New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators

Luong Dang Ky

NEW HARDY SPACES OF MUSIELAK-ORLICZ TYPE AND BOUNDEDNESS OF SUBLINEAR OPERATORS

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ABSTRACT. We introduce a new class of Hardy spaces $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$, called Hardy spaces of Musielak-Orlicz type, which generalize the Hardy-Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Here, $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt A_∞ weight. A function f belongs to $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ if and only if its maximal function f^* is so that $x \mapsto \varphi(x, |f^*(x)|)$ is integrable. Such a space arises naturally for instance in the description of the product of functions in $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ respectively (see [6]). We characterize these spaces via the grand maximal function and establish their atomic decomposition. We characterize also their dual spaces. The class of pointwise multipliers for $BMO(\mathbb{R}^n)$ characterized by Nakai and Yabuta can be seen as the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$ where $H^{\log}(\mathbb{R}^n)$ is the Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \dfrac{t}{\log(e+|x|) + \log(e+t)}$.

   Furthermore, under additional assumption on $\varphi(\cdot,\cdot)$ we prove that if T is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ to $\mathcal{B}$. These results are new even for the classical Hardy-Orlicz spaces on $\mathbb{R}^n$.

1. INTRODUCTION

Since Lebesgue theory of integration has taken a center stage in concrete problems of analysis, the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families naturally arose. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ when $p \in (0,1]$ are good substitutes of $L^p(\mathbb{R}^n)$ when studying the boundedness of operators: for example, the Riesz operators are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. The theory of Hardy spaces H^p on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [45]. Later, Fefferman and Stein [16] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$, which now plays an important role in various fields of analysis and partial differential equations;

see, for example, [12, 13, 39]. A key feature of the classical Hardy spaces is their atomic decomposition characterizations, which were obtained by Coifman [11] when $n = 1$ and Latter [31] when $n > 1$. Later, the theory of Hardy spaces and their dual spaces associated with Muckenhoupt weights have been extensively studied by García-Cuerva [18], Strömberg and Torchinsky [47] (see also [38, 10, 19]); there the weighted Hardy spaces was defined by using the nontangential maximal functions and the atomic decompositions were derived. On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum-Orlicz in [3] and Orlicz in [42], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [2, 24, 33]. Moreover, the Hardy-Orlicz spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis, say, the boundedness of operators.

Let Φ be a Orlicz function which is of positive lower type and (quasi-)concave. In [29], Janson has considered the Hardy-Orlicz space $H^\Phi(\mathbb{R}^n)$ the space of all tempered distributions f such that the nontangential grand maximal function of f is defined by

$$f^*(x) = \sup_{\phi \in \mathcal{A}_N} \sup_{|x-y|<t} |f * \phi_t(y)|,$$

for all $x \in \mathbb{R}^n$, here and in what follows $\phi_t(x) := t^{-n}\phi(t^{-1}x)$, with

$$\mathcal{A}_N = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial_x^\alpha \phi(x)| \leq 1 \text{ for } \alpha \in \mathbb{N}^n, |\alpha| \leq N \right\}$$

with $N = N(n, \Phi)$ taken large enough, belongs to the Orlicz space $L^\Phi(\mathbb{R}^n)$. Remark that these Hardy-Orlicz type spaces appear naturally when studying the theory of nonlinear PDEs (cf. [21, 25, 27]) since many cancellation phenomena for Jacobians cannot be observed in the usual Hardy spaces $H^p(\mathbb{R}^n)$. For instance, let $f = (f^1, ..., f^n)$ in the Sobolev class $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and the Jacobians $J(x, f)dx = df^1 \wedge \cdots \wedge df^n$, then (see [27, Theorem 10.2])

$$\mathcal{T}(J(x, f)) \in L^1(\mathbb{R}^n) + H^\Phi(\mathbb{R}^n)$$

where $\Phi(t) = t/\log(e + t)$ and $\mathcal{T}(f) = f \log |f|$, since $J(x, f) \in H^1(\mathbb{R}^n)$ (cf. [13]) and $\mathcal{T}$ is well defined on $H^1(\mathbb{R}^n)$. We refer readers to [43, 26] for this interesting nonlinear operator $\mathcal{T}$.

In this paper we want to allow generalized Hardy-Orlicz spaces related to generalized Orlicz functions that may vary in the spatial variables. More precisely the Orlicz function $\Phi(t)$ is replaced by a function $\varphi(x, t)$, called Musielak-Orlicz function (cf. [40, 14]). We then define Hardy spaces of Musielak-Orlicz type. Apart from interesting theoretical considerations, the motivation to

study function spaces of Musielak-Orlicz type comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs and the calculus of variation (cf. [14, 15]).

A particular case of Hardy spaces of Musielak-Orlicz type appears naturally when considering the products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [6]); and the endpoint estimates for the div-curl lemma (see [4, 6]). More precisely, in [6] the authors proved that product of a $BMO(\mathbb{R}^n)$ function and a $H^1(\mathbb{R}^n)$ function may be written as a sum of an integrable term and of a term in $H^{\log}(\mathbb{R}^n)$, a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$. Moreover, the corresponding bilinear operators are bounded. This result gives in particular a positive answer to the Conjecture 1.7 in [7]. By duality, one finds pointwise multipliers for $BMO(\mathbb{R}^n)$. Recall that a function g on $\mathbb{R}^n$ is called a pointwise multiplier for $BMO(\mathbb{R}^n)$, if the pointwise multiplication fg belongs to $BMO(\mathbb{R}^n)$ for all f in $BMO(\mathbb{R}^n)$. In [41], Nakai and Yabuta characterize the pointwise multipliers for $BMO(\mathbb{R}^n)$: they prove that g is a pointwise multiplier for $BMO(\mathbb{R}^n)$ if and only if g belong to $L^{\infty}(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$, where

$$BMO^{\log}(\mathbb{R}^n) =$$

$$= \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{BMO^{\log}} := \sup_{B(a,r)} \frac{|\log r| + \log(e+|a|)}{|B(a,r)|} \int_{B(a,r)} |f(x) - f_{B(a,r)}| dx < \infty \right\}.$$

By using the theory of these new Hardy spaces and dual spaces, we establish that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is just the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. Remark that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ have also recently been used by Lerner [32] for solving a conjecture of Diening (see [14]) on the boundedness of the Hardy-Littlewood maximal operator on the generalized Lebesgue spaces $L^{p(x)}(\mathbb{R}^n)$ (a special case of Musielak-Orlicz spaces, for the details see [14, 32]).

Motivated by all of the above mentioned facts, in this paper, we introduce a new class of Hardy spaces $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$, called Hardy spaces of Musielak-Orlicz type, which generalize the Hardy-Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Here, $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt weight A_{∞}. In the special case $\varphi(x,t) = w(x)\Phi(t)$ with w in the Muckenhoupt class and Φ an Orlicz function, our Hardy spaces are weighted Hardy-Orlicz spaces but they are different from the ones considered by Harboure, Salinas, and Viviani [22], [23].

As an example of our results, let us give the atomic decomposition with bounded atoms. Let φ be a *growth function* (see Section 2). A bounded function a is a φ-atom if it satisfies the following three conditions

i) supp $a \subset B$ for some ball B,

ii) $\|a\|_{L^\infty} \leq \|\chi_B\|_{L^\varphi}^{-1}$,

iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for any $|\alpha| \leq [n(\frac{q(\varphi)}{i(\varphi)} - 1)]$,

where $q(\varphi)$ and $i(\varphi)$ are the indices of φ (see Section 2). We next define the *atomic Hardy space of Musielak-Orlicz type* $H_{\mathrm{at}}^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ as those distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j b_j$ (in the sense of $\mathcal{S}'(\mathbb{R}^n)$), where b_j's are multiples of φ-atoms supported in the balls B_j's, with the property $\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty$; and define the norm of f by

$$\|f\|_{H_{\mathrm{at}}^{\varphi(\cdot,\cdot)}} = \inf\left\{ \Lambda_\infty(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n) \right\},$$

where $\Lambda_\infty(\{b_j\}) = \inf\left\{ \lambda > 0 : \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L^\infty}}{\lambda}\right) \leq 1 \right\}$ with $\varphi(B, t) := \int_B \varphi(x, t) dx$ for all $t \geq 0$ and B is measurable. Then we obtain

Theorem 1.1. $H_{\mathrm{at}}^{\varphi(\cdot,\cdot)}(\mathbb{R}^n) = H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ *with equivalent norms.*

The fact that $\Lambda_\infty(\{b_j\})$, which is the right expression for the (quasi-)norm in the atomic Hardy space of Musielak-Orlicz type, plays a central role in this paper. It should be emphasized that, even if the steps of the proof of such a theorem are standard, the adaptation to this context is not standard.

On the other hand, to establish the boundedness of operators on Hardy spaces, one usually appeals to the atomic decomposition characterization, see [11, 31, 48], which means that a function or distribution in Hardy spaces can be represented as a linear combination of functions of an elementary form, namely, atoms. Then, the boundedness of operators on Hardy spaces can be deduced from their behavior on atoms or molecules in principle. However, caution needs to be taken due to an example constructed in [8, Theorem 2]. There exists a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1, \infty, 0)$-atoms into bounded scalars, but however does not extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that the uniform boundedness of a linear operator T on atoms does not automatically guarantee the boundedness of T from $H^1(\mathbb{R}^n)$ to a Banach space $\mathcal{B}$. Nevertheless, by using the grand maximal function characterization of $H^p(\mathbb{R}^n)$, Meda, Sjögren, and Vallarino [34, 35] proved that if a sublinear operator T maps all (p, q, s)-atoms when $q < \infty$ and continuous (p, ∞, s)-atoms into uniformly bounded elements of $L^p(\mathbb{R}^n)$ (see also [50, 9] for quasi-Banach spaces), then T uniquely extends to a bounded sublinear operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In this paper, we study boundedness of sublinear operators in the context of new Hardy spaces of Musielak-Orlicz type which generalize the main results in [34, 35]. More precisely, under additional assumption on $\varphi(\cdot, \cdot)$, we prove that finite atomic norms on dense subspaces of $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ are equivalent with the standard infinite atomic decomposition norms. As an application, we prove that if T is a sublinear operator and maps all atoms into uniformly bounded

elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ to $\mathcal{B}$.

In a forecoming paper, using the theory of these new Hardy spaces and ideas from [6], we study and establish some new interesting estimates of endpoint type for the commutators of singular integrals and fractional integrals on Hardy-type spaces.

Our paper is organized as follows. In Section 2 we give the notation and definitions that we shall use in the sequel. For simplicity we write φ for $\varphi(\cdot,\cdot)$. One then introduces Hardy spaces of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ via grand maximal functions, atomic Hardy spaces $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$, finite atomic Hardy spaces $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ for any admissible triplet (φ,q,s), BMO-Musielak-Orlicz-type spaces $BMO^\varphi(\mathbb{R}^n)$, and generalized quasi-Banach spaces $\mathcal{B}_\gamma$ for $\gamma \in (0,1]$. In Section 3 we state the main results: the atomic decompositions (Theorem 3.1), the duality (Theorem 3.2), the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ (Theorem 3.3), the finite atomic decomposition (Theorem 3.4), and the criterion for boundedness of sublinear operators in $H^\varphi(\mathbb{R}^n)$ (Theorem 3.5). In Section 4 we present and prove the basic properties of the growth functions φ since they provide the tools for further work with this type of functions. In Section 5 we generalize the Calderón-Zygmund decomposition associated to the grand maximal function on $\mathbb{R}^n$ in the setting of the spaces of Musielak-Orlicz type. Applying this, we further prove that for any admissible triplet (φ,q,s), $H^\varphi(\mathbb{R}^n) = H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ with equivalent norms (Theorem 3.1). In Section 6 we prove the dual theorem. By Theorem 2 in [8], one has to be careful with the argument "the operator T is uniformly bounded in $H^p_w(\mathbb{R}^n)$ ($H^\varphi(\mathbb{R}^n)$ here $\varphi(x,t) = w(x).t^p$ in our context) on w-(p,∞)-atoms, and hence it extends to a bounded operator on $H^p_w(\mathbb{R}^n)$" which has been used in [18] and [10]. In Section 7 we introduce *log-atoms* and consider the particular case of $H^{\log}(\mathbb{R}^n)$. Finally, in Section 8 we prove that $\|\cdot\|_{H^{\varphi,q,s}_{\mathrm{fin}}}$ and $\|\cdot\|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ when $q < \infty$ and on $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q = \infty$, here and in what follows $C(\mathbb{R}^n)$ denotes the set of all continuous functions. Then, we consider generalized quasi-Banach spaces which generalize the notion of quasi-Banach spaces in [50] (see also [9]), and obtain criterious for boundedness of sublinear operators in $H^\varphi(\mathbb{R}^n)$.

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. The symbol $f \approx g$ means that f is equivalent to g (i.e. $C^{-1}f \leq g \leq Cf$), and $[\cdot]$ denotes the integer function. By X^* we denote the dual of the (quasi-)Banach space X. In $\mathbb{R}^n$, we denote by $B = B(x,r)$ an open ball with center x and radius $r > 0$. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^n \setminus E$.

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2. Notation and definitions

2.1. **Musielak-Orlicz-type functions.** First let us recall notations for Orlicz functions.

A function $\phi : [0, \infty) \to [0, \infty)$ is called *Orlicz* if it is nondecreasing and $\phi(0) = 0$; $\phi(t) > 0, t > 0$; $\lim_{t\to\infty} \phi(t) = \infty$. An Orlicz function ϕ is said to be of *lower type* (resp., *upper type*) p, $p \in (-\infty, \infty)$, if there exists a positive constant C so that

$$\phi(st) \le Cs^p\phi(t),$$

for all $t \ge 0$ and $s \in (0, 1)$ (resp., $s \in [1, \infty)$). One say that ϕ is of *positive lower type* (resp., *finite upper type*) if it is of lower type (resp., upper type) p for some $p > 0$ (resp., p finite).

