Ellipsoidal output-feedback sets for Hinf control of a class of stochastic hybrid systems with state-dependant noise

Samir Aberkane, Jean-Christophe Ponsart, Dominique Sauter

Ellipsoidal Output-Feedback Sets for $\mathcal{H}_\infty$ Control of a Class of Stochastic Hybrid Systems with State-Dependant Noise

Samir Aberkane

CRAN – CNRS UMR 7039
Université Henri Poincaré, Nancy 1, BP 239,
F-54506 Vandœuvre-lès-Nancy Cedex
Tel.: +33 3 83 68 44 74, Fax: +33 3 83 68 44 61
e-mail: samir.aberkane@cran.uhp-nancy.fr

RÉSUMÉ. *A définir par la commande* \resume{...}

ABSTRACT. *This paper deals with static output feedback $\mathcal{H}_\infty$ control of continuous time Active Fault Tolerant Control Systems with Markovian Parameters (AFTCSMP) and state-dependent noise. It adopts a new framework, based on the synthesis of ellipsoidal sets of controllers, introduced in (Peaucelle* et al., *2002, Peaucelle* et al., *2005). It is also shown that the obtained results can easily be applied to the problematic of mode-independent static output feedback $\mathcal{H}_\infty$ control of another class of stochastic hybrid systems known as Markovian Jump Linear Systems. Results are formulated as matrix inequalities one of which is nonlinear. A numerical algorithm based on nonconvex optimization is provided and its running is illustrated on classical examples from literature.*

MOTS-CLÉS : *A définir par la commande* \motscles{...}

KEYWORDS: *Active Fault Tolerant Control, Stochastic Hybrid Systems, Markovian Jumping Parameters, $\mathcal{H}_\infty$ Control, Static Output Feedback, Ellipsoidal Sets, Linear Matrix Inequalities (LMI).*

1. Introduction

As performance requirements increase in advanced technological systems, their associated control systems are becoming more and more complex. At the same time, complicated systems could have various consequences in the event of component failures. Therefore, it is very important to consider the safety and fault tolerance of such systems at the design stage. For these safety-critical systems, Fault Tolerant Control Systems (FTCS) have been developed to meet these essential objectives. FTCS have been a subject of great practical importance, which has attracted a lot of interest for the last three decades. A bibliographical review on reconfigurable fault tolerant control systems can be found in (Zhang *et al.,* 2003).
Active fault tolerant control systems are feedback control systems that reconfigure the control law in real time based on the response from an automatic fault detection and identification (FDI) scheme. The dynamic behaviour of Active Fault Tolerant Control Systems (AFTCS) is governed by stochastic differential equations and can be viewed as a general hybrid system (Srichander *et al.,* 1993). A major class of hybrid systems is Markovian Jump Linear Systems (MJLS). In MJLS, a single jump process is used to describe the random variations affecting the system parameters. This process is represented by a finite state Markov chain and is called the plant regime mode. The theory of stability, optimal control and $\mathcal{H}_2/\mathcal{H}_\infty$ control, as well as important applications of such systems, can be found in several papers in the current literature, for instance in (Boukas, 2006, Boukas, 2005, Boukas, 1999, Costa *et al.,* 1999, de Farias *et al.,* 2000, de Souza *et al.,* 1993, Ji *et al.,* 1990, Ji *et al.,* 1992).
To deal with AFTCS, another class of hybrid systems was defined, denoted as AFTCSMP. In this class of hybrid systems, two random processes are defined : the first random process represents system components failures and the second random process represents the FDI process used to reconfigure the control law. This model was proposed by Srichander and Walker (Srichander *et al.,* 1993). Necessary and sufficient conditions for stochastic stability of AFTCSMP were developed for a single component failure (actuator failures). The problem of stochastic stability of AFTCSMP in the presence of noise, parameter uncertainties, detection errors, detection delays and actuator saturation limits has also been investigated in (Mahmoud *et al.,* 1999a, Mahmoud *et al.,* 2001, Mahmoud *et al.,* 2003). Another issue related to the synthesis of fault tolerant control laws was also addressed by (Mahmoud *et al.,* 1999b, Shi *et al.,* 1997, Shi *et al.,* 2003). In (Mahmoud *et al.,* 1999b), the authors designed an optimal control law for AFTCSMP using the matrix minimum principle to minimize an equivalent deterministic cost function. The problem of $\mathcal{H}_\infty$ and robust $\mathcal{H}_\infty$ control was treated in (Shi *et al.,* 1997, Shi *et al.,* 2003) for both continuous and discrete time AFTCSMP. The authors showed that the state feedback control problem can be solved in terms of the solutions of a set of coupled Riccati inequalities. The dynamic/static output feedback counterpart was treated by (Aberkane *et al.,* 2005b, Aberkane *et al.,* 2005c, Aberkane *et al.,* 2005a) in a convex programming framework. Indeed, the authors provide an LMI characterization of dynamical/static output feedback compensators that stochastically stabilize (robustly stabilize) the AFTCSMP and ensures $\mathcal{H}_\infty$ (robust $\mathcal{H}_\infty$) constraints. In addition, it is important to

mention that the design problem in the framework of AFTCSMP remains an open and challenging problematic. This is due, particulary, to the fact that the controller only depends on the FDI process *i.e.* the number of controllers to be designed is less than the total number of the closed loop systems modes by combining both failure an FDI processes. The design problem involves searching feasible solutions of a problem where there are more constraints than variables to be solved. Generally speaking, there lacks tractable design methods for this stochastic FTC problem. Indeed, in (Aberkane *et al.,* 2005b, Aberkane *et al.,* 2005c, Mahmoud *et al.,* 2003, Shi *et al.,* 1997, Shi *et al.,* 2003), the authors make the assumption that the controller must access both failures and FDI processes. However, this assumption is too restrictive to be applicable in practical FTC systems. In this note, the assumption on the availability of failure processes, for the synthesis purposes, is stressed.
On the other hand, one of the most challenging open problems in control theory is the synthesis of fixed-order or static output feedback controllers that meet desired performances and specifications (Syrmos *et al.,* 1997). Among all variations of this problem, this note is concerned with the problem of static output feedback $\mathcal{H}_\infty$ control of continuous time AFTCSMP with state-dependent noise. This problematic is addressed under a new framework, based on the synthesis of ellipsoidal sets of controllers, introduced in (Peaucelle *et al.,* 2002, Peaucelle *et al.,* 2005). The problematic resulting from the fact that the controller only depends on the FDI process is shown to be naturally dealt with in this context. It is also shown that the obtained results can easily be applied to the problematic of mode-independent static output feedback $\mathcal{H}_\infty$ control of MJLS. Results are formulated as matrix inequalities one of which is nonlinear. A numerical algorithm based on nonconvex optimization is provided and its running is illustrated on classical examples from literature.
This paper is organized as follows : Section 2 describes the dynamical model of the system with appropriately defined random processes. A brief summary of basic stochastic terms, results and definitions are given in Section 3. Section 4 addresses the internal stochastic stabilization of the AFTCSMP. Sections 5 considers the $\mathcal{H}_\infty$ control problem for the output feedback. In Section 6, a numerical algorithm based on nonconvex optimization is provided and its running is illustrated on classical examples from literature. Finally, a conclusion is given in Section 7.

