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# Dynnikov coordinates on virtual braid groups

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## Abstract

We define Dynnikov coordinates on virtual braid groups. We prove that they are faithful invariants of virtual 2-braids, and present evidence that they are also very powerful invariants for general virtual braids.

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## 1. Virtual braid groups

The group of virtual braids  $VB_n$ ,  $n \geq 2$ , on  $n$  strings was introduced by Kauffman [3] as a generalization of the classical braid group  $B_n$ . The most useful system of generators and defining relations of  $VB_n$  was introduced by Vershinin in [5]. The generators of  $VB_n$  are

$$\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}, \quad (1)$$

and the defining relations are the following:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1, \quad (2)$$

$$\rho_i^2 = 1, i = 1, \dots, n-1, \quad \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \quad \rho_i \rho_j = \rho_j \rho_i \text{ if } |i - j| > 1, \quad (3)$$

$$\sigma_i \rho_j = \rho_j \sigma_i \text{ if } |i - j| > 1, \quad \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}. \quad (4)$$

Thus, the group generated by  $\sigma_1, \dots, \sigma_{n-1}$  with relations (2) is the braid group  $B_n$ ; the group generated by  $\rho_1, \dots, \rho_{n-1}$  with relations (3) is the symmetric group  $S_n$ ; the relations (4) will be referred as *mixed relations*. The last presented relation is equivalent to

$$\rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i. \quad (5)$$

We remark that the relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \rho_i \sigma_{i+1} \sigma_i \quad (6)$$

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do not hold in  $VB_n$ , and these relations will be referred to as *forbidden* relations.

There is a natural epimorphism  $\pi : VB_n \rightarrow S_n$  defined by

$$\pi(\sigma_i) = \pi(\rho_i) = \rho_i, \quad i = 1, \dots, n-1.$$

The kernel  $\ker(\pi)$  is called the *virtual pure braid group* and is denoted by  $VP_n$ . Generators and relations for  $VP_n$  are described in [1]. It is easy to see that  $VB_n$  is a semidirect product:  $VB_n = VP_n \rtimes S_n$ .

## 2. Coordinates on braid groups

In [2, Ch. 8], an action of the braid group  $B_n$  on the integer lattice  $\mathbb{Z}^{2n}$  by piecewise-linear bijections is defined. This action is induced by the  $B_n$ -action on integer laminations. For the reader's convenience we recall the definition. For  $x \in \mathbb{Z}$  denote  $x^+ = \max\{0, x\}$  and  $x^- = \min\{x, 0\}$ . Define actions

$$\sigma, \sigma^{-1} : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4$$

on  $(a, b, c, d) \in \mathbb{Z}^4$  as follows:

$$(a, b, c, d) \cdot \sigma = (a + b^+ + (d^+ - e)^+, d - e^+, c + d^- + (b^- + e)^-, b + e^+), \quad (7)$$

$$(a, b, c, d) \cdot \sigma^{-1} = (a - b^+ - (d^+ + f)^+, d + f^-, c - d^- - (b^- - f)^-, b - f^-), \quad (8)$$

where

$$e = a - b^- - c + d^+, \quad f = a + b^- - c - d^+. \quad (9)$$

For a given vector  $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{Z}^{2n}$  we define the action by  $\sigma_i^\varepsilon \in B_n$ , where  $i = 1, \dots, n-1$ :

$$(a_1, b_1, \dots, a_n, b_n) \cdot \sigma_i^\varepsilon = (a'_1, b'_1, \dots, a'_n, b'_n), \quad (10)$$

where  $a'_k = a_k, b'_k = b_k$  if  $k \neq i, i+1$ , and

$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = \begin{cases} (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \sigma, & \text{if } \varepsilon = 1, \\ (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \sigma^{-1}, & \text{if } \varepsilon = -1. \end{cases} \quad (11)$$

