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# The space of generalized $G_2$ -theta functions

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**Abstract.** Let  $G_2$  be the exceptional Lie group of automorphisms of the complex Cayley algebra and  $C$  be a smooth, connected, projective curve of genus at least 2. Using the map obtained from extension of structure groups, we prove explicit links between the space of generalized  $G_2$ -theta functions over  $C$  and spaces of generalized theta functions associated to the classical Lie groups  $SL_2$  and  $SL_3$ .

**Key words.** Generalized theta functions, moduli stacks, principal bundles.

**Mathematics Subject Classification.** 14H60

## 1. INTRODUCTION

Throughout this paper we fix a smooth, connected, projective curve  $C$  of genus at least 2. For a complex Lie group  $G$  we denote by  $\mathcal{M}_C(G)$  the moduli stack of principal  $G$ -bundles and by  $\mathcal{L}$  the ample line bundle that generates the Picard group  $\text{Pic}(\mathcal{M}_C(G))$ . The spaces  $H^0(\mathcal{M}_C(G), \mathcal{L}^l)$  of generalized  $G$ -theta functions of level  $l$  are well-known for classical Lie groups but less understood for exceptional Lie groups. Let  $G_2$  be the smallest exceptional Lie group: the group of automorphisms of the complex Cayley algebra. Our aim is to relate the space of generalized  $G_2$ -theta functions  $H^0(\mathcal{M}_C(G_2), \mathcal{L})$  of level one to other spaces of generalized theta functions associated to classical Lie groups.

Using the Verlinde formula, which gives the dimension of the space of generalized  $G$ -theta functions for any simple and simply-connected Lie group  $G$ , we observe a numerical coincidence:

$$\dim H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \dim H^0(\mathcal{M}_C(G_2), \mathcal{L}).$$

In addition, we link together  $\dim H^0(\mathcal{M}_C(G_2), \mathcal{L})$  and  $\dim H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ . Our aim is to give a geometric interpretation of these dimension equalities.

According to the Borel-De Siebenthal classification [BDS49], the groups  $SL_3$  and  $SO_4$  appear as the two maximal subgroups of  $G_2$  among the connected subgroups of  $G_2$  of maximal rank. We define two linear maps by pull-back of the corresponding extension maps: on the one hand

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})$$

and on the other hand using the isogeny  $SL_2 \times SL_2 \rightarrow SO_4$ :

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3).$$

These maps take values in the invariant part by the duality involution for the first map and by the action of 2-torsion elements of the Jacobian for the second one. We denote these invariant spaces by  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$  and  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$  respectively.

Using the natural isomorphism proved in [BNR89]:

$$H^0(\mathcal{M}_C(SL_3), \mathcal{L})^* \simeq H^0(\text{Pic}^{g-1}(C), 3\Theta)$$

where  $\Theta = \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, L) > 0\}$ , we prove the following theorem:

**Theorem A.**

The linear map

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$$

obtained by pull-back of the extension map  $\mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$  is surjective for a general curve and is an isomorphism when the genus of the curve equals 2.

A curve is said satisfying the cubic normality when the multiplication map  $\text{Sym}^3 H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$  is surjective. Using an explicit basis of  $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)$  described in [Bea91], we prove the theorem:

**Theorem B.**

The linear map

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

obtained by pull-back of the extension map  $\mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$  is an isomorphism for a general curve satisfying the cubic normality.

**Notation.** We use the following notations:

- $C$  a smooth, connected, projective curve of genus  $g$  at least 2,
- $G$  a connected and simply-connected simple complex Lie group,
- $\mathcal{M}_C(G)$  the moduli stack of principal  $G$ -bundles on  $C$ ,
- $K_C$  the canonical bundle on  $C$ ,
- $\text{Pic}^{g-1}(C)$  the Picard group parametrizing line bundles on  $C$  of degree  $g-1$ ,
- $h^0(X, \mathcal{L}) = \dim H^0(X, \mathcal{L})$ .

