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THE SPECTRUM OF THE LAPLACE OPERATOR ON THE MANIFOLD $\mathrm{Sp}(n)/\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$

FIDA EL CHAMI

Abstract: In this paper, we compute the Laplace spectrum on the forms of the manifold $\mathrm{Sp}(n)/\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$. The method is based on the representation theory of compact Lie groups and the "identification" of the Laplace operator with the Casimir operator in symmetric spaces.

Key Words: Laplace spectrum, differential forms, representation theory, Casimir operator.

1. INTRODUCTION

Let (G, K) be a compact symmetric pair with a compact connected semisimple Lie group G and $M = G/K$. We suppose that the Riemannian metric on M is induced from the Killing form sign changed. This is a G -invariant Riemannian metric on M . We consider the Laplace operator Δ_p acting on the space of differential p -forms and its spectrum $\mathrm{Spec}^p(M)$. The operator Δ_p is G -invariant when we consider the space of p -forms $C^\infty(\wedge^p M)$ as a G -module. Ikeda and Taniguchi [3] computed the spectrum on the forms for $M = S^n$ and $P^n(\mathbb{C})$ using representation theory. They showed that $\Delta_p = C$, the Casimir operator on G . On the other hand, Freudenthal's formula gives the eigenvalues of C on irreducible G -modules and Weyl's dimension formula gives their multiplicities. Then, it suffices to decompose $C^\infty(\wedge^p M)$ into irreducible G -submodules. Generally, this decomposition is not easy. Frobenius reciprocity law enables us to reduce the problem into the two followings: first, we decompose an irreducible G -module (as a K -module by restriction) into irreducible K -submodules, second, we decompose the p -th exterior power of the adjoint representation of the group K into irreducible K -submodules. C. Tsukamoto [6] uses this method to compute the spectra of the spaces $\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$. We note that in the case of functions, similar methods are used in [4] and [5] to compute the Laplace spectrum for the manifolds $\mathrm{SO}(2p+2q+1)/\mathrm{SO}(2p) \times \mathrm{SO}(2q+1)$ and

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$\mathrm{Sp}(n)/\mathrm{SU}(n)$.

In [1] or [2], I generalized the result of [6] to calculate the spectrum on the forms of Grassmann manifolds.

This work is devoted to compute the spectrum on the forms of the manifold $\mathrm{Sp}(n)/\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$. In the second section, we give a branching law to decompose the restriction of any irreducible $\mathrm{Sp}(n)$ -module into a sum of irreducible $\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$ -modules. In section three, we decompose the p -th exterior powers of the adjoint representation into irreducible $\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$ -modules.

2. BRANCHING LAW

Let $G = \mathrm{Sp}(n)$ and $K = \mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$. We denote by \mathfrak{g} (resp. \mathfrak{k}) the complexified Lie algebra of G (resp. K). Precisely,

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} ; A, B, C \in M_n(\mathbb{C}), {}^tB = B, {}^tC = C \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & -{}^tA_1 & 0 \\ 0 & C_2 & 0 & -{}^tA_2 \end{pmatrix} ; A_1, B_1, C_1 \in M_q(\mathbb{C}), A_2, B_2, C_2 \in M_{n-q}(\mathbb{C}), {}^tB_i = B_i, {}^tC_i = C_i, i = 1, 2 \right\}.$$

We choose the following Cartan subalgebra of \mathfrak{g} and \mathfrak{k} :

$$\mathfrak{t} = \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n); \lambda_j \in \mathbb{C}$$

where λ_j is considered to be an element of \mathfrak{t}^* .

We recall the following results:

- The roots of G :

$$\Delta_G = \{\pm\lambda_i \pm \lambda_j; 1 \leq i < j \leq n\} \cup \{\pm 2\lambda_i; 1 \leq i \leq n\}.$$

- The positive roots of G :

$$\Delta_G^+ = \{\lambda_i \pm \lambda_j; 1 \leq i < j \leq n\} \cup \{2\lambda_i; 1 \leq i \leq n\}.$$

- The simple roots of G :

$$\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, \dots, \alpha_{n-1} = \lambda_{n-1} - \lambda_n, \alpha_n = 2\lambda_n.$$

