$  be a 2-dominating set of  $G$ . If the subgraph induced by  $D$ ,  $G[D]$ , has no isolated edge,  $D$  is an identifying code of  $G$ .*

*Proof.* First observe that  $D$  is dominating since it is 2-dominating. Let us check that  $D$  is also separating.

Note that all the vertices that do not belong to  $D$  are separated because they are dominated at least twice each and  $g(G) > 4$ .

Similarly, a vertex  $x \in D$  and a vertex  $y \in V(G) \setminus D$  are separated since  $y$  has two vertices which dominate it, but they cannot both dominate  $x$  (otherwise there would be a triangle or a 4-cycle in  $G$ ).

Finally, consider two vertices of  $D$ . If they are not adjacent they are separated by themselves. Otherwise, by the assumption that  $G[D]$  has no isolated edge and that  $G$  has no triangles, we know that at least one of them has a neighbour in  $D$ , which separates them since it is not a neighbour of the other.  $\square$

The following theorem makes use of Lemma 14. The idea of the proof is inspired by a classic proof of a result on dominating sets which can be found in the first chapter of [1].

**Theorem 15.** *Let  $G$  be a graph on  $n$  vertices with minimum degree  $\delta$  and girth at least 5. Then  $\gamma^{ID}(G) \leq (1 + o_\delta(1)) \frac{3 \log \delta}{2\delta} n$ . Moreover if  $G$  has average degree  $\bar{d} = O_\delta(\delta(\log \delta)^2)$  then  $\gamma^{ID}(G) \leq \frac{\log \delta + \log \log \delta + O_\delta(1)}{\delta} n$ .*

*Proof.* Let  $S \subseteq V(G)$  be a random subset of vertices, where each vertex  $v \in V(G)$  is added to  $S$  uniformly at random with probability  $p$  (where  $p$  will be determined later). For every vertex  $v \in V(G)$ , we define the random variable  $X_v$  as follows:

$$X_v = \begin{cases} 0 & \text{if } |N[v] \cap S| \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

Let  $T$  be the set of vertices which are not 2-dominated, i.e.  $T = \{v \mid X_v = 1\}$ . Note that  $|T| = \sum X_v$ . Let us estimate the size of  $T$ . Observing that  $|N[v] \cap S| \sim \text{Bin}(\delta + 1, p)$  we obtain:

$$\begin{aligned} \mathbb{E}(|T|) &= \sum_{v \in V(G)} \mathbb{E}(X_v) \\ &= n \left( (1-p)^{\delta+1} + (\delta+1)p(1-p)^\delta \right) \\ &= n(1-p)^\delta \left( (1-p) + (\delta+1)p \right) \\ &\leq n(1+\delta p)e^{-\delta p}. \end{aligned}$$

where we have used the fact that  $1-x \leq e^{-x}$ . Now, note that the set  $D = S \cup T$  is a 2-dominating set of  $G$ . We have  $|D| \leq |S| + |T|$ . Hence

$$\begin{aligned} \mathbb{E}(|D|) &\leq \mathbb{E}(|S|) + \mathbb{E}(|T|) \\ &\leq np + n(1+\delta p)e^{-\delta p} \end{aligned} \tag{11}$$

Let us set  $p = \frac{\log \delta + \log \log \delta}{\delta}$ . Plugging this into Equation (11), we obtain:

$$\mathbb{E}(|D|) \leq \frac{\log \delta + \log \log \delta}{\delta} n + \frac{1 + \log \delta + \log \log \delta}{\delta \log \delta} n = \frac{\log \delta + \log \log \delta + O_\delta(1)}{\delta} n$$

This shows that there exists at least one 2-dominating set  $D$  having this size.

**Case 1:** (general case) Note that we can use Lemma 14 by considering all pairs  $u, v$  of vertices of  $D$  forming an isolated edge in  $G[D]$ , and add an arbitrary neighbour of either one of them to  $D$ . Observe that such a vertex exists, otherwise  $u$  and  $v$  would be twins in  $G$ . Since there are at most  $\frac{|D|}{2}$  such pairs, we obtain a 2-dominating set of size at most  $|D| + \frac{|D|}{2} = (1 + o_\delta(1)) \frac{3 \log \delta}{2\delta} n$  having the desired property. Now applying Lemma 14 completes Case 1.