For a word  $w$  in the alphabet  $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$  we define an action by  $w$ :

$$(a_1, b_1, \dots, a_n, b_n) \cdot w = \begin{cases} (a_1, b_1, \dots, a_n, b_n), & \text{if } w = 1, \\ ((a_1, b_1, \dots, a_n, b_n) \cdot \sigma_i^\varepsilon) \cdot w', & \text{if } w = \sigma_i^\varepsilon w'. \end{cases} \quad (12)$$

It can be shown that the above action by  $B_n$  on  $\mathbb{Z}^{2n}$  is correctly defined, i.e. if two words  $w_1$  and  $w_2$  present the same element of the braid group  $B_n$  then

$$(a_1, b_1, \dots, a_n, b_n) \cdot w_1 = (a_1, b_1, \dots, a_n, b_n) \cdot w_2$$

for any vector  $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{Z}^{2n}$ . By *Dynnikov coordinates* of a word  $w$  we will mean the vector  $(0, 1, \dots, 0, 1) \cdot w$ .

It is also shown in [2] that Dynnikov coordinates are faithful invariants of braids, i.e. if  $(0, 1, \dots, 0, 1) \cdot w_1 = (0, 1, \dots, 0, 1) \cdot w_2$  then  $w_1 = w_2$  in  $B_n$ ; thus, Dynnikov coordinates are very useful for solving the words problem in  $B_n$ . The proof uses the fact that the vector  $(0, 1, \dots, 0, 1)$  encodes a certain lamination of the  $n$ -times punctured disk whose complement consists only of disks and once-punctured disks.

**Example 1.** Actions by some elements of  $B_2$  on  $(0, 1, 0, 1) \in \mathbb{Z}^2$  are as follows:

$$\begin{aligned} (0, 1, 0, 1) \cdot \sigma_1 &= (1, 0, 0, 2), & (0, 1, 0, 1) \cdot \sigma_1^{-1} &= (-1, 0, 0, 2), \\ (0, 1, 0, 1) \cdot \sigma_1^2 &= (1, -1, 0, 3), & (0, 1, 0, 1) \cdot \sigma_1^{-2} &= (-1, -1, 0, 3), \\ &\vdots & &\vdots \\ (0, 1, 0, 1) \cdot \sigma_1^k &= (1, -k + 1, 0, k + 1), & (0, 1, 0, 1) \cdot \sigma_1^{-k} &= (-1, -k + 1, 0, k + 1), \end{aligned}$$

where  $k \in \mathbb{N}$ . Also, acting by some elements of  $B_3$  on  $(0, 1, 0, 1, 0, 1) \in \mathbb{Z}^6$  we have

$$\begin{aligned} (0, 1, 0, 1, 0, 1) \cdot \sigma_1 \sigma_2^{-1} &= (1, 0, -2, 0, 0, 3), \\ (0, 1, 0, 1, 0, 1) \cdot \sigma_1 \sigma_2 \sigma_1 &= (2, 0, 1, 0, 0, 3). \end{aligned}$$

**Question 1.** As mentioned above, two braids  $\beta$  and  $\beta'$  from  $B_n$  are equal if and only if their actions on the vector  $(0, 1, \dots, 0, 1)$  coincide. Which other vectors have this property? For instance, the vector  $(0, 0, 0, 1) \in \mathbb{Z}$  does not have it, because  $\sigma_1 \in B_2$  acts trivially on it. Is it true that any vector  $(x_1, y_1, \dots, x_n, y_n)$  which does not have subvectors  $(x_i, y_i)$  of the form  $(0, 0)$  has this property?