Notations. The notations in this paper are quite standard. $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices and $\mathbb{S}^n$ is the subset of symmetric matrices in $\mathbb{R}^{n\times n}$. A' is the transpose of the matrix A. The notation $X \geq Y$ ($X > Y$, respectively), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (positive definite, respectively) ; $\mathbb{I}$ and $\mathbf{0}$ are identity and zero matrices of appropriate dimensions, respectively ; $\mathcal{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure P ; $L^2[0,\infty)$ stands for the space of square-integrable vector functions over the interval $[0,\infty)$; $\|\cdot\|$ refers to either the Euclidean vector norm or the matrix norm, which is the operator norm induced by the standard vector norm ; $\|\cdot\|_2$ stands for the norm in $L^2[0,\infty)$; while $\|\cdot\|_{\mathcal{E}_2}$ denotes the norm in $L^2((\Omega,\mathcal{F},P),[0,\infty))$;

$(\Omega, \mathcal{F}, P)$ is a probability space. In block matrices, $\star$ indicates symmetric terms : $\begin{bmatrix} A & B \\ B' & C \end{bmatrix} = \begin{bmatrix} A & \star \\ B' & C \end{bmatrix} = \begin{bmatrix} A & B \\ \star & C \end{bmatrix}$.

2. Dynamical Model of the AFTCSMP with Wiener Process

To describe the class of linear systems with Markovian jumping parameters that we deal with in this paper, let us fix a complete probability space $(\Omega, \mathcal{F}, P)$. This class of systems owns a hybrid state vector. The first component vector is continuous and represents the system states, and the second one is discrete and represents the failure processes affecting the system. The dynamical model of the uncertain AFTCSMP with Wiener Process, defined in the fundamental probability space $(\Omega, \mathcal{F}, P)$, is described by the following differential equations :

$$\varphi : \begin{cases} dx(t) = A(\xi(t))x(t)dt + B(\eta(t))u(y(t), \psi(t), t)dt \\ +E(\xi(t), \eta(t))w(t)dt + \sum_{l=1}^{v} \mathbb{W}_l(\xi(t), \eta(t))x(t)d\varpi_l(t) \\ y(t) = C_2x(t) + D_2(\xi(t), \eta(t))w(t) \\ z(t) = C_1x(t) + D_1(\eta(t))u(y(t), \psi(t), t) \end{cases} \quad [1]$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(y(t), \psi(t), t) \in \mathbb{R}^r$ is the system input, $y(t) \in \mathbb{R}^q$ is the system measured output, $z(t) \in \mathbb{R}^p$ is the controlled output, $w(t) \in \mathbb{R}^m$ is the system external disturbance, $\xi(t)$, $\eta(t)$ and $\psi(t)$ represent the plant component failure process, the actuator failure process and the FDI process, respectively. $\xi(t)$, $\eta(t)$ and $\psi(t)$ are separable and mesurable Markov processes with finite state spaces $Z = \{1, 2, ..., z\}$, $S = \{1, 2, ..., s\}$ and $R = \{1, 2, ..., r\}$, respectively. $\varpi(t) = [\varpi_1(t) \dots \varpi_v(t)]'$ is a v-dimensional standard Wiener process on a given probability space $(\Omega, \mathcal{F}, P)$, that is assumed to be independent of the Markov processes. The matrices $A(\xi(t))$, $B(\eta(t))$, $E(\xi(t), \eta(t))$, $D_2(\xi(t), \eta(t))$, $D_1(\eta(t))$ and $\mathbb{W}_l(\xi(t), \eta(t))$ are properly dimensioned matrices which depend on random parameters.

Remark 1 : For the existence and uniqueness of the solution of (1), we refer the reader to [], and the references therein.

In AFTCS, we consider that the control law is only a function of the mesurable FDI process $\psi(t)$. Therefore, we introduce a static output feedback compensator (φ_s) of the form :

$$\varphi_s : \Big\{ u(t) = \mathcal{K}(\psi(t))y(t) \quad [2]$$

Applying the controller φ_s to the AFTCSMP φ, we obtain the following closed loop system :

$$\varphi_{cl} : \begin{cases} dx(t) = \bar{A}(\xi(t), \eta(t), \psi(t))x(t)dt + \bar{E}(\xi(t), \eta(t), \psi(t))w(t)dt \\ + \sum_{l=1}^{v} \mathbb{W}_l(\xi(t), \eta(t))x(t)d\varpi_l(t) \\ y(t) = C_2x(t) + D_2(\xi(t), \eta(t))w(t) \\ z(t) = \bar{C}_1(\eta(t), \psi(t))x(t) + \bar{D}_1(\xi(t), \eta(t), \psi(t))w(t) \end{cases} \quad [3]$$

where

$$\begin{bmatrix} \bar{A}(\xi(t),\eta(t),\psi(t)) & \bar{E}(\xi(t),\eta(t),\psi(t)) \\ \bar{C}_1(\eta(t),\psi(t)) & \bar{D}_1(\xi(t),\eta(t),\psi(t)) \end{bmatrix} = \begin{bmatrix} A(\xi(t)) & E(\xi(t),\eta(t)) \\ C_1 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} B(\eta(t)) \\ D_1(\eta(t)) \end{bmatrix} \mathcal{K}(\psi(t)) \begin{bmatrix} C_2 & D_2(\xi(t),\eta(t)) \end{bmatrix}$$

2.1. *The FDI and the Failure Processes*

$\xi(t)$, $\eta(t)$ and $\psi(t)$ being homogeneous Markov processes with finite state spaces, we can define the transition probability of the plant components failure process as (Mahmoud *et al.,* 2003, Srichander *et al.,* 1993) :

$$\begin{cases} p_{ij}(\Delta t) = \pi_{ij}\Delta t + o(\Delta t) & (i \neq j) \\ p_{ii}(\Delta t) = 1 - \sum\limits_{i\neq j} \pi_{ij}\Delta t + o(\Delta t) & (i = j) \end{cases}$$

The transition probability of the actuator failure process is given by :

$$\begin{cases} p_{kl}(\Delta t) = \nu_{kl}\Delta t + o(\Delta t) & (k \neq l) \\ p_{kk}(\Delta t) = 1 - \sum\limits_{k\neq l} \nu_{kl}\Delta t + o(\Delta t) & (k = l) \end{cases}$$

where π_{ij} is the plant components failure rate, and ν_{kl} is the actuator failure rate.
Given that $\xi = k$ and $\eta = l$, the conditional transition probability of the FDI process $\psi(t)$ is :

$$\begin{cases} p^{kl}_{iv}(\Delta t) = \lambda^{kl}_{iv}\Delta t + o(\Delta t) & (i \neq v) \\ p^{kl}_{ii}(\Delta t) = 1 - \sum\limits_{i\neq v} \lambda^{kl}_{iv}\Delta t + o(\Delta t) & (i = v) \end{cases}$$

Here, λ^{kl}_{iv} represents the transition rate from i to v for the Markov process $\psi(t)$ conditioned on $\xi = k \in Z$ and $\eta = l \in S$. Depending on the values of $i, v \in R$, $k \in Z$ and $l \in S$, various interpretations, such as rate of false detection and isolation, rate of correct detection and isolation, false alarm recovery rate, etc, can be given to λ^{kl}_{iv} (Mahmoud *et al.,* 2003, Srichander *et al.,* 1993).
For notational simplicity, we will denote $\bullet(\xi(t)) = \bullet_i$ when $\xi(t) = i \in Z$, $\bullet(\eta(t)) = \bullet_j$ when $\eta(t) = j \in S$, $\bullet(\xi(t),\eta(t)) = \bullet_{ij}$, when $\xi(t) = i \in Z, \eta(t) = j \in S$ and $\bullet(\psi(t)) = \bullet_k$ when $\psi(t) = k \in R$. We also denote $\bullet(t) = \bullet_t$ and the initial conditions $\bullet(t_0) = \bullet_0$.