- Any dominant weight for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of G has the form

$$\begin{cases} \Lambda = h_1\lambda_1 + h_2\lambda_2 + \dots + h_n\lambda_n \\ h_i \in \mathbb{Z} \\ h_1 \geq h_2 \geq \dots \geq h_n. \end{cases} \quad (1)$$

• The Weyl group of G : $W_G = \{\phi = (\varepsilon_1, \dots, \varepsilon_n, \sigma) / \varepsilon_i = \pm 1, \sigma \in S_n\}$, with $\phi(a_1\lambda_1 + \dots + a_n\lambda_n) = \sum_{i=1}^n \varepsilon_i a_i \sigma(\lambda_i)$, $\det(\phi) = \mathrm{sign}(\sigma)$ and S_n is the group of all permutations of $\{1, \dots, n\}$.

• The roots of K :

$$\Delta_K = \{\pm\lambda_i \pm \lambda_j; 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq n\} \cup \{\pm 2\lambda_i; 1 \leq i \leq n\}.$$

• The positive roots of K :

$$\Delta_K^+ = \{\lambda_i \pm \lambda_j; 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq n\} \cup \{2\lambda_i; 1 \leq i \leq n\}.$$

• The simple roots of K :

$$\begin{aligned} &\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{q-1} - \lambda_q, 2\lambda_q, \\ &\lambda_{q+1} - \lambda_{q+2}, \lambda_{q+2} - \lambda_{q+3}, \dots, \lambda_{n-1} - \lambda_n, 2\lambda_n. \end{aligned}$$

• Any dominant weight for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of K can be written:

$$\begin{cases} \Lambda' = k_1\lambda_1 + \dots + k_q\lambda_q + k_{q+1}\lambda_{q+1} + \dots + k_n\lambda_n \\ k_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n \\ k_1 \geq k_2 \geq \dots \geq k_q \geq 0 \\ k_{q+1} \geq k_{q+2} \geq \dots \geq k_m \geq 0. \end{cases} \quad (2)$$

• The Weyl group of K : $W_K = W_{\mathrm{Sp}(q)} \times W_{\mathrm{Sp}(n-q)}$.

Notations :

(i) We denote by:

$$\begin{aligned} e(\Lambda) &= e^{2\pi i \Lambda}, \quad s(\Lambda) = e(\Lambda) - e(-\Lambda), \quad c(\Lambda) = e(\Lambda) + e(-\Lambda), \\ \alpha_{ij} &= \frac{\lambda_i + \lambda_j}{2}, \quad \beta_{ij} = \frac{\lambda_i - \lambda_j}{2}. \end{aligned}$$

(ii) For r and s integers such that $1 \leq r \leq s$, we designate by $[a_{ij}]_{r:s}$ a square matrix with i, j between r and s .

Remark 1. If r is an integer, then we have $\frac{s(rx)}{s(x)} = \sum_{k=0}^{r-1} e((2k - r + 1)x)$.

The demonstration of the next lemma is similar to that of the lemma 9 page 403 in [2] (or lemme 2.2.4 page 52 in [1]).

Lemma 2. Let H_1, \dots, H_n be integers verifying $H_1 > \dots > H_n > 0$. We have for all $q \in \{1, \dots, n\}$:

$$\frac{\det[s(H_i\lambda_j)]_{1:n}}{\prod_{i=1}^q \prod_{j=i+1}^n s(\alpha_{ij})s(\beta_{ij})} = \sum_{K_{1,v}} \dots \sum_{K_{q,v}} \left\{ \prod_{r=1}^q \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) s(l_{r,n}\lambda_r) \right\} \det[s(K_{q,i}\lambda_j)]_{q+1:n},$$

where the summations are taken over all the sets of integers $K_{u,v}$ ($1 \leq u \leq q$ and $r+1 \leq v \leq n$) satisfying:

$$\begin{cases} K_{u-1,v+1} < K_{u,v} < K_{u-1,v-1} & \text{for } u+1 \leq v \leq n-1 \\ K_{u,n} < K_{u-1,n-1} \\ 0 < K_{u,n} < K_{u,n-1} < \dots < K_{u,u+1}, \end{cases} \quad (3)$$

$K_{0,v} = H_v$ and for all $1 \leq r \leq q$ and $r \leq s \leq n$, the integers $l_{r,s}$ are given by:

$$\begin{cases} l_{r,r} = K_{r-1,r} - \max(K_{r-1,r+1}, K_{r,r+1}) \\ l_{r,s} = \min(K_{r-1,s}, K_{r,s}) - \max(K_{r-1,s+1}, K_{r,s+1}) & \text{for } r+1 \leq s \leq n-1 \\ l_{r,n} = \min(K_{r-1,n}, K_{r,n}). \end{cases}$$