**Case 2:** (sparse case) Whenever  $\bar{d} = O_\delta(\delta(\log \delta)^2)$ , we can get a better bound by estimating the number of isolated edges of  $G[D]$ . For each edge  $uv$  of  $G$ , let  $Y_{uv}$  denote the random variable defined as follows:

$$Y_{uv} = \begin{cases} 1 & \text{if } uv \text{ is isolated in } G[D] \\ 0 & \text{otherwise} \end{cases}$$

Note that  $Y_{uv} = 1$  when  $N[u]\Delta N[v] \subseteq V(G) \setminus S$ . Using the facts that  $p = \frac{\log \delta + \log \log \delta}{\delta}$  and  $1 - x \leq e^{-x}$ , let us calculate the expected value of  $Y = \sum_{uv \in E} Y_{uv}$ .

$$\mathbb{E}(Y) = \sum_{uv \in E(G)} \mathbb{E}(Y_{uv}) \leq \frac{n\bar{d}}{2}(1-p)^{2\delta-2}p^2 \leq \frac{n\bar{d}}{2}e^{-(2\delta-2)p} = \frac{n\bar{d}e^{-2(\log \delta + \log \log \delta)}}{2} = \frac{n\bar{d}}{2\delta^2(\log \delta)^2}$$

We construct  $U$  by picking an arbitrary neighbour of either  $u$  or  $v$  for each edge  $uv$  such that  $Y_{uv} = 1$ . We have  $|U| \leq Y$ . The final set  $\mathcal{C} = S \cup T \cup U$  is an identifying code. Now we have:

$$\mathbb{E}(|\mathcal{C}|) \leq \mathbb{E}(|S|) + \mathbb{E}(|T|) + \mathbb{E}(|U|) \leq \frac{\log \delta + \log \log \delta + O_\delta(1)}{\delta}n + \frac{\bar{d}}{2\delta^2(\log \delta)^2}n$$

Using that  $\bar{d} = O_\delta(\delta(\log \delta)^2)$ ,

$$\mathbb{E}(|\mathcal{C}|) \leq \frac{\log \delta + \log \log \delta + O_\delta(1)}{\delta}n$$

Then there exists some choice of  $S$  such that  $|\mathcal{C}|$  has the desired size, and completes the proof.  $\square$

In fact, it is shown in the next section (Corollary 19) that Theorem 15 is asymptotically tight.

Moreover, note that Theorem 15 cannot be extended much in the sense that if we drop the condition on girth 5, we know arbitrarily large  $d$ -regular triangle-free graphs having large minimum identifying codes. For instance, Construction 22 of Section 7, provides a graph  $G$  which satisfies  $\gamma^{\text{ID}}(G) = n - \frac{n}{d}$ . Similarly, we cannot drop the minimum degree condition. Indeed it is known that any  $(d-1)$ -ary complete tree  $T_{d,h}$  of height  $h$ , which is of maximum degree  $d$ , minimum degree 1 and has infinite girth, also has a large identifying code number (i.e.  $\gamma^{\text{ID}}(T_{d,h}) = n - \frac{n}{d-1+o_d(1)}$  [5]).

## 6 Identifying codes of random regular graphs

From the study of regular graphs arises the question of the value of the identifying code number for most regular graphs. We know some lower and upper bounds for this parameter, but is it concentrated around some value? A good way to study this question is to look at random regular graphs.