Obviously, if ϕ is both of lower type p_1 and of upper type p_2, then $p_1 \le p_2$. Moreover, if ϕ is of lower type (resp., upper type) p then it is also of lower type (resp., upper) $\widetilde{p}$ for $-\infty < \widetilde{p} < p$ (resp., $p < \widetilde{p} < \infty$). We thus write

$$i(\phi) := \sup\{p \in (-\infty, \infty) : \phi \text{ is of lower type } p\}$$

$$I(\phi) := \inf\{p \in (-\infty, \infty) : \phi \text{ is of upper type } p\}$$

to denote the critical lower type and the critical upper type of the function ϕ.

Let us generalize these notions to functions $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$.

Given a function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ so that for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is Orlicz. We say that φ is of *uniformly lower type* (resp., *upper type*) p if there exists a positive constant C so that

$$(2.1) \qquad\qquad \varphi(x, st) \le Cs^p\varphi(x, t),$$

for all $x \in \mathbb{R}^n$ and $t \ge 0, s \in (0, 1)$ (resp., $s \in [1, \infty)$). We say that φ is of *positive uniformly lower type* (resp., *finite uniform upper type*) if it is of uniformly lower type (resp., uniform upper type) p for some $p > 0$ (resp., p finite), and denote

$$i(\varphi) := \sup\{p \in (-\infty, \infty) : \varphi \text{ is of uniformly lower type } p\}$$

$$I(\varphi) := \inf\{p \in (-\infty, \infty) : \varphi \text{ is of uniformly upper type } p\}.$$

We next need to recall notations for Muckenhoupt weights.

Let $1 \le q < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class A_q*, say $w \in A_q$, if there exists a positive constant C so that

$$\frac{1}{|B|}\int_B w(x)dx\Big(\frac{1}{|B|}\int_B (w(x))^{-1/(q-1)}dx\Big)^{q-1} \le C, \quad \text{if } 1 < q < \infty,$$

and

$$\frac{1}{|B|}\int_B w(x)dx \leq C \operatorname*{ess-inf}_{x\in B} w(x), \quad \text{if } q=1,$$

for all balls B in $\mathbb{R}^n$. We say that $w \in A_\infty$ if $w \in A_q$ for some $q \in [1,\infty)$.

It is well known that $w \in A_q$, $1 \leq q < \infty$, implies $w \in A_r$ for all $r > q$. Also, if $w \in A_q$, $1 < q < \infty$, then $w \in A_r$ for some $r \in [1,q)$. One thus write $q_w := \inf\{q \geq 1 : w \in A_q\}$ to denote the critical index of w.

Now, let us generalize these notions to functions $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$.

Let $\varphi : \mathbb{R}^n \times [0,\infty) \to \mathbb{C}$ be so that $x \mapsto \varphi(x,t)$ is measurable for all $t \in [0,\infty)$. We say that $\varphi(\cdot,t)$ is *uniformly locally integrable* if for all compact set K in $\mathbb{R}^n$, the following holds

$$\int_K \sup_{t>0} \frac{|\varphi(x,t)|}{\int_K |\varphi(y,t)|dy}dx < \infty$$

whenever the integral exists. A simple example for such *uniformly locally integrable* functions is $\varphi(x,t) = w(x)\Phi(t)$ with w a locally integrable function on $\mathbb{R}^n$ and Φ an arbitrary function on $[0,\infty)$. Our interesting examples are *uniformly locally integrable* functions $\varphi(x,t) = \frac{t^p}{(\log(e+|x|)+\log(e+t^p))^p}, 0 < p \leq 1$, since they arise naturally in the study of pointwise product of functions in $H^p(\mathbb{R}^n)$ with functions in $BMO(\mathbb{R}^n)$ (cf. [6]).

Given $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a *uniformly locally integrable* function. We say that the function $\varphi(\cdot,t)$ satisfies the *uniformly Muckenhoupt* condition $\mathbb{A}_q$, say $\varphi \in \mathbb{A}_q$, for some $1 \leq q < \infty$ if there exists a positive constant C so that

$$\frac{1}{|B|}\int_B \varphi(x,t)dx.\left(\frac{1}{|B|}\int_B \varphi(x,t)^{-1/(q-1)}dx\right)^{q-1} \leq C, \quad \text{if } 1 < q < \infty,$$

and

$$\frac{1}{|B|}\int_B \varphi(x,t)dx \leq C \operatorname*{ess-inf}_{x\in B} \varphi(x,t), \quad \text{if } q=1,$$

for all $t > 0$ and balls B in $\mathbb{R}^n$. We also say that $\varphi \in \mathbb{A}_\infty$ if $\varphi \in \mathbb{A}_q$ for some $q \in [1,\infty)$, and denote

$$q(\varphi) := \inf\{q \geq 1 : \varphi \in \mathbb{A}_q\}.$$

Now, we are able to introduce the *growth functions* which are the basis for our new Hardy spaces.

Definition 2.1. *We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a* **growth function** *if the following conditions are satisfied.*

    (1) The function φ is a **Musielak-Orlicz function** *that is*

        (a) the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$,

     (b) the function $\varphi(\cdot, t)$ is a Lebesgue measurable function for all $t \in [0, \infty)$.

  (2) The function φ belongs to $\mathbb{A}_\infty$.

  (3) The function φ is of positive uniformly lower type and of uniformly upper type 1.

For φ a growth function, we denote $m(\varphi) := \left[n \left(\frac{q(\varphi)}{i(\varphi)} - 1 \right) \right]$.

Clearly, $\varphi(x,t) = w(x)\Phi(t)$ is a *growth function* if $w \in A_\infty$ and Φ is of positive lower type and of upper type 1. Of course, there exists *growth functions* which are not of that form for instance $\varphi(x,t) = \frac{t^\alpha}{[\log(e+|x|)]^\beta + [\log(e+t)]^\gamma}$ for $\alpha \in (0,1]; \beta, \gamma \in (0, \infty)$. More precisely, $\varphi \in \mathbb{A}_1$ and φ is of uniformly upper type α with $i(\varphi) = \alpha$. In this paper, we are especially interested in the *growth functions* $\varphi(x,t) = \frac{t^p}{(\log(e+|x|) + \log(e+t^p))^p}, 0 < p \leq 1$, since the Hardy spaces of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ arise naturally in the study of pointwise product of functions in $H^p(\mathbb{R}^n)$ with functions in $BMO(\mathbb{R}^n)$ (see also [5] in the setting of holomorphic functions in convex domains of finite type or strictly pseudoconvex domains in $\mathbb{C}^n$).

2.2. Hardy spaces of Musielak-Orlicz type.

Throughout the whole paper, we always assume that φ is a *growth function*.

Let us now introduce the *Musielak-Orlicz-type spaces*.

The Musielak-Orlicz-type space $L^\varphi(\mathbb{R}^n)$ is the set of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)dx < \infty$ for some $\lambda > 0$, with Luxembourg (quasi-)norm

$$\|f\|_{L^\varphi} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)dx \leq 1 \right\}.$$

As usual, $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of test functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions (or distributions for brevity). For $m \in \mathbb{N}$, we define

$$\mathcal{S}_m(\mathbb{R}^n) = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \|\phi\|_m = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m+1} (1 + |x|)^{(m+2)(n+1)} |\partial_x^\alpha \phi(x)| \leq 1 \right\}.$$

For each distribution f, we define the nontangential grand maximal function f_m^* of f by

$$f_m^*(x) = \sup_{\phi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t} |f * \phi_t(y)|, \ x \in \mathbb{R}^n.$$

When $m = m(\varphi)$ we write f^* instead of $f_{m(\varphi)}^*$.

Definition 2.2. *The Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ is the space of all distributions f such that $f^* \in L^\varphi(\mathbb{R}^n)$ with the (quasi-)norm*

$$\|f\|_{H^\varphi} := \|f^*\|_{L^\varphi}.$$

Observe that, when $\varphi(x,t) = w(x)\Phi(t)$ with w a Muckenhoupt weight and Φ an Orlicz function, our Hardy spaces are weighted Hardy-Orlicz spaces which include the classical Hardy-Orlicz spaces of Janson [29] ($w \equiv 1$ in this context) and the classical weighted Hardy spaces of García-Cuerva [18], Strömberg and Torchinsky [47] ($\Phi(t) \equiv t^p$ in this context), see also [38, 10, 19] . Recently, the weighted anisotropic Hardy spaces (see [9]) and the Hardy-Orlicz spaces associated with operators (see [30]) have also been studied.

Next, to introduce the *atomic Hardy spaces of Musielak-Orlicz type* below, we need the following new spaces.

Definition 2.3. *For each ball B in $\mathbb{R}^n$, we denote $L^q_\varphi(B), 1 \leq q \leq \infty$, the set of all measurable functions f on $\mathbb{R}^n$ supported in B such that*

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup_{t>0} \left(\frac{\int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)dx}{\varphi(B,t)} \right)^{1/q} < \infty & , \quad 1 \leq q < \infty, \\ \|f\|_{L^\infty} < \infty & , \quad q = \infty, \end{cases}$$

here and in the future $\varphi(B,t) := \int_B \varphi(x,t)dx$.

Then, it is straightforward to verify that $(L^q_\varphi(B), \|\cdot\|_{L^q_\varphi(B)})$ is a Banach space.

Now, we are able to introduce the *atomic Hardy spaces of Musielak-Orlicz type*.

Definition 2.4. *A triplet (φ, q, s) is called admissible, if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{N}$ satisfies $s \geq m(\varphi)$. A measurable function a is a (φ, q, s)-atom if it satisfies the following three conditions*
i) $a \in L^q_\varphi(B)$ for some ball B,
ii) $\|a\|_{L^q_\varphi(B)} \leq \|\chi_B\|_{L^\varphi}^{-1}$,
iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $|\alpha| \leq s$.

In this setting we define the *atomic Hardy space of Musielak-Orlicz type* $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ as those distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be represented as a sum of multiples of (φ, q, s)-atoms, that is,

$$f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where b_j's are multiples of (φ, q, s)-atoms supported in the balls B_j's, with the property

$$\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty.$$

We introduce a (quasi-)norm in $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$. Given a sequence of multiples of (φ, q, s)-atoms, $\{b_j\}_j$, we denote

$$\Lambda_q(\{b_j\}) = \inf\left\{ \lambda > 0 : \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \leq 1 \right\}$$

and define

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at}}} = \inf\left\{\Lambda_q(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n)\right\}.$$

Let (φ, q, s) be an admissible triplet. We denote $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ the vector space of all finite linear combinations of (φ, q, s)-atoms, that is,

$$f = \sum_{j=1}^{k} b_j,$$

where b_j's are multiples of (φ, q, s)-atoms supported in balls B_j's. Then, the norm of f in $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{fin}}} = \inf\left\{\Lambda_q(\{b_j\}_{j=1}^{k}) : f = \sum_{j=1}^{k} b_j\right\}.$$

Obviously, for any admissible triplet (φ, q, s), the set $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\varphi,q,s}_{\mathrm{at}}}$.

We should point out that the theory of atomic Hardy-Orlicz spaces have been first introduced by Viviani [49] in the setting of spaces of homogeneous type. Later, Serra [44] generalized it to the context of the Euclidean space $\mathbb{R}^n$ and obtained the molecular characterization. In the particular case, when $\varphi(x, t) \equiv \Phi(t)$ the space $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ is the space considered in [44]. We also remark that when $\varphi(x, t) \equiv w(x).t^p, 0 < p \leq 1$, w a Muckenhoupt weight, the space $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ is just the classical weighted atomic Hardy space $H^{p,q,s}_w(\mathbb{R}^n)$ which has been considered by García-Cuerva [18], Strömberg and Torchinsky [47].

2.3. **BMO-Musielak-Orlicz-type spaces.** We also need BMO type spaces, which will be in duality of the Hardy spaces of Musielak-Orlicz type defined above. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to $BMO^\varphi(\mathbb{R}^n)$ if

$$\|f\|_{BMO^\varphi} := \sup_B \frac{1}{\|\chi_B\|_{L^\varphi}} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

Our typical example is $BMO^\varphi(\mathbb{R}^n)$, called $BMO^{\log}(\mathbb{R}^n)$, related to $\varphi(x, t) = \frac{t}{\log(e+|x|)+\log(e+t)}$. Clearly, when $\varphi(x, t) \equiv t$, then $BMO^\varphi(\mathbb{R}^n)$ is just the well-known $BMO(\mathbb{R}^n)$ of John and Nirenberg. We remark that when $\varphi(x, t) = w(x).t$ with $w \in A_{(n+1)/n}$, then $BMO^\varphi(\mathbb{R}^n)$ is just $BMO_w(\mathbb{R}^n)$ was first introduced by Muckenhoupt and Wheeden [37, 38]. There, they proved that $BMO_w(\mathbb{R}^n)$ is the dual of $H^1_w(\mathbb{R}^n)$ (see also [10]).

2.4. **Quasi-Banach valued sublinear operators.** Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant κ no less than 1 such that for all $f, g \in \mathcal{B}$, we have $\|f + g\|_{\mathcal{B}} \leq \kappa(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Definition 2.5. *Let* $\gamma \in (0, 1]$. *A quasi-Banach space* $\mathcal{B}_{\gamma}$ *with the quasi-norm* $\|\cdot\|_{\mathcal{B}_{\gamma}}$ *is said to be a* γ-*quasi-Banach space if there exists a positive constant* κ *no less than 1 such that for all* $f_j \in \mathcal{B}_{\gamma}, j = 1, 2, ..., m$, *we have*

$$\Big\| \sum_{j=1}^{m} f_j \Big\|_{\mathcal{B}_{\gamma}}^{\gamma} \leq \kappa \sum_{j=1}^{m} \|f_j\|_{\mathcal{B}_{\gamma}}^{\gamma}.$$

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces $\ell^p, L_w^p(\mathbb{R}^n)$ and $H_w^p(\mathbb{R}^n)$ with $p \in (0, 1]$ are typical p-quasi-Banach spaces. Also, when φ is of uniformly lower type $p \in (0, 1]$, the space $H^{\varphi}(\mathbb{R}^n)$ is a p-quasi-Banach space.