Theorem 3. Let $V = V(\Lambda)$ be an irreducible G -module of highest weight $\Lambda = h_1\lambda_1 + \dots + h_n\lambda_n$ satisfying (1). Then the irreducible decomposition of V as a K -module contains an irreducible K -submodule $V' = V'(\Lambda')$ with highest weight $\Lambda' = k_1\lambda_1 + \dots + k_q\lambda_q + k_{q+1}\lambda_{q+1} + \dots + k_n\lambda_n$ satisfying (2), if and only if:

1. $\begin{cases} h_{i+q} \leq k_i \leq h_{i-q} & \text{for } q+1 \leq i \leq n-q \\ k_i \leq h_{i-q} & \text{for } n-q+1 \leq i \leq n. \end{cases}$
2. The multiplicity $m_{\Lambda'}$ of $V' = V'(\Lambda')$ in the decomposition is the coefficient, when it does not vanish, of $e((k_1+q)\lambda_1 + \dots + (k_q+1)\lambda_q)$ in:

$$\prod_{i=1}^{q-1} \prod_{j=i+1}^q s(\alpha_{ij}) s(\beta_{ij}) \sum_{k_{1,v}} \dots \sum_{k_{q-1,v}} \left\{ \prod_{r=1}^q \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) \right\} s(l_{1,n}\lambda_1) \dots s(l_{q,n}\lambda_q).$$

where the summations are taken over all the sets of integers $k_{u,v}$, $1 \leq u \leq q-1$ and $u+1 \leq v \leq n$ such that:

- if $2u < 3q - n + 1$:

$$\begin{cases} \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq k_{u-1,v-1} & \text{for } u+1 \leq v \leq n-q+u \\ k_{u-1,v+1} \leq k_{u,v} \leq k_{u-1,v-1} & \text{for } n-q+u+1 \leq v \leq 2q-u \\ k_{u-1,v+1} \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) & \text{for } 2q-u+1 \leq v \leq n-1 \\ k_{u,n} \leq \min(k_{u-1,n-1}, k_{q,n-q+u}) \\ 0 \leq k_{u,n} \leq \dots \leq k_{u,u+1}, \end{cases} \quad (4)$$

- if $2u \geq 3q - n + 1$:

$$\begin{cases} \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq k_{u-1,v-1} & \text{for } u+1 \leq v \leq 2q-u \\ \max(k_{u-1,v+1}, k_{q,v+q-u}) \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) & \text{for } 2q-u+1 \leq v \leq n-q+u \\ k_{u-1,v+1} \leq k_{u,v} \leq \min(k_{u-1,v-1}, k_{q,v-q+u}) & \text{for } n-q+u+1 \leq v \leq n-1 \\ k_{u,n} \leq \min(k_{u-1,n-1}, k_{q,n-q+u}) \\ 0 \leq k_{u,n} \leq \dots \leq k_{u,u+1}, \end{cases} \quad (5)$$

with $k_{0,v} = h_v$ and $k_{q,v} = k_v$. The integers $l_{r,s}$ are given by:

$$\begin{cases} l_{r,r} = k_{r-1,r} - \max(k_{r-1,r+1}, k_{r,r+1}) + 1 \\ l_{r,s} = \min(k_{r-1,s}, k_{r,s}) - \max(k_{r-1,s+1}, k_{r,s+1}) + 1 \text{ for } r+1 \leq s \leq n-1 \\ l_{r,n} = \min(k_{r-1,n}, k_{r,n}) + 1. \end{cases} \quad (6)$$

Proof: To decompose an irreducible G -module of highest weight Λ into irreducible K -modules, we will determine the set E of highest weights of K such that:

$$\chi_G(\Lambda) = \sum_{\Lambda' \in E} \chi_K(\Lambda'),$$

where $\chi_G(\Lambda)$ (resp. $\chi_K(\Lambda')$) is the character of $V(\Lambda)$ (resp. $V'(\Lambda')$). Using the Weyl character formula, we obtain:

$$\frac{\xi_G(\Lambda + \delta_G)}{\xi_G(\delta_G)} = \sum_{\Lambda' \in E} \frac{\xi_K(\Lambda' + \delta_K)}{\xi_K(\delta_K)}.$$