Consider the Configuration Model, where a  $d$ -regular multigraph on  $n$  vertices is obtained by selecting some perfect matching of  $K_{nd}$  at random (see [7] for further reference). We will denote by  $\mathcal{G}^*(n, d)$  the former probability space and by  $\mathcal{G}(n, d)$  the same probability space conditioned on the event that  $G$  is simple. It is well-known (see e.g. [7]) that

$$\Pr(G \in \mathcal{G}(n, d) \mid G \in \mathcal{G}^*(n, d)) = e^{\frac{1-d^2}{4}}$$

Then any property that holds with probability tending to 1 for  $\mathcal{G}^*(n, d)$  as  $n \rightarrow \infty$ , will also hold with probability tending to 1 for  $\mathcal{G}(n, d)$ . In this case we will say that the property holds *with high probability* (w.h.p.). In fact our bounds include asymptotic terms in  $d$ , which means they are meaningful for sufficiently large  $d$ .

**Theorem 16.** *Let  $G \in \mathcal{G}(n, d)$  then w.h.p.  $\gamma^{\text{ID}}(G) \leq \frac{\log d + \log \log d + O_d(1)}{d}n$ .*

*Proof.* First of all we have to show that almost all random regular graphs are twin-free.

Observe that the number of perfect matchings of  $K_{2m}$  is  $(2m-1)!! = (2m-1)(2m-3)(2m-5)\dots 1$ . Fix a vertex  $u$  of  $G$  and let  $N(u) = \{v_1, \dots, v_d\}$ . We compute the probability that  $u$  and  $v_1$  are twins, i.e.  $N[u] = N[v_1]$ . The number of perfect matchings of  $K_{nd}$  such that in the resulting graph  $G$  of  $\mathcal{G}(n, d)$ ,  $v_1$  and  $v_2$  are adjacent, is at most  $(d-1)(d-1)(nd-2d-3)!!$ . Indeed, there must be an edge between  $v_1$  and  $v_2$ , which gives  $(d-1)(d-1)$  possibilities. Since  $u$  has  $d$  neighbours, the number of possibilities for the remaining graph is the number of perfect matchings of  $K_{nd-2d-2}$ .

Analogously the number of perfect matchings with  $v_2, v_3 \in N(v_1)$  is at most  $(d-1)(d-1)(d-2)(d-1)(nd-2d-5)!!$ . Thus we have:

$$\begin{aligned}
\Pr(N[u] = N[v_1]) &= \frac{(d-1)(d-1)(d-2)(d-1)\dots 2(d-1)1(d-1)(nd-4d+1)!!}{(nd-2d-1)!!} \\
&\leq \frac{d^{d-1}(d-1)!}{(nd-2d-1)\dots(nd-4d+3)} \\
&\leq \left(\frac{d}{n}\right)^{d-1}
\end{aligned}$$

As we have at most  $\frac{nd}{2}$  possible pairs of twins (one for each edge), the probability of having some twins is:

$$\Pr(G \text{ has twins}) \leq \frac{nd}{2} \left(\frac{d}{n}\right)^{d-1} \rightarrow 0$$

since  $d \geq 3$ . Therefore, random regular graphs are twin-free w.h.p.

By the proof of Theorem 15, for any  $G \in \mathcal{G}(n, d)$ , we have a set  $\mathcal{C}$  with

$$|\mathcal{C}| \leq \frac{\log d + \log \log d + O_d(1)}{d} n$$

that separates any pair of vertices except from the ones where both vertices belong to a triangle or a 4-cycle. We have to add some vertices to  $\mathcal{C}$  in order to separate the vertices of these small cycles.

Classical results on random regular graphs (independently, [6] and [25]) state that the random variables that count the number of cycles of length  $k$ ,  $X_k$ , tend in distribution to independent Poisson variables with parameter  $\lambda_k = \frac{1}{2k}(d-1)^k$ .

Observe that:

$$\mathbb{E}(X_3) = \frac{(d-1)^3}{6} \quad \mathbb{E}(X_4) = \frac{(d-1)^4}{8}$$

i.e. a constant number of triangles and 4-cycles are expected.

Using Markov's inequality we can bound the probability of having too small cycles:

$$\Pr(X_3 > t) \leq \frac{(d-1)^3}{6t} \quad \Pr(X_4 > t) \leq \frac{(d-1)^4}{8t}$$

For  $t = \log n$ , the probability of having more than  $t$  triangles is  $o(1)$ . Then w.h.p., we have at most  $\log n$  cycles of length 3 and  $\log n$  cycles of length 4.