### 3. Coordinates on virtual braid groups

Let us define an action by elements of  $VB_n$  on  $\mathbb{Z}^{2n}$ . Consider the actions on  $\mathbb{Z}^4$  by  $\sigma$  and  $\sigma^{-1}$  as defined in (7) and (8), and define the action by  $\rho$  as the following permutation of coordinates:

$$(a, b, c, d) \cdot \rho = (c, d, a, b). \quad (13)$$

For a given vector  $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{Z}^{2n}$  we define the action by  $\rho_i \in VB_n$ ,  $i = 1, \dots, n - 1$ :

$$(a_1, b_1, \dots, a_n, b_n) \cdot \rho_i = (a'_1, b'_1, \dots, a'_n, b'_n), \quad (14)$$

where  $a'_k = a_k$ ,  $b'_k = b_k$  for  $k \neq i, i + 1$ , and

$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \rho. \quad (15)$$

The action by  $\sigma_i^\varepsilon \in VB_n$  on  $\mathbb{Z}^{2n}$  is defined according to (10) and (11).

Suppose that  $w$  is a word in the alphabet  $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, \rho_1, \dots, \rho_{n-1}\}$  representing an element of the group  $VB_n$ . Then we define

$$(a_1, b_1, \dots, a_n, b_n) \cdot w = \begin{cases} ((a_1, b_1, \dots, a_n, b_n) \cdot \rho_i) \cdot w', & \text{if } w = \rho_i w', \\ ((a_1, b_1, \dots, a_n, b_n) \cdot \sigma_i^\varepsilon) \cdot w', & \text{if } w = \sigma_i^\varepsilon w'. \end{cases} \quad (16)$$

To show that the action by  $VB_n$  on  $\mathbb{Z}^{2n}$  is correctly defined we will verify that the defining relations of the group  $VB_n$  are satisfied. Since  $\rho_i$  acts by permuting pairs of coordinates, the relations of the group  $S_n$  are obviously satisfied. The fact that relations of the group  $B_n$  are satisfied follows from [2]. So, we need to check only the case of mixed relations, i.e. that for any  $v \in \mathbb{Z}^{2n}$  the relations

$$v \cdot (\sigma_i \rho_j) = v \cdot (\rho_j \sigma_i), \quad |i - j| > 1, \quad (17)$$

$$v \cdot (\rho_i \rho_{i+1} \sigma_i) = v \cdot (\sigma_{i+1} \rho_i \rho_{i+1}) \quad (18)$$

hold. Relations (17) hold obviously, because  $\sigma_i$  acts non-trivial only on the subvector  $(a_i, b_i, a_{i+1}, b_{i+1})$  and  $\rho_j$  acts non-trivially only on the subvector  $(a_j, b_j, a_{j+1}, b_{j+1})$ . In order to verify (18) it is enough to consider the case  $i = 1$  in the group  $VB_3$ . Denote

$$(x, y, z, t) \cdot \sigma = (a'(x, y, z, t), b'(x, y, z, t), c'(x, y, z, t), d'(x, y, z, t))$$

From

$$\begin{aligned} (a_1, b_1, a_2, b_2, a_3, b_3) \cdot (\rho_1 \rho_2 \sigma_1) &= (a_2, b_2, a_1, b_1, a_3, b_3) \cdot (\rho_2 \sigma_1) \\ &= (a_2, b_2, a_3, b_3, a_1, b_1) \cdot \sigma_1 = ((a_2, b_2, a_3, b_3) \cdot \sigma, a_1, b_1) \end{aligned}$$

and

$$\begin{aligned} (a_1, b_1, a_2, b_2, a_3, b_3) \cdot (\sigma_2 \rho_1 \rho_2) &= (a_1, b_1, (a_2, b_2, a_3, b_3) \cdot \sigma) \cdot \rho_1 \rho_2 \\ &= (a'(a_2, b_2, a_3, b_3), b'(a_2, b_2, a_3, b_3), a_1, b_1, c'(a_2, b_2, a_3, b_3), d'(a_2, b_2, a_3, b_3)) \cdot \rho_2 \\ &= (a'(a_2, b_2, a_3, b_3), b'(a_2, b_2, a_3, b_3), c'(a_2, b_2, a_3, b_3), d'(a_2, b_2, a_3, b_3), a_1, b_1) \\ &= ((a_2, b_2, a_3, b_3) \cdot \sigma, a_1, b_1) \end{aligned}$$

we see that (18) holds.