3. Definitions

In this section, we will first give some basic definitions related to stochastic stability notions and then we will summarize some results about exponential stability in the mean square sense of the AFTCSMP with Wiener process. Without loss of generality, we assume that the equilibrium point, $x = 0$, is the solution at which stability properties are examined.

3.1. *Stochastic Stability*

Definition 1 :
System (3) is said to be

(i) stochastically stable (SS) if there exists a finite positive constant $K(x_0, \xi_0, \eta_0, \psi_0)$ *such that the following holds for any initial conditions* $(x_0, \xi_0, \eta_0, \psi_0)$:

$$\mathcal{E}\left\{\int_0^\infty \parallel x_t \parallel^2 dt\right\} \leq K(x_0, \xi_0, \eta_0, \psi_0) \qquad [4]$$

(ii) internally exponentially stable in the mean square sense (IESS) if it is exponentially stable in the mean square sense for $w_t = 0$, *i.e. for any* ξ_0, η_0, ψ_0 *and some* $\gamma(\xi_0, \eta_0, \psi_0)$, *there exists two numbers* $a > 0$ *and* $b > 0$ *such that when* $\|x_0\| \leq \gamma(\xi_0, \eta_0, \psi_0)$, *the following inequality holds* $\forall t \geq t_0$ *for all solution of (3) with initial condition* x_0 :

$$\mathcal{E}\left\{\|x_t\|^2\right\} \leq b\|x_0\|^2 \exp\left[-a(t - t_0)\right] \qquad [5]$$

The following theorem gives a sufficient condition for internal exponential stability in the mean square sense for the system (φ) coupled with (φ_s).

Theorem 1 : *The solution* $x = 0$ *of the system* (φ) *coupled with* (φ_s) *is internally exponentially stable in the mean square for* $t \geq t_0$ *if there exists a Lyapunov function* $\vartheta(x_t, \xi_t, \eta_t, \psi_t, t)$ *such that*

$$K_1\|x_t\|^2 \leq \vartheta(x_t, \xi_t, \eta_t, \psi_t, t) \leq K_2\|x_t\|^2 \qquad [6]$$

and

$$\mathcal{L}\vartheta(x_t, \xi_t, \eta_t, \psi_t, t) \leq -K_3\|x_t\|^2 \qquad [7]$$

for some positive constants K_1, K_2 *and* K_3.

A necessary condition for internal exponential stability in the mean square for the system (φ) coupled with (φ_c) is given by theorem 2.

Theorem 2 : *If the solution* $x = 0$ *of the system* (φ) *coupled with* (φ_s) *is internally exponentially stable in the mean square, then for any given quadratic positive definite function* $W(x_t, \xi_t, \eta_t, \psi_t, t)$ *in the variables* x *which is bounded and continuous*

$\forall t \geq t_0$, $\forall \xi_t \in Z$, $\forall \eta_t \in S$ and $\forall \psi_t \in R$, there exists a quadratic positive definite function $\vartheta(x_t, \xi_t, \eta_t, \psi_t, t)$ in x that satisfies the conditions in theorem 1 and is such that $\mathcal{L}\vartheta(x_t, \xi_t, \eta_t, \psi_t, t) = -W(x_t, \xi_t, \eta_t, \psi_t, t)$.

Remark 2 : The proofs of these theorems follow the same arguments as in (Mahmoud *et al.,* 2003, Srichander *et al.,* 1993) for their proposed stochastic Lyapunov functions, so they are not shown in this paper to avoid repetition.

The following proposition gives a necessary and sufficient condition for internal exponential stability in the mean square sense for the system (3).

Proposition 1 : A necessary and sufficient condition for internal exponential stability in the mean square of the system (3) is that there exist symmetric positive-definite matrices $\mathcal{P}_{ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that :

$$\tilde{A}'_{ijk}\mathcal{P}_{ijk} + \mathcal{P}_{ijk}\tilde{A}_{ijk} + \sum_{l=1}^{v} \mathbb{W}'_{lij}\mathcal{P}_{ijk}\mathbb{W}_{lij} + \sum_{h\in Z} \pi_{ih}\mathcal{P}_{hjk} + \sum_{l\in S} \nu_{jl}\mathcal{P}_{ilk} + \sum_{v\in R} \lambda_{kv}^{ij}\mathcal{P}_{ijv} = \beth_{ijk} < 0 \qquad [8]$$

$\forall i \in Z$, $j \in S$ and $k \in R$, where

$$\tilde{A}_{ijk} = A_i + B_j\mathcal{K}_k C_2 \qquad [9]$$

□

Proof : The proof of this proposition is easily deduced from theorems 1 and 2. ■

Proposition 2 : If the system (3) is internally exponentially stable in the mean square sense, then it is stochastically stable. □

Proof : The proof of this proposition follows the same lines as for the proof of proposition 4 in (Aberkane *et al.,* 2005a). ■

3.2. *Matrix Ellipsoids*

Through this note, a particular set of matrices is used. Due to the notations and by extension of the notion of $\mathbb{R}^n$ ellipsoids, these sets are referred to as matrix ellipsoids of $\mathbb{R}^{(m\times p)}$.

Definition 2 : (Peaucelle, 2000, Peaucelle *et al.,* 2002, Peaucelle *et al.,* 2005) Given three matrices $\mathbb{X} \in \mathbb{S}^q$, $\mathbb{Y} \in \mathbb{R}^{q\times r}$ and $\mathbb{Z} \in \mathbb{S}^r$, the $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}\}$-ellipsoid of $\mathbb{R}^{r\times q}$ is the set of matrices $\mathcal{K}$ satisfying the following matrix inequalities :

$$\mathbb{Z} > 0 \quad \begin{bmatrix} \mathbb{I} & \mathcal{K}' \end{bmatrix} \begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ \star & \mathbb{Z} \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ \mathcal{K} \end{bmatrix} \leq 0 \qquad [10]$$

By definition, $\mathcal{K}_0 = -\mathbb{Z}^{-1}\mathbb{Y}'$ is the center of the ellipsoid and $R = \mathcal{K}_0'\mathbb{Z}\mathcal{K}_0 - \mathbb{X}$ is the radius. Inequalities (8) can also be written as

$$\mathbb{Z} > 0 \quad (\mathcal{K} - \mathcal{K}_0)'\mathbb{Z}(\mathcal{K} - \mathcal{K}_0) \leq R \qquad [11]$$

This definition shows that matrix ellipsoids are special cases of matrix sets defined by quadratic matrix inequality. Some properties of these sets are

i) A matrix ellipsoid is a convex set ;

ii) the $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}\}$-ellipsoid is nonempty iff the radius ($R \geq 0$) is positive semi definite. This property can also be expressed as

$$\mathbb{X} \leq \mathbb{Y}\mathbb{Z}^{-1}\mathbb{Y}' \qquad [12]$$

4. Stochastic Stabilization

In this section, we shall address the problem of finding all static compensators (φ_s), as defined in section 2, such that the system (φ) coupled with (φ_s) becomes internally exponentially stochastically stable in the mean square. To this end, we use proposition 1 to get the following necessary and sufficient conditions for the internal exponential stability in the mean square of the system (3).

