Multi-Objective Output Feedback Control of a Class of Stochastic Hybrid Systems with State Dependent Noise

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This paper deals with dynamic output feedback control of continuous time Active Fault Tolerant Control Systems with Markovian Parameters (AFTCSMP) and state-dependent noise. The main contribution is to formulate conditions for multi-performance design, related to this class of stochastic hybrid systems, that take into account the problematic resulting from the fact that the controller only depends on the FDI (Fault Detection and Isolation) process. The specifications and objectives under consideration include stochastic stability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ (or more generally, *Stochastic Integral Quadratic Constraints*) performances. Results are formulated as matrix inequalities. The theoretical results are illustrated using a classical example from literature.

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1 Introduction

Control engineers are faced with increasingly complex systems where dependability considerations are sometimes more important than performance. Sensor, actuator or process (plant) failures may drastically change the system behavior, resulting in performance degradation or even instability. Thus, fault tolerance is essential for modern, highly complex control systems. Fault Tolerant Control Systems (FTCS) are needed in order to preserve or maintain the performance objectives, or if that turns out to be impossible, to assign new (achievable) objectives so as to avoid catastrophic failures. FTCS have been a subject of great practical importance, which has attracted a lot of interest for the last three decades. A bibliographical review on reconfigurable fault tolerant control systems can be found in [39].

Active fault tolerant control systems are feedback control systems that reconfigure the control law in real time based on the response from an automatic fault detection and identification (FDI) scheme. The dynamic behaviour of active fault tolerant control systems (AFTCS) is governed by stochastic differential equations (because the failures and failure detection occur randomly) and can be viewed as a general hybrid system [34]. A major class of hybrid systems is jump linear systems (JLS). In JLS, a single jump process is used to describe the random variations affecting the system parameters. This process is represented by a finite state Markov chain and is called the plant regime mode. The theory of stability, optimal control and $\mathcal{H}_2/\mathcal{H}_\infty$ control, as well as important applications of such systems, can be found in several papers in the current literature, for instance in [5, 6, 7, 8, 9, 12, 13, 14, 22, 23]. To deal with AFTCS, another class of hybrid systems was defined, denoted as active fault tolerant control systems with Markovian parameters (AFTCSMP). In this class of hybrid systems, two random processes are defined: the first random process represents system components failures and the second random process represents the FDI process used to reconfigure the control law. This model was proposed by Srichander and Walker [34]. Necessary and sufficient conditions for stochastic stability of AFTCSMP were developed for a single component failure (actuator failures). In [26], the authors proposed a dynamical model that takes into account multiple failures occurring at different locations in the system, such as in control actuators and plant components. The authors derived necessary and sufficient conditions for the stochastic stability in the mean square sense. The problem of stochastic

stability of AFTCSMP in the presence of noise, parameter uncertainties, detection errors, detection delays and actuator saturation limits has also been investigated in [26, 29, 30, 31]. Another issue related to the synthesis of fault tolerant control laws was also addressed by [27, 32, 33]. In [27], the authors designed an optimal control law for AFTCSMP using the matrix minimum principle to minimize an equivalent deterministic cost function. The problem of $\mathcal{H}_\infty$ and robust $\mathcal{H}_\infty$ control (in the presence of norm bounded parameter uncertainties) was treated in [32, 33] for both continuous and discrete time AFTCSMP. The authors defined a single failure process to characterize random failures affecting the system, and they showed that the state feedback control problem can be solved in terms of the solutions of a set of coupled Riccati inequalities. The dynamic/static output feedback counterpart was treated by [1, 2, 3, 4] in a convex programming framework. Indeed, the authors provide an LMI characterization of dynamical/static output feedback compensators that stochastically stabilize (robustly stabilize) the AFTCMP and ensures $\mathcal{H}_\infty$ (robust $\mathcal{H}_\infty$) constraints. In addition, it is important to mention that the design problem in the framework of AFTCSMP remains an open and challenging problematic. This is due, particulary, to the fact that the controller only depends on the FDI process *i.e.* the number of controllers to be designed is less than the total number of the closed loop systems modes by combining both failure an FDI processes. The design problem involves searching feasible solutions of a problem where there are more constraints than variables to be solved. Generally speaking, there lacks tractable design methods for this stochastic FTC problem. Indeed, in [1, 2, 31, 32, 33], the authors make the assumption that the controller must access both failures and FDI processes. However, this assumption is too restrictive to be applicable in practical FTC systems. In this note, and inspired by the work of [15] on *mode-independent* $\mathcal{H}_\infty$ filtering for Markovian jumping linear systems, the assumption on the availability of failure processes, for the synthesis purposes, is stressed. The results are based on a version of the well known Finsler's lemma and a special parametrization of the Lyapunov matrices. This note is concerned with dynamic output feedback control of continuous time AFTCSMP with state-dependent noise. The main contribution is to formulate conditions for multi-performance design, related to this class of stochastic hybrid systems, that take into account the problematic resulting from the fact that the controller only depends on the FDI process. The specifications and objectives under consideration include stochastic stability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ (or more generally, *Stochastic Integral Quadratic Constraints*) performances. Results are formulated as matrix inequalities. A coordinate descent-type algorithm is provided and its running is illustrated on a VTOL helicopter example.
This paper is organized as follows: Section 2 describes the dynamical model of the system with appropriately defined random processes. A brief summary of basic stochastic terms, results and definitions are given in Section 3. Section 4 considers the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. In Section 5, a coordinate descent-type algorithm is provided and its running is illustrated on a classical example from literature. Finally, a conclusion is given in Section 6.

Notations. The notations in this paper are quite standard. $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices. A^T is the transpose of the matrix A. The notation $X \geq Y$ ($X > Y$, respectively), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (positive definite, respectively); $\mathbb{I}$ and $\mathbf{0}$ are identity and zero matrices of appropriate dimensions, respectively; $\mathcal{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure P; $L^2[0,\infty)$ stands for the space of square-integrable vector functions over the interval $[0,\infty)$; $\|\cdot\|$ refers to either the Euclidean vector norm or the matrix norm, which is the operator norm induced by the standard vector norm; $\|\cdot\|_2$ stands for the norm in $L^2[0,\infty)$; while $\|\cdot\|_{\mathcal{E}_2}$ denotes the norm in $L^2((\Omega,\mathcal{F},P),[0,\infty))$; $(\Omega,\mathcal{F},P)$ is a probability space. In block matrices, $\star$ indicates symmetric terms: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & \star \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ \star & C \end{bmatrix}$.

2 Dynamical Model of AFTCSMP with State Dependent Noise

To describe the class of linear systems with Markovian jumping parameters that we deal with in this paper, let us fix a complete probability space $(\Omega, \mathcal{F}, P)$. This class of systems owns a hybrid state vector. The first component vector is continuous and represents the system states, and the second one is discrete and represents the failure processes affecting the system. The dynamical model of the AFTCSMP with Wiener Process, defined in the fundamental probability space $(\Omega, \mathcal{F}, P)$, is described by the following differential equations:

$$\varphi : \begin{cases} dx(t) = A(\xi(t))x(t)dt + B(\eta(t))u(y(t), \psi(t), t)dt + E(\xi(t), \eta(t))w(t)dt + \sum_{l=1}^{v} \mathbb{W}_l(\xi(t), \eta(t))x(t)d\varpi_l(t) \\ y(t) = C_2 x(t) + D_2(\xi(t), \eta(t))w(t) \\ z(t) = C_1 x(t) + D_1(\eta(t))u(y(t), \psi(t), t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(y(t), \psi(t), t) \in \mathbb{R}^r$ is the system input, $y(t) \in \mathbb{R}^q$ is the system measured output, $z(t) \in \mathbb{R}^p$ is the controlled output, $w(t) \in \mathbb{R}^m$ is the system external disturbance, $\xi(t)$, $\eta(t)$ and $\psi(t)$ represent the plant component failure process, the actuator failure process and the FDI process, respectively. $\xi(t)$, $\eta(t)$ and $\psi(t)$ are separable and mesurable Markov processes with finite state spaces $Z = \{1, 2, ..., z\}$, $S = \{1, 2, ..., s\}$ and $R = \{1, 2, ..., \sigma\}$, respectively. $\varpi(t) = [\varpi_1(t) \dots \varpi_v(t)]^T$ is a v-dimensional standard Wiener process on a given probability space $(\Omega, \mathcal{F}, P)$, that is assumed to be independent of the Markov processes. The matrices $A(\xi(t))$, $B(\eta(t))$, $E(\xi(t), \eta(t))$, $D_2(\xi(t), \eta(t))$, $D_1(\eta(t))$ and $\mathbb{W}_l(\xi(t), \eta(t))$ are properly dimensioned matrices which depend on random parameters.
In AFTCS, we consider that the control law is only a function of the mesurable FDI process $\psi(t)$. Therefore, we introduce a full order dynamic output feedback compensator (φ_d) of the form:

$$\varphi_d : \begin{cases} dv(t) = A_c(\psi(t))v(t)dt + B_c(\psi(t))y(t)dt \\ u(t) = C_c(\psi(t))v(t) \end{cases} \tag{2}$$