**Example 2.** Actions by some elements of  $VB_2$  on  $(0, 1, 0, 1) \in \mathbb{Z}^4$  are as follows:

$$\begin{aligned} (0, 1, 0, 1) \cdot \sigma_1 \rho_1 &= (0, 2, 1, 0), \\ (0, 1, 0, 1) \cdot \sigma_1 \rho_1 \sigma_1 &= (3, 0, 0, 2), \\ (0, 1, 0, 1) \cdot \sigma_1 \rho_1 \sigma_1^{-1} &= (-2, -1, 1, 3). \end{aligned}$$

Let us demonstrate that the forbidden relations are not satisfied. More exactly, show that for  $v = (0, 1, 0, 1, 0, 1)$  we get

$$v \cdot (\rho_1 \sigma_2 \sigma_1) \neq v \cdot (\sigma_2 \sigma_1 \rho_2), \quad (19)$$

$$v \cdot (\rho_2 \sigma_1 \sigma_2) \neq v \cdot (\sigma_1 \sigma_2 \rho_1). \quad (20)$$

Indeed, (19) holds because

$$(0, 1, 0, 1, 0, 1) \cdot (\rho_1 \sigma_2 \sigma_1) = (0, 1, 0, 1, 0, 1) \cdot \sigma_2 \sigma_1 = (0, 1, 1, 0, 0, 2) \cdot \sigma_1 = (2, 0, 0, 1, 0, 2),$$

but

$$(0, 1, 0, 1, 0, 1) \cdot (\sigma_2 \sigma_1 \rho_2) = (2, 0, 0, 1, 0, 2) \cdot \rho_2 = (2, 0, 0, 2, 0, 1).$$

Analogously, (20) holds because

$$(0, 1, 0, 1, 0, 1) \cdot (\rho_2 \sigma_1 \sigma_2) = (0, 1, 0, 1, 0, 1) \cdot \sigma_1 \sigma_2 = (1, 0, 0, 2, 0, 1) \cdot \sigma_2 = (1, 0, 2, 0, 0, 3),$$

but

$$(0, 1, 0, 1, 0, 1) \cdot (\sigma_1 \sigma_2 \rho_1) = (1, 0, 2, 0, 0, 3) \cdot \rho_1 = (2, 0, 1, 0, 0, 3).$$

**Question 2.** Is there any relation between our coordinates on  $VB_n$  and the invariant of virtual braids defined by Manturov in [4]?

#### 4. Faithfulness of the $VB_n$ -action on $\mathbb{Z}^{2n}$

In this section we will be concerned with the following

**Question 3.** Is the  $VB_n$ -action on  $\mathbb{Z}^{2n}$  faithful? In other words, is it true that only the trivial element of  $VB_n$  acts as the identity on  $\mathbb{Z}^{2n}$ ?

In computer experiments, we have tested several billion ( $10^9$ ) random virtual braids with 3, 4 and 5 strands, but failed to find one that would provide a negative answer. The programs used for these tests (written in Scilab) can be obtained from B. Wiest's web page [6].

It should be stressed that nontrivial elements of  $VB_n$  may very well act trivially on individual vectors. Let us, for instance, look at the  $VB_3$ -action on  $(0, 1, 0, 1, 0, 1)$ . The actions of  $\rho_1$  and  $\rho_2$  obviously fix this vector, but those of many other braids do, too. Here is one particularly striking example:

**Example 3.** The virtual 3 strand braid

$$\beta = \sigma_1 \rho_2 \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} \rho_1 \sigma_2 \rho_1 \sigma_1 \rho_2 \sigma_1^{-1} \rho_2 \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \rho_2 \sigma_1^{-1}$$

acts trivially on the vector  $(0, 1, 0, 1, 0, 1)$ , and indeed in computer experiments, we found that it acted nontrivially on only about 0.25% of randomly generated vectors of  $\mathbb{Z}^6$  with integer coefficients between  $-100$  and  $100$ . In this sense,  $\beta$  is “nearly a negative answer” to Question 3. Another one is the virtual braid  $\sigma_2^{-1} \sigma_1 \rho_2 \sigma_2 \sigma_1 \sigma_2^{-1} \rho_2 \sigma_1 \rho_2 \sigma_2 \rho_1 \sigma_2^{-1} \rho_1 \sigma_1^{-1} \sigma_2^{-1} \rho_2 \sigma_1^{-1} \sigma_2$  which also moves only about 0.6% of random vectors.