Proposition 3 : System (3) is internally exponentially stabilisable in the mean square by static output-feedback if and only if there exist matrices $\mathcal{P}_{ijk} = \mathcal{P}_{ijk}' > 0$, $\mathbb{X}_k \in \mathbb{S}^q$, $\mathbb{Y}_k \in \mathbb{R}^{q\times r}$ and $\mathbb{Z}_k \in \mathbb{S}^r$ that simultaneously satisfy the following LMI constraints

$$\mathbb{Z}_k > 0 \quad \mathcal{P}_{ijk} > 0 \qquad [13]$$

$$\begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix}' \begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \\ \mathcal{P}_{ijk} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix} < \begin{bmatrix} C_2 & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix}' \begin{bmatrix} \mathbb{X}_k & \mathbb{Y}_k \\ \star & \mathbb{Z}_k \end{bmatrix} \begin{bmatrix} C_2 & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \qquad [14]$$

and the nonlinear inequalities constraints

$$\mathbb{X}_k \leq \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}_k' \qquad [15]$$

$\forall i \in Z$, $j \in S$ and $k \in R$, where

$$\Theta_{ijk} = \sum_{l=1}^{v} \mathbb{W}_{lij}'\mathcal{P}_{ijk}\mathbb{W}_{lij} + \sum_{h\in Z}\pi_{ih}\mathcal{P}_{hjk} + \sum_{l\in S}\nu_{jl}\mathcal{P}_{ilk} + \sum_{v\in R}\lambda_{kv}^{ij}\mathcal{P}_{ijv} \qquad [16]$$

Let $\{\mathcal{P}_{ijk}, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k\}$ be a solution, then the nonempty $\{\mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k\}$-ellipsoids are sets of stabilizing gains. □

Proof : The proof of this proposition follows essentially the same arguments as in (Peaucelle *et al.,* 2005) :
Sufficiency Assume that the constraints (13)-(15) are satisfied for some $\{\mathcal{P}_{ijk}, \mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k\}$ matrices. Due to the properties of matrix ellipsoids, the $\{\mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k\}$-ellipsoids are nonempty. Take any element $\mathcal{K}_k$. The LMI (14) implies that for all $\begin{pmatrix} x_t' & u_t' \end{pmatrix} \neq 0$

$$\begin{pmatrix} x_t \\ A_i x_t + B_j u_t \end{pmatrix}' \begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \\ \mathcal{P}_{ijk} & \mathbf{0} \end{bmatrix} \begin{pmatrix} x_t \\ A_i x_t + B_j u_t \end{pmatrix} < \begin{pmatrix} C_2 x_t \\ u_t \end{pmatrix}' \begin{bmatrix} \mathbb{X}_k & \mathbb{Y}_k \\ \star & \mathbb{Z}_k \end{bmatrix} \begin{pmatrix} C_2 x_t \\ u_t \end{pmatrix} \quad [17]$$

Definition 2 implies that for all nonzero trajectories

$$x_t' \Gamma_{ijk} x_t < y_t' \begin{bmatrix} \mathbb{I} & \mathcal{K}_k' \end{bmatrix} \begin{bmatrix} \mathbb{X}_k & \mathbb{Y}_k \\ \star & \mathbb{Z}_k \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ \mathcal{K}_k \end{bmatrix} y_t \leq 0 \quad [18]$$

$\forall i \in Z, j \in S$ and $k \in R$.
Then, the closed-loop exponential stochastic stability is assessed by proposition 1 for the quadratic stochastic Lyapunov function $\vartheta(\xi_t, \eta_t, \psi_t) = x_t' P(\xi_t, \eta_t, \psi_t) x_t$.

Necessity Assume $\mathcal{K}_k$ are stabilizing static output feedback gains and $\vartheta(\xi_t, \eta_t, \psi_t) = x_t' P(\xi_t, \eta_t, \psi_t) x_t$ is a stochastic Lyapunov fonction. Then from proposition 1, we have

$$\begin{bmatrix} \mathcal{K}_k C_2 & -\mathbb{I} \end{bmatrix} \begin{pmatrix} x_t \\ u_t \end{pmatrix} = 0$$
$$\Rightarrow \begin{pmatrix} x_t \\ u_t \end{pmatrix}' \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix}' \begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \\ \mathcal{P}_{ijk} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix} \begin{pmatrix} x_t \\ u_t \end{pmatrix} < 0 \quad [19]$$

$\forall i \in Z, j \in S$ and $k \in R$.
Applying the well known Finsler Lemma (Skelton *et al.,* 1998), there exist scalars τ_{ijk} such that

$$\begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix}' \begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \\ \mathcal{P}_{ijk} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_j \end{bmatrix} < \tau_{ijk} \begin{bmatrix} \mathcal{K}_k C_2 & -\mathbb{I} \end{bmatrix}' \begin{bmatrix} \mathcal{K}_k C_2 & -\mathbb{I} \end{bmatrix} \leq \varepsilon_k \begin{bmatrix} \mathcal{K}_k C_2 & -\mathbb{I} \end{bmatrix}' \begin{bmatrix} \mathcal{K}_k C_2 & -\mathbb{I} \end{bmatrix} \quad [20]$$

where $\varepsilon_k = \max_{i,j}(\tau_{ijk})$. The inequality (14) is obtained with

$$\mathbb{X}_k = \varepsilon_k \mathcal{K}_k' \mathcal{K}_k, \quad \mathbb{Y}_k = -\varepsilon_k \mathcal{K}_k', \quad \mathbb{Z}_k = \varepsilon_k \mathbb{I}$$

The bottom-right block implies $\mathbf{0} < \mathbb{Z}_k$. Hence the proof is complete. ■

Remark 3 : The results developed above can be easily applied to the *mode-independent* static output feedback stochastic stabilization of MJLS. Indeed, let us consider the following closed loop dynamical model

$$\varphi_{cl} : \begin{cases} dx(t) = \bar{A}(\phi_t)x(t)dt + \bar{E}(\phi_t)w(t)dt + \sum_{l=1}^{v} \mathbb{W}_l(\phi_t)x(t)d\varpi_l(t) \\ y(t) = C_2(\phi_t)x(t) + D_2(\phi_t)w(t) \\ z(t) = \bar{C}_1(\phi_t)x(t) + \bar{D}_1(\phi_t)w(t) \end{cases} \tag{21}$$

where

$$\begin{bmatrix} \bar{A}(\phi_t) & \bar{E}(\phi_t) \\ \bar{C}_1(\phi_t) & \bar{D}_1(\phi_t) \end{bmatrix} = \begin{bmatrix} A(\phi_t) & E(\phi_t) \\ C_1(\phi_t) & 0 \end{bmatrix} + \begin{bmatrix} B(\phi_t) \\ D_1(\phi_t) \end{bmatrix} \mathcal{K} \begin{bmatrix} C_2(\phi_t) & D_2(\phi_t) \end{bmatrix}$$