Then we have to determine the set E such that:

$$\frac{\xi_G(\Lambda + \delta_G)}{\xi_G(\delta_G)/\xi_K(\delta_K)} = \sum_{\Lambda' \in E} \xi_K(\Lambda' + \delta_K), \quad (7)$$

It is well known that:

$$\xi_G(\delta_G) = \prod_{\alpha \in \Delta_G^+} (e(\alpha/2) - e(-\alpha/2)) \quad \text{and} \quad \xi_K(\delta_K) = \prod_{\alpha \in \Delta_K^+} (e(\alpha/2) - e(-\alpha/2)),$$

then

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{\alpha \in \Delta_G^+ - \Delta_K^+} (e(\alpha/2) - e(-\alpha/2)).$$

Writing Λ in the form (1), we have $\Lambda + \delta_G = H_1\lambda_1 + H_2\lambda_2 + \dots + H_n\lambda_n$, where $H_i = h_i + n - i + 1$ for all $1 \leq i \leq n$. The H_i are integers verifying $H_1 > H_2 > \dots > H_n > 0$.

In the same way we have $\Lambda' + \delta_K = K_1\lambda_1 + \dots + K_q\lambda_q + K_{q+1}\lambda_{q+1} + \dots + K_n\lambda_n$, where $K_i = k_i + q - i + 1$ for all $1 \leq i \leq q$ and $K_i = k_i + n - i + 1$ for all $q+1 \leq i \leq n$. The K_i are integers verifying:

$$K_1 > K_2 > \dots > K_q > 0 \quad \text{and} \quad K_{q+1} > K_{q+2} > \dots > K_n > 0.$$

Then we obtain:

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{i=1}^q \prod_{j=q+1}^n s(\alpha_{ij})s(\beta_{ij}).$$

On the other hand, we show that

$$\xi_G(\Lambda + \delta_G) = \det[s(H_i\lambda_j)]_{1:n},$$

and

$$\xi_K(\Lambda' + \delta_K) = \det[s(K_i\lambda_j)]_{1:q} \times \det[s(K_i\lambda_j)]_{q+1:n}.$$

To determine the set E in the equality (7), it suffices to determine the integers K_1, \dots, K_n such that

$$\frac{\det[s(H_i \lambda_j)]_{1:n}}{\prod_{i=1}^q \prod_{j=q+1}^n s(\alpha_{ij}) s(\beta_{ij})} = \sum_{\substack{K_1 > \dots > K_q > 0 \\ K_{q+1} > \dots > K_n > 0}} \det[s(K_i \lambda_j)]_{1:q} \times \det[s(K_i \lambda_j)]_{q+1:n}$$

Using lemma 2, we have to determine the integers K_1, \dots, K_n such that

$$\begin{aligned} & \prod_{i=1}^{q-1} \prod_{j=i+1}^q s(\alpha_{ij}) s(\beta_{ij}) \times \sum_{K_{1,v}} \dots \sum_{K_{q,v}} \left\{ \prod_{r=1}^q \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s} \lambda_r)}{s(\lambda_r)} \right) s(l_{r,n} \lambda_r) \right\} \det[s(K_{q,i} \lambda_j)]_{q+1:n} \\ &= \sum_{\substack{K_1 > \dots > K_q > 0 \\ K_{q+1} > \dots > K_n > 0}} \det[s(K_i \lambda_j)]_{1:q} \times \det[s(K_i \lambda_j)]_{q+1:n}. \end{aligned}$$

We permute successively the summations on the $K_{1,v}, \dots, K_{q,v}$ satisfying (3) to get the first one on $K_{q,v}$ which verify:

$$\begin{cases} H_{v+q} + q \leq K_{q,v} \leq H_{v-q} - q & \text{for } q+1 \leq v \leq n-q \\ K_{q,v} \leq H_{v-q} - q & \text{for } n-q+1 \leq v \leq n \\ 0 < K_{q,n} < \dots < K_{q,q+1}, \end{cases}$$

and the other ones on $K_{1,v}, \dots, K_{q-1,v}$ such that

- if $2u > n - q - 3$:

$$\begin{cases} a_{q,u,v} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for } q-u \leq v \leq n-u-1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for } n-u \leq v \leq q+u+1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < b_{q,u,v} & \text{for } q+u+2 \leq v \leq n-1 \\ K_{q-u-1,n} < b_{q,u,n} \\ 0 < K_{q-u-1,n} < \dots < K_{q-u-1,q-u}, \end{cases} \quad (8)$$