Applying the controller φ_d to the AFTCSMP φ, we obtain the following closed loop system:

$$\varphi_{cl} : \begin{cases} d\chi(t) = \Lambda(\xi(t), \eta(t), \psi(t))\chi(t)dt + \bar{E}(\xi(t), \eta(t), \psi(t))w(t)dt + \sum_{l=1}^{v} \overline{\mathbb{W}}_l(\xi(t), \eta(t))\chi(t)d\varpi_l(t) \\ \bar{y}(t) = \bar{C}_2(\psi(t))\chi(t) + \bar{D}_2(\xi(t), \eta(t))w(t) \\ z(t) = \bar{C}_1(\eta(t), \psi(t))\chi(t) \end{cases} \tag{3}$$

where:

$$\chi(t) = [x(t)^T, v(t)^T]^T; \bar{y}(t) = [y(t)^T, u(t)^T]^T; \Lambda(\xi(t), \eta(t), \psi(t)) = \begin{bmatrix} A(\xi(t)) & B(\eta(t))C_c(\psi(t)) \\ B_c(\psi(t))C_2 & A_c(\psi(t)) \end{bmatrix};$$

$$\bar{E}(\xi(t), \eta(t), \psi(t)) = \begin{bmatrix} E(\xi(t), \eta(t)) \\ B_c(\psi(t))D_2(\xi(t), \eta(t)) \end{bmatrix}; \bar{C}_2(\psi(t)) = \begin{bmatrix} C_2 & 0 \\ 0 & C_c(\psi(t)) \end{bmatrix};$$

$$\bar{D}_2(\xi(t), \eta(t)) = \begin{bmatrix} D_2(\xi(t), \eta(t)) \\ 0 \end{bmatrix}; \bar{C}_1(\eta(t), \psi(t)) = \begin{bmatrix} C_1 & D_1(\eta(t))C_c(\psi(t)) \end{bmatrix}; \overline{\mathbb{W}}_l(\xi(t), \eta(t)) = \begin{bmatrix} \mathbb{W}_l(\xi(t), \eta(t)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Our goal is to compute dynamical output feedback controllers φ_d that meet various specifications on the closed loop behavior. The specifications and objectives under consideration include stochastic stability, $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ performance (or more generally, *Stochastic Integral Quadratic Constraints* (SIQC)).

The FDI and the Failure Processes

$\xi(t)$, $\eta(t)$ and $\psi(t)$ being homogeneous Markov processes with finite state spaces, we can define the transition probability of the plant components failure process as [31, 34]:

$$\begin{cases} p_{ij}(\Delta t) = \pi_{ij}\Delta t + o(\Delta t) & (i \neq j) \\ p_{ii}(\Delta t) = 1 - \sum\limits_{i \neq j} \pi_{ij}\Delta t + o(\Delta t) & (i = j) \end{cases}$$

The transition probability of the actuator failure process is given by:

$$\begin{cases} p_{kl}(\Delta t) = \nu_{kl}\Delta t + o(\Delta t) & (k \neq l) \\ p_{kk}(\Delta t) = 1 - \sum\limits_{k \neq l} \nu_{kl}\Delta t + o(\Delta t) & (k = l) \end{cases}$$

where π_{ij} is the plant components failure rate, and ν_{kl} is the actuator failure rate.
Given that $\xi = k$ and $\eta = l$, the conditional transition probability of the FDI process $\psi(t)$ is:

$$\begin{cases} p_{iv}^{kl}(\Delta t) = \lambda_{iv}^{kl}\Delta t + o(\Delta t) & (i \neq v) \\ p_{ii}^{kl}(\Delta t) = 1 - \sum\limits_{i \neq v} \lambda_{iv}^{kl}\Delta t + o(\Delta t) & (i = v) \end{cases}$$

Here, λ_{iv}^{kl} represents the transition rate from i to v for the Markov process $\psi(t)$ conditioned on $\xi = k \in Z$ and $\eta = l \in S$. Depending on the values of i, $v \in R$, $k \in Z$ and $l \in S$, various interpretations, such as rate of false detection and isolation, rate of correct detection and isolation, false alarm recovery rate, etc, can be given to λ_{iv}^{kl} [31, 34].
For notational simplicity, we will denote $A(\xi(t)) = A_i$ when $\xi(t) = i \in Z$, $B(\eta(t)) = B_j$ and $D_1(\eta(t)) = D_{1j}$ when $\eta(t) = j \in S$, $E(\xi(t), \eta(t)) = E_{ij}$, $D_2(\xi(t), \eta(t)) = D_{2ij}$ and $\mathbb{W}_l(\xi(t), \eta(t)) = \mathbb{W}_{lij}$ when $\xi(t) = i \in Z, \eta(t) = j \in S$ and $A_c(\psi(t)) = A_{ck}$, $B_c(\psi(t)) = B_{ck}$, $C_c(\psi(t)) = C_{ck}$ when $\psi(t) = k \in R$. We also denote $x(t) = x_t$, $y(t) = y_t$, $z(t) = z_t$, $w(t) = w_t$, $\xi(t) = \xi_t$, $\eta(t) = \eta_t$, $\psi(t) = \psi_t$ and the initial conditions $x(t_0) = x_0$, $\xi(t_0) = \xi_0$, $\eta(t_0) = \eta_0$ and $\psi(t_0) = \psi_0$.

3 Definitions and Basic Results

In this section, we will first give a basic definition related to stochastic stability notion and then we will summarize some results about exponential stability in the mean square sense of the AFTCSMP.

3.1 Stochastic Stability

For system (1), when $u_t \equiv 0$ for all $t \geq 0$, we have the following definition.

Definition 1: System (1) is said to be *internally exponentially stable in the mean square sense (IESS)*, if there exist positive constants α and β such that the solution of

$$dx_t = A(\xi_t)x_t dt + \sum_{l=1}^{v} \mathbb{W}_l(\xi_t, \eta_t)x_t d\varpi_{lt}$$

satisfies the following inequality

$$\mathcal{E}\left\{\|x_t\|^2\right\} \leq \beta\|x_0\|^2 \exp\left[-\alpha(t - t_0)\right] \tag{4}$$

for arbitrary initial conditions $(x_0, \xi_0, \eta_0, \psi_0)$.

The following theorem gives a sufficient condition for internal exponential stability in the mean square sense for the closed loop system (3).

Theorem 1: *The closed loop system* (3) *is IESS for* $t \geq t_0$ *if there exists a Lyapunov function* $\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t)$ *such that*

$$K_1\|\chi_t\|^2 \leq \vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) \leq K_2\|\chi_t\|^2 \tag{5}$$

and

$$\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) \leq -K_3\|\chi_t\|^2 \tag{6}$$

for some positive constants K_1, K_2 and K_3, where $\mathcal{L}$ is the weak infinitesimal operator of the joint Markov process $\{\chi_t, \xi_t, \eta_t, \psi_t\}$.

Remark 1: The proof of the Markovian property of the joint process $\{\chi_t, \xi_t, \eta_t, \psi_t\}$ is given for example in [25, 31, 34].

A necessary condition for internal exponential stability in the mean square sense for the closed loop system φ_{cl} is given by theorem 2.

Theorem 2: *If the system* (3) *is IESS, then for any given quadratic positive definite function $W(\chi_t, \xi_t, \eta_t, \psi_t, t)$ in the variables χ_t which is bounded and continuous $\forall t \geq t_0$, $\forall \xi_t \in Z$, $\forall \eta_t \in S$ and $\forall \psi_t \in R$, there exists a quadratic positive definite function $\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t)$ in χ_t that satisfies the conditions in theorem 1 and is such that $\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t, t) = -W(\chi_t, \xi_t, \eta_t, \psi_t, t)$.*

Remark 2: The proofs of these theorems follow the same arguments as in [31, 34] for their proposed stochastic Lyapunov functions, so they are not shown in this paper to avoid repetition.

The following proposition gives a necessary and sufficient condition for internal exponential stability in the mean square sense for the system (3).

Proposition 1: A necessary and sufficient condition for IESS of the system (3) is that there exists symmetric positive-definite matrices $\mathcal{P}_{ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that:

$$\tilde{\Lambda}_{ijk}^T \mathcal{P}_{ijk} + \mathcal{P}_{ijk}\tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{ijk} \overline{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih}\mathcal{P}_{hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl}\mathcal{P}_{ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij}\mathcal{P}_{ijv} < 0 \tag{7}$$

$\forall i \in Z$, $j \in S$ and $k \in R$, where

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} - 0.5\mathbb{I}\left(\sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij}\right) \tag{8}$$

□

Proof: The proof of this proposition is easily deduced from theorems 1 and 2. ■

Proposition 2: If the system (3) is IESS, for every $w = \{w_t; t \geq 0\} \in L_2[0, \infty)$, we have that $\chi = \{\chi_t; t \geq 0\} \in L_2((\Omega, \mathcal{F}, P), [0, \infty))$, i.e., $\mathcal{E}\left\{\int_0^\infty \chi_t^T \chi_t dt\right\} < \infty$, for any initial conditions $(\chi_0, \xi_0, \eta_0, \psi_0)$. □

Proof: The proof of this proposition follows the same lines as for the proof of proposition 4 in [3].■

We conclude this section by recalling a version of the well known Finsler's lemma that will be used in the derivation of the main results of this paper.