For any given γ-quasi-Banach space $\mathcal{B}_{\gamma}$ with $\gamma \in (0, 1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_{\gamma}$ is called $\mathcal{B}_{\gamma}$-sublinear if there exists a positive constant κ no less than 1 such that for all $f_j \in \mathcal{Y}, \lambda_j \in \mathbb{C}, j = 1, ..., m$, we have

$$\Big\| T\Big(\sum_{j=1}^{m} \lambda_j f_j \Big) \Big\|_{\mathcal{B}_{\gamma}}^{\gamma} \leq \kappa \sum_{j=1}^{m} |\lambda_j|^{\gamma} \|T(f_j)\|_{\mathcal{B}_{\gamma}}^{\gamma}.$$

We remark that if T is linear, then T is $\mathcal{B}_{\gamma}$-sublinear. We should point out that if the constant κ, in Definition 2.5, equal 1, then we obtain the notion of γ-quasi-Banach spaces introduced in [50] (see also [9]).

3. STATEMENT OF THE RESULTS

Our main theorems are the following.

Theorem 3.1. *Let* (φ, q, s) *be admissible. Then* $H^{\varphi}(\mathbb{R}^n) = H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n)$ *with equivalent norms.*

Denote by $L_0^{\infty}(\mathbb{R}^n)$ the set of all bounded functions with compact support and zero average. As a consequence of Theorem 3.1, we have the following

Lemma 3.1. *Let* φ *be a growth function satisfying* $nq(\varphi) < (n+1)i(\varphi)$. *Then,* $L_0^{\infty}(\mathbb{R}^n)$ *is dense in* $H^{\varphi}(\mathbb{R}^n)$.

We now can present our dual theorem as follows

Theorem 3.2. *Let* φ *be a growth function satisfying* $nq(\varphi) < (n+1)i(\varphi)$. *Then, the dual space of* $H^{\varphi}(\mathbb{R}^n)$ *is* $BMO^{\varphi}(\mathbb{R}^n)$ *in the following sense*

i) Suppose $\mathfrak{b} \in BMO^{\varphi}(\mathbb{R}^n)$. *Then the linear functional* $L_{\mathfrak{b}} : f \to L_{\mathfrak{b}}(f) := \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx$, *initially defined for* $L_0^{\infty}(\mathbb{R}^n)$, *has a bounded extension to* $H^{\varphi}(\mathbb{R}^n)$.

ii) Conversely, every continuous linear functional on $H^{\varphi}(\mathbb{R}^n)$ *arises as the above with a unique element* $\mathfrak{b}$ *of* $BMO^{\varphi}(\mathbb{R}^n)$. *Moreover* $\|\mathfrak{b}\|_{BMO^{\varphi}} \approx \|L_{\mathfrak{b}}\|_{(H^{\varphi})^*}$.

Next result concerns the class of pointwise multipliers for $BMO(\mathbb{R}^n)$.

Theorem 3.3. *The class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$ where $H^{\log}(\mathbb{R}^n)$ is a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$.*

In order to obtain the finite atomic decomposition, we need the notion of *uniformly locally dominated convergence condition.* A growth function φ is said to be satisfy *uniformly locally dominated convergence condition* if the following holds:

Given K compact set in $\mathbb{R}^n$. Let $\{f_m\}_{m \geq 1}$ be a sequence of measurable functions s.t $f_m(x)$ tends to $f(x)$ a.e $x \in \mathbb{R}^n$. If there exists a nonnegative measurable function g s.t $|f_m(x)| \leq g(x)$ and $\sup_{t>0} \int_K g(x) \frac{\varphi(x,t)}{\int_K \varphi(y,t)dy} dx < \infty$, then $\sup_{t>0} \int_K |f_m(x) - f(x)| \frac{\varphi(x,t)}{\int_K \varphi(y,t)dy} dx$ tends 0.

We remark that the growth functions $\varphi(x,t) = w(x)\Phi(t)$ and $\varphi(x,t) = \frac{t^p}{(\log(e+|x|)+\log(e+t^p))^p}$, $0 < p \leq 1$, satisfy the *uniformly locally dominated convergence condition.*

Theorem 3.4. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, and (φ, q, s) be an admissible triplet.*

i) If $q \in (q(\varphi), \infty)$ then $\| \cdot \|_{H^{\varphi,q,s}_{\mathrm{fin}}}$ and $\| \cdot \|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$.

ii) $\| \cdot \|_{H^{\varphi,\infty,s}_{\mathrm{fin}}}$ and $\| \cdot \|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

As an application, we obtain criterions for boundedness of quasi-Banach valued sublinear operators in $H^\varphi(\mathbb{R}^n)$.

Theorem 3.5. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, (φ, q, s) be an admissible triplet, φ be of uniformly upper type $\gamma \in (0,1]$, and $\mathcal{B}_\gamma$ be a quasi-Banach space. Suppose one of the following holds:*

i) $q \in (q(\varphi), \infty)$, and $T : H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that

$$A = \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \text{ is a } (\varphi,q,s)-\text{atom}\} < \infty;$$

ii) T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous (φ, ∞, s)-atoms such that

$$A = \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \text{ is a continuous } (\varphi, \infty, s)-\text{atom}\} < \infty.$$

Then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\widetilde{T}$ from $H^\varphi(\mathbb{R}^n)$ to $\mathcal{B}_\gamma$ which extends T.

4. SOME BASIC LEMMAS ON GROWTH FUNCTIONS

We start by the following lemma.

Lemma 4.1. *i) Let φ be a growth function. Then φ is uniformly σ-quasi-subadditive on $\mathbb{R}^n \times [0, \infty)$, i.e. there exists a constant $C > 0$ such that*

$$\varphi(x, \sum_{j=1}^{\infty} t_j) \leq C \sum_{j=1}^{\infty} \varphi(x, t_j),$$

for all $(x, t_j) \in \mathbb{R}^n \times [0, \infty)$, $j = 1, 2, ...$

ii) Let φ be a growth function and $\widetilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x,s)}{s} ds$ for $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Then $\widetilde{\varphi}$ is a growth function equivalent to φ, moreover, $\widetilde{\varphi}(x, \cdot)$ is continuous and strictly increasing.

iii) A Musielak-Orlicz function φ is a growth function if and only if φ is of positive uniformly lower type and uniformly quasi-concave, i.e. there exists a constant $C > 0$ such that

$$\lambda \varphi(x, t) + (1 - \lambda)\varphi(x, s) \leq C\varphi(x, \lambda t + (1 - \lambda)s),$$

for all $x \in \mathbb{R}^n, t, s \in [0, \infty)$ and $\lambda \in [0, 1]$.

Proof. i) We just need to consider the case when $\sum_{j=1}^{\infty} t_j > 0$. Then it follows from the fact that

$$\frac{t_k}{\sum_{j=1}^{\infty} t_j} \varphi(x, \sum_{j=1}^{\infty} t_j) \leq C\varphi(x, t_k)$$

by φ is of uniformly upper type 1.

ii) Since φ is a growth function, it is easy to see that $\widetilde{\varphi}(x, \cdot)$ is continuous and strictly increasing. Moreover, there exists $p > 0$ such that φ is of uniformly lower type p. Hence,

$$(4.1) \qquad \widetilde{\varphi}(x, t) = \int_0^t \frac{\varphi(x, s)}{s} ds \leq C \frac{\varphi(x, t)}{t^p} \int_0^t \frac{1}{s^{1-p}} ds \leq C\varphi(x, t).$$

On the other hand, since φ is of uniformly upper type 1, we get

$$(4.2) \qquad \widetilde{\varphi}(x, t) = \int_0^t \frac{\varphi(x, s)}{s} ds \geq C^{-1} \int_0^t \frac{\varphi(x, t)}{t} ds \geq C^{-1}\varphi(x, t).$$

Combining (4.1) and (4.2), we obtain $\widetilde{\varphi} \approx \varphi$, and thus $\widetilde{\varphi}$ is a growth function.

iii) Suppose φ is a growth function. By (ii), φ is equivalent to $\widetilde{\widetilde{\varphi}}$. On the other hand, $\frac{\partial \widetilde{\widetilde{\varphi}}}{\partial t}(x, t) = \frac{\widetilde{\varphi}(x,t)}{t}$ is uniformly quasi-decreasing in t. Hence, $\widetilde{\widetilde{\varphi}}$ is uniformly quasi-concave, and thus is φ.

The converse is easy by taking $s = 0$. We omit the details. $\square$

Remark 4.1. *Let us observe that the results stated in Section 3 are invariant under change of equivalent growth functions. By Lemm 4.1, in the future, we always consider a growth function φ of positive uniformly lower type, of*

uniformly upper type 1 (or, equivalently, uniformly quasi-concave), and so that $\varphi(x,\cdot)$ is continuous and strictly increasing for all $x \in \mathbb{R}^n$.

Lemma 4.2. *Let φ be a growth function. Then*

i) $\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^\varphi}}\right) dx = 1$ *for all $f \in L^\varphi(\mathbb{R}^n) \setminus \{0\}$.*

ii) $\lim_{k\to\infty} \|f_k\|_{L^\varphi} = 0$ *if and only if $\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x, |f_k(x)|)dx = 0$.*

Proof. Statement *i)* follows from the fact that the function

$$\vartheta(t) := \int_{\mathbb{R}^n} \varphi(x, t|f(x)|)dx,$$

$t \in [0, \infty)$, is continuous by the dominated convergence theorem since $\varphi(x, \cdot)$ is continuous.

Statement *ii)* follows from the fact that

$$\|f\|_{L^\varphi} \leq C \max\left\{ \int_{\mathbb{R}^n} \varphi(x, |f(x)|)dx, \left(\int_{\mathbb{R}^n} \varphi(x, |f(x)|)dx\right)^{1/p}\right\},$$

and

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|)dx \leq C \max\left\{\|f\|_{L^\varphi}, (\|f\|_{L^\varphi})^p\right\}$$

for some $p \in (0, i(\varphi))$. $\qquad\square$

Lemma 4.3. *Given c is a positive constant. Then, there exists a constant $C > 0$ such that*

i) The inequality $\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leq c$, for $\lambda > 0$, implies

$$\|f\|_{L^\varphi} \leq C\lambda.$$

ii) The inequality $\sum_j \varphi\left(B_j, \frac{t_j}{\lambda}\right) \leq c$, for $\lambda > 0$, implies

$$\inf\left\{\alpha > 0 : \sum_j \varphi\left(B_j, \frac{t_j}{\alpha}\right) \leq 1\right\} \leq C\lambda.$$

Proof. The proofs are simple since we may take $C = (1 + c.C_p)^{1/p}$, for some $p \in (0, i(\varphi))$, where C_p is such that (2.1) holds. $\qquad\square$

Lemma 4.4. *Let (φ, q, s) be an admissible triplet. Then there exists a positive constant C such that*

$$\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi} \leq C\Lambda_q(\{b_j\}),$$

for all $f = \sum_{j=1}^{\infty} b_j \in H^{\varphi,q,s}_{at}(\mathbb{R}^n)$ where b_j's are multiples of (φ, q, s)-atoms supported in balls B_j's.

Proof. Since φ is of uniformly upper type 1, there exists a positive constant $c > 0$ such that

$$\varphi\Big(x, \frac{\|b_i\|_{L_\varphi^q(B_i)}}{\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c\frac{\|b_i\|_{L_\varphi^q(B_i)}\|\chi_{B_i}\|_{L^\varphi}}{\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi}}\varphi\Big(x, \frac{1}{\|\chi_{B_i}\|_{L^\varphi}}\Big)$$

for all $x \in \mathbb{R}^n, i \geq 1$. Hence, for all $i \geq 1$,

$$\varphi\Big(B_i, \frac{\|b_i\|_{L_\varphi^q(B_i)}}{\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c\frac{\|b_i\|_{L_\varphi^q(B_i)}\|\chi_{B_i}\|_{L^\varphi}}{\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi}}$$

since $\int_{B_i} \varphi\Big(x, \frac{1}{\|\chi_{B_i}\|_{L^\varphi}}\Big) dx = 1$ by Lemma 4.2. It follows that

$$\sum_{i=1}^\infty \varphi\Big(B_i, \frac{\|b_i\|_{L_\varphi^q(B_i)}}{\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c.$$

We deduce from the above that

$$\sum_{j=1}^\infty \|b_j\|_{L_\varphi^q(B_j)}\|\chi_{B_j}\|_{L^\varphi} \leq C\Lambda_q(\{b_j\}),$$

with $C = (C_p/c)^{1/p}$ for some $p \in (0, i(\varphi))$, where C_p is such that (2.1) holds. $\square$

Lemma 4.5. *Let* $\varphi \in \mathbb{A}_q, 1 < q < \infty$. *Then, there exists a positive constant* C *such that*

i) For all ball $B(x_0, r), \lambda > 0,$ *and* $t \in [0, \infty)$, *we have*

$$\varphi(B(x_0, \lambda r), t) \leq C\lambda^{nq}\varphi(B(x_0, r), t).$$

ii) For all ball $B(x_0, r)$ *and* $t \in [0, \infty)$, *we have*

$$\int_{B^c} \frac{\varphi(x, t)}{|x - x_0|^{nq}} dx \leq C\frac{\varphi(B, t)}{r^{nq}}.$$

iii) For all ball B, f *measurable and* $t \in (0, \infty)$, *we have*

$$\Big(\frac{1}{|B|} \int_B |f(x)|\Big)^q \leq C\frac{1}{\varphi(B, t)} \int_B |f(x)|^q\varphi(x, t)dx.$$

iii) For all f *measurable and* $t \in [0, \infty)$, *we have*

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)^q\varphi(x, t)dx \leq C \int_{\mathbb{R}^n} |f(x)|^q\varphi(x, t)dx,$$

where $\mathcal{M}$ *is the classical Hardy-Littlewood maximal operator defined by*

$$\mathcal{M}f(x) = \sup_{x \in B-\text{ball}} \frac{1}{|B|} \int_B |f(y)|dy, \quad x \in \mathbb{R}^n.$$

In the setting $\varphi(x,t) = w(x)\Phi(t)$, $w \in A_\infty$ and Φ a Orlicz function, the above lemma is well-known as a classical result in the theory of Muckenhoupt weight (cf. [20]). Since φ satisfies uniformly Muckenhoupt condition, the proof of Lemma 4.5 is a slight modification of the classical result. We omit the details.