The process ϕ_t represents a continuous time discrete state Markov process with values in a finite set $H = \{1, ..., h\}$ with transition probability rate matrix $\Xi = [\Phi]_{i,j=1,...,h}$. In this case, the transition probability for the jump process, ϕ_t, can be defined as :

$$p_{kj}(\Delta t) = \Phi_{kj}\Delta t + \circ(\Delta t) \;\; (k \neq j) \tag{22}$$

with $\sum\limits_{\substack{j \in H \\ j \neq i}} \Phi_{ij} = -\Phi_{ii} = \Phi_i$.
Then, the following corollary can be stated

Corollary 1 : System (21) is internally exponentially stabilisable in the mean square by static output-feedback if and only if there exist matrices $\mathcal{P}_i = \mathcal{P}_i' > 0$, $\mathbb{X} \in \mathbb{S}^q$, $\mathbb{Y} \in \mathbb{R}^{q \times r}$ and $\mathbb{Z} \in \mathbb{S}^r$ that simultaneously satisfy the following LMI constraints

$$\mathbb{Z} > 0 \quad \mathcal{P}_i > 0 \tag{23}$$

$$\begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_i \end{bmatrix}' \begin{bmatrix} \Theta_i & \mathcal{P}_i \\ \mathcal{P}_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ A_i & B_i \end{bmatrix} < \begin{bmatrix} C_{2i} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix}' \begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ \star & \mathbb{Z} \end{bmatrix} \begin{bmatrix} C_{2i} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} \tag{24}$$

and the nonlinear inequalities constraints

$$\mathbb{X} \leq \mathbb{Y}\mathbb{Z}^{-1}\mathbb{Y}' \tag{25}$$

$\forall i \in H$, where

$$\Theta_i = \sum_{l=1}^{v} \mathbb{W}_{li}' \mathcal{P}_i \mathbb{W}_{li} + \sum_{v \in H} \Phi_{iv}\mathcal{P}_v \tag{26}$$

Let $\{\mathcal{P}_i, \mathbb{X}, \mathbb{Y}, \mathbb{Z}\}$ be a solution, then the nonempty $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}\}$-ellipsoid is a set of stabilizing gains. $\Diamond$

5. The $\mathcal{H}_\infty$ Control Problem

Let us consider the system (3) with

$$z(t) = z_\infty(t) = C_{\infty 1}x(t) + D_{\infty 1}(\eta(t))u(y(t), \psi(t), t)$$

$z_\infty(t)$ stands for the controlled output related to $\mathcal{H}_\infty$ performance.
In this section, we deal with the design of controllers that stochastically stabilize the closed-loop system and guarantee the disturbance rejection, with a certain level $\gamma_\infty > 0$. Mathematically, we are concerned with the characterization of compensators φ_c that stochastically stabilize the system (3) and guarantee the following for all $w \in L^2[0, \infty)$:

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{\int_0^\infty z'_{\infty t} z_{\infty t} dt\right\}^{1/2} < \gamma_\infty \| w \|_2 \qquad [27]$$

where $\gamma_\infty > 0$ is a prescribed level of disturbance attenuation to be achieved. To this end, we need the auxiliary result given by the following proposition.

Proposition 4 : If there exist symmetric positive-definite matrices $\mathcal{P}_{\infty ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\begin{bmatrix} \Upsilon_{ijk} & \bar{C}'_{1jk}\bar{D}_{1ijk} + \mathcal{P}_{\infty ijk}\bar{E}_{ijk} \\ \star & \bar{D}'_{1ijk}\bar{D}_{1ijk} - \gamma_\infty^2 \mathbb{I} \end{bmatrix} = \Phi_{ijk} < 0 \qquad [28]$$

where

$$\Upsilon_{ijk} = \tilde{A}'_{ijk}\mathcal{P}_{\infty ijk} + \mathcal{P}_{\infty ijk}\tilde{A}_{ijk} + \sum_{l=1}^{v} \mathbb{W}'_{lij}\mathcal{P}_{\infty ijk}\mathbb{W}_{lij} + \bar{C}'_{1jk}\bar{C}_{1jk} + \sum_{h \in Z} \pi_{ih}\mathcal{P}_{\infty hjk} + \sum_{l \in S} \nu_{jl}\mathcal{P}_{\infty ilk} + \sum_{v \in R} \lambda_{kv}^{ij}\mathcal{P}_{\infty ijv}$$

$\forall i \in Z$, $j \in S$ and $k \in R$.
then the system (3) is stochastically stable and satisfies

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{\int_0^\infty z'_{\infty t} z_{\infty t} dt\right\}^{1/2} < \gamma_\infty \| w \|_2 \qquad [29]$$

□

Proof : See (Aberkane *et al.,* 2005a). ■

Using the previous proposition, the following $\mathcal{H}_\infty$ control result can be stated.

Proposition 5 : If there exist matrices $\mathcal{P}_{\infty ijk} = \mathcal{P}'_{\infty ijk} > 0$, $\mathbb{X}_k \in \mathbb{S}^q$, $\mathbb{Y}_k \in \mathbb{R}^{q \times r}$ and $\mathbb{Z}_k \in \mathbb{S}^r$ that simultaneously satisfy the following LMI constraints

$$\mathbb{Z}_k > 0; \quad \mathcal{P}_{\infty ijk} > 0 \qquad [30]$$

$$\mathbb{M}'_{1ij}\begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{\infty ijk} \\ \mathcal{P}_{\infty ijk} & \mathbf{0} \end{bmatrix}\mathbb{M}_{1ij} < \mathbb{M}'_{2j}\begin{bmatrix} -\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \gamma_\infty^2\mathbb{I} \end{bmatrix}\mathbb{M}_{2j} + \mathbb{M}'_{3ij}\begin{bmatrix} \mathbb{X}_k & \mathbb{Y}_k \\ \star & \mathbb{Z}_k \end{bmatrix}\mathbb{M}_{3ij} \quad [31]$$

and the nonlinear inequalities constraints

$$\mathbb{X}_k \leq \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}'_k \quad [32]$$

$\forall i \in Z$, $j \in S$ and $k \in R$, where

$$\mathbb{M}_{1ij} = \begin{bmatrix} \mathbb{I} & \mathbf{0} & \mathbf{0} \\ A_i & E_{ij} & B_j \end{bmatrix}, \quad \mathbb{M}_{2j} = \begin{bmatrix} C_{\infty 1} & \mathbf{0} & D_{\infty 1j} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{bmatrix},$$
$$\mathbb{M}_{3ij} = \begin{bmatrix} C_2 & D_{2ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{bmatrix}.$$

then the $\{\mathbb{X}_k, \mathbb{Y}_k, \mathbb{Z}_k\}$-ellipsoids are sets of stabilizing gains such that

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{\int_0^\infty z'_{\infty t} z_{\infty t} dt\right\}^{1/2} < \gamma_\infty \| w \|_2 \quad [33]$$

□

Proof : The proof of this proposition follows the same arguments as for the proof of proposition 3. ■

Remark 4 : As for the internal stochastic stabilization problematic, the *mode-independent* static output feedback $\mathcal{H}_\infty$ control of MJLS can be solved in the same way as for AFTCSMP. This result is illustrated by corollary 2.