Lemma 1 [15]: Given matrices $\Psi_{ijk} = \Psi_{ijk}^T \in \mathbb{R}^{n \times n}$ and $H_{ijk} \in \mathbb{R}^{m \times n}$, $\forall i \in Z$, $j \in S$ and $k \in R$, then

$$x_t^T \Psi_{ijk} x_t < 0, \ \forall x_t \in \mathbb{R}^n : \ H_{ijk} x_t = 0, \ x_t \neq 0; \tag{9}$$

if and only if there exist matrices $L_{ijk} \in \mathbb{R}^{n \times m}$ such that:

$$\Psi_{ijk} + L_{ijk}H_{ijk} + H_{ijk}^T L_{ijk}^T < 0, \ \forall i \in Z, \ j \in S, \ k \in R. \tag{10}$$

Note that conditions (10) remain sufficient for (9) to hold even when arbitrary constraints are imposed to the scaling matrices L_{ijk}.

4 The Control Problem

4.1 $\mathcal{H}_\infty$ Control

Let us consider the system (3) with

$$z_t = z_{\infty t} = C_{\infty 1} x_t + D_{\infty 1}(\eta_t) u(y_t, \psi_t, t)$$

$z_{\infty t}$ stands for the controlled output related to $\mathcal{H}_\infty$ performance.
In this section, we deal with the design of controllers that stochastically stabilize the closed-loop system and guarantee the disturbance rejection, with a certain level $\mu > 0$. Mathematically, we are concerned with the characterization of compensators φ_d that stochastically stabilize the system (3) and guarantee the following for all $w \in L^2[0, \infty)$:

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{ \int_0^\infty z_{\infty t}^T z_{\infty t} dt \right\}^{1/2} < \mu \| w \|_2 \tag{11}$$

where $\mu > 0$ is a prescribed level of disturbance attenuation to be achieved. To this end, we need the auxiliary result given by the following proposition.

Proposition 3: If there exists symmetric positive-definite matrices $\mathcal{P}_{\infty ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\begin{aligned} \tilde{\Lambda}_{ijk}^T \mathcal{P}_{\infty ijk} + \mathcal{P}_{\infty ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{\infty ijk} \overline{\mathbb{W}}_{lij} + \bar{C}_{\infty 1jk}^T \bar{C}_{\infty 1jk} + \mu^{-2} \mathcal{P}_{\infty ijk} \bar{E}_{ijk} \bar{E}_{ijk}^T \mathcal{P}_{\infty ijk} \\ + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{\infty hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{\infty ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{\infty ijv} < 0 \end{aligned} \tag{12}$$

$\forall i \in Z$, $j \in S$ and $k \in R$.
then the system (3) is IESS and satisfies

$$\| z_\infty \|_{\mathcal{E}_2} = \mathcal{E}\left\{ \int_0^\infty z_{\infty t}^T z_{\infty t} dt \right\}^{1/2} < \mu \| w \|_2 \tag{13}$$

□

Proof The proof of this proposition follows the same arguments as in [2]. ■

Remark 3: Given fixed matrices $\mathbb{U} \geq 0$, $\mathbb{V} = \mathbb{V}^T$ and $\mathbb{Q}$, the previous characterization extends to more general quadratic constraints on w_t and z_t of the form

$$J_{SIQC} = \mathcal{E}\left\{ \int_{t_0}^{t_f} \begin{pmatrix} z_t \\ w_t \end{pmatrix}^T \begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix} \begin{pmatrix} z_t \\ w_t \end{pmatrix} dt \right\} < 0 \tag{14}$$

These constraints are known as *stochastic quadratic integral constraints* [11].

The following proposition gives a sufficient condition to (14) to be hold.

Proposition 4: If there exists symmetric positive-definite matrices $\mathcal{P}_{ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\begin{bmatrix} \Theta_{ijk} & \mathcal{P}_{ijk} \overline{E}_{ijk} + \overline{C}_{1jk}^T \mathbb{Q} & \overline{C}_{1jk}^T \Delta \\ \star & \mathbb{V} & \mathbf{0} \\ \star & \star & -\Sigma \end{bmatrix} < 0 \tag{15}$$

where

$$\begin{cases} \mathbb{U} = \Delta\Sigma^{-1}\Delta^T \\ \Theta_{ijk} = \tilde{\Lambda}_{ijk}^T\mathcal{P}_{ijk} + \mathcal{P}_{ijk}\tilde{\Lambda}_{ijk} + \sum_{l=1}^{v}\overline{\mathbb{W}}_{lij}^T\mathcal{P}_{ijk}\overline{\mathbb{W}}_{lij} + \sum\limits_{\substack{h\in Z\\ h\neq i}}\pi_{ih}\mathcal{P}_{hjk} + \sum\limits_{\substack{l\in S\\ l\neq j}}\nu_{jl}\mathcal{P}_{ilk} + \sum\limits_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}\mathcal{P}_{ijv} \\ \end{cases}$$

then

$$J_{SIQC} < 0$$

□

Proof: Let us consider the following quadratic Lyapunov function

$$\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) = \chi_t^T\mathcal{P}(\xi_t, \eta_t, \psi_t)\chi_t \tag{16}$$

then:

$$\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t) = \begin{pmatrix} \chi_t \\ w_t \end{pmatrix}^T \begin{pmatrix} \Theta(\xi_t, \eta_t, \psi_t) & \mathcal{P}(\xi_t, \eta_t, \psi_t\overline{E}(\xi_t, \eta_t, \psi_t \\ \star & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi_t \\ w_t \end{pmatrix} \tag{17}$$

adding and subtracting $\mathcal{E}\Big\{\int_{t_0}^{t_f}\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t)dt\Big\}$ to J_{SIQC}, we get

$$\begin{aligned} J_{SIQC} = \mathcal{E}\left\{\int_{t_0}^{t_f}\begin{pmatrix} \chi_t \\ w_t \end{pmatrix}^T\left(\Phi(\xi_t, \eta_t, \psi_t) + \begin{pmatrix} \overline{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}^T\begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix}\begin{pmatrix} \overline{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}\right)\begin{pmatrix} \chi_t \\ w_t \end{pmatrix}dt\right\} \\ - \mathcal{E}\left\{\int_{t_0}^{t_f}\mathcal{L}\vartheta(\chi_t, \xi_t, \eta_t, \psi_t)dt\right\} \end{aligned} \tag{18}$$

where

$$\Phi(\xi_t, \eta_t, \psi_t) = \begin{pmatrix} \Theta(\xi_t, \eta_t, \psi_t) & \mathcal{P}(\xi_t, \eta_t, \psi_t\overline{E}(\xi_t, \eta_t, \psi_t \\ \star & \mathbf{0} \end{pmatrix}$$

From Dynkin's formula, we have

$$\mathcal{E}\left\{\vartheta(\chi_{t_f}, \xi_{t_f}, \eta_{t_f}, \psi_{t_f})\right\} - \vartheta(\chi_{t_0}, \xi_{t_0}, \eta_{t_0}, \psi_{t_0}) = \mathcal{E}\left\{\int_{t_0}^{t_f}\mathcal{L}\vartheta(\xi_t, \eta_t, \psi_t)dt\right\} \tag{19}$$

Assuming, without loss of generality, that $\vartheta(\chi_{t_0}, \xi_{t_0}, \eta_{t_0}, \psi_{t_0}) = 0$, it follows from (18) and (19) that if

$$\Phi(\xi_t, \eta_t, \psi_t) + \begin{pmatrix} \overline{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}^T\begin{pmatrix} \mathbb{U} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{V} \end{pmatrix}\begin{pmatrix} \overline{C}_1(\eta_t, \psi_t) & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{pmatrix} < 0 \tag{20}$$

then (14) holds.
Finally, by factorizing $\mathbb{U} \geq 0$ as

$$\mathbb{U} = \Delta\Sigma^{-1}\Delta^T$$

and by Schur complement property, (20) is equivalent to (15). Hence, the proof is complete. ■

The $\mathcal{H}_\infty$ output feedback fault tolerant control method to be developed in this paper is based on Lemma 1 and with the following parametrization of the Lyapunov matrices $\mathcal{P}_{\infty ijk}$:

$$\mathcal{P}_{\infty ijk} = \mathcal{M}_k^T\mathcal{N}_{\infty ijk}^{-1}\mathcal{M}_k > 0 \tag{21}$$

where $\mathcal{N}_{\infty ijk}$ are symmetric positive definite matrices and $\mathcal{M}_k$ are nonsingular matrices. This parametrization is inspired by the work of [15] on *mode-independent* $\mathcal{H}_\infty$ filtering for Markovian jumping linear systems.
The next proposition presents the main result of this section, which is derived from proposition 3 with $\mathcal{P}_{\infty ijk}$ as in (21) and using Finsler's lemma together with appropriate parametrization of matrices

$\mathcal{M}_k$ and the dynamical controller matrices.