5. Atomic decompositions

The purpose of this section is prove the atomic decomposition theorem (Theorem 3.1). The construction is by now standard, but the estimates require the preliminary lemmas. For the reader convenience, we give all steps of the proof, even if only the generalization to our framework is new.

We first introduce a class of Hardy spaces that the Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ containing as a particular case.

Definition 5.1. *For $m \in \mathbb{N}$, we denote by $H_m^\varphi(\mathbb{R}^n)$ the space of all distributions f such that $f_m^* \in L^\varphi(\mathbb{R}^n)$ with the (quasi-)norm*

$$\|f\|_{H_m^\varphi} := \|f_m^*\|_{L^\varphi}.$$

Clearly, $H^\varphi(\mathbb{R}^n)$ is a special case associated with $m = m(\varphi)$.

5.1. **Some basic properties concerning $H_m^\varphi(\mathbb{R}^n)$ and $H_{at}^{\varphi,q,s}(\mathbb{R}^n)$.** We start by the following proposition.

Proposition 5.1. *For $m \in \mathbb{N}$, we have $H_m^\varphi(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Let $f \in H_m^\varphi(\mathbb{R}^n)$. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in B(0,1)$, we write

$$\langle f, \phi \rangle = f * \widetilde{\phi}(0) = f * \psi(x),$$

where $\psi(y) = \widetilde{\phi}(y - x) = \phi(x - y)$ for all $y \in \mathbb{R}^n$.

It is easy to verify that $\sup\limits_{x \in B(0,1), y \in \mathbb{R}^n} \frac{1+|y|}{1+|y-x|} \leq 2$. Consequently,

$$
\begin{aligned}
|\langle f, \phi \rangle| = |f * \psi(x)| &\leq 2^{(m+2)(n+1)} \|\phi\|_{\mathcal{S}_m} \inf_{x \in B(0,1)} f_m^*(x) \\
&\leq 2^{(m+2)(n+1)} \|\phi\|_{\mathcal{S}_m} \|\chi_{B(0,1)}\|_{L^\varphi}^{-1} \|f\|_{H_m^\varphi}.
\end{aligned}
$$

This implies that $f \in \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous. $\square$

The following proposition gives the completeness of $H_m^\varphi(\mathbb{R}^n)$.

Proposition 5.2. *The space $H_m^\varphi(\mathbb{R}^n)$ is complete.*

Proof. In order to prove the completeness of $H_m^\varphi(\mathbb{R}^n)$, it suffices to prove that for every sequence $\{f_j\}_{j \geq 1}$ with $\|f_j\|_{H_m^\varphi} \leq 2^{-j}$ for any $j \geq 1$, the series $\sum_j f_j$

converges in $H_m^\varphi(\mathbb{R}^n)$. Let us now take $p > 0$ such that φ is of uniformly lower type p. Then, for any $j \geq 1$,

$$(5.1) \qquad \int_{\mathbb{R}^n} \varphi(x, (f_j)_m^*(x))dx \leq C(2^{-j})^p \int_{\mathbb{R}^n} \varphi\left(x, \frac{(f_j)_m^*(x)}{2^{-j}}\right)dx \leq C2^{-jp}.$$

Since $\{\sum_{i=1}^j f_i\}_{j\geq 1}$ is a Cauchy sequence in $H_m^\varphi(\mathbb{R}^n)$, by Proposition 5.1 and the completeness of $\mathcal{S}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j\geq 1}$ is also a Cauchy sequence in $\mathcal{S}'(\mathbb{R}^n)$ and thus converges to some $f \in \mathcal{S}'(\mathbb{R}^n)$. This implies that, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, the series $\sum_j f_j * \phi$ converges to $f * \phi$ pointwisely. Therefore $f_m^*(x) \leq \sum_j (f_j)_m^*(x)$ and $(f - \sum_{j=1}^k f_j)_m^*(x) \leq \sum_{j\geq k+1}(f_j)_m^*(x)$ for all $x \in \mathbb{R}^n, k \geq 1$. Combining this and (5.1), we obtain

$$\int_{\mathbb{R}^n} \varphi(x, (f - \sum_{j=1}^k f_j)_m^*(x))dx \leq C \sum_{j\geq k+1}\int_{\mathbb{R}^n} \varphi(x, (f_j)_m^*(x))dx$$

$$\leq C \sum_{j\geq k+1} 2^{-jp} \to 0,$$

as $k \to \infty$, here we used Lemma 4.1. Thus, the series $\sum_j f_j$ converges to f in $H_m^\varphi(\mathbb{R}^n)$ by Lemma 4.2. This completes the proof. $\square$

Corollary 5.1. *The Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ is complete.*

The following lemma and its corollary show that (φ, q, s)-atoms are in $H^\varphi(\mathbb{R}^n)$. Furthermore, it is the necessary estimate for proving that $H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n)$ and the inclusion is continuous, see Theorem 5.1 below.

Lemma 5.1. *Let (φ, q, s) be an admissible triplet and $m \geq s$. Then, there exists a constant $C = C(\varphi, q, s, m)$ such that*

$$\int_{\mathbb{R}^n} \varphi(x, f_m^*(x))dx \leq C\varphi(B, \|f\|_{L_\varphi^q(B)}),$$

for all f multiples of (φ, q, s)-atom associated with ball $B = B(x_0, r)$.

Proof. The case $q = \infty$ is easy and will be omitted. We just consider $q \in (q(\varphi), \infty)$. Now let us set $\widetilde{B} = B(x_0, 9r)$, and write

$$\int_{\mathbb{R}^n} \varphi(x, f_m^*(x))dx = \int_{\widetilde{B}} \varphi(x, f_m^*(x))dx + \int_{(\widetilde{B})^c} \varphi(x, f_m^*(x))dx$$

$$= I + II.$$

Since φ is of uniformly upper type 1, by Hölder inequality, we get

$$
\begin{aligned}
I &= \int_{\widetilde{B}} \varphi(x, f_m^*(x)) dx \leq C \int_{\widetilde{B}} \Big(\frac{f_m^*(x)}{\|f\|_{L_\varphi^q(B)}} + 1\Big) \varphi(x, \|f\|_{L_\varphi^q(B)}) dx \\
&\leq C\varphi(\widetilde{B}, \|f\|_{L_\varphi^q(B)}) \\
&+ C\frac{1}{\|f\|_{L_\varphi^q(B)}} \Big(\int_{\widetilde{B}} |f_m^*(x)|^q \varphi(x, \|f\|_{L_\varphi^q(B)}) dx\Big)^{1/q} \varphi(\widetilde{B}, \|f\|_{L_\varphi^q(B)})^{(q-1)/q} \\
&\leq C\varphi(B, \|f\|_{L_\varphi^q(B)}) + C\frac{1}{\|f\|_{L_\varphi^q(B)}} \|f\|_{L_\varphi^q(\widetilde{B})} \varphi(\widetilde{B}, \|f\|_{L_\varphi^q(B)}) \\
&\leq C\varphi(B, \|f\|_{L_\varphi^q(B)}).
\end{aligned}
$$

We used the fact $f_m^*(x) \leq C(m)\mathcal{M}f(x)$ and Lemma 4.5.

To estimate II, we note that since $m \geq s$, there exists a constant $C = C(m)$ such that

$$
\Big| \phi\Big(\frac{x-y}{t}\Big) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(\frac{x-x_0}{t})}{\alpha!} \Big(\frac{x_0-y}{t}\Big)^\alpha \Big| \leq C t^n \frac{|y-x_0|^{s+1}}{|x-x_0|^{n+s+1}}
$$

for all $\phi \in \mathcal{S}_m(\mathbb{R}^n), t > 0, x \in (\widetilde{B})^c, y \in B$. Therefore

$$
\begin{aligned}
|f * \phi_t(x)| &= \frac{1}{t^n} \Big| \int_B f(y) \Big[\phi\Big(\frac{x-y}{t}\Big) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(\frac{x-x_0}{t})}{\alpha!} \Big(\frac{x_0-y}{t}\Big)^\alpha\Big] dy \Big| \\
&\leq C \int_B |f(y)| \frac{|y-x_0|^{s+1}}{|x-x_0|^{n+s+1}} dy \\
&\leq C\frac{r^{s+1}}{|x-x_0|^{n+s+1}} \Big(\int_B |f(y)|^q \varphi(y,\lambda) dy\Big)^{1/q} \Big(\int_B [\varphi(y,\lambda)]^{-1/(q-1)} dy\Big)^{(q-1)/q} \\
&\leq C\|f\|_{L_\varphi^q(B)} \Big(\frac{r}{|x-x_0|}\Big)^{n+s+1}.
\end{aligned}
$$

For any $\lambda > 0$, we used that $\int_B \varphi(y,\lambda) dy (\int_B [\varphi(y,\lambda)]^{-1/(q-1)} dy)^{q-1} \leq C|B|^q$ since $\varphi \in \mathbb{A}_q$. As a consequence, we get

$$
f_m^*(x) \leq C(m) \sup_{\phi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{t>0} |f * \phi_t(x)| \leq C\|f\|_{L_\varphi^q(B)} \Big(\frac{r}{|x-x_0|}\Big)^{n+s+1}.
$$

By $s \geq m(\varphi)$, there exists $p \in (0, i(\varphi))$ such that $(n + s + 1)p > nq(\varphi)$. Hence, by Lemma 4.5,

$$
\begin{aligned}
II = \int\limits_{(\widetilde{B})^c} \varphi(x, f_m^*(x)) dx &\leq C \int\limits_{(\widetilde{B})^c} \left(\frac{r}{|x - x_0|} \right)^{(n+s+1)p} \varphi(x, \|f\|_{L_\varphi^q(B)}) dx \\
&\leq Cr^{(n+s+1)p} \frac{\varphi(\widetilde{B}, \|f\|_{L_\varphi^q(B)})}{(9r)^{(n+s+1)p}} \\
&\leq C\varphi(B, \|f\|_{L_\varphi^q(B)}).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Corollary 5.2. *There exists a constant $C = C(\varphi, q, s) > 0$ such that*
$$
\|a\|_{H^\varphi} \leq C,
$$
for all (φ, q, s)-atom a.

Theorem 5.1. *Let (φ, q, s) be an admissible triplet and $m \geq s$. Then*
$$
H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n) \subset H_m^\varphi(\mathbb{R}^n),
$$
moreover, the inclusion is continuous.

Proof. For any $0 \neq f \in H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n)$. Let $f = \sum_j b_j$ be an atomic decomposition of f, with supp $b_j \subset B_j$, $j = 1, 2, \ldots$ For all $\phi \in \mathcal{S}(\mathbb{R}^n)$, the series $\sum_j b_j * \phi$ converges to $f * \phi$ pointwise since $f = \sum_j b_j$ in $\mathcal{S}'$. Hence $f_m^*(x) \leq \sum_j (b_j)_m^*(x)$. By applying Lemma 5.1, we obtain

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} \varphi\left(x, \frac{f_m^*(x)}{\Lambda_q(\{b_j\})}\right) dx &\leq C \sum_j \int\limits_{\mathbb{R}^n} \varphi\left(x, \frac{(b_j)_m^*(x)}{\Lambda_q(\{b_j\})}\right) dx \\
&\leq C \sum_j \varphi\left(B_j, \frac{\|b_j\|_{L_\varphi^q(B_j)}}{\Lambda_q(\{b_j\})}\right) \\
&\leq C.
\end{aligned}
$$

This implies that $\|f\|_{H_m^\varphi} \leq C\Lambda_q(\{b_j\})$ (see Lemma 4.3) for any atomic decomposition $f = \sum_j b_j$, and thus, $\|f\|_{H_m^\varphi} \leq C\|f\|_{H_{\mathrm{at}}^{\varphi,q,s}}$. $\qquad\square$

5.2. **Calderón-Zygmund decompositions.** Throughout this subsection, we fix m and s so that $m, s \geq m(\varphi)$. For a given $\lambda > 0$, we set $\Omega = \{x \in \mathbb{R}^n : f_m^*(x) > \lambda\}$. Observe that Ω is open. Hence by Whitney's Lemma, there exists $x_1, x_2, \ldots$ in Ω and $r_1, r_2, \ldots > 0$ such that
   (i) $\Omega = \cup_j B(x_j, r_j)$,
   (ii) the balls $B(x_j, r_j/4)$, $j = 1, 2, \ldots$, are disjoint,
   (iii) $B(x_j, 18r_j) \cap \Omega^c = \emptyset$, but $B(x_j, 54r_j) \cap \Omega^c \neq \emptyset$, for any $j = 1, 2, \ldots$,
   (iv) there exists $L \in \mathbb{N}$ (depending only on n) such that no point of Ω lies in more than L of the balls $B(x_j, 18r_j)$, $j = 1, 2, \ldots$

We fix once for all, a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that supp $\theta \subset B(0,2)$, $0 \le \theta \le 1$, $\theta = 1$ on $B(0,1)$, and set $\theta_j(x) = \theta((x - x_j)/r_j)$, for j=1,2,... Obviously, supp $\theta_j \subset B(x_j, 2r_j)$, $j = 1, 2, ...$, and $1 \le \sum_j \theta_j \le L$ for all $x \in \Omega$. Hence if we set $\zeta_j(x) = \theta_j(x)/\sum_{i=1}^\infty \theta_i(x)$ if $x \in \Omega$ and $\zeta_j(x) = 0$ if $x \in \Omega^c$, $j = 1, 2, ...$, then supp $\zeta_j \subset B(x_j, 2r_j)$, $0 \le \zeta_j \le 1$, $\sum_j \zeta_j = \chi_\Omega$, and $L^{-1} \le \zeta_j \le 1$ on $B(x_j, r_j)$. The family $\{\zeta_j\}_j$ forms a smooth partition of unity of Ω. let $s \in \mathbb{N}$ be some fixed natural number and $\mathcal{P}_s(\mathbb{R}^n)$ (or simply $\mathcal{P}_s$) denote the linear space of polynomials in n variables of degree less than s. For each j, we consider the inner product $\langle P, Q \rangle_j = \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} P(x)Q(x)\zeta_j(x)dx$ for $P, Q \in \mathcal{P}_s$. Then $(\mathcal{P}_s, \langle \cdot, \cdot \rangle_j)$ is a finite dimensional Hilbert space. Let $f \in \mathcal{S}'$. Since f induces a linear functional on $\mathcal{P}_s$ via $Q \to \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} f(x)Q(x)\zeta_j(x)dx$, by the Riesz theorem, there exists a unique polynomial $P_j \in \mathcal{P}_s$ such that for all $Q \in \mathcal{P}_s$, $\langle P_j, Q \rangle_j = \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} f(x)Q(x)\zeta_j(x)dx$. For each j, $j = 1, 2, ...$, we define $b_j = (f - P_j)\zeta_j$, and note $B_j = B(x_j, r_j)$, $\widetilde{B}_j = B(x_j, 9r_j)$. Then, it is easy to see that $\int_{\mathbb{R}^n} b_j(x)Q(x)dx = 0$ for all $Q \in \mathcal{P}_s$. It turns out, in the case of interest, that the series $\sum_j b_j$ converges in $\mathcal{S}'$. In this case, we set $g = f - \sum_j b_j$, and we call the representation $f = g + \sum_j b_j$ a Calderón-Zygmund decomposition of f of degree s and height λ associated to f_m^*.