- if $2u \leq n - q - 3$:

$$\begin{cases} a_{q,u,v} < K_{q-u-1,v} < K_{q-u-2,v-1} & \text{for } q-u \leq v \leq q+u+1 \\ a_{q,u,v} < K_{q-u-1,v} < b_{q,u,v} & \text{for } q+u+2 \leq v \leq n-u-1 \\ K_{q-u-2,v+1} < K_{q-u-1,v} < b_{q,u,v} & \text{for } n-u \leq v \leq n-1 \\ K_{q-u-1,n} < b_{q,u,n} \\ 0 < K_{q-u-1,n} < \dots < K_{q-u-1,q-u}, \end{cases} \quad (9)$$

where

$$\begin{aligned} a_{q,u,v} &= \max(K_{q-u-2,v+1}, K_{q,v+u+1} + u) \\ b_{q,u,v} &= \min(K_{q-u-2,v-1}, K_{q,v-u-1} - u). \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} & \prod_{i=1}^{q-1} \prod_{j=i+1}^q s(\alpha_{ij})s(\beta_{ij}) \times \sum_{K_{q,v}} \sum_{K_{1,v}} \dots \sum_{K_{q-1,v}} \left\{ \prod_{r=1}^q \left(\prod_{i=s}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) \right\} \\ & s(l_{1,n}\lambda_1) \dots s(l_{q,n}\lambda_q) \det[s(K_{q,i}\lambda_j)]_{q+1:n} \\ & = \sum_{\substack{K_1 > \dots > K_q > 0 \\ K_{q+1} > \dots > K_n > 0}} \det[s(K_i\lambda_j)]_{1:q} \times \det[s(K_i\lambda_j)]_{q+1:n}. \end{aligned}$$

By identifying the left and right terms of the last equality, we get:

$K_i = K_{q,i}$ for all $q+1 \leq i \leq n$ and

$$\begin{aligned} & \sum_{K_1 > \dots > K_q > 0} \det[s(K_i\lambda_j)]_{1:q} \\ & = \prod_{i=1}^{q-1} \prod_{j=i+1}^q s(\alpha_{ij})s(\beta_{ij}) \times \sum_{K_{1,v}} \dots \sum_{K_{q-1,v}} \left\{ \prod_{r=1}^q \left(\prod_{s=r}^{n-1} \frac{s(l_{r,s}\lambda_r)}{s(\lambda_r)} \right) \right\} s(l_{1,n}\lambda_1) \dots s(l_{q,n}\lambda_q), \end{aligned}$$

where the conditions on $K_{u,v}$ for $1 \leq u \leq q-1$, are (8) and (9). We find:

$$\begin{cases} h_{i+q} \leq k_i \leq h_{i-q} & \text{for } q+1 \leq i \leq n-q \\ k_i \leq h_{i-q} & \text{for } n-q+1 \leq i \leq n \\ 0 \leq k_n \leq \dots \leq k_q. \end{cases}$$

If we denote by:

$$k_{u,v} = K_{u,v} - n + v - 1, \quad \text{for all } 0 \leq u \leq q-1 \text{ and } u+1 \leq v \leq n,$$

we obtain the result. \square

3. DECOMPOSITION OF $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$

We identify the complexified cotangent space of $M = G/K$ at $o = [K]$ with $(\mathfrak{g}/\mathfrak{k})^*$, the dual space of $\mathfrak{g}/\mathfrak{k}$.

The K -module $(\mathfrak{g}/\mathfrak{k})^*$ is irreducible of highest weight $\lambda_1 + \lambda_{q+1}$.

Notations: Let H and L be two groups, V a H -module and W a L -module. The space $V \otimes W$ has a structure of $H \times L$ -module by the action of H on V and L on W . We denote by $V \boxtimes W$ the obtained $H \times L$ -module.

Thus, the $\mathrm{Sp}(q) \times \mathrm{Sp}(n-q)$ -module $(\mathfrak{g}/\mathfrak{k})^*$ is isomorphic to $V(\lambda_1) \boxtimes V(\lambda_{q+1})$.