Proposition 5: If there exists matrices $\mathbb{R}_k$, $\mathbb{S}_k$, $\mathbb{D}_k$, $\mathbb{X}_k$, $\mathbb{Y}_k$, $\mathbb{Z}_k$ and symmetric matrices $\mathbb{N}_{\infty ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\begin{bmatrix} \Omega_k + \Omega_k^T & \star & \star & \star & \star \\ \mathbb{A}_{ijk} - \mathbb{N}_{\infty ijk} & \delta_{ijk}\mathbb{N}_{\infty ijk} & \star & \star & \star \\ \mathbf{0} & \mathbb{E}_{ijk}^T & -\mu^2\mathbb{I} & \star & \star \\ \mathbb{C}_{ijk} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & \star \\ \Xi_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{bmatrix} < 0 \tag{22}$$

where

$$\Omega_k = \begin{bmatrix} \mathbb{R}_k & 0 \\ \mathbb{S}_k + \mathbb{D}_k & \mathbb{D}_k \end{bmatrix} \tag{23}$$

$$\mathbb{A}_{ijk} = \begin{bmatrix} \mathbb{R}_k A_i + \mathbb{R}_k B_j \mathbb{Z}_k & \mathbb{R}_k B_j \mathbb{Z}_k \\ \mathbb{S}_k A_i + \mathbb{Y}_k C_2 + \mathbb{X}_k + \mathbb{S}_k B_j \mathbb{Z}_k & \mathbb{X}_k + \mathbb{S}_k B_j \mathbb{Z}_k \end{bmatrix} \tag{24}$$

$$\mathbb{E}_{ijk} = \begin{bmatrix} \mathbb{R}_k E_{ij} \\ \mathbb{S}_k E_{ij} + \mathbb{Y}_k D_{2ij} \end{bmatrix} \tag{25}$$

$$\mathbb{C}_{ijk} = \begin{bmatrix} C_{\infty 1} + D_{\infty 1}\mathbb{Z}_k & D_{\infty 1}\mathbb{Z}_k \end{bmatrix} \tag{26}$$

$$\Gamma_{ijk} = \operatorname{diag}\{\urcorner_{1ijk}, \urcorner_{2ijk}, \urcorner_{3ijk}, \urcorner_{4ijk}\} \tag{27}$$

$$\begin{cases} \urcorner_{1ijk} = \left[\mathbb{N}_{\infty 1jk}, \dots \mathbb{N}_{\infty (i-1)jk}, \mathbb{N}_{\infty (i+1)jk}, \dots \mathbb{N}_{\infty zjk}\right] \\ \urcorner_{2ijk} = \left[\mathbb{N}_{\infty i1k}, \dots \mathbb{N}_{\infty i(j-1)k}, \mathbb{N}_{\infty i(j+1)k}, \dots \mathbb{N}_{\infty isk}\right] \\ \urcorner_{3ijk} = \left[\mathbb{N}_{\infty ij1}, \dots \mathbb{N}_{\infty ij(k-1)}, \mathbb{N}_{\infty ij(k+1)}, \dots \mathbb{N}_{\infty ijr}\right] \\ \urcorner_{4ijk} = \left[\mathbb{N}_{\infty ijk}, \dots, \mathbb{N}_{\infty ijk}\right] \\ \Xi_{ijk} = \left[\digamma_{1ijk}, \digamma_{2ijk}, \digamma_{3ijk}, \digamma_{4ijk}\right]^T \\ \digamma_{1ijk} = \left[\sqrt{\pi_{i1}}\Omega_k^T, \dots, \sqrt{\pi_{i(i-1)}}\Omega_k^T, \sqrt{\pi_{i(i+1)}}\Omega_k^T, \dots, \sqrt{\pi_{iz}}\Omega_k^T\right] \\ \digamma_{2ijk} = \left[\sqrt{\nu_{j1}}\Omega_k^T, \dots, \sqrt{\nu_{j(j-1)}}\Omega_k^T, \sqrt{\nu_{j(j+1)}}\Omega_k^T, \dots, \sqrt{\nu_{js}}\Omega_k^T\right] \\ \digamma_{3ijk} = \left[\sqrt{\lambda_{k1}^{ij}}\Omega_k^T, \dots, \sqrt{\lambda_{k(k-1)}^{ij}}\Omega_k^T, \sqrt{\lambda_{k(k+1)}^{ij}}\Omega_k^T, \dots, \sqrt{\lambda_{kr}^{ij}}\Omega_k^T\right] \\ \digamma_{4ijk} = \left[\Phi_{1ijk}^T, \dots, \Phi_{vijk}^T\right] \\ \Phi_{lijk} = \begin{bmatrix} \mathbb{R}_k\mathbb{W}_{lij} & \mathbf{0} \\ \mathbb{S}_k\mathbb{W}_{lij} & \mathbf{0} \end{bmatrix} \\ \delta_{ijk} = -\left(\sum_{\substack{h\in Z \\ h\neq i}} \pi_{ih} + \sum_{\substack{l\in S \\ l\neq j}} \nu_{jl} + \sum_{\substack{v\in R \\ v\neq k}} \lambda_{kv}^{ij}\right) \end{cases} \tag{28}$$

and $\urcorner_{4ijk}$, $\digamma_{4ijk}$ contain v elements. Then, the system (3) is IESS and satisfies (13).
Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k\left(s\mathbb{I} - \mathbb{D}_k^{-1}\mathbb{X}_k\right)^{-1}\mathbb{D}_k^{-1}\mathbb{Y}_k, \quad \forall k \in R. \tag{29}$$

□

Proof: It will be shown that if the inequalities (22) hold, then the controllers (29) ensure that conditions (12) of proposition 3 are satisfied with a matrix $\mathcal{P}_{\infty ijk} > 0$ as in (21).
First, note that with a matrix $\mathcal{P}_{\infty ijk} > 0$ as in (21), the inequalities (12) are equivalent to

$$\Pi_{ijk}^T \Upsilon_{ijk} \Pi_{ijk} < 0 \tag{30}$$

where

$$\Pi_{ijk} = \begin{bmatrix} \mathbb{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathcal{N}_{\infty ijk}^{-1}\mathcal{M}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{I} \end{bmatrix} \tag{31}$$

$$\Upsilon_{ijk} = \begin{bmatrix} \mathbf{0} & \star & \mathbf{0} & \overline{C}_{\infty 1jk}^T & \mathcal{R}_{\infty ijk} \\ \mathcal{M}_k\Lambda_{ijk} & \delta_{ijk}\mathcal{N}_{\infty ijk} & \mathcal{M}_k\overline{E}_{ijk} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mu^2\mathbb{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\mathbb{I} & \mathbf{0} \\ \star & \star & \star & \star & -\mathcal{S}_{\infty ijk} \end{bmatrix} \tag{32}$$

$$\begin{cases} \mathcal{R}_{\infty ijk} = [\mathcal{R}_{\infty 1ijk}, \mathcal{R}_{\infty 2ijk}, \mathcal{R}_{\infty 3ijk}, \mathcal{R}_{\infty 4ijk}] \\ \mathcal{R}_{\infty 1ijk} = \left[\sqrt{\pi_{i1}}\mathcal{M}_k^T, \ldots, \sqrt{\pi_{i(i-1)}}\mathcal{M}_k^T, \sqrt{\pi_{i(i+1)}}\mathcal{M}_k^T, \ldots, \sqrt{\pi_{iz}}\mathcal{M}_k^T\right] \\ \mathcal{R}_{\infty 2ijk} = \left[\sqrt{\nu_{j1}}\mathcal{M}_k^T, \ldots, \sqrt{\nu_{j(j-1)}}\mathcal{M}_k^T, \sqrt{\nu_{j(j+1)}}\mathcal{M}_k^T, \ldots, \sqrt{\nu_{js}}\mathcal{M}_k^T\right] \\ \mathcal{R}_{\infty 3ijk} = \left[\sqrt{\lambda_{k1}^{ij}}\mathcal{M}_k^T, \ldots, \sqrt{\lambda_{k(k-1)}^{ij}}\mathcal{M}_k^T, \sqrt{\lambda_{k(k+1)}^{ij}}\mathcal{M}_k^T, \ldots, \sqrt{\lambda_{kr}^{ij}}\mathcal{M}_k^T\right] \\ \mathcal{R}_{\infty 4ijk} = \left[\overline{\mathbb{W}}_{1ij}^T\mathcal{M}_k^T, \ldots, \overline{\mathbb{W}}_{vij}^T\mathcal{M}_k^T\right] \\ \mathcal{S}_{\infty ijk} = \mathrm{diag}\,[\mathcal{S}_{\infty 1ijk}, \mathcal{S}_{\infty 2ijk}, \mathcal{S}_{\infty 3ijk}, \mathcal{S}_{\infty 4ijk}] \\ \mathcal{S}_{\infty 1ijk} = \left[\mathcal{N}_{\infty 1jk}, \ldots \mathcal{N}_{\infty (i-1)jk}, \mathcal{N}_{\infty (i+1)jk}, \ldots \mathcal{N}_{\infty zjk}\right] \\ \mathcal{S}_{\infty 2ijk} = \left[\mathcal{N}_{\infty i1k}, \ldots \mathcal{N}_{\infty i(j-1)k}, \mathcal{N}_{\infty i(j+1)k}, \ldots \mathcal{N}_{\infty isk}\right] \\ \mathcal{S}_{\infty 3ijk} = \left[\mathcal{N}_{\infty ij1}, \ldots \mathcal{N}_{\infty ij(k-1)}, \mathcal{N}_{\infty ij(k+1)}, \ldots \mathcal{N}_{\infty ijr}\right] \\ \mathcal{S}_{\infty 4ijk} = [\mathcal{N}_{\infty ijk}, \ldots, \mathcal{N}_{\infty ijk}] \end{cases} \tag{33}$$

$\mathcal{S}_{\infty 4ijk}$ contains v elements.