Corollary 2 : If there exist matrices $\mathcal{P}_{\infty i} = \mathcal{P}'_{\infty i} > 0$, $\mathbb{X} \in \mathbb{S}^q$, $\mathbb{Y} \in \mathbb{R}^{q\times r}$ and $\mathbb{Z} \in \mathbb{S}^r$ that simultaneously satisfy the following LMI constraints

$$\mathbb{Z} > 0 \quad \mathcal{P}_{\infty i} > 0 \quad [34]$$

$$\mathbb{M}'_{1i}\begin{bmatrix} \Theta_i & \mathcal{P}_{\infty i} \\ \mathcal{P}_{\infty i} & \mathbf{0} \end{bmatrix}\mathbb{M}_{1i} < \mathbb{M}'_{2i}\begin{bmatrix} -\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \gamma_\infty^2\mathbb{I} \end{bmatrix}\mathbb{M}_{2i} + \mathbb{M}'_{3i}\begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ \star & \mathbb{Z} \end{bmatrix}\mathbb{M}_{3i} \quad [35]$$

and the nonlinear inequalities constraints

$$\mathbb{X} \leq \mathbb{Y}\mathbb{Z}^{-1}\mathbb{Y}' \quad [36]$$

$\forall i \in H$, where

$$\mathbb{M}_{1i} = \begin{bmatrix} \mathbb{I} & \mathbf{0} & \mathbf{0} \\ A_i & E_i & B_i \end{bmatrix}, \quad \mathbb{M}_{2i} = \begin{bmatrix} C_{\infty 1i} & \mathbf{0} & D_{\infty 1i} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{bmatrix},$$
$$\mathbb{M}_{3i} = \begin{bmatrix} C_{2i} & D_{2i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{bmatrix}.$$

then the $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}\}$-ellipsoid is a set of stabilizing gains such that

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{\int_0^\infty z'_{\infty t} z_{\infty t} dt\right\}^{1/2} < \gamma_\infty \| w \|_2 \qquad [37]$$

$\Diamond$

6. Computational Issues and Examples

6.1. *A Cone Complementary Algorithm*

The numerical examples are solved using a first order iterative algorithm. It is based on a cone complementary technique (Ghaoui *et al.,* 1997), that allows to concentrate the non convex constraint in the criterion of some optimisation problem.

Lemma 1 : The problem (30)-(32) is feasible if and only if zero is the global optimum of the optimisation problem

$$\begin{cases} \min \quad \text{tr}(\mathbb{TS}) \\ \text{s.t.} \quad (30),\ (31) \\ \mathbb{X}_k \leq \widehat{\mathbb{X}}_k \quad \mathcal{S}_k = \begin{bmatrix} \widehat{\mathbb{X}}_k & \mathbb{Y}_k \\ \star & \mathbb{Z}_k \end{bmatrix} \geq 0 \\ \mathcal{T}_{1k} \geq \mathbb{I} \quad \mathcal{T}_k = \begin{bmatrix} \mathcal{T}_{1k} & \mathcal{T}_{2k} \\ \star & \mathcal{T}_{3k} \end{bmatrix} \geq 0 \end{cases} \qquad [38]$$

where

$$\mathbb{S} = \text{diag}\{\mathcal{S}_1, \ldots, \mathcal{S}_r\}, \quad \mathbb{T} = \text{diag}\{\mathcal{T}_1, \ldots, \mathcal{T}_r\}$$

$\blacklozenge$

Proof : The proof of this Lemma follows the same arguments as in (Peaucelle *et al.,* 2005). With the constraints $\mathcal{T}_k \geq 0$ and $\mathcal{S}_k \geq 0$, we have that $\mathbb{T} \geq 0$ and $\mathbb{S} \geq 0$ which induce the following implications

$$\text{tr}(\mathbb{TS}) = 0 \Rightarrow \mathbb{TS} = \mathbf{0} \Rightarrow \mathcal{T}_k \mathcal{S}_k = \mathbf{0}, \quad \forall k \in R \qquad [39]$$

Therefore, after some manipulations, one gets

$$\widehat{\mathbb{X}}_k = -\mathcal{T}_{1k}^{-1}\mathcal{T}_{2k}\mathbb{Y}'_k = -\mathcal{T}_{1k}^{-1}(-\mathcal{T}_{1k}\mathbb{Y}_k\mathbb{Z}_k^{-1})\mathbb{Y}'_k = \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}'_k$$

Thus the nonlinear constraints is satisfied

$$\mathbb{X}_k \leq \widehat{\mathbb{X}}_k = \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}'_k$$

The converse implication is proved taking $\widehat{\mathbb{X}}_k = \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}'_k$ and $\mathcal{T}_k$ such that $\mathcal{T}_k\mathcal{S}_k = 0,\ \forall k \in R$. ■

As in (Ghaoui *et al.,* 1997, Peaucelle *et al.,* 2005), the optimisation problem (38) can then be solved with a first order conditional gradient algorithm also known as the Franck and Wolfe feasible direction method. Its properties are not reminded here. Not only that the nonlinear objective tr($\mathbb{T}\mathbb{S}$) is relaxed as the linear objective tr($\mathbb{T}_h\mathbb{S} + \mathbb{T}\mathbb{S}_h$). The obtained LMI optimisation is repeated iteratively with matrices $\mathbb{T}_h$ and $\mathbb{S}_h$ computed from each previous optimisation step. The obtained sequence, tr($\mathbb{T}_h\mathbb{S}_h$), is strictly decreasing. However, there is no guarantee that the algorithm converges to the global optimum.

Remark 5 : (Peaucelle *et al.,* 2005) The stoping criteria of the usual gradient algorithm is either related to slow progress of the optimisation objective or to the achievement of tr($\mathbb{T}\mathbb{S}$) $= 0$. In the first case, the algorithm fails due to "*plateauing*" behavior or because it found a non satisfactory local optimum. The second case corresponds to the expected success of the algorithm. Unfortunately, due to the constraints $\mathbb{T} \geq \mathbf{0}$ and $\mathbb{S} \geq \mathbf{0}$ the algorithm is more often stopped while tr($\mathbb{T}\mathbb{S}$) $= \epsilon$ where ϵ is a chosen accuracy level. The exact non linear constraint may then not be exactly satisfied which is a significant weakness of the algorithm.
As a matter of fact, since the equality constraints involving $\widehat{\mathbb{X}}_k$ are not the goal of the original problem (30)-(32), in the numerical example below we adopted the following stoping criteria for the conditional gradient algorithm.

- If tr($\mathbb{T}_{h-1}\mathbb{S}_{h-1} - \mathbb{T}_h\mathbb{S}_h$) is below a chosen level, then STOP, the algorithm failed.
- As soon as $\mathbb{X}_k \leq \mathbb{Y}_k\mathbb{Z}_k^{-1}\mathbb{Y}_k'$, $\forall k \in R$, STOP, required ellipsoids are found.

6.2. *Numerical Examples*

a) Fault Tolerant Control

In this section, the proposed $\mathcal{H}_\infty$ static output feedback control of AFTCSMP is illustrated using a flight control example. Consider the nominal system with

$$A = \begin{bmatrix} -0.0565 & 29.072 & -175.610 & 9.6783 & 1.6022 \\ -0.0601 & -0.7979 & -0.2996 & 0 & 0 \\ 9.218 \times 10^{-3} & -0.0179 & -0.1339 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.1339 & 0.1339 & 2.0092 \\ 2.3491 & -2.3491 & 0.7703 \\ 0.0444 & -0.0444 & -1.3575 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; E = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \\ 0.1 & 0.1 \\ 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C_{\infty 1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbb{W}_1 = 0.1 \times \mathbb{I},$$

$$D_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},\ D_{\infty 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This model is adapted from (Maki *et al.,* 2004). It represents the lateral-directional dynamics of McDonnell F-4C Phantom flying at Mach 0.6 at an altitude of 35000 ft. The states $x_i,\ i = 1, \ldots, 5$ denote the lateral velocity (ft per second), the roll rate (radian per second), yaw rate (radian per second), roll angle (radian) and yaw angle (radian), respectively. The control inputs $u_1,\ u_2$ and u_3 correspond to the left aileron, the right aileron and the rudder surface displacement, respectively.
For illustration purposes, we will consider two faulty modes :

i) **Mode 2** : A 50% power loss on the left aileron ;

ii) **Mode 3** : Right aileron outage.