Let  $T = \{u_1, u_2, u_3\}$  be a triangle in  $G$ . As  $d \geq 3$  there exists at least one vertex  $v_i$  outside the triangle. Since our graph is twin-free, for each ordered pair  $(u_i, u_j)$  there exists some vertex  $v_{ij}$ , such that  $v_{ij} \in N(u_i) \setminus N(u_j)$ . Observe that we can add  $v_{12}$ ,  $v_{23}$  and  $v_{31}$  to  $\mathcal{C}$  and then any pair of vertices from  $T$  will be separated.

If  $T = \{u_1, u_2, u_3, u_4\}$  induces a  $K_4$ , each pair of vertices of  $T$  is contained in some triangle and is separated by the last step. If  $T$  induces a 4-cycle, adding  $T$  to  $\mathcal{C}$  separates all the elements in  $T$ . Otherwise,  $T$  induces two triangles and adding  $T$  to  $\mathcal{C}$  separates the two vertices which have not been separated in the last step.

After these two steps, we have added at most  $7 \log n$  vertices to  $\mathcal{C}$ . Hence, for any  $G \in \mathcal{G}(n, d)$  w.h.p. we obtain:

$$\gamma^{\text{ID}}(G) \leq \frac{\log d + \log \log d + O_d(1)}{d} n + 7 \log n = \frac{\log d + \log \log d + O_d(1)}{d} n$$

Observe that the  $\frac{O_d(1)}{d} n$  term contains the  $7 \log n$  term. □

Theorem 16 shows that despite the fact that for any  $d$ , we know infinitely many  $d$ -regular graphs having a very large identifying code number (e.g.  $n - \frac{n}{d}$  for the graphs of Construction 21 of Section 7), almost all  $d$ -regular graphs have very small identifying code.

Moreover,  $\gamma^{\text{ID}}(G)$  is concentrated, as the following theorem and its corollary show. In fact the following result might be already known, since a similar result is stated for independent dominating sets in [16]. However we could not find it in the literature and decided to give a proof for the sake of completeness.

**Theorem 17.** *Let  $G \in \mathcal{G}(n, d)$ , then w.h.p. all the dominating sets of  $G$  have size at least  $\frac{\log d - 2 \log \log d}{d} n$ .*

*Proof.* We will proceed by contradiction. Given a set of vertices  $D$  of size  $m$ , we will compute the probability that  $D$  dominates  $Y = V(G) \setminus D$ . Recall that  $G$  has been obtained from the configuration model by selecting a random perfect matching of  $K_{nd}$ . Let  $y \in Y$  fixed, then let  $A_y = \{N(D) \cap y \neq \emptyset\}$  be the event that  $y$  is dominated by  $D$ . Its complementary corresponds to the situation where none of the edges of the perfect matching of  $K_{nd}$  connects the points corresponding to  $y$  to the ones corresponding to any vertex of  $D$ . Hence:

$$\begin{aligned} \Pr(\overline{A_y}) &= \left(1 - \frac{d}{nd-1}\right) \left(1 - \frac{d}{nd-3}\right) \cdots \left(1 - \frac{d}{nd-(2md-1)}\right) \\ &= \prod_{i=1}^{md} \left(1 - \frac{d}{nd-(2i-1)}\right) \\ &\geq \prod_{i=1}^{md} \left(1 - \frac{1}{n-2m}\right) \end{aligned}$$

Since  $1 - x = e^{-x+(\log(1-x)+x)}$  (here we take  $x = \frac{1}{n-2m}$ ) and  $\log(1-x) + x = O(x^2)$  (by the Taylor expansion of the logarithm in  $x=0$ ), we obtain:

$$\begin{aligned} \Pr(\overline{A_y}) &\geq \exp \left\{ -\sum_{i=1}^{md} \frac{1}{n-2m} + O\left(\frac{1}{(n-2m)^2}\right) \right\} \\ &= \exp \left\{ -(1+o(1)) \frac{md}{n-2m} \right\} \end{aligned}$$