We now turn our attention to the case  $n = 2$ .

**Example 4.** It is known [4] that  $\beta = (\sigma_1^2 \rho_1 \sigma_1^{-1} \rho_1 \sigma_1^{-1} \rho_1)^2 \in VB_2$  and the identity element  $\mathbf{e} \in VB_2$  can not be distinguished by the Burau presentation. Considering actions by these elements on  $(0, 1, 0, 1)$  we get  $(0, 1, 0, 1) \cdot \beta = (85, 49, -90, -47)$ , while  $(0, 1, 0, 1) \cdot \mathbf{e} = (0, 1, 0, 1)$ . Thus, these elements are distinguished by Dynnikov coordinates.

In fact, in the case  $n = 2$  we have a positive answer to Question 3.

**Theorem 1.** *The  $VB_2$ -action on  $\mathbb{Z}^4$  given by the above formulae is faithful.*

**PROOF.** We will, in fact show a stronger result than Theorem 1, namely that the only element of  $VB_2$  which acts trivially on a vector  $(0, x, 0, y)$ , where  $x$  and  $y$  are different positive integers, is the trivial one. As a simplest vector of this type one can take  $(0, 2, 0, 1)$ .

Consider the set of symbols  $S = \{0, +, -, +0, -0\}$ . With  $s \in S$  we associate the following subset of  $\mathbb{Z}$ :

$$s = 0 \longleftrightarrow \{x \in \mathbb{Z} : x = 0\}, \quad s = + \longleftrightarrow \{x \in \mathbb{Z} : x > 0\}, \quad s = - \longleftrightarrow \{x \in \mathbb{Z} : x < 0\},$$

$$s = +0 \iff \{x \in \mathbb{Z} : x \geq 0\}, \quad s = -0 \iff \{x \in \mathbb{Z} : x \leq 0\}.$$

A quadruple  $(s_1, s_2, s_3, s_4)$ , where  $s_i \in S$ , indicates the set all quadruples  $(a, b, c, d) \in \mathbb{Z}^4$  such that  $a, b, c, d$  belongs to subset of  $\mathbb{Z}$  associated with symbols  $s_1, s_2, s_3, s_4$ , respectively. For example,

$$(+, +0, 0, -) = \{(a, b, c, d) \in \mathbb{Z}^4 : a > 0, b \geq 0, c = 0, d < 0\}.$$

For instance, the vector  $(-1, -1, 0, 4)$  belongs to sets associated with  $(-, -, 0, +)$  and  $(-, -0, +0, +)$ , as well as with some other quadruples of symbols.

Let  $\sigma, \sigma^{-1}, \rho : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4$  be the transformations defined by (7), (8), (13). Let us apply a sequence of such transformations (without fragments  $\sigma\sigma^{-1}$ , or  $\sigma^{-1}\sigma$ , or  $\rho\rho$ ) to the initial vector, say  $(0, 2, 0, 1)$ . The statement of the theorem will follow from the fact that we trace out a path in the diagram shown in Figure 1, where an arrow with labels  $\beta$  and  $i\beta$ , where  $i \in \{1, 2, \dots, 14\}$  and  $\beta \in \{\sigma, \sigma^{-1}, \rho\}$  from box  $(s_1, s_2, s_3, s_4)$  to box  $(t_1, t_2, t_3, t_4)$  means that the image of any element of the subset of  $\mathbb{Z}^4$  associated with  $(s_1, s_2, s_3, s_4)$  under transformation  $\beta$  belongs to the subset of  $\mathbb{Z}^4$  associated with  $(t_1, t_2, t_3, t_4)$ . The numbering of arrows by  $1, 2, \dots, 14$  is done for the reader's convenience in the further discussion of cases.