3.1. Particular case $K = \mathbf{Sp}(2) \times \mathbf{Sp}(n-2)$. Let H be the subgroup $T \times T$ of $\mathrm{Sp}(2)$ where T is a torus of $\mathrm{Sp}(1)$. We begin by decomposing the restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $H \times \mathrm{Sp}(n-2)$. To restrict $(\mathfrak{g}/\mathfrak{k})^*$, i.e. $V(\lambda_1) \boxtimes V(\lambda_3)$, to

$H \times \mathrm{Sp}(n-2)$, we restrict the $\mathrm{Sp}(2)$ -module $V(\lambda_1)$ to H . The decomposition of the $\mathrm{Sp}(2)$ -module $V(\lambda_1)$ into irreducible H -submodules:

$$V(\lambda_1)|_H \cong V(\lambda_1) \oplus V(-\lambda_1) \oplus V(\lambda_2) \oplus V(-\lambda_2).$$

We denote by $V_1 = V(\lambda_1) \boxtimes V(\lambda_3)$, $V_2 = V(-\lambda_1) \boxtimes V(\lambda_3)$, $V_3 = V(\lambda_2) \boxtimes V(\lambda_3)$ and $V_4 = V(-\lambda_2) \boxtimes V(\lambda_3)$. Then

$$(\mathfrak{g}/\mathfrak{k})^* \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4 \quad (\text{irreducible } H \times \mathrm{Sp}(n-2)\text{-modules}).$$

Using the notation $\wedge^{a,b,c,d} = \wedge^a V_1 \otimes \wedge^b V_2 \otimes \wedge^c V_3 \otimes \wedge^d V_4$ ($H \times \mathrm{Sp}(n-2)$ -module), we get the $H \times \mathrm{Sp}(n-2)$ -decomposition

$$\wedge^p(\mathfrak{g}/\mathfrak{k})^* \cong \sum \wedge^{a,b,c,d} \quad \text{with } a + b + c + d = p. \quad (10)$$

On the other hand, the restriction to $\mathrm{Sp}(n-2)$ of V_1, V_2, V_3 or V_4 is isomorphic to $V = V(\lambda_3)$. Also, the $H \times \mathrm{Sp}(n-2)$ -module, $\wedge^{a,b,c,d}$, is isomorphic to:

$$V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes (\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V). \quad (11)$$

It means that it suffices to decompose the $\mathrm{Sp}(n-2)$ -module $(\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V)$ into irreducible $\mathrm{Sp}(n-2)$ -submodules to obtain the decomposition of the $H \times \mathrm{Sp}(n-2)$ -module $\wedge^{a,b,c,d}$. We suppose that:

$$\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V \cong \sum V(\mu), \quad (\text{irreducible } \mathrm{Sp}(n-2)\text{-modules}). \quad (12)$$

We obtain:

$$\wedge^{a,b,c,d} \cong \sum_{\mu} V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes V(\mu), \quad (H \times \mathrm{Sp}(n-2)\text{-modules}). \quad (13)$$

Notations: We set $\gamma_{j-2} = \lambda_j$ for $3 \leq j \leq n$, and:

$$\begin{aligned} \Gamma_0 &= 0 \\ \Gamma_j &= \gamma_1 + \dots + \gamma_j \quad \text{for } 1 \leq j \leq n-2 \end{aligned}$$

The Γ_j for $1 \leq j \leq n-2$ are the fundamental weights of the group $\mathrm{Sp}(n-2)$. With these notations, the restriction of $\wedge^{a,b,c,d}$ to $\mathrm{Sp}(n-2)$ is isomorphic to:

$$\wedge^a V(\Gamma_1) \otimes \wedge^b V(\Gamma_1) \otimes \wedge^c V(\Gamma_1) \otimes \wedge^d V(\Gamma_1).$$

Proposition 4. [6]