For any $j = 1, 2, ...$, we denote $B_j = B(x_j, r_j)$ and $\widetilde{B}_j = B(x_j, 9r_j)$. Then we have the following lemma

Lemma A (see [17, Chapter 3]) There are four constant c_1, c_2, c_3, c_4, independent of f, j, and λ, such that

i)
$$\sup_{|\alpha| \le N, x \in \mathbb{R}^n} r_j^{|\alpha|} |\partial^\alpha \zeta_j(x)| \le c_1.$$

ii)
$$\sup_{x \in \mathbb{R}^n} |P_j(x)\zeta_j(x)| \le c_2 \lambda.$$

iii)
$$(b_j)_m^*(x) \le c_3 f_m^*(x), \quad \text{for all } x \in \widetilde{B}_j.$$

iv)
$$(b_j)_m^*(x) \le c_4 \lambda (r_j/|x - x_j|)^{n+m_s}, \quad \text{for all } x \notin \widetilde{B}_j,$$

where $m_s = \min\{s + 1, m + 1\}$.

Lemma 5.2. *For all $f \in H_m^\varphi(\mathbb{R}^n)$, there exists a geometric constant C, independent of f, j, and λ, such that,*

$$\int_{\mathbb{R}^n} \varphi\Big(x, (b_j)_m^*(x)\Big)dx \le C \int_{\widetilde{B}_j} \varphi(x, f_m^*(x))dx.$$

Moreover, the series $\sum_j b_j$ converges in $H_m^\varphi(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \varphi\Big(x, (\sum_j b_j)^*_m(x)\Big) dx \leq C \int_{\Omega} \varphi(x, f^*_m(x)) dx.$$

Proof. As $m, s \geq m(\varphi)$, $m_s = \min\{s+1, m+1\} > n(q(\varphi)/i(\varphi) - 1)$. Hence, there exist $q > q(\varphi)$ and $0 < p < i(\varphi)$ such that $m_s > n(q/p - 1)$, deduce that $(n+m_s)p > nq$. Therefore, $\varphi \in \mathbb{A}_{(n+m_s)p/n}$ and φ is of uniformly lower type p. Thus, there exists a positive constant C, independent of f, j, and λ, such that

$$
\begin{aligned}
\int_{(\widetilde{B}_j)^c} \varphi(x, \lambda(r_j/|x-x_j|)^{n+m_s}) dx &\leq C \int_{(\widetilde{B}_j)^c} \Big(\frac{r_j}{|x-x_j|}\Big)^{(n+m_s)p} \varphi(x, \lambda) dx \\
&\leq C(r_j)^{(n+m_s)p} \frac{\varphi(\widetilde{B}_j, \lambda)}{(9r_j)^{(n+m_s)p}} \\
&\leq C \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx,
\end{aligned}
$$

since $r_j/|x-x_j| < 1$ and $f^*_m > \lambda$ on $\widetilde{B}_j$. Combining this and Lemma A, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} \varphi\Big(x, (b_j)^*_m(x)\Big) dx &\leq C\Big[\int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx + \int_{(\widetilde{B}_j)^c} \varphi(x, \lambda(r_j/|x-x_j|)^{n+m_s}) dx\Big] \\
&\leq C \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx.
\end{aligned}
$$

As a consequence of the above estimate, since $\sum_j \chi_{\widetilde{B}_j} \leq L$ and $\Omega = \cup_j \widetilde{B}_j$, we obtain

$$
\begin{aligned}
\sum_j \int_{\mathbb{R}^n} \varphi\Big(x, (b_j)^*_m(x)\Big) dx &\leq C \sum_j \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx \\
&\leq C \int_{\Omega} \varphi(x, f^*_m(x)) dx.
\end{aligned}
$$

This implies that the series $\sum_j b_j$ converges in $H^\varphi_m(\mathbb{R}^n)$ by completeness of $H^\varphi_m(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} \varphi\Big(x, (\sum_j b_j)^*_m(x)\Big) dx \leq C \int_{\Omega} \varphi(x, f^*_m(x)) dx.$$

$\square$

Let $q \in [1, \infty]$. We denote by $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$ the usually weighted Lebesgue space with the Muckenhoupt weight $\varphi(x, 1)$. Then we have the following lemma (see [9, Lemma 4.8]).

Lemma B. Let $q \in (q(\varphi), \infty]$. Assume that $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$, then the series $\sum_j b_j$ converges in $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and there exists a constant C, independent of f, j, and λ such that $\left\| \sum_j |b_j| \right\|_{L^q_{\varphi(\cdot,1)}} \leq C\|f\|_{L^q_{\varphi(\cdot,1)}}$.

Remark 5.1. *By Lemma B, the series $\sum_j |b_j|$, and thus the series $\sum_j b_j$, converges a.e on $\mathbb{R}^n$.*

Lemma C. (see [17, Lemma 3.19]) Suppose that the series $\sum_j b_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Then, there exists a positive constant C, independent of f, j, and λ, such that for all $x \in \mathbb{R}^n$,

$$g_m^*(x) \leq C\lambda \sum_j \left(\frac{r_j}{|x - x_j| + r_j} \right)^{n+m_s} + f_m^*(x)\chi_{\Omega^c}(x).$$

Lemma 5.3. *For any $q \in (q(\varphi), \infty)$ and $f \in H^\varphi_m(\mathbb{R}^n)$. Then $g_m^* \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$, and there exists a positive constant C, independent of f, j, and λ, such that*

$$\int_{\mathbb{R}^n} [g_m^*(x)]^q \varphi(x, 1) dx \leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int_{\mathbb{R}^n} \varphi(x, f_m^*(x)) dx.$$

Proof. For any $j = 1, 2, ...$ and $x \in \mathbb{R}^n$, we have

$$\left(\frac{r_j}{|x - x_j| + r_j} \right)^n = \frac{1}{|B(x_j, |x - x_j| + r_j)|} \int_{B(x_j, |x-x_j|+r_j)} \chi_{B_j}(y) dy \leq \mathcal{M}(\chi_{B_j})(x)$$

since $B_j \subset B(x_j, |x - x_j| + r_j)$.

Therefore, by $L^{rq}_{\varphi(\cdot,1)}$-boundedness of vector-valued maximal functions [1, Theorem 3.1], where $r := (n + m_s)/n > 1$, we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Big[\sum_j \Big(\frac{r_j}{|x - x_j| + r_j} \Big)^{n+m_s} \Big]^q \varphi(x, 1) dx &\leq \int_{\mathbb{R}^n} \Big[\Big(\sum_j (\mathcal{M}(\chi_{B_j})(x))^r \Big)^{1/r} \Big]^{rq} \varphi(x, 1) dx \\
&\leq C_{s,q} \int_{\mathbb{R}^n} \Big[\Big(\sum_j (\chi_{B_j}(x))^r \Big)^{1/r} \Big]^{rq} \varphi(x, 1) dx \\
&\leq C_{s,q} L \int_\Omega \varphi(x, 1) dx \\
&\leq C \max\{1/\lambda, 1/\lambda^p\} \int_{\mathbb{R}^n} \varphi(x, f_m^*(x)) dx
\end{aligned}
$$

for some $p \in (0, i(\varphi))$ since $\varphi \in \mathbb{A}_q \subset \mathbb{A}_{rq}$ and $f_m^* > \lambda$ on Ω. Combine this, Lemma C and Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} [g_m^*(x)]^q \varphi(x, 1) dx &\leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int_{\mathbb{R}^n} \varphi(x, f_m^*(x)) dx + C \int_{\Omega^c} [f_m^*(x)]^q \varphi(x, 1) dx \\
&\leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int_{\mathbb{R}^n} \varphi(x, f_m^*(x)) dx,
\end{aligned}
$$

since $f_m^* \leq \lambda$ on Ω^c, here one used $\varphi(x, \lambda)/\lambda^q \leq C\varphi(x, f_m^*(x))/[f_m^*(x)]^q$ on Ω^c. $\qquad \square$

Proposition 5.3. *For any $q \in (q(\varphi), \infty)$ and $m \geq m(\varphi)$. The subspace $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$ is dense in $H^\varphi_m(\mathbb{R}^n)$.*

Proof. Let f be an arbitrary element in $H^\varphi_m(\mathbb{R}^n)$. For each $\lambda > 0$, let $f = g^\lambda + \sum_j b_j^\lambda$ be the Calderon-Zygmund decomposition of f of degree $m(\varphi)$, and height λ associated with f_m^*. Then by Lemma 5.2 and Lemma 5.3, $g^\lambda \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$, moreover,

$$
\int_{\mathbb{R}^n} \varphi(x, (g^\lambda - f)_m^*(x)) dx \leq C \int_{f_m^*(x) > \lambda} \varphi(x, f_m^*(x)) dx \to 0,
$$

as $\lambda \to \infty$. Consequently, $\|g^\lambda - f\|_{H^\varphi_m} \to 0$ as $\lambda \to \infty$ by Lemma 4.2. Thus $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$ is dense in $H^\varphi_m(\mathbb{R}^n)$. $\qquad \square$

5.3. **The atomic decompositions $H^\varphi_m(\mathbb{R}^n)$.** Recall that $m, s \geq m(\varphi)$, and f is a distribution such that $f_m^* \in L^\varphi(\mathbb{R}^n)$. For each $k \in \mathbb{Z}$, let $f = g^k + \sum_j b_j^k$ be the Calderón-Zygmund decomposition of f of degree s and height 2^k associated with f_m^*. We shall label all the ingredients in this construction as in subsection 5.2, but with superscript $k's$: for example,

$$
\Omega^k = \{x \in \mathbb{R}^n : f_m^*(x) > 2^k\}, \qquad b_j^k = (f - P_j^k)\zeta_j^k, \qquad B_j^k = B(x_j^k, r_j^k).
$$

Moreover, for each $k \in \mathbb{Z}$, and i, j, let $P_{i,j}^{k+1}$ be the orthogonal projection of $(f - P_j^{k+1})\zeta_i^k$ onto $\mathcal{P}_s$ with respect to the norm associated to ζ_j^{k+1}, namely, the unique element of $\mathcal{P}_s$ such that for all $Q \in \mathcal{P}_s$,

$$
\int_{\mathbb{R}^n} (f(x) - P_j^{k+1}(x))\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)dx = \int_{\mathbb{R}^n} P_{i,j}^{k+1}(x)Q(x)\zeta_j^{k+1}(x)dx.
$$

For convenience, we set $\hat{B}_j^k = B(x_j^k, 2r_j^k)$. Then we have the following lemma.
Lemma D. (see [17, Chapter 3])
i) If $\hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset$ the $r_j^{k+1} < 4r_i^k$ and $\hat{B}_j^{k+1} \subset B(x_i^k, 18r_i^k)$.
ii) For each j there are at most L (depending only on n as in last section)

values of i such that $\hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset$.

iii) There is a constant $C > 0$, independent of f, i, j, and k, such that

$$\sup_{x \in \mathbb{R}^n} |P_{i,j}^{k+1}(x)\zeta_j^{k+1}(x)| \leq C2^{k+1}.$$

iv) For every $k \in \mathbb{Z}$, $\sum_i(\sum_j P_{i,j}^{k+1}\zeta_j^{k+1}) = 0$, where the series converges pointwise and in $\mathcal{S}'(\mathbb{R}^n)$.

We now give the necessary estimates for proving that $H_m^\varphi(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$, $m \geq s \geq m(\varphi)$, and the inclusion is continuous.