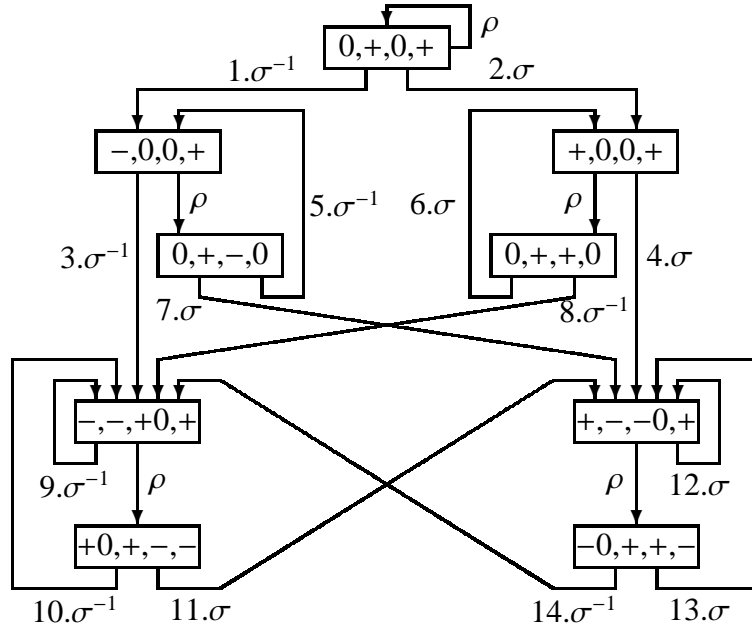


Figure 1: The diagram of actions of  $VB_2$ .

It is easy to see from (7), (8) and (13) that for any  $\beta \in VB_2$  there is the following invariant: if  $(a^*, b^*, c^*, d^*) = (a, b, c, d) \cdot \beta$  then  $b^* + d^* = b + d$ . In particular, if the initial vector is taken to be  $(0, 2, 0, 1)$  then for any vector in the diagram the sum of its second and fourth coordinates is equal to 3.

Below to simplify expressions we will use the following notations:

$$(a', b', c', d') = (a, b, c, d) \cdot \sigma, \quad (a'', b'', c'', d'') = (a, b, c, d) \cdot \sigma^{-1}.$$

By (7) and (8) we have

$$\begin{cases} a' = a + b^+ + (-a + b^- + c)^+, \\ b' = d - (a - b^- - c + d^+)^+, \\ c' = c + d^- + (a - c + d^+)^-, \\ d' = b + (a - b^- - c + d^+)^+, \end{cases} \quad \begin{cases} a'' = a - b^+ - (a + b^- - c)^+, \\ b'' = d + (a + b^- - c - d^+)^-, \\ c'' = c - d^- - (-a + c + d^+)^-, \\ d'' = b - (a + b^- - c - d^+)^-, \end{cases} \quad (21)$$

First of all we remark that the arrows of the diagram related to the action by  $\rho$  hold obviously. The other actions we are considering case by case according to the numeration of arrows.

1. Consider the action by  $\sigma^{-1}$  on  $(0, +, 0, +)$ . Each element of  $(0, +, 0, +)$  has a form  $(0, b, 0, d)$  for some  $b, d > 0$ . Since its image  $(0, b, 0, d) \cdot \sigma^{-1} = (-b, 0, 0, b + d)$  belongs to  $(-, 0, 0, +)$ , the corresponding arrow of the diagram is proven.