The probability that  $D$  is dominating  $Y = \{y_1, \dots, y_{n-m}\}$  is:

$$\Pr(\cap_{y \in Y} A_y) = \Pr(A_{y_1}) \Pr(A_{y_2} | A_{y_1}) \cdots \Pr(A_{y_{n-m}} | \cap_{j=1}^{n-m-1} A_{y_j})$$

We claim that  $\Pr(A_{y_i} | \cap_{j=1}^{i-1} A_{y_j}) \leq \Pr(A_{y_i})$ . Suppose that  $y_1, \dots, y_{i-1}$  are dominated. This means that the corresponding perfect matching of  $K_{nd}$  has an edge between one of the points corresponding to  $y_j$  ( $1 \leq j \leq i-1$ ) and one of the points corresponding to the vertices of  $D$ . The probability that  $y_i$  is not dominated by  $D$  is now the probability that none of the remaining edges of the perfect matching connect any vertex of  $D$  with  $y_i$ . Hence:

$$\begin{aligned} \Pr(\overline{A_{y_i}} | \cap_{j=1}^{i-1} A_{y_j}) &= \left(1 - \frac{d}{nd-2(i-1)-1}\right) \left(1 - \frac{d}{nd-2(i-1)-3}\right) \cdots \left(1 - \frac{d}{nd-2md+1}\right) \\ &\geq \left(1 - \frac{d}{nd-1}\right) \left(1 - \frac{d}{nd-3}\right) \cdots \left(1 - \frac{d}{nd-2md+1}\right) \\ &= \Pr(\overline{A_{y_i}}) \end{aligned}$$

By considering the complementary events,  $\Pr(A_{y_i} | \cap_{j=0}^{i-1} A_{y_j}) \leq \Pr(A_{y_i})$ . Hence these events are negatively correlated, and:

$$\Pr(\cap_{y \in Y} A_y) \leq \prod_{i=1}^{n-m} \Pr(A_{y_i}) \leq \left(1 - e^{-\frac{md}{n-2m}}\right)^{n-m} \leq \exp \left\{ -(n-m)e^{-\frac{md}{n-2m}} \right\}$$

For the sake of contradiction, let  $m \leq \frac{\log d - c \log \log d}{d} n$  for some  $c > 2$ . Then:

$$\begin{aligned} \Pr(\cap_{y \in Y} A_y) &\leq \exp \left\{ -\left(1 - \frac{\log d - c \log \log d}{d}\right) n \exp \left\{ -\frac{\log d - c \log \log d}{1 - 2\frac{\log d - c \log \log d}{d}} \right\} \right\} \\ &= \exp \left\{ -(1+o_d(1)) n \exp \left\{ -\frac{\log d - c \log \log d}{1+o_d(1)} \right\} \right\} \\ &= (1+o_d(1)) e^{-\frac{(\log d)^c}{d} n} \end{aligned}$$

Note that if no set of size  $m$  dominates  $Y$ , neither will do a smaller one. So we have to look just at the sets of size  $m$ . The number of these sets can be bounded by

$$\binom{n}{m} \leq \frac{n^m}{m!} \leq \left(\frac{en}{m}\right)^m = \left(\frac{de}{\log d - c \log \log d}\right)^{\frac{\log d - c \log \log d}{d}n} = (1 + o_d(1)) \left(\frac{de}{\log d}\right)^{\frac{\log d - c \log \log d}{d}n}$$

where we have used  $m! \geq \left(\frac{m}{e}\right)^m$ .