Define $n \times n$ nonsingular matrices $\mathcal{U}_k$ and $\mathcal{V}_k$ such that $\mathcal{U}_k\mathcal{V}_k = \mathbb{D}_k$ and let the matrix $\mathcal{M}_k$ be parameterized as follows

$$\mathcal{M}_k = \begin{bmatrix} \mathbb{R}_k^{-1} & \mathbb{R}_k^{-1}\mathcal{V}_k^T \\ -\mathcal{U}_k^{-1}\mathbb{S}_k\mathbb{R}_k^{-1} & \mathcal{U}_k^{-1}\mathcal{V}_k^T - \mathcal{U}_k^{-1}\mathbb{S}_k\mathbb{R}_k^{-1}\mathcal{V}_k^T \end{bmatrix}^{-1} \tag{34}$$

Moreover, let the transformation matrix

$$\mathcal{T}_k = \begin{bmatrix} \mathbb{R}_k^T & \mathbb{S}_k^T \\ \mathbf{0} & \mathcal{U}_k^T \end{bmatrix} \tag{35}$$

Note that

$$\mathcal{M}_k^{-T}\mathcal{T}_k = \beth_k = \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathcal{V}_k & \mathcal{V}_k \end{bmatrix} \tag{36}$$

and $\mathcal{M}_k$ and $\mathcal{T}_k$ as defined above are nonsingular. Indeed, from (22) it follows that $\Omega_k + \Omega_k^T < 0$, which implies that the matrices $\mathbb{R}_k$ and $\mathbb{D}_k$ are nonsingular. Therefore, in view of the definition of the matrices $\mathcal{U}_k$ and $\mathcal{V}_k$, the matrices $\mathcal{T}_k$ and $\mathcal{M}_k^{-T}\mathcal{T}_k$ are nonsingular and thus $\mathcal{M}_k^{-T}$ are nonsingular matrices as well.

Next, introduce the matrices

$$\mathbb{F}_k = \mathrm{diag}\left\{\beth_k, \beth_k, \mathbb{I}, \mathbb{I}, \bar{\beth}_k\right\} \tag{37}$$

$$\bar{\beth}_k = \{\beth_k, \ldots, \beth_k\} \tag{38}$$

where $\bar{\beth}_k$ contains $((z-1)+(s-1)+(r-1)+v)$ blocks $\beth_k$.

Performing the congruence transformation $\mathbb{F}_k^T(\cdot)\mathbb{F}_k$ on Υ_{ijk}, inequality (30) is equivalent to

$$\kappa^T\mathbb{F}_k^T\Upsilon_{ijk}\mathbb{F}_k\kappa < 0,\ \kappa = \mathbb{F}_k^{-1}\Pi_{ijk}\varsigma,\ \varsigma \neq 0 \tag{39}$$

considering that

$$\left\{\kappa : \kappa = \mathbb{F}_k^{-1}\Pi_{ijk}\varsigma,\ \varsigma \neq 0\right\} = \left\{\kappa : \mathbb{H}_{ijk}\mathbb{F}_k\kappa = 0, \kappa \neq 0\right\} \tag{40}$$

where

$$\mathbb{H}_{ijk} = \left[\begin{array}{cccccc} \mathcal{M}_k & -\mathcal{N}_{\infty ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{41}$$

we have that (39) is equivalent to

$$\kappa^T\mathbb{F}_k^T\Upsilon_{ijk}\mathbb{F}_k\kappa < 0,\ \forall\kappa \neq 0 : \mathbb{H}_{ijk}\mathbb{F}_k\kappa = 0 \tag{42}$$

By lemma 1, (42) holds iff the following inequalities are feasible for some matrices $\overline{\mathbb{L}}_{ijk}$ of appropriate dimensions

$$\mathbb{F}_k^T\Upsilon_{ijk}\mathbb{F}_k + \overline{\mathbb{L}}_{ijk}\mathbb{H}_{ijk}\mathbb{F}_k + \mathbb{F}_k^T\mathbb{H}_{ijk}^T\overline{\mathbb{L}}_{ijk}^T < 0 \tag{43}$$

Without loss of generality, let the matrices $\overline{\mathbb{L}}_{ijk}$ be rewritten as $\overline{\mathbb{L}}_{ijk} = \mathbb{F}_k^T\mathbb{L}_{ijk}$, then (43) is equivalent to

$$\mathbb{F}_k^T\left(\Upsilon_{ijk} + \mathbb{L}_{ijk}\mathbb{H}_{ijk} + \mathbb{H}_{ijk}^T\mathbb{L}_{ijk}^T\right)\mathbb{F}_k < 0 \tag{44}$$

Setting

$$\mathbb{L}_{ijk} = \left[\begin{array}{cccccc} \mathbb{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{45}$$

inequalities (44) become

$$\mathbb{F}_k^T\overline{\Upsilon}_{ijk}\mathbb{F}_k < 0 \tag{46}$$

where

$$\overline{\Upsilon}_{ijk} = \left[\begin{array}{ccccc} \mathcal{M}_k + \mathcal{M}_k^T & \star & \star & \star & \star \\ \mathcal{M}_k\Lambda_{ijk} - \mathcal{N}_{\infty ijk}^T & \delta_{ijk}\mathcal{N}_{\infty ijk} & \star & \star & \star \\ \mathbf{0} & \overline{E}_{ijk}^T\mathcal{M}_k^T & -\mu^2\mathbb{I} & \star & \star \\ \overline{C}_{\infty 1jk} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & \star \\ \mathcal{R}_{\infty ijk}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathcal{S}_{\infty ijk} \end{array}\right] \tag{47}$$

Consider the following state-space realization for the controllers (2)

$$A_{ck} = \mathcal{V}_k\mathbb{D}_k^{-1}\mathbb{X}_k\mathcal{V}_k^{-1},\ B_{ck} = \mathcal{V}_k\mathbb{D}_k^{-1}\mathbb{Y}_k,\quad C_{ck} = \mathbb{Z}_k\mathcal{V}_k^{-1} \tag{48}$$

and let the matrix $\mathbb{N}_{\infty ijk}$ be defined as

$$\mathbb{N}_{\infty ijk} = \mathcal{T}_k^T\mathcal{M}_k^{-1}\mathcal{N}_{\infty ijk}\mathcal{M}_k^{-T}\mathcal{T}_k \tag{49}$$

By performing straightforward matrix manipulations, it can be easily shown that

$$\mathcal{T}_k^T\Lambda_{ijk}\mathcal{M}_k^{-T}\mathcal{T}_k = \mathbb{A}_{ijk},\quad \mathcal{T}_k^T\overline{E}_{ijk} = \mathbb{E}_{ijk},\quad \overline{C}_{\infty 1jk}\mathcal{M}_k^{-T}\mathcal{T}_k = \mathbb{C}_{ijk} \tag{50}$$

$$\mathcal{T}_k^T\mathcal{M}_k^{-T}\mathcal{T}_k = \Omega_k,\quad \mathbb{J}_k^T\mathcal{R}_{\infty ijk}\mathbb{J}_k = \Xi_{ijk}^T \tag{51}$$

$$\overline{\mathbb{J}}_k^T\mathcal{S}_{\infty ijk}\overline{\mathbb{J}}_k = \Gamma_{ijk} \tag{52}$$

Next, taking into account (37), (38) and (49)-(52), it can be readily verified that (46) is identical to (22). Thus (12) is satisfied with $\mathcal{P}_{\infty ijk} = \mathcal{M}_k^T\mathcal{N}_{\infty ijk}^{-1}\mathcal{M}_k$. Finally, the controller transfer matrix of (29) is readily obtained from (48). ■

From practical point of view, the controller that stochastically stabilizes the AFTCMP and at the same time guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$\mathcal{O}_\infty : \left\{\begin{array}{l} \min\limits_{\tau>0,\ \mathbb{N}_{\infty ijk}=\mathbb{N}_{\infty ijk}^T>0,\ \mathbb{R}_k,\ \mathbb{S}_k,\ \mathbb{D}_k,\ \mathbb{X}_k,\ \mathbb{Y}_k,\ \mathbb{Z}_k} \tau \\ \text{s.t :} \\ \left[\begin{array}{ccccc} \Omega_k + \Omega_k^T & \star & \star & \star & \star \\ \mathbb{A}_{ijk} - \mathbb{N}_{\infty ijk} & \delta_{ijk}\mathbb{N}_{\infty ijk} & \star & \star & \star \\ \mathbf{0} & \mathbb{E}_{ijk}^T & -\tau\mathbb{I} & \star & \star \\ \mathbb{C}_{ijk} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & \star \\ \Xi_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{array}\right] < 0 \end{array}\right.$$

where the matrices inequalities in the constraints are obtained from (22) by replacing μ^2 by τ. This leads to the following Corollary:

Corollary 1: Let $\tau > 0$, $\mathbb{N}_{\infty ijk} = \mathbb{N}_{\infty ijk}^T > 0$, $\mathbb{R}_k$, $\mathbb{S}_k$, $\mathbb{D}_k$, $\mathbb{X}_k$, $\mathbb{Y}_k$, $\mathbb{Z}_k$ be the solution of the optimization problem $\mathcal{O}_\infty$. Then, the controller (2) stochastically stabilizes the AFTCSMP we are considering and moreover the closed loop system satisfies the disturbance rejection of level $\sqrt{\tau}$. $\Diamond$

Remark 4: The above result can be easily extended to general SIQ constraints. This is illustrated by the following corollary:

Corollary 2: If there exists matrices $\mathbb{R}_k$, $\mathbb{S}_k$, $\mathbb{D}_k$, $\mathbb{X}_k$, $\mathbb{Y}_k$, $\mathbb{Z}_k$ and symmetric matrices $\mathbb{N}_{ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\begin{bmatrix} \Omega_k + \Omega_k^T & \star & \star & \star & \star \\ \mathbb{A}_{ijk} - \mathbb{N}_{ijk} & \delta_{ijk}\mathbb{N}_{ijk} & \star & \star & \star \\ \mathcal{C}_{ijk} & \mathbb{E}_{ijk}^T & \mathbb{V} & \star & \star \\ \overline{\mathbb{C}}_{ijk} & \mathbf{0} & \mathbf{0} & -\Sigma & \star \\ \Xi_{ijk} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\Gamma_{ijk} \end{bmatrix} < 0 \tag{53}$$

where

$$\begin{cases} \mathcal{C}_{ijk} = \mathbb{Q}^T \mathbb{C}_{ijk} \\ \overline{\mathbb{C}}_{ijk} = \Delta^T \mathbb{C}_{ijk} \end{cases}$$

Then, the SIQ contraints (14) are verified.
Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k \left(s\mathbb{I} - \mathbb{D}_k^{-1}\mathbb{X}_k\right)^{-1} \mathbb{D}_k^{-1}\mathbb{Y}_k, \quad \forall k \in R. \tag{54}$$

$\Diamond$

4.2 $\mathcal{H}_2$ Control

Before introducing the main results of this section, let us consider the following definition which represents a generalization of the $\mathcal{H}_2$-norm from Markovian jump linear systems [12, 16, 17] to AFTCSMP. Let us consider the system (3) with

$$z_t = z_{2t} = C_{21}x_t + D_{21}(\eta_t)u(y_t, \psi_t, t)$$

z_{2t} stands for the controlled output related to $\mathcal{H}_2$ performance.

Definition 2: *We define the $\mathcal{H}_2$-norm of the IESS system (φ_{cl}) as*

$$\| \varphi_{cl} \|_2^2 = \sum_{d=1}^{m} \sum_{i,j,k} \mu_{ijk} \| z_{d,i,j,k} \|_{\mathcal{E}_2}^2$$

where $z_{d,i,j,k}$ represents the output $\{z_t; t \geq 0\}$ when:

a) *the input is given by $w_t = \{w_t; t \geq 0\}$, $w_t = e_d\delta_t$, δ_t the unitary impulse, and e_d the m-dimensional unitary vector formed by 1 at the dth position and zero elsewhere*;

b) *$\chi_0 = 0$, $\xi_0 = i$, $\eta_0 = j$, $\psi_0 = k$ and $\mu = (\mu_{111}, \ldots, \mu_{szr})$ is the initial distribution of the joint Markov process.*

From the definition above and using the same arguments as in [12, 16, 17], we can state the following corollary.

Corollary 3: Assume that (φ_{cl}) is *IESS* then

i) $\| \varphi_{cl} \|_2^2 = \sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\overline{E}_{ij}^T \mathcal{P}_{oijk} \overline{E}_{ij})$, where $\mathcal{P}_o = \{\mathcal{P}_{o111}, \ldots, \mathcal{P}_{oszr}\}$ denotes the observability Gramian, *i.e.*, $\mathcal{P}_{oijk}$ are the unique positive semidefinite solutions of the following equations

$$\tilde{\Lambda}_{ijk}^T \mathcal{P}_{oijk} + \mathcal{P}_{oijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{oijk} \overline{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{ijv} + \overline{C}_{21}^T \overline{C}_{21} = 0 \tag{55}$$

$\forall i \in Z$, $j \in S$ and $k \in R$.

ii) $\| \varphi_{cl} \|_2^2 < \sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\overline{E}_{ij}^T \mathcal{P}_{2ijk} \overline{E}_{ij})$, where $\mathcal{P}_{2ijk}$ is a positive definite solution of the following matrix inequality

$$\tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \overline{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \overline{C}_{21}^T \overline{C}_{21} < 0 \tag{56}$$

$\forall i \in Z$, $j \in S$ and $k \in R$.

iii) If there exists positive definite matrices $\mathcal{P}_{2ijk}$, and matrices A_{ck}, B_{ck} and C_{ck} such that

$$\sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\overline{E}_{ij}^T \mathcal{P}_{2ijk} \overline{E}_{ij}) < \gamma^2$$

$$\tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \overline{\mathbb{W}}_{lij} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \overline{C}_{21}^T \overline{C}_{21} < 0$$

$\forall i \in Z$, $j \in S$ and $k \in R$. Then φ_{dk} are stabilizing controllers such that $\| \varphi_{cl} \|_2 < \gamma$.

iv) The $\mathcal{H}_2$ output feedback control problem is solved by the following optimization problem

$$\begin{cases} \min \sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\mathcal{Z}_{ijk}) \\ \text{s.t:} \\ \overline{E}_{ij}^T \mathcal{P}_{2ijk} \overline{E}_{ij} < \mathcal{Z}_{ijk} \\ \tilde{\Lambda}_{ijk}^T \mathcal{P}_{2ijk} + \mathcal{P}_{2ijk} \tilde{\Lambda}_{ijk} + \sum_{l=1}^{v} \overline{\mathbb{W}}_{lij}^T \mathcal{P}_{2ijk} \overline{\mathbb{W}}_{lij} + \sum\limits_{\substack{h \in Z \\ h \neq i}} \pi_{ih} \mathcal{P}_{2hjk} + \sum\limits_{\substack{l \in S \\ l \neq j}} \nu_{jl} \mathcal{P}_{2ilk} + \sum\limits_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \mathcal{P}_{2ijv} + \overline{C}_{21}^T \overline{C}_{21} < 0 \end{cases} \tag{57}$$

$\Diamond$

From definition 2 and corollary 3, we can state the following result which solves the $\mathcal{H}_2$ output feedback control problem.

Proposition 6: The $\mathcal{H}_2$ output feedback control problem is solved by the following optimization

problem

$$\mathcal{O}_2 : \begin{cases} \min\limits_{\mathcal{Z}_{ijk},\, \mathbb{N}_{2ijk},\, \mathbb{R}_k,\, \mathbb{S}_k,\, \mathbb{D}_k,\, \mathbb{X}_k,\, \mathbb{Y}_k,\, \mathbb{Z}_k} \ \sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\mathcal{Z}_{ijk}) \\ \text{s.t :} \\ \begin{bmatrix} \mathcal{Z}_{ijk} & \star \\ \mathbb{E}_{ijk} & \mathbb{N}_{2ijk} \end{bmatrix} > 0 \\ \begin{bmatrix} \Omega_k + \Omega_k^T & \star & \star & \star \\ \mathbb{A}_{ijk} - \mathbb{N}_{2ijk} & \delta_{ijk}\mathbb{N}_{2ijk} & \mathbf{0} & \mathbf{0} \\ \mathbb{C}_{ijk} & \star & -\mathbb{I} & \mathbf{0} \\ \Xi_{ijk} & \star & \star & -\Gamma_{ijk} \end{bmatrix} < 0 \end{cases} \tag{58}$$

$\forall i \in Z$, $j \in S$ and $k \in R$.
If (58) hold then $\| \varphi_{cl} \|_2^2 < \sum_{i,j,k} \mu_{ijk} \mathrm{tr}(\overline{E}_{ij}^T \mathcal{P}_{2ijk} \overline{E}_{ij})$. Moreover, the transfer functions matrices of suitable controllers are given by

$$H_k(s) = \mathbb{Z}_k \left(s\mathbb{I} - \mathbb{D}_k^{-1}\mathbb{X}_k \right)^{-1} \mathbb{D}_k^{-1}\mathbb{Y}_k, \quad \forall k \in R. \tag{59}$$

□

Proof: The proof of this proposition follows the same arguments as for the proof of proposition 5.■