2. Consider the action by  $\sigma$  on  $(0, +, 0, +)$ . Each element of  $(0, +, 0, +)$  has a form  $(0, b, 0, d)$  for some  $b, d > 0$ . Since its image  $(0, b, 0, d) \cdot \sigma = (b, 0, 0, b + d)$  belongs to  $(+, 0, 0, +)$ , the corresponding arrow of the diagram is proven.

3. Consider the action by  $\sigma^{-1}$  on  $(-, 0, 0, +)$ . Let  $a < 0, d > 0$  then  $(a, 0, 0, d) \cdot \sigma^{-1} = (a, d + (a - d)^-, 0, -(a - d)^-) = (a, a, 0, -a + d) \in (-, -, 0, +) \subset (-, -, +0, +)$ .

4. Consider the action by  $\sigma$  on  $(+, 0, 0, +)$ . Let  $a, d > 0$  then  $(a, 0, 0, d) \cdot \sigma = (a, -a, 0, a + d) \in (+, -, 0, +) \subset (+, -, -0, +)$ .

5. Consider the action by  $\sigma^{-1}$  on  $(0, +, -, 0)$ . Let  $b > 0, c < 0$  then  $(0, b, c, 0) \cdot \sigma^{-1} = (-b - (-c)^+, 0, c - (-c)^-, b - (-c)^-) = (-b + c, 0, 0, b) \in (-, 0, 0, +)$ .

6. Consider the action by  $\sigma$  on  $(0, +, +, 0)$ . Let  $b, c > 0$  then  $(0, b, c, 0) \cdot \sigma = (b + c, 0, 0, b) \in (+, 0, 0, +)$ .

7. Consider the action by  $\sigma$  on  $(0, +, -, 0)$ . Let  $b > 0, c < 0$  then  $(0, b, c, 0) \cdot \sigma = (b, c, c, b - c) \in (+, -, -, +) \subset (+, -, -0, +)$ .

8. Consider the action by  $\sigma^{-1}$  on  $(0, +, +, 0)$ . Let  $b, c > 0$  then  $(0, b, c, 0) \cdot \sigma^{-1} = (-b, -c, c, b + c) \in (-, -, +, +) \subset (-, -, +0, +)$ .

9. Let us demonstrate that  $(-, -, +0, +) \cdot \sigma^{-1} \in (-, -, +0, +)$ . Since  $a < 0, b < 0, c \geq 0, d > 0$ , the action is given by formulae

$$\begin{cases} a'' = a - (a + b - c)^+, \\ b'' = d + (a + b - c - d)^-, \\ c'' = c - (-a + c + d)^-, \\ d'' = b - (a + b - c - d)^-. \end{cases}$$

Moreover,  $a + b - c < 0$  and  $a + b - c - d < 0$  imply that  $(a, b, c, d) \cdot \sigma^{-1} = (a, a + b - c, c, -a + c + d) \in (-, -, +0, +)$ .

10. Let us demonstrate that  $(+0, +, -, -) \cdot \sigma^{-1} \in (-, -, +0, +)$ . Since  $a \geq 0, b > 0, c < 0, d < 0$ , the action is given by formulae

$$\begin{cases} a'' = a - b - (a - c)^+, \\ b'' = d + (a - c)^-, \\ c'' = c - d - (-a + c)^-, \\ d'' = b - (a - c)^-. \end{cases}$$



Moreover,  $a - c > 0$  implies that  $(a, b, c, d) \cdot \sigma^{-1} = (-b + c, d, a - d, b) \in (-, -, +, +) \subset (-, -, +0, +)$ .

11. Let us demonstrate that  $(+0, +, -, -) \cdot \sigma \in (+, -, -0, +)$ . Since  $a \geq 0, b > 0, c < 0, d < 0$ , the action is given by formulae

$$\begin{cases} a' = a + b + (-a + c)^+, \\ b' = d - (a - c)^+, \\ c' = c + d + (a - c)^-, \\ d' = b + (a - c)^+. \end{cases}$$

Moreover,  $a - c > 0$  implies that  $(a, b, c, d) \cdot \sigma = (a + b, d - a + c, c + d, b + a - c) \in (+, -, -, +) \subset (+, -, -0, +)$ .