Let  $E_{DS}$  be the event that  $G$  has a dominating set of size  $m$ . Applying the union bound, we obtain:

$$\begin{aligned} \Pr(E_{DS}) &\leq (1 + o_d(1)) \left(\frac{de}{\log d}\right)^{\frac{\log d - c \log \log d}{d}n} e^{-\frac{(\log d)^c}{d}n} \\ &= (1 + o_d(1)) \exp \left\{ \frac{\log d - c \log \log d}{d}(\log d + 1 - \log \log d)n - \frac{(\log d)^c}{d}n \right\} \\ &= (1 + o_d(1)) \exp \left\{ \left( \frac{(\log d)^2}{d} - \frac{(\log d)^c}{d} + o_d\left(\frac{(\log d)^2}{d}\right) \right)n \right\} \rightarrow 0 \end{aligned}$$

since  $c > 2$ . This shows that w.h.p. no set of size at most  $\frac{\log d - c \log \log d}{d}n$  can dominate the whole graph and completes the proof.  $\square$

Since any identifying code is also a dominating set, we obtain the following immediate corollary.

**Corollary 18.** *Let  $G \in \mathcal{G}(n, d)$ , then w.h.p.  $\gamma^{ID}(G) \geq \frac{\log d - 2 \log \log d}{d}n$ .*

Plugging together Theorems 16 and 17, we obtain the following result.

**Corollary 19.** *Let  $G \in \mathcal{G}(n, d)$ , then w.h.p.*

$$\frac{\log d - 2 \log \log d}{d}n \leq \gamma^{ID}(G) \leq \frac{\log d + \log \log d + O_d(1)}{d}n$$

## 7 Extremal constructions

This section gathers some constructions which show the tightness of some of our upper bounds. Some of these constructions can be found in [10].

**Construction 20.** *Given any  $d_H$ -regular graph  $H$  on  $n_H$  vertices, let  $\mathcal{C}_1(H)$  be the graph on  $n = n_H(d_H + 1)$  and maximum degree  $d = d_H + 1$  constructed as follows:*

1. *Replace each vertex  $v$  of  $H$  by a clique  $K(v)$  of  $d_H + 1$  vertices*
2. *For each vertex  $v$  of  $H$ , let  $N(v) = \{v_1, \dots, v_{d_H}\}$  and  $K(v) = \{k_0(v), \dots, k_{d_H}(v)\}$ . For each  $k_i(v)$  but one ( $1 \leq i \leq d_H$ ), connect it with an edge in  $\mathcal{C}_1(H)$ , to a unique vertex of  $K(v_i)$ , denoted  $f(k_i(v))$ .*

One can see that the graphs  $\mathcal{C}_1(H)$  given by Construction 20 are twin-free. Moreover, for each vertex  $v$  of  $H$  and for each  $1 \leq i \leq d_H$ , note that  $f(k_i(v))$  is  $k_0(v)k_i(v)$ -forced. Therefore  $\mathcal{C}_1(H)$  has  $d_H n_H = n - \frac{n}{d}$  forced vertices. In fact these forced vertices form an identifying code, therefore  $\gamma^{ID}(\mathcal{C}_1(H)) = n - \frac{n}{d}$ . An example of this construction is given in Figure 2, where  $H$  is the hypercube of dimension 3,  $H_3$ , and the black vertices are those which belong to a minimum identifying code of  $\mathcal{C}_1(H_3)$ .

The following construction is very similar, but yields regular graphs.

**Construction 21.** [10] *Given any  $d_H$ -regular graph  $H$  on  $n_H$  vertices, let  $\mathcal{C}_2(H)$  be the  $d$ -regular graph on  $n = n_H d_H$  vertices (where  $d = d_H$ ) constructed as follows:*

1. *Replace each vertex  $v$  of  $H$  by a clique  $K(v)$  of  $d_H$  vertices.*
2. *For each vertex  $v$  of  $H$ , let  $N(v) = \{v_1, \dots, v_{d_H}\}$  and  $K(v) = \{k_1(v), \dots, k_{d_H}(v)\}$ . For each  $k_i(v)$  ( $1 \leq i \leq d_H$ ), connect it with an edge in  $\mathcal{C}_2(H)$ , to a unique vertex of  $K(v_i)$ , denoted  $f(k_i(v))$ .*