Remark 5: We can see from definition 2 that the considered $\mathcal{H}_2$-norm depends on the the initial distribution of the joint Markov process. However, these distributions are, in general, unknown. This inconvenient could be avoided by replacing, for example, the performance criteria by the one as in (19) in [17] (when the input is the unit variance white noise). Proposition 6 could be then easily applied for this $\mathcal{H}_2$-norm definition. The only difference is that, in the optimization problem (58), the initial distribution μ_{ijk} is replaced by $\varepsilon_{ijk} = \sum_{l,h,v} \tilde{p}_{(lhv)(ijk)}$ where

$$\tilde{P} = [\tilde{p}_{(lhv)(ijk)}] = \lim_{t \to \infty} P(t)$$

and $P(t)$ is the probability transition matrix of the joint Markov process.
Indeed, the $\mathcal{H}_2$-norm definition considered in this case is given by

$$\| \varphi_{cl} \|_2^2 = \lim_{T \to \infty} \sum_{i,j,k} \mathcal{E} \left[\int_0^T \| z_s \|^2 \, ds \mid \xi_0 = i, \eta_0 = j, \psi_0 = k \right]$$

Using the same arguments as in [17], the following characterization of the $\mathcal{H}_2$-norm in terms of observability Gramian can be given:

$$\| \varphi_{cl} \|_2^2 = \sum_{i,j,k} \varepsilon_{ijk} \mathrm{tr}(\overline{E}_{ij}^T \mathcal{P}_{oijk} \overline{E}_{ij}) \tag{60}$$

4.3 Multi-Objective Synthesis

The multi-objective synthesis problem amounts to find common controllers that stochastically stabilize the system and ensure $\mathcal{H}_2/\mathcal{H}_\infty$ performances. This multi-performance synthesis problem can be stated as follows:

Given positive scalars α_2 and α_∞, find stabilizing dynamic output feedback controllers φ_d that solve the following constrained optimization problem

$$\begin{cases} \min\limits_{\gamma_2, \gamma_\infty, A_{ck}, B_{ck}, C_{ck}} \ \alpha_2\gamma_2 + \alpha_\infty\gamma_\infty \\ \text{s.t :} \\ \| z_\infty \|_{\mathcal{E}_2} < \gamma_\infty \| w \|_2, \quad \| \varphi_{cl} \|_2 < \gamma_2 \end{cases} \tag{61}$$

The result is straightforward. It amounts to the collection of all related matrix inequality constraints.

Remark 6: Notice that the developed synthesis conditions are only sufficient. This is due to the fact that the controllers only depend on the FDI process, *i.e.*, the number of controllers to be designed is less than the total number of the closed loop system modes by combining both failures an FDI processes.

5 Computational Issues and Example

The inequality conditions in propositions 5 and 6 are not linear in the variables and it is difficult to verify these conditions directly. However, the characterizations given in these propositions have the following nice properties:

i) The given parametrization enables us to express these norm minimization problems in closed form, *i.e.*, all variables explicitly appear in constraints (22) and (58), which is not possible with projection-like conditions;

ii) The $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem with fixed matrices $C_c(\psi_t)$ (synthesis problem similar to the one considered by [36] in the case of linear time invariant systems) is a LMI problem;

iii) With $\mathbb{Z}_k$ fixed, conditions (22) and (58) are LMIs in $\mathcal{Z}_{ijk}$, $\mathbb{N}_{2ijk}$, $\mathbb{N}_{\infty ijk}$, $\mathbb{R}_k$, $\mathbb{S}_k$, $\mathbb{D}_k$, $\mathbb{X}_k$, $\mathbb{Y}_k$.

iV) With $\mathbb{R}_k$ and $\mathbb{S}_k$ fixed, conditions (22) and (58) are LMIs in $\mathcal{Z}_{ijk}$, $\mathbb{N}_{2ijk}$, $\mathbb{N}_{\infty ijk}$, $\mathbb{D}_k$, $\mathbb{X}_k$, $\mathbb{Y}_k$, $\mathbb{Z}_k$.

From the two last properties, one can see that the synthesis problem is expressed as a BMI (Bilinear Matrix Inequalities) problem. BMI problems are known to be generally nonconvex and NP-hard [35]. This means that any algorithm which is guaranteed to find a global optimum cannot be expected to have a polynomial time complexity. There exists different approaches to the solution of this problem, which can be classified into global [18, 37, 38] and local [20, 21, 19]. Most of the global algorithms to the BMI problem are varaiations of of the Branch and Bound algorithm [18, 37]. Although the major focus of global search algorithms is the computational complexity, none of them is polynomial time due to the NP-hardness of the problem. As a result, these approaches can currently be applied only to problems with modest size.
Most of the existing local approaches, on the other hand, are computationally fast but, depending on the initial condition, may not converge to the global optimum. The simplest local approach makes use of the fact that by fixing some of the variables, x, the BMI problem becomes convex in the remaining variables y, and vice versa, and iterates between them [21]. The algorithm (coordinate descent-type algorithm) used in this paper belongs to this class of methods. Nevertheless, these types of algorithms, called coordinate descent methods in [21], alternating SDP method in [18], and the dual iteration in [21], are not guaranteed to converge to a local solution [18]. Such an algorithm is given by **Algorithm 1**. For the purposes of its initialization, we adopt the same methodology as in [24]. These can be summarized as follows:

Algorithm 1.

- **Step 0**. Initialization: Set $q = 0$. Design state feedback matrices $\mathbb{Z}_{kq}$ that solves the following optimization problem

$$\begin{cases} \min\limits_{\gamma_{2q},\gamma_{\infty q},\mathbb{Z}_{kq}} \alpha_2\gamma_{2q} + \alpha_\infty\gamma_{\infty q} \\ \text{s.t :} \\ \| z_\infty \|_{\mathcal{E}_2} < \gamma_{\infty q} \| w \|_2, \quad \| \varphi_{cls} \|_2 < \gamma_{2q} \end{cases} \tag{62}$$

  where φ_{cls} is the closed loop system obtained by applying the controller φ_s given by:

$$\varphi_s : \Big\{ u_t = \mathbb{Z}_{kq} x_t$$

to the system (1).

- **Step 1**. Fix the matrices $\mathbb{Z}_{kq}$ and search for a solution to the multi-objective control problem, defined in equation (60), in terms of the remaining unknown matrices $\mathcal{Z}_{ijkq}$, $\mathbb{N}_{2ijkq}$, $\mathbb{N}_{\infty ijkq}$, $\mathbb{R}_{kq}$, $\mathbb{S}_{kq}$, $\mathbb{D}_{kq}$, $\mathbb{X}_{kq}$, $\mathbb{Y}_{kq}$. Set the current cost value $\upsilon_q = \alpha_2\gamma_{2q} + \alpha_\infty\gamma_{\infty q}$.
- **Step 2**. Fix the matrices $\mathbb{R}_{kq}$ and $\mathbb{S}_{kq}$ and search for a solution to the multi-objective control problem, defined in equation (60), in terms of the remaining unknown matrices $\mathcal{Z}_{ijkq}$, $\mathbb{N}_{2ijkq}$, $\mathbb{N}_{\infty ijkq}$, $\mathbb{D}_{kq}$, $\mathbb{X}_{kq}$, $\mathbb{Y}_{kq}$, $\mathbb{Z}_{kq}$. Set the current cost value $\omega_q = \alpha_2\gamma_{2q} + \alpha_\infty\gamma_{\infty q}$.
- **Step 3**. If $\upsilon_q - \omega_q < \epsilon$, $\epsilon > 0$, Stop. Otherwise, set $q \leftarrow q + 1$ and go back to Step 1.

As in the usual coordinate descent methods, the above algorithm generates a non-increasing sequence of the objective function values, and thus the convergence is guaranteed. Note, however, that the limit of the sequence may not be optimal.

A VTOL Example

In this section, the proposed multi-objective dynamic output feedback control of AFTCSMP is illustrated using a VTOL helicopter model adapted from [13]. Consider the nominal system with

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}, \; E = \begin{bmatrix} 0.0468 & 0 \\ 0.0457 & 0.0099 \\ 0.0437 & 0.0011 \\ -0.0218 & 0 \end{bmatrix}$$

$$\mathbb{W}_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_{\infty 1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D_{\infty 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The state vector $x_t \in \mathbb{R}^4$ is composed by the following:

x_1: longitudinal velocity;

x_2: vertical velocity;

x_3: rate of pitch;

x_4: pitch angle.

and the components of command vector are:

u_1: general cyclic command;

u_2: longitudinal cyclic command.

For illustration purposes, we will consider two faulty modes:

i) **Mode 2**: A 50% power loss on the first actuator;

ii) **Mode 3**: A 50% power loss on both actuators.