12. Let us demonstrate that  $(+, -, -0, +) \cdot \sigma = (+, -, -0, +)$ . Since  $a > 0, b < 0, c \leq 0, d > 0$ , the action is given by formulae

$$\begin{cases} a' = a + (-a + b + c)^+, \\ b' = d - (a - b - c + d)^+, \\ c' = c + (a - c + d)^-, \\ d' = b + (a - b - c + d)^+. \end{cases}$$

Moreover,  $-a + b + c < 0, a - b - c + d > 0$ , and  $a - c + d > 0$  imply that  $(a, b, c, d) \cdot \sigma = (a, -a + b + c, c, a - c + d) \in (+, -, -0, +)$ .

13. Let us demonstrate that  $(-0, +, +, -) \cdot \sigma \in (+, -, -0, +)$ . Since  $a \leq 0, b > 0, c > 0, d < 0$ , the action is given by formulae

$$\begin{cases} a' = a + b + (-a + c)^+, \\ b' = d - (a - c)^+, \\ c' = c + d + (a - c)^-, \\ d' = b + (a - c)^+. \end{cases}$$

Moreover,  $a - c < 0$  implies that  $(a, b, c, d) \cdot \sigma = (b + c, d, d + a, b) \in (+, -, -, +) \subset (+, -, -0, +)$ .

14. Let us demonstrate that  $(-0, +, +, -) \cdot \sigma^{-1} = (-, -, +0, +)$ . Since  $a \leq 0, b > 0, c > 0, d < 0$ , the action is given by

$$\begin{cases} a'' = a - b - (a - c)^+, \\ b'' = d + (a - c)^-, \\ c'' = c - d - (-a + c)^-, \\ d'' = b - (a - c)^-. \end{cases} \quad (22)$$

Moreover,  $a - c < 0$  implies that  $(a, b, c, d) \cdot \sigma^{-1} = (a - b, d + a - c, c - d, b - a + c) \in (-, -, +, +) \subset (-, -, +0, +)$ . The proof is completed.

**Remark 1.** Theorem 1 allows us to introduce on  $VB_2$  various ‘‘coordinates systems’’. For example, taking  $(0, 2, 0, 1)$  as the initial vector, for any  $\beta \in VB_2$  one can define  $(0, 2, 0, 1) \cdot \beta$  as its coordinates. In this sense Theorem 1 gives an analog of Dynnikov coordinates originally defined for braid groups.

**Remark 2.** It is shown in the proof of Theorem 1 that the  $VB_2$ -action is faithful on any vector of the form  $(0, x, 0, y)$ , where  $x$  and  $y$  are different positive integers. Note that the action on some other vectors of  $\mathbb{Z}^4$  can fail to be faithful. For example,  $(0, 0, 0, 1) \cdot \sigma = (0, 0, 0, 1)$  and  $(0, 1, 0, 1) \cdot \rho = (0, 1, 0, 1)$ .

**Remark 3.** Let us define the norm of a quadruple  $\|(a, b, c, d)\| = |a| + |b| + |c| + |d|$ . Obviously, the norm is invariant under the  $\rho$ -action. One can easily see from the proof of Theorem 1 that all the arrows labelled  $\sigma$  or  $\sigma^{-1}$  in Figure 1 increase the norm. For example, considering case 13, one gets  $\|(a, b, c, d) \cdot \sigma\| = \|(b + c, d, d + a, b)\| = |b + c| + |d| + |d + a| + |b| > |a| + |b| + |c| + |d|$ , because  $a \leq 0, b > 0, c > 0, d < 0$ . However, this property doesn't hold if one takes an arbitrary vector from  $\mathbb{Z}^4$ :  $\|(7, 4, 1, 1)\| = 13$ , but  $\|(7, 4, 1, 1) \cdot \sigma\| = \|(3, 1, 6, 1)\| = 11$ .

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