From above, we have that $S = \{1, 2, 3\}$, where the **mode 1** represents the nominal case. The failure process is assumed to have Markovian transition characteristics. The FDI process is also Markovian with three states $R = \{1, 2, 3\}$.
The actuator failure rates are assumed to be :

$$[\pi_{ij}] = \begin{bmatrix} -0.002 & 0.0010 & 0.0010 \\ 0.0010 & -0.002 & 0.0010 \\ 0.0010 & 0.0010 & -0.002 \end{bmatrix}$$

The FDI conditional transition rates are :

$$[\lambda^1_{ij}] = \begin{bmatrix} -0.02 & 0.01 & 0.01 \\ 1.00 & -1.01 & 0.01 \\ 1.00 & 0.01 & -1.01 \end{bmatrix},\ [\lambda^2_{ij}] = \begin{bmatrix} -1.01 & 1.00 & 0.01 \\ 0.01 & -0.02 & 0.01 \\ 0.01 & 1.00 & -1.01 \end{bmatrix},$$

$$[\lambda^3_{ij}] = \begin{bmatrix} -1.01 & 0.01 & 1.00 \\ 0.01 & -1.01 & 1.00 \\ 0.01 & 0.01 & -0.02 \end{bmatrix}.$$

For the above AFTCSMP, several numerical experiments are performed using the cone complementary algorithm. These tests are realised for various specifications on the $\mathcal{H}_\infty$ performance (γ_∞). Here are presented some cases described in Table 1, where `iter` is the number of the algorithms iterations, `time` is the computation time (LMIs solved with LMI toolbox, Matlab 6.5.1), `Tr(`$\mathbb{TS}$`)` is the value of the optimisation criteria trace ($\mathbb{T}_k\mathbb{S}_k$) at the step when the algorithm stopped, and $\mathcal{K}_{k0},\ k = 1, 2, 3$ are the controllers obtained as the centers of the stabilising ellipsoids.

b) Mode-Independent Control of MJLS

test	γ_∞^2	iter	time(s)	Tr(TS)	$\mathcal{K}'_{10}$	$\mathcal{K}'_{20}$	$\mathcal{K}'_{30}$
1	20	3	28.9060	615.7093	$\begin{bmatrix} -0.7589 & 0.9560 & 0.6559 \end{bmatrix}$	$\begin{bmatrix} -0.7007 & 1.1969 & 0.6718 \end{bmatrix}$	$\begin{bmatrix} -1.2359 & -0.1512 & 0.6826 \end{bmatrix}$
2	10	3	28.8750	810.2702	$\begin{bmatrix} -0.7694 & 1.0038 & 0.6897 \end{bmatrix}$	$\begin{bmatrix} -0.7056 & 1.2640 & 0.7075 \end{bmatrix}$	$\begin{bmatrix} -1.2744 & -0.1605 & 0.7129 \end{bmatrix}$
3	5	3	28.0620	4.1924e+003	$\begin{bmatrix} -0.7843 & 1.1103 & 0.7451 \end{bmatrix}$	$\begin{bmatrix} -0.7187 & 1.3857 & 0.7645 \end{bmatrix}$	$\begin{bmatrix} -1.3583 & -0.1397 & 0.7706 \end{bmatrix}$
4	1	10	126.6410	1.4087e+006	fails	fails	fails

Tableau 1. *Numerical experiments*

We applied the proposed static output feedback $\mathcal{H}_\infty$ control to a VTOL helicopter model adapted from (de Farias *et al.,* 2000). The dynamics can be written as

$$\begin{cases} dx_t = A(\phi_t)x_t dt + B(\phi_t)u(y_t,t)dt + Ew_t dt + \mathbb{W}_1 x_t d\varpi_t \\ y_t = C_2 x_t + D_2 w_t \\ z_{\infty t} = C_{\infty 1} x_t + D_{\infty 1} u(y_t,t) \end{cases}$$

where ϕ_t indicates the airspeed. The parameters are given by

$$A(\phi_t) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & a_{32}(\phi_t) & -0.707 & a_{34}(\phi_t) \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B(\phi_t) = \begin{bmatrix} 0.4422 & 0.1761 \\ b_{21}(\phi_t) & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix},\ E = \begin{bmatrix} 0.0468 & 0 \\ 0.0457 & 0.0099 \\ 0.0437 & 0.0011 \\ -0.0218 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},\ C_{\infty 1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$D_{\infty 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbb{W}_1 = 0.1 \times \mathbb{I}.$$

The behavior of ϕ_t is modelled as a Markov chain with three different states, corresponding to airspeeds of 135 (nominal value), 60, 170, Knots. The values of parameters a_{32}, a_{34}, and b_{32} are shown in Table 2.

Airspeed (Knots)	a_{32}	a_{34}	b_{21}
135	0.3681	1.4200	3.5446
60	0.0664	0.1198	0.9775
170	0.5047	2.5460	5.1120

Tableau 2. *Parameters*

The transition matrix is given by

$$\Xi = \begin{bmatrix} -0.0907 & 0.0671 & 0.0236 \\ 0.0671 & -0.0671 & 0 \\ 0.0236 & 0 & -0.0236 \end{bmatrix}$$

As for the previous example, several numerical experiments are performed using the cone complementary algorithm. These tests are realised for various specifications on the $\mathcal{H}_\infty$ performance (γ_∞). Here are presented some cases described in Table 3, where $\mathcal{K}_0$ is the controller obtained as the center of the stabilising ellipsoid.

test	γ_∞^2	`iter`	`time`(s)	`Tr`($\mathbb{TS}$)	$\mathcal{K}_0'$
1	10	3	2.2810	15.2404	$\begin{bmatrix} 0.4321 & -0.4037 \end{bmatrix}$
2	5	3	1.8750	3.6171	$\begin{bmatrix} 0.2692 & -0.3942 \end{bmatrix}$
3	1	3	4.1250	0.2235	$\begin{bmatrix} 0.2465 & -0.3116 \end{bmatrix}$
4	0.5	11	10.9060	751.6071	fails

Tableau 3. *Numerical experiments*

7. Conclusion

In this paper, the static output feedback $\mathcal{H}_\infty$ control of continuous time AFTCSMP was considered within a new framework. This last one is based on the synthesis of ellipsoidal sets of controllers and was introduced by (Peaucelle *et al.,* 2002, Peaucelle *et al.,* 2005). The problematic resulting from the fact that the controller only depends on the FDI process is shown to be naturally dealt with in this context. It was also shown that the obtained results could easily be applied to the problem of mode-independent static output feedback $\mathcal{H}_\infty$ control of Markovian Jump Linear Systems. The numerical resolution of the obtained results was done using a cone complementary algorithm and its running was illustrated on classical examples from literature.