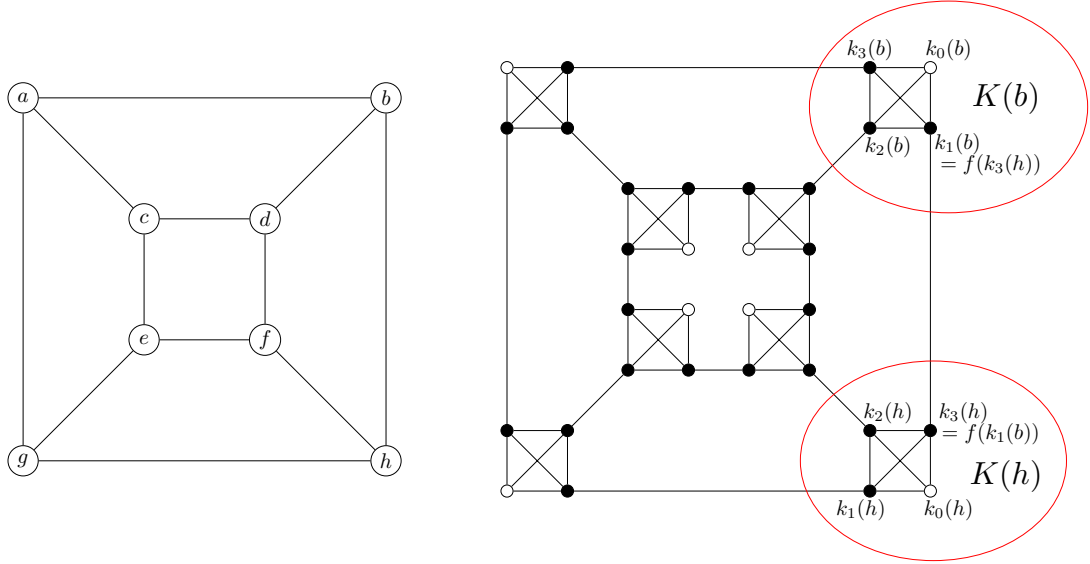


Figure 2: The graphs  $H_3$  and  $\mathcal{C}_1(H_3)$

Note that for some vertex  $v$  of  $H$ , in order to separate each pair of vertices  $k_i(v), k_j(v)$  of  $K(v)$  in  $\mathcal{C}_2(H)$ , either  $f(k_i(v))$  or  $f(k_j(v))$  must belong to any identifying code. Repeating this argument for each pair shows that at least  $d - 1$  such vertices are needed in the code. Since for any two cliques  $K(u)$  and  $K(v)$ , the set of these neighbours are disjoint, this shows that at least  $n_H(d - 1)$  vertices are needed in an identifying code of  $\mathcal{C}_2(H)$ . In fact it is easy to construct an identifying code of this size. This shows that despite the fact that  $\mathcal{C}_2(H)$  has no forced vertices,  $\gamma^{\text{ID}}(\mathcal{C}_2(H)) = n - \frac{n}{d}$ . An example of this construction is given in Figure 3, where  $H$  is the complete graph  $K_5$ , and the black vertices form a minimum identifying code of  $\mathcal{C}_2(K_5)$ .