Lemma 5.4. *Let $f \in H_m^\varphi(\mathbb{R}^n)$, and for each $k \in \mathbb{Z}$, set*

$$\Omega^k = \{x \in \mathbb{R}^n : f_m^*(x) > 2^k\}.$$

Then for any $\lambda > 0$, there exists a constant C, independent of f and λ, such that

$$\sum_{k=-\infty}^{\infty} \varphi\Big(\Omega^k, \frac{2^k}{\lambda}\Big) \leq C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx.$$

Proof. Let $p \in (0, i(\varphi))$ and C_p is such that (2.1) holds. We now set $N_0 = [(\log_2 C_p)/p] + 1$ so that $2^{N_0 p} > C_p$. For each $\ell \in \mathbb{N}, 0 \leq \ell \leq N_0 - 1$, we consider the sequence $U_m^\ell = \sum_{k=-m}^{m} \varphi\Big(\Omega^{N_0 k+\ell}, \frac{2^{N_0 k+\ell}}{\lambda}\Big)$. Obviously, $\{U_m^\ell\}_{m \in \mathbb{N}}$ is an increasing sequence. Moreover, for any $m \in \mathbb{N}$,

$$
\begin{aligned}
U_m^\ell &= \sum_{k=-m}^{m} \varphi\Big(\Omega^{N_0(k+1)+\ell}, \frac{2^{N_0 k+\ell}}{\lambda}\Big) + \sum_{k=-m}^{m} \Big\{\varphi\Big(\Omega^{N_0 k+\ell}, \frac{2^{N_0 k+\ell}}{\lambda}\Big) - \varphi\Big(\Omega^{N_0(k+1)+\ell}, \frac{2^{N_0 k+\ell}}{\lambda}\Big)\Big\} \\
&\leq C_p \frac{1}{2^{N_0 p}}\Big\{U_m^\ell + \varphi\Big(\Omega^{N_0(m+1)+\ell}, \frac{2^{N_0(m+1)+\ell}}{\lambda}\Big) + \varphi\Big(\Omega^{N_0(-m)+\ell}, \frac{2^{N_0(-m)+\ell}}{\lambda}\Big)\Big\} + \\
&\qquad\qquad\qquad + \sum_{k=-m}^{m} \int_{\Omega^{N_0 k+\ell}\setminus\Omega^{N_0(k+1)+\ell}} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx \\
&\leq \frac{C_p}{2^{N_0 p}}U_m^\ell + \Big(2\frac{C_p}{2^{N_0 p}} + 1\Big)\int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx.
\end{aligned}
$$

This implies that $U_m^\ell \leq \frac{3}{1-C_p/(2^{N_0 p})}\int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx$. Consequently,

$$\sum_{k=-\infty}^{\infty} \varphi\Big(\Omega^k, \frac{2^k}{\lambda}\Big) = \sum_{\ell=0}^{N_0-1} \lim_{m\to\infty} U_m^\ell \leq C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx,$$

where $C = \frac{3N_0}{1-C_p/(2^{N_0 p})}$ independent of f and λ. $\qquad\square$

Theorem 5.2. *Let $m \geq s \geq m(\varphi)$. Then, $H_m^\varphi(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Suppose first that $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$ for some $q \in (q(\varphi), \infty)$. Let $f = g^k + \sum_j b^k_j$ be the Calderón-Zygmund decompositions of f of degree s with height 2^k, for $k \in \mathbb{Z}$ associated with f^*_m. By Proposition 5.3, $g^k \to f$ in $H^\varphi_m(\mathbb{R}^n)$ as $k \to \infty$, while by [9, Lemma 4.10], $g^k \to 0$ uniformly as $k \to -\infty$ since $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Therefore, $f = \sum_{-\infty}^{\infty}(g^{k+1} - g^k)$ in $\mathcal{S}'(\mathbb{R}^n)$. Using [17, Lemma 3.27] together with the equation $\sum_i \zeta^k_i b^{k+1}_j = \chi_{\Omega^k} b^{k+1}_j = b^{k+1}_j$ by $\mathrm{supp} b^{k+1}_j \subset \Omega^{k+1} \subset \Omega^k$, we get

$$
\begin{aligned}
g^{k+1} - g^k &= (f - \sum_j b^{k+1}_j) - (f - \sum_i b^k_i) \\
&= \sum_i b^k_i - \sum_j b^{k+1}_j + \sum_i \sum_j P^{k+1}_{i,j} \zeta^{k+1}_j \\
&= \sum_i \left[b^k_i - \sum_j \left(\zeta^k_i b^{k+1}_j - P^{k+1}_{i,j} \zeta^{k+1}_j \right) \right] \\
&= \sum_i h^k_i
\end{aligned}
$$

where all the series converge in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Furthermore,

$$
(5.2) \qquad h^k_i = (f - P^k_i)\zeta^k_i - \sum_j \left((f - P^{k+1}_j)\zeta^k_i - P^{k+1}_{i,j} \right)\zeta^{k+1}_j.
$$

From this formula it is obvious that $\int_{\mathbb{R}^n} h^k_i(x)P(x)dx = 0$ for all $P \in \mathcal{P}_s$. Moreover, $h^k_i = \zeta^k_i f \chi_{(\Omega^{k+1})^c} - P^k_i \zeta^k_i + \zeta^k_i \sum_j P^{k+1}_j \zeta^{k+1}_j + \sum_j P^{k+1}_{i,j} \zeta^{k+1}_j$, by $\sum_j \zeta^{k+1}_j = \chi_{\Omega^{k+1}}$. But $|f(x)| \leq C(m)f^*_m(x) \leq C2^{k+1}$ for almost every $x \in (\Omega^{k+1})^c$, so by [17, Lemma 3.8 and 3.26], and $\sum_j \zeta^{k+1}_j \leq L$,

$$
(5.3) \qquad \|h^k_i\|_{L^\infty} \leq C2^{k+1} + C2^k + CL2^{k+1} + CL2^{k+1} \leq C2^k,
$$

Lastly, since $P^{k+1}_{i,j} = 0$ unless $\hat{B}^k_i \cap \hat{B}^{k+1}_j \neq \emptyset$, it follows from (5.2) and [17, Lemma 3.24] that h^k_i is supported in $B(x^k_i, 18r^k_i)$. Thus h^k_i is a multiple of (φ, ∞, s)-atom. Moreover, by (5.3) and Lemma 5.4, for any $\lambda > 0$,

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}} \sum_i \varphi\left(B(x^k_i, 18r^k_i), \frac{\|h^k_i\|_{L^\infty}}{\lambda} \right) &\leq \sum_{k \in \mathbb{Z}} L\varphi(\Omega^k, C2^k/\lambda) \\
&\leq C \int_{\mathbb{R}^n} \varphi\left(x, \frac{f^*_m(x)}{\lambda} \right)dx < \infty.
\end{aligned}
$$

Thus the series $\sum_{k\in\mathbb{Z}}\sum_i h_i^k$ converges in $H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$ and defines an atomic decomposition of f. Moreover,

$$
\sum_{k\in\mathbb{Z}}\sum_i \varphi\Big(B(x_i^k,18r_i^k),\frac{\|h_i^k\|_{L^\infty}}{\|f\|_{H_m^\varphi}}\Big) \;\leq\; C\int_{\mathbb{R}^n}\varphi\Big(x,\frac{f_m^*(x)}{\|f\|_{H_m^\varphi}}\Big)dx
$$
$$
\leq\; C.
$$

Consequently, $\|f\|_{H_{\mathrm{at}}^{\varphi,\infty,s}} \leq \Lambda_\infty(\{h_i^k\}) \leq C\|f\|_{H_m^\varphi}$ by Lemma 4.3.

Now, let f be an arbitrary element of $H_m^\varphi(\mathbb{R}^n)$. There exists a sequence $\{f_\ell\}_{\ell\geq 1} \subset L_{\varphi(\cdot,1)}^q(\mathbb{R}^n) \cap H_m^\varphi(\mathbb{R}^n)$ such that $f = \sum_{\ell=1}^\infty f_\ell$ in $H_m^\varphi(\mathbb{R}^n)$ (thus in $\mathcal{S}'(\mathbb{R}^n)$) and $\|f_\ell\|_{H_m^\varphi} \leq 2^{2-\ell}\|f\|_{H_m^\varphi}$ for any $\ell \geq 1$. For any $\ell \geq 1$, let $f_\ell = \sum_j b_{j,\ell}$ be the atomic decomposition of f_ℓ, with supp $b_{j,\ell} \subset B_{j,\ell}$ constructed above. Then $f = \sum_{\ell=1}^\infty \sum_j b_{j,\ell}$ is an atomic decomposition of f, and

$$
\sum_{\ell=1}^\infty \sum_j \varphi\Big(B_{j,\ell},\frac{\|b_{j,\ell}\|_{L^\infty}}{\|f\|_{H_m^\varphi}}\Big) \;\leq\; \sum_{\ell=1}^\infty \sum_i \varphi\Big(B_{j,\ell},\frac{\|b_{j,\ell}\|_{L^\infty}}{2^{\ell-2}\|f_\ell\|_{H_m^\varphi}}\Big)
$$
$$
\leq\; \sum_{\ell=1}^\infty C_p \frac{1}{(2^{\ell-2})^p} =: C,
$$

where C_p is such that (2.1) holds. Thus $f \in H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$, moreover,

$$
\|f\|_{H_{\mathrm{at}}^{\varphi,\infty,s}} \leq \Lambda_\infty(\{b_{j,\ell}\}) \leq C\|f\|_{H_m^\varphi}
$$

by Lemma 4.3. This completes the proof. $\qquad\square$

Proof of Theorem 3.1. By Theorem 5.1 and Theorem 5.2, we obtain

$$
H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,q,m(\varphi)}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n) \subset H_s^\varphi(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)
$$

and the inclusions are continuous. Thus $H^\varphi(\mathbb{R}^n) = H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n)$ with equivalent norms. $\qquad\square$

6. Dual spaces

In this Section, we give the proof of Theorem 3.2. In order to do this, we need the lemma, which can be seen as a consequence of the fact that $\varphi(\cdot,t)$ is *uniformly locally integrable*, below. We omit the details here.

Lemma 6.1. *Given a ball B, and $\{B_j\}_j$ be a sequence of measurable subsets of B such that $\lim_{j\to\infty}|B_j| = 0$. Then the following holds*

$$
\lim_{j\to\infty}\sup_{t>0}\frac{\varphi(B_j,t)}{\varphi(B,t)} = 0.
$$

We next note that if $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ is real-valued and

$$\mathfrak{b}_N(x) = \begin{cases} N & \text{if} \quad \mathfrak{b}(x) > N, \\ \mathfrak{b}(x) & \text{if} \quad |\mathfrak{b}(x)| \le N, \\ -N & \text{if} \quad \mathfrak{b}(x) < -N, \end{cases}$$

then by using the fact

$$\|f\|_{BMO^\varphi} \le \sup_{B-\text{ball}} \frac{1}{\|\chi_B\|_{L^\varphi}} \frac{1}{|B|} \int_{B \times B} |f(x) - f(y)| dx dy \le 2\|f\|_{BMO^\varphi},$$

we obtain that $\|\mathfrak{b}_N\|_{BMO^\varphi} \le 2\|\mathfrak{b}\|_{BMO^\varphi}$ for all $N > 0$.

Proof of Theorem 3.2. i) It is sufficient to prove it for $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ real-valued since $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ iff $\mathfrak{b} = \mathfrak{b}_1 + i\mathfrak{b}_2$ with $\mathfrak{b}_j \in BMO^\varphi(\mathbb{R}^n)$ real-valued, $j = 1, 2$, moreover

$$\|\mathfrak{b}\|_{BMO^\varphi} \approx \|\mathfrak{b}_1\|_{BMO^\varphi} + \|\mathfrak{b}_2\|_{BMO^\varphi}.$$

Suppose first that $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, the functional

$$L_\mathfrak{b}(f) = \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x) dx$$

is well defined for any $f \in L_0^\infty(\mathbb{R}^n)$ since $\mathfrak{b} \in L_{\text{loc}}^1(\mathbb{R}^n)$.

Furthermore, since $f \in L_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, we remark that the atomic decomposition $f = \sum_{k \in \mathbb{Z}} \sum_i h_i^k$ in the proof of Theorem 5.2 is also the classical atomic decomposition of f in $H^1(\mathbb{R}^n)$, so that the series converge in $H^1(\mathbb{R}^n)$ and thus in $L^1(\mathbb{R}^n)$. Combining this with the fact $\mathfrak{b} \in L^\infty(\mathbb{R}^n)$, we obtain

$$L_\mathfrak{b}(f) = \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x) dx = \sum_{k \in \mathbb{Z}} \sum_i \int_{\mathbb{R}^n} h_i^k(x)\mathfrak{b}(x) dx.$$

Therefore, by Lemma 4.4 and the proof of Theorem 5.2,

$$\begin{aligned} |L_\mathfrak{b}(f)| = \Big| \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x) dx \Big| &\le \sum_{k \in \mathbb{Z}} \sum_i \Big| \int_{\mathbb{R}^n} h_i^k(x)\mathfrak{b}(x) dx \Big| \\ &= \sum_{k \in \mathbb{Z}} \sum_i \Big| \int_{B(x_i^k, 18r_i^k)} h_i^k(x)(\mathfrak{b}(x) - \mathfrak{b}_{B(x_i^k, 18r_i^k)}(x)) dx \Big| \\ &\le \|\mathfrak{b}\|_{BMO^\varphi} \sum_{k \in \mathbb{Z}} \sum_i \|h_i^k\|_{L^\infty} \|\chi_{B(x_i^k, 18r_i^k)}\|_{L^\varphi} \\ &\le C\|\mathfrak{b}\|_{BMO^\varphi} \Lambda_\infty(\{h_i^k\}) \\ &\le C\|\mathfrak{b}\|_{BMO^\varphi} \|f\|_{H^\varphi}. \end{aligned}$$

Now, let $\mathfrak{b}$ be an arbitrary element in $BMO^\varphi(\mathbb{R}^n)$. For any $f \in L_0^\infty(\mathbb{R}^n)$, it is clear that $|f|.|\mathfrak{b}_\ell| \le |f|.|\mathfrak{b}| \in L^1(\mathbb{R}^n)$, $\ell \ge 1$, and $f(x)\mathfrak{b}_\ell(x) \to f(x)\mathfrak{b}(x)$,

as $\ell \to \infty$, a.e $x \in \mathbb{R}^n$. Therefore, by the Lebesgue dominated convergence theorem, we obtain

$$|L_{\mathfrak{b}}(f)| = \Big| \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx \Big| = \lim_{\ell \to \infty} \Big| \int_{\mathbb{R}^n} f(x)\mathfrak{b}_\ell(x)dx \Big| \leq C\|\mathfrak{b}\|_{BMO^\varphi}\|f\|_{H^\varphi},$$

since $\|\mathfrak{b}_\ell\|_{BMO^\varphi} \leq 2\|\mathfrak{b}\|_{BMO^\varphi}$ for all $\ell \geq 1$.