8. Bibliographie

Aberkane S., Ponsart J., Sauter D., « Output Feedback $\mathcal{H}_\infty$ Control of a Class of Stochastic Hybrid Systems with Wiener Process via Convex Analysis », *Submitted*, 2005a.

Aberkane S., Ponsart J., Sauter D., « Output Feedback Stochastic Stabilization of Active Fault Tolerant Control Systems : LMI Formulation », *16th IFAC World Congress, Prague, Czech Republic*, 2005b.

Aberkane S., Sauter D., Ponsart J., « $\mathcal{H}_\infty$ Stochastic Stabilization of Active Fault Tolerant Control Systems : Convex Approach », *44th IEEE Conference on Decision and Control and European Control Conference ECC 2005, Seville, Spain*, 2005c.

Boukas E. K., « Exponential stabilizability of stochastic systems with Markovian jumping parameters », *Automatica*, vol. 35, p. 1437-1441, 1999.

Boukas E. K., « Stabilization of Stochastic Nonlinear Hybrid Systems », *Int. J. Innovative Computing, Information and Control*, vol. 1, p. 131-141, 2005.

Boukas E. K., « Static Output Feedback Control for Stochastic Hybrid Systems : LMI Approach », *Automatica*, vol. 42, p. 183-188, 2006.

Costa O. L. V., do Val J. B. R., Geromel J. C., « Continuous-time state-feedback $\mathcal{H}_2$-control of Markovian jump linear systems via convex analysis », *Automatica*, vol. 35, p. 259-268, 1999.

de Farias D. P., Geromel J. C., do Val J. B. R., Costa O. L. V., « Output Feedback Control of Markov Jump Linear Systems in Continuous-Time », *IEEE Transactions on Automatic Control*, vol. 45, p. 944-949, 2000.

de Souza C. E., Fragoso M. D., « $\mathcal{H}_\infty$ Control For Linear Systems With Markovian Jumping Parameters », *Control Theory and Advanced Technology*, vol. 9, p. 457-466, 1993.

Ghaoui L. E., Oustry F., AitRami M., « A Cone Complementary Linearization Algorithm for Static Output-Feedback and related Problems », *IEEE Transactions on Automatic Control*, vol. 42, p. 1171-1176, 1997.

Ji Y., Chizeck H. J., « Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control », *IEEE Transactions on Automatic Control*, vol. 35, p. 777-788, 1990.

Ji Y., Chizeck H. J., « Jump linear quadratic Gaussian control in continuous time », *IEEE Transactions on Automatic Control*, vol. 375, p. 1884-1892, 1992.

Mahmoud M., Jiang J., Zhang Y., « Analysis of the Stochastic Stability for Active Fault Tolerant Control Systems », *Proceedings of the* 38^{th} *IEEE Conference on Decision & Control, Phoenix, Arizona USA*, 1999a.

Mahmoud M., Jiang J., Zhang Y., « Optimal Control Law for Fault Tolerant Control Systems », *Proceedings of the* 39^{th} *IEEE Conference on Decision & Control, Sydney, Australia*, 1999b.

Mahmoud M., Jiang J., Zhang Y., « Stochastic Stability Analysis of Active Fault-Tolerant Control Systems in the Presence of Noise », *IEEE Transactions on Automatic Control*, vol. 46, p. 1810-1815, 2001.

Mahmoud M., Jiang J., Zhang Y., *Active Fault Tolerant Control Systems : Stochastic Analysis and Synthesis*, Springer, 2003.

Maki M., Jiang J., Hagino K., « A Stability Guaranteed Active Fault Tolerant Control System Against Actuator Failures », *International Journal of Robust and Nonlinear Control*, vol. 14, p. 1061-1077, 2004.

Peaucelle D., « Formulation générique de problèmes en analyse et commande robuste par des fonctions de Lyapunov dépendant des paramètres », *Ph.D. dissertation, Université Toulouse III-Paul Sabatier, Toulouse, France*, 2000.

Peaucelle D., Arzelier D., « Ellipsoidal Sets for Resilient and Robust Static Output-Feedback », *IEEE Transaction on Automatic Control*, vol. 50, p. 899-904, 2005.

Peaucelle D., Arzelier D., Bertrand R., « Ellipsoidal Sets for Static Output-Feedback », *15th IFAC World Congress, Barcelona, Spain*, 2002.

Shi P., Boukas E. K., « $\mathcal{H}_\infty$-Control for Markovian Jumping Linear Systems with Parametric Uncertainty », *Journal of Optimization Theory and Applications*, vol. 95, p. 75-99, 1997.

Shi P., Boukas E. K., Nguang S. K., Guo X., « Robust disturbance attenuation for discrete-time active fault tolerant control systems with uncertainties », *Optimal Control Applications and Methods*, vol. 24, p. 85-101, 2003.

Skelton R. E., Iwasaki T., Grigoriadis K., *A Unified Algebraic Approach to Linear Control Design*, Taylor and Francis, 1998.

Srichander R., Walker B. K., « Stochastic stability analysis for continuous-time fault tolerant control systems », *International Journal of Control*, vol. 57, p. 433-452, 1993.

Syrmos V. L., Abdallah C. T., Dorato P., Grigoriadis K., « Static Output Feedback : A Survey », *Automatica*, vol. 33, p. 125-137, 1997.

Zhang Y., Jiang J., « Bibliographical review on reconfigurable fault-tolerant control systems », *IFAC SAFEPROCESS*p. 265-276, 2003.

ANNEXE POUR LE SERVICE FABRICATION
A FOURNIR PAR LES AUTEURS AVEC UN EXEMPLAIRE PAPIER
DE LEUR ARTICLE ET LE COPYRIGHT SIGNE PAR COURRIER
LE FICHIER PDF CORRESPONDANT SERA ENVOYE PAR E-MAIL

1. ARTICLE POUR LA REVUE :

L'objet. xx

2. AUTEURS :

Samir Aberkane

3. TITRE DE L'ARTICLE :

Ellipsoidal Output-Feedback Sets for $\mathcal{H}_\infty$ Control of a Class of Stochastic Hybrid Systems with State-Dependant Noise

4. TITRE ABRÉGÉ POUR LE HAUT DE PAGE MOINS DE 40 SIGNES :

$\mathcal{H}_\infty$ Control of Stochastic Hybrid Systems

5. DATE DE CETTE VERSION :

18 mars 2011

6. COORDONNÉES DES AUTEURS :

– adresse postale :
CRAN – CNRS UMR 7039
Université Henri Poincaré, Nancy 1, BP 239,
F-54506 Vandœuvre-lès-Nancy Cedex
Tel.: +33 3 83 68 44 74, Fax: +33 3 83 68 44 61
e-mail: *samir.aberkane@cran.uhp-nancy.fr*
– téléphone : 00 00 00 00 00
– télécopie : 00 00 00 00 00
– e-mail : Roger.Rousseau@unice.fr

7. LOGICIEL UTILISÉ POUR LA PRÉPARATION DE CET ARTICLE :

LaTeX, avec le fichier de style `article-hermes.cls`,
version 1.2 du 03/03/2005.

8. FORMULAIRE DE COPYRIGHT :

Retourner le formulaire de copyright signé par les auteurs, téléchargé sur :
`http://www.revuesonline.com`

SERVICE ÉDITORIAL – HERMES-LAVOISIER
14 rue de Provigny, F-94236 Cachan cedex
Tél : 01-47-40-67-67
E-mail : revues@lavoisier.fr
Serveur web : http://www.revuesonline.com