(1) For $0 \leq r \leq n-2$, we have

$$\wedge^r V(\Gamma_1) \cong V(\Gamma_r) \oplus V(\Gamma_{r-2}) \oplus \dots \oplus V(\Gamma_1) \quad \text{when } r \text{ is odd}$$

and

$$\wedge^r V(\Gamma_1) \cong V(\Gamma_r) \oplus V(\Gamma_{r-2}) \oplus \dots \oplus V(\Gamma_0) \quad \text{when } r \text{ is even}$$

with

$$\wedge^r V(\Gamma_1) \cong \wedge^{2n-4-r} V(\Gamma_1).$$

- (2) For $0 \leq r \leq s \leq n - 2$, the $\mathrm{Sp}(n - 2)$ -module $V(\Gamma_r) \otimes V(\Gamma_s)$ can be decomposed into irreducible modules as follows:

$$V(\Gamma_r) \otimes V(\Gamma_s) \cong \sum_{i,j} V(\Gamma_i + \Gamma_j),$$

where the indices of the summation (i, j) are non-negative integers satisfying:

$$\begin{cases} s - r \leq j - i \leq 2n - 4 - s - r \\ i + j \leq r + s \\ i + j \equiv r + s \pmod{2}. \end{cases}$$

Conclusion

- The previous proposition allows us to decompose $\wedge^r V(\Gamma_1) \otimes \wedge^s V(\Gamma_1)$ into irreducible $\mathrm{Sp}(n - 2)$ -modules.
- The decomposition of $\wedge^{a,b,c,d}$ is reduced to that of $V(\Gamma_i + \Gamma_j) \otimes V(\Gamma_k + \Gamma_l)$ into irreducible $\mathrm{Sp}(n - 2)$ -modules which can be done using the Steinberg multiplicity formula.
- The decomposition of $\wedge^{a,b,c,d}$ into $\mathrm{Sp}(2) \times \mathrm{Sp}(n - 2)$ -modules can be done by gathering the irreducible H -modules in irreducible $\mathrm{Sp}(2)$ -modules.

3.2. General case. We consider now the general case $K = \mathrm{Sp}(q) \times \mathrm{Sp}(n - q)$. We consider a torus T of $\mathrm{Sp}(1)$. To decompose the K -module $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ into irreducible K -submodules, we begin by decomposing the restriction of $(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q - 1) \times \mathrm{Sp}(n - q)$, then the restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q - 1) \times \mathrm{Sp}(n - q)$ and finally, we come back to K as the case $q = 2$.

As $(\mathfrak{g}/\mathfrak{k})^* \cong V(\lambda_1 + \lambda_{q+1})$, it suffices to study the restriction of the $\mathrm{Sp}(q)$ -module $V(\lambda_1)$ to $T \times \mathrm{Sp}(q - 1)$. It is easy to show that

$$V(\lambda_1)|_{T \times \mathrm{Sp}(q-1)} \cong V(\lambda_1) \oplus V(-\lambda_1) \oplus V(\lambda_2),$$

where $V(\lambda_1)$ and $V(-\lambda_1)$ are trivial and $V(\lambda_2)$ is the standard representation of $\mathrm{Sp}(q - 1)$. Then:

$$V(\lambda_1 + \lambda_{q+1})|_{T \times \mathrm{Sp}(q-1) \times \mathrm{Sp}(n-q)} \cong U_1 \oplus U_2 \oplus U_3,$$

where U_1, U_2, U_3 are the irreducible $T \times \mathrm{Sp}(q - 1) \times \mathrm{Sp}(n - q)$ -modules of highest weights $\lambda_1 + \lambda_{q+1}$, $-\lambda_1 + \lambda_{q+1}$ and $\lambda_2 + \lambda_{q+1}$ respectively.

The decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ into irreducible K -submodules can be made recursively as follow:

- (i) The first step is given by the previous conclusion.
- (ii) The restriction of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ to $T \times \mathrm{Sp}(q - 1) \times \mathrm{Sp}(n - q)$ can be decomposed as follow:

•

$$\wedge^p(\mathfrak{g}/\mathfrak{k})^* \cong \sum_{i+j+k=p} \wedge^i U_1 \otimes \wedge^j U_2 \otimes \wedge^k U_3.$$

- The decomposition of $\wedge^i U_1 \otimes \wedge^j U_2$ is determined by applying the proposition 4.
 - We decompose $\wedge^k U_3$ recursively.
- (iii) To obtain the decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*$ as $\mathrm{Sp}(q) \times \mathrm{Sp}(n - q)$ -module, we regroup the irreducible $T \times \mathrm{Sp}(q - 1)$ -modules occurring in the decomposition into irreducible $\mathrm{Sp}(q)$ -modules.

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