Because of the density of $L_0^\infty(\mathbb{R}^n)$ in $H^\varphi(\mathbb{R}^n)$, the functional $L_{\mathfrak{b}}$ can be extended to a bounded functional on $H^\varphi(\mathbb{R}^n)$, moreover, $\|L_{\mathfrak{b}}\|_{(H^\varphi)^*} \leq C\|\mathfrak{b}\|_{BMO^\varphi}$.

ii) Conversely, suppose L is a continuous linear functional on $H^\varphi(\mathbb{R}^n) \equiv H_{\mathrm{at}}^{\varphi,q,0}(\mathbb{R}^n)$ for some $q \in (q(\varphi), \infty)$. For any ball B, denote by $L_{\varphi,0}^q(B)$ the subspace of $L_\varphi^q(B)$ defined by

$$L_{\varphi,0}^q(B) := \Big\{ f \in L_\varphi^q(B) : \int_{\mathbb{R}^n} f(x)dx = 0 \Big\}.$$

Obviously, if $B_1 \subset B_2$ then

(6.1) $$L_\varphi^q(B_1) \subset L_\varphi^q(B_2) \quad \text{and} \quad L_{\varphi,0}^q(B_1) \subset L_{\varphi,0}^q(B_2).$$

Moreover, when $f \in L_{\varphi,0}^q(B) \setminus \{0\}$, $a(x) = \|\chi_B\|_{L^\varphi}^{-1}\|f\|_{L_\varphi^q(B)}^{-1}f(x)$ is a $(\varphi, q, 0)$-atom, thus $f \in H_{\mathrm{at}}^{\varphi,q,0}(\mathbb{R}^n)$ and

$$\|f\|_{H_{\mathrm{at}}^{\varphi,q,0}} \leq \|\chi_B\|_{L^\varphi}\|f\|_{L_\varphi^q(B)}.$$

Since $L \in (H_{\mathrm{at}}^{\varphi,q,0}(\mathbb{R}^n))^*$, by the above,

$$|L(f)| \leq \|L\|_{(H_{\mathrm{at}}^{\varphi,q,0})^*}\|f\|_{H_{\mathrm{at}}^{\varphi,q,0}} \leq \|L\|_{(H_{\mathrm{at}}^{\varphi,q,0})^*}\|\chi_B\|_{L^\varphi}\|f\|_{L_\varphi^q(B)},$$

for all $f \in L_{\varphi,0}^q(B)$. Therefore, L provides a bounded linear functional on $L_{\varphi,0}^q(B)$ which can be extended by the Hahn-Banach theorem to the whole space $L_\varphi^q(B)$ without increasing its norm. On the other hand, by Lemma 6.1 and Lebesgue-Nikodym theorem, there exists $h \in L^1(B)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x)h(x)dx,$$

for all $f \in L_{\varphi,0}^q(B)$, and thus $f \in L_{\varphi,0}^\infty(B)$ since $L_{\varphi,0}^\infty(B) \subset L_{\varphi,0}^q(B)$.

We now take a sequence of balls $\{B_j\}_{j \geq 1}$ such that $B_1 \subset B_2 \subset \cdots \subset B_j \subset \cdots$ and $\cup_j B_j = \mathbb{R}^n$. Then, there exists a sequence $\{h_j\}_{j \geq 1}$ such that

$$h_j \in L^1(B_j) \quad \text{and} \quad L(f) = \int_{\mathbb{R}^n} f(x)h_j(x)dx,$$

for all $f \in L^\infty_{\varphi,0}(B_j), j = 1, 2, ...$ Hence, for all $f \in L^\infty_{\varphi,0}(B_1) \subset L^\infty_{\varphi,0}(B_2)$ (by 6.1),

$$\int\limits_{\mathbb{R}^n} f(x)(h_1(x) - h_2(x))dx = \int\limits_{\mathbb{R}^n} f(x)h_1(x)dx - \int\limits_{\mathbb{R}^n} f(x)h_2(x)dx = L(f) - L(f) = 0.$$

As $f_{B_1} = 0$ if $f \in L^\infty_{\varphi,0}(B_1)$, we have

$$\int\limits_{\mathbb{R}^n} f(x)\Big((h_1(x) - h_2(x)) - (h_1 - h_2)_{B_1}\Big)dx = 0$$

for all $f \in L^\infty_{\varphi,0}(B_1)$, and thus for $f \in L^\infty_\varphi(B_1)$. Hence,

$$h_1(x) - h_2(x) = (h_1 - h_2)_{B_1}, \ a.e \ x \in B_1.$$

By the similar arguments, we also obtain

(6.2) $$h_j(x) - h_{j+1}(x) = (h_j - h_{j+1})_{B_j}$$

a.e $x \in B_j, j = 2, 3, ...$ Consequently, if we define the sequence $\{\widetilde{h}_j\}_{j \geq 1}$ by

$$\begin{cases} \widetilde{h}_1 = h_1 \\ \widetilde{h}_{j+1} = h_{j+1} + (\widetilde{h}_j - h_{j+1})_{B_j} \end{cases}, \quad j = 1, 2, ...$$

then it follows from (6.2) that

$$\widetilde{h}_j \in L^1(B_j) \quad \text{and} \quad \widetilde{h}_{j+1}(x) = \widetilde{h}_j(x)$$

a.e $x \in B_j, j = 1, 2, ...$ Thus, we can be define the function $\mathfrak{b}$ on $\mathbb{R}^n$ by

$$\mathfrak{b}(x) = \widetilde{h}_j(x)$$

if $x \in B_j$ for some $j \geq 1$ since $B_1 \subset B_2 \subset \cdots \subset B_j \subset \cdots$ and $\cup_j B_j = \mathbb{R}^n$.

Let us now show that $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ and

$$L(f) = \int\limits_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx,$$

for all $f \in L^\infty_0(\mathbb{R}^n)$.

Indeed, for any $f \in L^\infty_0(\mathbb{R}^n)$, there exists $j \geq 1$ such that $f \in L^\infty_{\varphi,0}(B_j)$. Hence,

$$L(f) = \int\limits_{\mathbb{R}^n} f(x)\widetilde{h}_j(x)dx = \int\limits_{\mathbb{B}_j} f(x)\widetilde{h}_j(x)dx = \int\limits_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx.$$

On the other hand, for all ball B, one consider $f = \text{sign}(\mathfrak{b} - \mathfrak{b}_B)$ where $\text{sign}\xi = \overline{\xi}/|\xi|$ if $\xi \neq 0$ and $\text{sign}0 = 0$. Then,

$$a = \frac{1}{2}\|\chi_B\|_{L^\varphi}^{-1}(f - f_B)\chi_B$$

is a $(\varphi, \infty, 0)$-atom. Consequently,

$$
\begin{aligned}
|L(a)| &= \frac{1}{2}\|\chi_B\|_{L^\varphi}^{-1}\left|\int_{\mathbb{R}^n} \mathfrak{b}(x)(f(x)-f_B)\chi_B(x)dx\right| \\
&= \frac{1}{2}\frac{1}{\|\chi_B\|_{L^\varphi}}\left|\int_B (\mathfrak{b}(x)-\mathfrak{b}_B)f(x)dx\right| \\
&= \frac{1}{2}\frac{1}{\|\chi_B\|_{L^\varphi}}\int_B |\mathfrak{b}(x)-\mathfrak{b}_B|dx \\
&\leq \|L\|_{(H^\varphi)^*}\|a\|_{H^\varphi} \leq C\|L\|_{(H^\varphi)^*}
\end{aligned}
$$

since $L \in (H^\varphi(\mathbb{R}^n))^*$ and Corollary 5.2. As B is arbitrary, the above implies $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ and

$$
\|\mathfrak{b}\|_{BMO^\varphi} \leq C\|L\|_{(H^\varphi)^*}.
$$

The uniqueness (in the sense $\mathfrak{b} = \widetilde{\mathfrak{b}}$ if $\mathfrak{b} - \widetilde{\mathfrak{b}} = $ const) of the function $\mathfrak{b}$ is clear. And thus the proof is finished. $\qquad\square$

7. THE CLASS OF POINTWISE MULTIPLIERS FOR $BMO(\mathbb{R}^n)$

In this subsection, we give as an interesting application that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is just the dual of $L^1(\mathbb{R}^n)+H^{\log}(\mathbb{R}^n)$ where $H^{\log}(\mathbb{R}^n)$ is a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$.

We first introduce *log-atoms*. A measurable function a is said to be *log-atom* if it satisfies the following three conditions

- a supported in B for some ball B in $\mathbb{R}^n$,
- $\|a\|_{L^\infty} \leq \dfrac{\log(e + \frac{1}{|B|}) + \sup_{x \in B} \log(e + |x|)}{|B|}$,
- $\int_{\mathbb{R}^n} a(x)dx = 0$.

To prove Theorem 3.3, we need the following two propositions.

Proposition 7.1. *There exists a positive constant C such that if f is a θ-atom (resp., log-atom) then $C^{-1}f$ is a log-atom (resp., θ-atom).*

Proposition 7.2. *On $BMO^{\log}(\mathbb{R}^n)$, we have*

$$
\|f\|_{BMO^{\log}} \approx \sup_{B-\text{ball}} \frac{\log(e + \frac{1}{|B|}) + \sup_{x \in B} \log(e + |x|)}{|B|}\int_B |f(x) - f_B|dx < \infty.
$$

We first note that θ is a *growth function* that satisfies $nq(\theta) < (n + 1)i(\theta)$ in Theorem 3.2. More precisely, $\theta \in \mathbb{A}_1$ and $\theta(x, \cdot)$ is concave with $i(\theta) = 1$.

Proof of Proposition 7.1. Let f be a *log-atom*. By the above remark, to prove that there exists a constant $C > 0$ (independent of f and which may change from line to line) such that $C^{-1}f$ is a θ-atom, it is sufficient to show that there exists a constant $C > 0$ such that

$$\int_B \theta\Big(x, \frac{\log(e + \frac{1}{|B|}) + \sup_{x \in B}\log(e + |x|)}{|B|}\Big)dx \leq C$$

or, equivalently,

$$\frac{\frac{\log(e+\frac{1}{|B|})+\sup_{x \in B}\log(e+|x|)}{|B|}}{\log(e + \frac{\log(e+\frac{1}{|B|})+\sup_{x \in B}\log(e+|x|)}{|B|}) + \sup_{x \in B}\log(e + |x|)}|B| \leq C,$$

since $\theta \in \mathbb{A}_1$. However, the last inequality is obvious.

Conversely, suppose that f is a θ-atom. Similarly, we need to show that there exists a constant $C > 0$ such that

$$\int_B \theta\Big(x, C\frac{\log(e + \frac{1}{|B|}) + \sup_{x \in B}\log(e + |x|)}{|B|}\Big)dx \geq 1$$

or, equivalently,

$$\frac{C\frac{\log(e+\frac{1}{|B|})+\sup_{x \in B}\log(e+|x|)}{|B|}}{\log(e + C\frac{\log(e+\frac{1}{|B|})+\sup_{x \in B}\log(e+|x|)}{|B|}) + \sup_{x \in B}\log(e + |x|)}|B| \geq 1.$$

However it is true. For instance we may take $C = 3$. $\qquad\square$

Proof of Proposition 7.2. It is sufficient to show that there exists a constant $C > 0$ such that

$$C^{-1}(|\log r|+\log(e+|x|)) \leq \log\Big(e+\frac{1}{|B(x,r)|}\Big)+ \sup_{y \in B(x,r)}\log(e+|y|) \leq C(|\log r|+\log(e+|x|)).$$

The first inequality is easy and shall be omited. For the second, one first consider the 1 dimensional case. Then by symmetry, we just need to prove that

$$\log(e + 1/(b - a)) + \sup_{x \in [a,b]}\log(e + |x|) \leq C(|\log(b-a)/2| + \log(e + |a + b|/2))$$

for all $b > 0, a \in [-b, b) \subset \mathbb{R}$. However, this follows from the basic two inequalities:

$$\log(e + 1/(b - a)) \leq 2(|\log(b-a)/2| + \log(e + |a + b|/2))$$

and

$$\log(e + b) \leq 5\log(e + b)/2 \leq 5(|\log(b-a)/2| + \log(e + |a + b|/2)).$$

For the general case $\mathbb{R}^n$, by the 1-dimensional result, we obtain

$$
\begin{aligned}
\log \left(e + \frac{1}{|B(x,r)|} \right) &\leq \frac{2^n}{c_n} \sum_{i=1}^{n} \log \left(e + \frac{1}{|[x_i - r, x_i + r]|} \right) \\
&\leq C \sum_{i=1}^{n} (|\log r| + \log(e + |x_i|)) \\
&\leq C(|\log r| + \log(e + |x|))
\end{aligned}
$$

where $c_n = |B(0,1)|$, and

$$
\begin{aligned}
\sup_{y \in B(x,r)} \log(e + |y|) &\leq \sum_{i=1}^{n} \sup_{y_i \in [x_i - r, x_i + r]} \log(e + |y_i|) \\
&\leq C \sum_{i=1}^{n} (|\log r| + \log(e + |x_i|)) \\
&\leq C(|\log r| + \log(e + |x|))
\end{aligned}
$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$. This finishes the proof. $\square$

Proof of Theorem 3.3. By Theorem 3.1, Theorem 3.2, Proposition 7.1, and Proposition 7.2, we obtain $(H^{\log}(\mathbb{R}^n))^* \equiv BMO^{\log}(\mathbb{R}^n)$. We deduce that, the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. $\square$

8. Finite atomic decompositions and their applications

We first prove the finite atomic decomposition theorem.