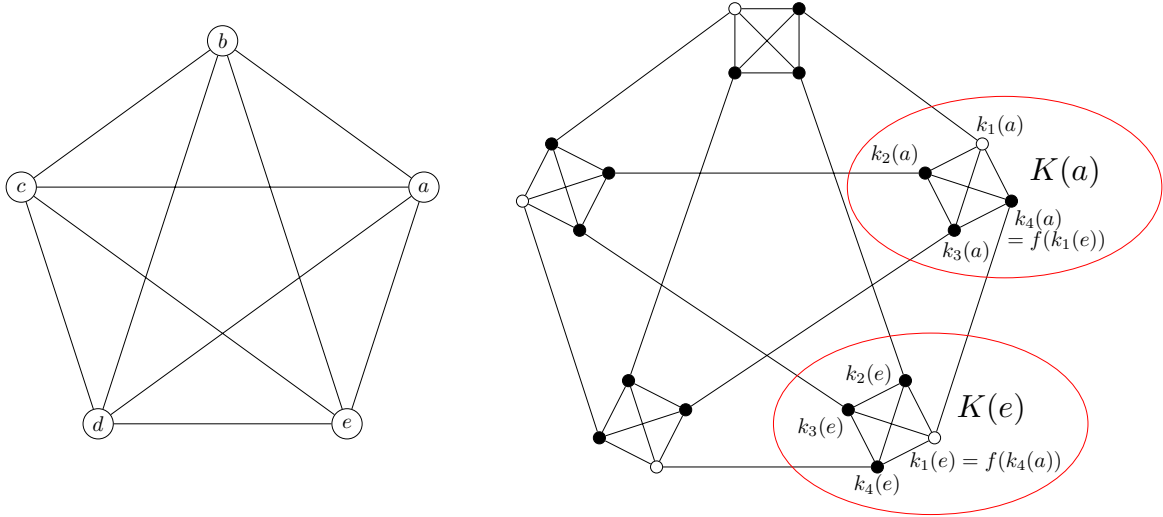


Figure 3: The graphs  $K_5$  and  $\mathcal{C}_2(K_5)$

**Construction 22.** [10] Given an even number  $2k$  and an integer  $d \geq 3$ , we construct a twin-free  $d$ -regular triangle-free graph  $\mathcal{C}_3(2k, d)$  on  $n = 2kd$  vertices as follows.

1. Let  $\{c_0, \dots, c_{2k-1}\}$  be a set of  $2k$  vertices and add the edges of the perfect matching  $\{c_i c_{i+1 \bmod 2k} \mid i \text{ is odd}\}$ .
2. For each even  $i$  ( $0 \leq i \leq 2k - 2$ ), build a copy  $K(i)$  of the complete bipartite graph  $K_{d-1, d-1}$ . Join vertex  $c_i$  to all vertices of one part of the bipartition of  $K(i)$ , and join vertex  $c_{i+1}$  to all other



vertices of  $K(i)$ .

Consider an identifying code of  $\mathcal{C}_3(2k, d)$ . Note that in each copy  $K(i)$  of  $K_{d-1, d-1}$ , at least  $2d - 4$  vertices belong to the code in order to separate the vertices being in the same part of the bipartition of  $K(i)$ . Now if exactly  $2d - 4$  vertices of  $K(i)$  belong to the code, in order to separate the two remaining vertices, either  $c_i$  or  $c_{i+1}$  belongs to the code. Hence for each odd  $i$ , at most three vertices from  $\{c_i, c_{i+1}\} \cup V(K(i))$  do not belong to a code of  $\mathcal{C}_3(2k, d)$ . On the other hand, taking all vertices  $c_i$  such that  $i$  is even together with  $d - 2$  vertices of each part of the bipartition of each copy of  $K_{d-1, d-1}$  yields an identifying code of this size. Hence  $\gamma^{\text{ID}}(\mathcal{C}_3(2k, d)) = k + 2k(d - 2) = n - \frac{n}{2d/3}$ . An example of this construction is given in Figure 4, where  $2k = 8$ ,  $d = 3$ , and the black vertices form a minimum identifying code of  $\mathcal{C}_3(8, 3)$ .

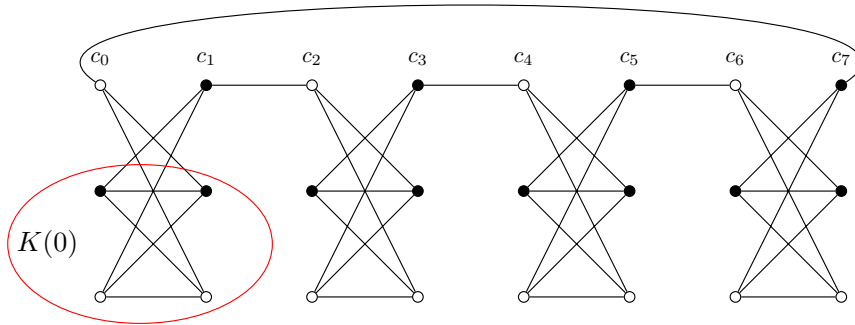


Figure 4: The graph  $\mathcal{C}_3(8, 3)$

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