From above, we have that $S = \{1, 2, 3\}$, where the **mode 1** represents the nominal case. The failure process is assumed to have Markovian transition characteristics. The FDI process is also Markovian with three states $R = \{1, 2, 3\}$.
The actuator failure rates are assumed to be:

$$[\pi_{ij}] = \begin{bmatrix} -0.002 & 0.0010 & 0.0010 \\ 0.0010 & -0.002 & 0.0010 \\ 0.0010 & 0.0010 & -0.002 \end{bmatrix}$$

The FDI conditional transition rates are:

$$[\lambda_{ij}^1] = \begin{bmatrix} -0.02 & 0.01 & 0.01 \\ 1.00 & -1.01 & 0.01 \\ 1.00 & 0.01 & -1.01 \end{bmatrix}, \; [\lambda_{ij}^2] = \begin{bmatrix} -1.01 & 1.00 & 0.01 \\ 0.01 & -0.02 & 0.01 \\ 0.01 & 1.00 & -1.01 \end{bmatrix}, \; [\lambda_{ij}^3] = \begin{bmatrix} -1.01 & 0.01 & 1.00 \\ 0.01 & -1.01 & 1.00 \\ 0.01 & 0.01 & -0.02 \end{bmatrix}.$$

For the above AFTCSMP, and using Algorithm 1 with $\alpha_2 = \alpha_\infty = 1$, we obtain the following $\mathcal{H}_2/\mathcal{H}_\infty$ performances from w_t to z_2 and z_∞ respectively: $\gamma_2 = 1.3813$, $\gamma_\infty = 7.0430$.
The corresponding dynamical controllers are given as follows:

$$\left[\begin{array}{cccc|cc} -4.5605 & 6.1141 & -0.7661 & -2.0089 & 4.6913 & -4.9146 \\ 1.2505 & -13.949 & 2.1057 & 2.8241 & -2.0707 & 10.8691 \\ -0.8586 & 9.0361 & -3.6110 & -6.1882 & 4.1034 & -6.6348 \\ 1.7167 & 0.8173 & 0.8173 & 0.8173 & -1.8753 & -1.2901 \\ \hline -0.5951 & -0.1871 & 0.3333 & 0.4416 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 0.0519 & 0.8206 & -0.6220 & -1.5290 & \multicolumn{2}{c}{} \end{array}\right]_{\psi_t=1}$$

$$\left[\begin{array}{cccc|cc} -1.8554 & -0.8810 & -0.0395 & -0.5837 & 1.6635 & 0.9851 \\ -2.7850 & -8.6549 & 4.1078 & 3.8934 & -1.7314 & 5.2828 \\ 0.2180 & 2.2321 & -2.8481 & -4.2791 & 3.0771 & -1.2354 \\ 1.3433 & 3.3313 & 1.1581 & 0.1551 & -1.4002 & -3.4934 \\ \hline -0.6729 & 0.0591 & 0.1496 & 0.2176 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 2.4528 & 1.4950 & -2.1517 & -3.1308 & \multicolumn{2}{c}{} \end{array}\right]_{\psi_t=2}$$

$$\left[\begin{array}{cccc|cc} -2.0190 & -0.9464 & 0.0742 & -0.4221 & 1.7047 & 0.9938 \\ -0.7222 & -6.3567 & 2.2273 & 1.2006 & -1.3945 & 4.5383 \\ -0.5881 & 0.9657 & -2.1838 & -3.2807 & 2.9401 & -0.5600 \\ 1.3238 & 3.2111 & 1.2362 & 0.1852 & -1.3741 & -3.4574 \\ \hline -0.7787 & 0.0063 & 0.2350 & 0.3908 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 1.8829 & 1.2999 & -1.9038 & -3.0831 & \multicolumn{2}{c}{} \end{array}\right]_{\psi_t=3}$$

where

$$\left[\begin{array}{c|c} \star & \star \\ \hline \star & \mathbf{0} \end{array}\right]_{\psi_t=i}$$

is a realization of the controller (φ_d) for $\psi_t = i$.

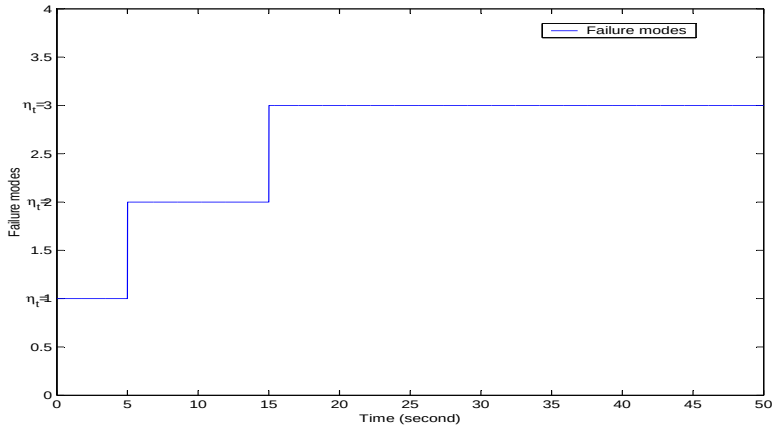

Figure 1: Failure modes

The state trajectories of the closed loop system resulting from the obtained controllers are shown in Figure 3. These trajectories represent a single sample path simulation corresponding to a realization of the failure process η_t and the FDI process ψ_t given by Figure 1 and Figure 2 respectively. Figure 4 represents the evolution of the controlled outputs $z_{\infty t}$. It can be seen that the closed-loop system is stochastically stable and that the disturbance attenuation is achieved.

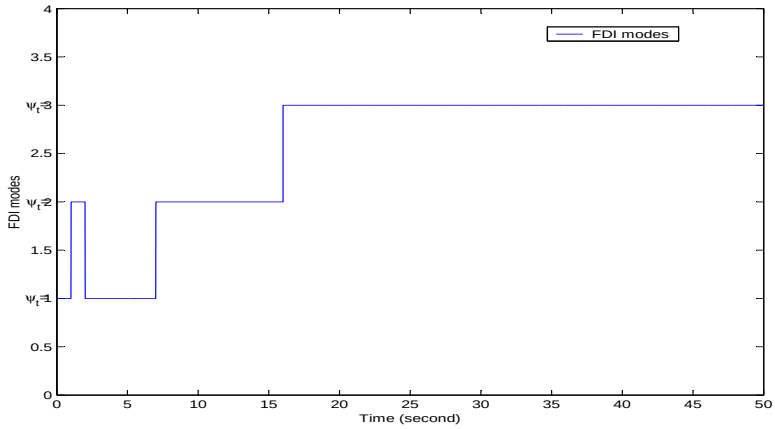

Figure 2: FDI modes

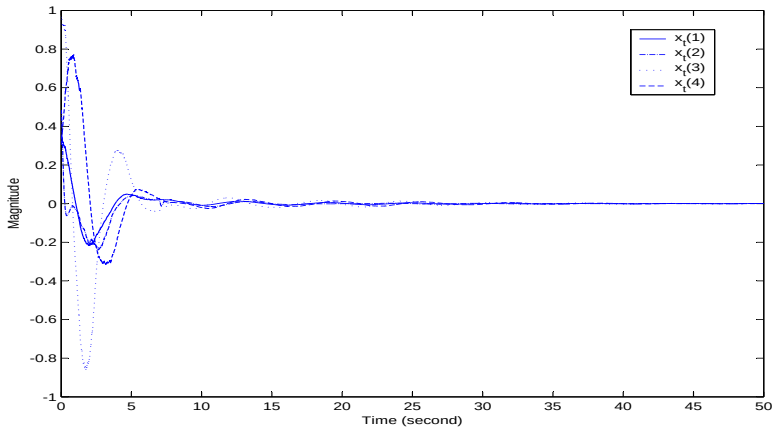

Figure 3: State variables evolution: single sample path simulation

6 Conclusion

In this paper, the dynamic output feedback multi-objective control of continuous time AFTCSMP was considered within a framework that allows to take into account the problematic resulting from the fact that the controller only depends on the FDI process. The specifications and objectives considered, include stochastic stability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances. The main results were derived using a version of the well known Finsler's lemma and a parametrization of the Lyapunov matrices. The numerical resolution of the obtained results was done using a coordinate descent-type algorithm. The effectiveness of the developed method was illustrated on a VTOL helicopter example.

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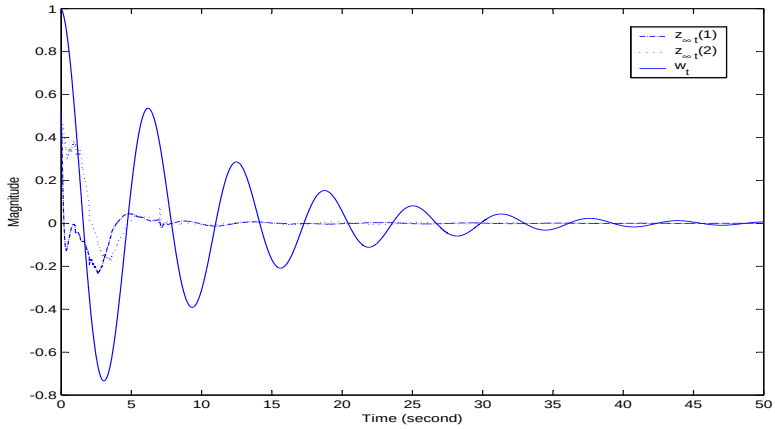

Figure 4: Evolution of the variables $z_{\infty t}$: single sample path simulation

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