Proof of Theorem 3.4. Obviously, $H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n)$ and for all $f \in H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n)$,

$$
\|f\|_{H^\varphi} \leq C \|f\|_{H_{\text{fin}}^{\varphi,q,s}}.
$$

Thus, we have to show that for every $q \in (q(\varphi), \infty)$ there exists a constant $C > 0$ such that

$$
\|f\|_{H_{\text{fin}}^{\varphi,q,s}} \leq C \|f\|_{H^\varphi}
$$

for all $f \in H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n)$ and that a similar estimate holds for $q = \infty$ and all $f \in H_{\text{fin}}^{\varphi,\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Assume that $q \in (q(\varphi), \infty]$, and by homogeneity, $f \in H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n)$ with $\|f\|_{H^\varphi} = 1$. Notice that f has compact support. Suppose that supp $f \subset B = B(x_0, r)$ for some ball B. Recall that, for each $k \in \mathbb{Z}$,

$$
\Omega_k = \{x \in \mathbb{R}^n : f^*(x) > 2^k\}.
$$

Clearly, $f \in L_{\varphi(\cdot,1)}^{\overline{q}}(\mathbb{R}^n) \cap H^\varphi(\mathbb{R}^n)$ where $\overline{q} = q$ if $q < \infty$, $\overline{q} = q(\varphi) + 1$ if $q = \infty$. Hence, by Theorem 5.2, there exists a atomic decomposition $f =$

$\sum_{k \in \mathbb{Z}} \sum_i h_i^k \in H_{\mathrm{at}}^{\varphi, \infty, s}(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi, q, s}(\mathbb{R}^n)$ where the series converges in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Moreover,

$$(8.1) \qquad \Lambda_q(\{h_i^k\}) \leq \Lambda_\infty(\{h_i^k\}) \leq C\|f\|_{H^\varphi} = C.$$

On the other hand, it follows from the second step in the proof of Theorem 6.2 in [9] that there exists a constant $\widetilde{C} > 0$, depending only on $m(\varphi)$, such that $f^*(x) \leq \widetilde{C} \inf_{y \in B} f^*(y)$ for all $x \in B(x_0, 2r)^c$. Hence, we have

$$f^*(x) \leq \widetilde{C} \inf_{y \in B} f^*(y) \leq \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}\|f^*\|_{L^\varphi} \leq \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}$$

for all $x \in B(x_0, 2r)^c$. We now denote by k' the largest integer k such that $2^k < \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}$. Then,

$$(8.2) \qquad \Omega_k \subset B(x_0, 2r) \quad \text{for all } k > k'.$$

Next we define the functions g and ℓ by

$$g = \sum_{k \leq k'} \sum_i h_i^k \quad \text{and} \quad \ell = \sum_{k > k'} \sum_i h_i^k,$$

where the series converge in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Clearly, $f = g + \ell$ and $\operatorname{supp} \ell \subset \cup_{k>k'}\Omega_k \subset B(x_0, 2r)$ by (8.2). Therefore, $g = f = 0$ in $B(x_0, 2r)^c$, and thus $\operatorname{supp} g \subset B(x_0, 2r)$.

Let $1 < \widetilde{q} < \frac{q}{q(\varphi)}$, then $\varphi \in \mathbb{A}_{q/\widetilde{q}}$. Consequently,

$$\left(\frac{1}{|B|} \int_B |f(x)|^{\widetilde{q}}dx\right)^{1/\widetilde{q}} \leq C \left(\frac{1}{\varphi(B, 1)} \int_B |f(x)|^q \varphi(x, 1)dx\right)^{1/q} < \infty$$

by Lemma 4.5 if $q < \infty$ and it is trivial if $q = \infty$. Observe that $\operatorname{supp} f \subset B$ and that f has vanishing moments up to order s. By the above, we obtain that f is a multiple of a classical $(1, \widetilde{q}, 0)$-atom and thus $f^* \in L^1(\mathbb{R}^n)$. Hence, it follows from (8.2) that

$$\int_{\mathbb{R}^n} \sum_{k>k'} \sum_i |h_i^k(x)x^\alpha|dx \leq C(|x_0| + 2r)^s \sum_{k>k'} 2^k |\Omega_k| \leq C(|x_0| + 2r)^s\|f^*\|_{L^1} < \infty,$$

for all $|\alpha| \leq s$. This together with the vanishing moments of h_i^k implies that ℓ has vanishing moments up to order s and thus so does g by $g = f - \ell$.

In order to estimate the size of g in $B(x_0, 2r)$, we recall that

$$(8.3) \qquad \|h_i^k\|_{L^\infty} \leq C2^k, \quad \operatorname{supp} h_i^k \subset B(x_i^k, 18r_i^k) \text{ and } \sum_i \chi_{B(x_i^k, 18r_i^k)} \leq C.$$

Combining the above and the fact $\|\chi_B\|_{L^\varphi} \approx \|\chi_{B(x_0, 2r)}\|_{L^\varphi}$, we obtain

$$\|g\|_{L^\infty} \leq C \sum_{k \leq k'} 2^k \leq C2^{k'} \leq C\widetilde{C}\|\chi_B\|_{L^\varphi}^{-1} \leq C\|\chi_{B(x_0, 2r)}\|_{L^\varphi}^{-1}.$$

This proves that (see Definition 2.4)

(8.4) $$C^{-1}g \text{ is a } (\varphi, \infty, s)-\text{atom.}$$

Now, we assume that $q \in (q(\varphi), \infty)$ and conclude the proof of (i). We first verify $\sum_{k>k'} \sum_i h_i^k \in L_\varphi^q(B(x_0, 2r))$. For any $x \in \mathbb{R}^n$, since $\mathbb{R}^n = \cup_{k \in \mathbb{Z}}(\Omega_k \setminus \Omega_{k+1})$, there exists $j \in \mathbb{Z}$ such that $x \in \Omega_j \setminus \Omega_{j+1}$. Since supp $h_i^k \subset \Omega_k \subset \Omega_{j+1}$ for $k \geq j+1$, it follows from (8.3) that

$$\sum_{k>k'} \sum_i |h_i^k(x)| \leq C \sum_{k \leq j} 2^k \leq C2^j \leq Cf^*(x).$$

Since $f \in L_\varphi^q(B) \subset L_\varphi^q(B(x_0, 2r))$, we have $f^* \in L_\varphi^q(B(x_0, 2r))$. As φ satisfies uniformly locally dominated convergence condition, we further obtain $\sum_{k>k'} \sum_i h_i^k$ converges to ℓ in $L_\varphi^q(B(x_0, 2r))$.

Now, for any positive integer K, set $F_K = \{(i, k) : k > k', |i| + |k| \leq K\}$ and $\ell_K = \sum_{(i,k) \in F_K} h_i^k$. Observe that since $\sum_{k>k'} \sum_i h_i^k$ converges to ℓ in $L_\varphi^q(B(x_0, 2r))$, for any $\varepsilon > 0$, if K is large enough, we have $\varepsilon^{-1}(\ell - \ell_K)$ is a (φ, q, s)-atom. Thus, $f = g + \ell_K + (\ell - \ell_K)$ is a finite linear atom combination of f. Then, it follows from (8.1) and (8.4) that

$$\|f\|_{H_{\text{fin}}^{\varphi,q,s}} \leq C(C + \Lambda_q(\{h_i^k\}_{(i,k) \in F_K}) + \varepsilon) \leq C,$$

which ends the proof of (i).

To prove (ii), assume that f is a continuous function in $H_{\text{fin}}^{\varphi,\infty,s}(\mathbb{R}^n)$, and thus f is uniformly continuous. Then, h_i^k is continuous by examining its definition. Since f is bounded, there exists a positive integer $k'' > k'$ such that $\Omega_k = \emptyset$ for all $k > k''$. Consequently, $\ell = \sum_{k' < k \leq k''} \sum_i h_i^k$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Write $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$ with

$$\ell_1^\varepsilon \equiv \sum_{(i,k) \in F_1} h_i^k \quad \text{and} \quad \ell_2^\varepsilon \equiv \sum_{(i,k) \in F_2} h_i^k$$

where $F_1 = \{(i, k) : Cr_i^k \geq \delta, k' < k \leq k''\}$ and $F_2 = \{(i, k) : Cr_i^k < \delta, k' < k \leq k''\}$ with $C > 36$ the geometric constant (see [35]). Notice that the remaining part ℓ_1^ε will then be a finite sum. Since the atoms are continuous, ℓ_1^ε will be a continuous function. Furthermore, $\|\ell_2^\varepsilon\|_{L^\infty} \leq C(k'' - k')\varepsilon$ (see also [35]). This means that one can write ℓ as the sum of one continuous term and of one which is uniformly arbitrarily small. Hence, ℓ is continuous, and so is $g = f - \ell$.

To find a finite atomic decomposition of f, we use again the splitting $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$. By (8.1), the part ℓ_1^ε is a finite sum of multiples of (φ, ∞, s)-atoms, and

(8.5) $$\|\ell_1^\varepsilon\|_{H_{\text{fin}}^{\varphi,\infty,s}} \leq \Lambda_\infty(\{h_i^k\}) \leq C\|f\|_{H^\varphi} = C.$$

By ℓ, ℓ_1^ε are continuous and have vanishing moments up to order s, and thus so does $\ell_2^\varepsilon = \ell - \ell_1^\varepsilon$. Moreover, supp $\ell_2^\varepsilon \subset B(x_0, 2r)$ and $\|\ell_2^\varepsilon\|_{L^\infty} \leq C(k'' - k')\varepsilon$.

So we can choose ε small enough such that ℓ_2^ε into an arbitrarily small multiple of a continuous (φ, ∞, s)-atom. Therefore, $f = g + \ell_1^\varepsilon + \ell_2^\varepsilon$ is a finite linear continuous atom combination of f. Then, it follows from (8.4) and (8.5) that

$$\|f\|_{H_{\text{fin}}^{\varphi,\infty,s}} \leq C(\|g\|_{H_{\text{fin}}^{\varphi,\infty,s}} + \|\ell_1^\varepsilon\|_{H_{\text{fin}}^{\varphi,\infty,s}} + \|\ell_2^\varepsilon\|_{H_{\text{fin}}^{\varphi,\infty,s}}) \leq C.$$

This finishes the proof of (ii) and hence, the proof of Theorem 3.4. $\qquad\square$

Next we give the proof for Theorem 3.5.

Proof of Theorem 3.5. Suppose that the assumption (i) holds. For any $f \in H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n)$, by Theorem 3.4, there exists a finite atomic decomposition $f = \sum_{j=1}^k \lambda_j a_j$, where a_j's are multiples of (φ, q, s)-atoms with supported in balls B_j's, such that

$$\Lambda_q(\{\lambda_j a_j\}_{j=1}^k) = \inf\left\{\lambda > 0 : \sum_{j=1}^k \varphi\left(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{\lambda}\right) \leq 1\right\} \leq C\|f\|_{H^\varphi}.$$

Recall that, since φ is of uniformly upper type γ, there exists a constant $C_\gamma > 0$ such that

(8.6) $\qquad \varphi(x, st) \leq C_\gamma s^\gamma \varphi(x, t)$ for all $x \in \mathbb{R}^n, s \geq 1, t \in [0, \infty)$.

If there exist $j_0 \in \{1, ..., k\}$ such that $C_\gamma |\lambda_{j_0}|^\gamma \geq \sum_{j=1}^k |\lambda_j|^\gamma$, then

$$\sum_{j=1}^k \varphi\left(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{C_\gamma^{-1/\gamma}(\sum_{j=1}^k |\lambda_j|^\gamma)^{1/\gamma}}\right) \geq \varphi(B_{j_0}, \|\chi_{B_{j_0}}\|_{L^\varphi}^{-1}) = 1.$$

Otherwise, it follows from (8.6) that

$$\sum_{j=1}^k \varphi\left(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{C_\gamma^{-1/\gamma}(\sum_{j=1}^k |\lambda_j|^\gamma)^{1/\gamma}}\right) \geq \sum_{j=1}^k \frac{|\lambda_j|^\gamma}{\sum_{j=1}^k |\lambda_j|^\gamma} \varphi(B_j, \|\chi_{B_j}\|_{L^\varphi}^{-1}) = 1.$$

The above means that

$$\left(\sum_{j=1}^k |\lambda_j|^\gamma\right)^{1/\gamma} \leq C_\gamma^{1/\gamma} \Lambda_q(\{\lambda_j a_j\}_{j=1}^k) \leq C\|f\|_{H^\varphi}.$$

Therefore, by assumption (i), we obtain that

$$\|Tf\|_{\mathcal{B}_\gamma} = \left\|T\left(\sum_{j=1}^k \lambda_j a_j\right)\right\|_{\mathcal{B}_\gamma} \leq C\left(\sum_{j=1}^k |\lambda_j|^\gamma\right)^{1/\gamma} \leq C\|f\|_{H^\varphi}.$$

Since $H_{\text{fin}}^{\varphi,q,s}(\mathbb{R}^n)$ is dense in $H^\varphi(\mathbb{R}^n)$, a density argument gives the desired result.

The case (ii) is similar by using the fact that $H_{\text{fin}}^{\varphi,\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H_{\text{fin}}^{\varphi,\infty,s}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H^\varphi}$, see the lemma below. $\qquad\square$

We end the paper by the following lemma.

Lemma 8.1. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, and (φ, ∞, s) be an admissible triplet. Then, $H_{\mathrm{fin}}^{\varphi,\infty,s}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ is dense in $H_{\mathrm{fin}}^{\varphi,\infty,s}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H^\varphi}$.*

Proof. We take $q \in (q(\varphi), \infty)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying supp $\phi \subset B(0,1)$, $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Then, the proof of the lemma is simple since it follows from the fact that for every (φ, ∞, s)-atom a supported in ball $B(x_0, r)$,

$$\lim_{t \to 0} \|a - a * \phi_t\|_{L_\varphi^q(B(x_0, 2r))} = 0$$

as φ satisfies uniformly locally dominated convergence condition. $\square$

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MAPMO-UMR 6628, Département de Mathématiques, Université d'Orleans, 45067 Orléans Cedex 2, France

E-mail address: dangky@math.cnrs.fr