## 2. PRINCIPAL $G_2$ -BUNDLES ARISING FROM VECTOR BUNDLES OF RANK TWO AND THREE

**2.1. The octonions algebra.** Let  $\mathbb{O}$  be the complex algebra of the octonions,  $\text{Im}(\mathbb{O})$  the 7-dimensional subalgebra of the imaginary part of  $\mathbb{O}$  and  $\mathcal{B}_0 = (e_1, \dots, e_7)$  the canonical basis of  $\text{Im}(\mathbb{O})$  (see [Bae02]). The exceptional Lie group  $G_2$  is the group of automorphisms of the octonions.

In Appendix A we give the multiplication table in  $\text{Im}(\mathbb{O})$  by the Fano diagram and we introduce another basis  $\mathcal{B}_1 = (y_1, \dots, y_7)$  of  $\text{Im}(\mathbb{O})$  so

that  $\langle y_1, y_2, y_3 \rangle$  and  $\langle y_4, y_5, y_6 \rangle$  are isotropic and orthogonal and  $y_7$  is orthogonal to both of these subspaces. This basis is defined in Appendix A, as well as two other basis obtained by permutation of elements of  $\mathcal{B}_1$ :  $\mathcal{B}_2 = (y_2, y_3, y_4, y_5, y_6, y_1, y_7)$  and  $\mathcal{B}_3 = (y_1, y_2, y_4, y_5, y_3, y_6, y_7)$ .

In the following paragraphs, we use local sections of rank-7 vector bundles satisfying multiplication rules of  $\mathcal{B}_2$  or  $\mathcal{B}_3$ .

**2.2. Principal  $G_2$ -bundles admitting a reduction.** We introduce the notion of non-degenerated trilinear form on  $\text{Im}(\mathbb{O})$  as Engel did it in [Enga] and [Engb]. A trilinear form  $\omega$  on  $\text{Im}(\mathbb{O})$  is said non-degenerated if the associated bilinear symmetric form  $B_\omega$  is non-degenerated where  $B_\omega(x, y) = \omega(x, \cdot, \cdot) \wedge \omega(y, \cdot, \cdot) \wedge \omega(\cdot, \cdot, \cdot) \forall x, y \in \text{Im}(\mathbb{O})$ .

**Lemma 2.1.**

Giving a principal  $G_2$ -bundle is equivalent to giving a rank-7 vector bundle with a non-degenerated trilinear form.

*Proof.* Let  $P$  be a principal  $G_2$ -bundle on  $C$  and  $V$  the associated rank-7 vector bundle and let  $\omega$  be any non-degenerated trilinear form on  $\text{Im}(\mathbb{O})$ . By construction,  $V$  has a reduction to  $G_2$ , *i.e.* it exists a section  $\sigma : C \rightarrow GL_7/G_2$ . The Lie group  $G_2$  is the stabilizer  $\text{Stab}_{SL_7}(\omega)$  under the action of  $SL_7$ . Besides, under the action of  $GL_7$ ,  $\text{Stab}_{GL_7}(\omega_0) \simeq G_2 \times \mathbb{Z}/3\mathbb{Z}$ . Then

$$\sigma : C \xrightarrow{\sigma} GL_7/G_2 \twoheadrightarrow GL_7/(G_2 \times \mathbb{Z}/3\mathbb{Z}) \simeq GL_7/\text{Stab}_{GL_7}(\omega) \simeq \text{Orb}_{GL_7}(\omega).$$

In addition, the orbit  $\text{Orb}_{GL_7}(\omega)$  is the set of all the non-degenerated trilinear form on  $\text{Im}(\mathbb{O})$ . So,  $V$  is fitted with a non-degenerated trilinear form. Reciprocally any rank-7 vector bundle fitted with a non-degenerated trilinear form defines a  $G_2$ -vector bundle.  $\square$

For a principal  $G_2$ -bundle, we use the non-degenerate trilinear form  $\omega$  on  $\mathcal{V}$ , locally defined by  $\omega(x, y, z) = -\text{Re}[(xy)z]$ .

According to the Borel-De Siebenthal classification (see [BDS49]),  $SL_3$  and  $SO_4$  are, up to conjugation, the two maximal subgroups of  $G_2$  among the connected subgroups of  $G_2$  of maximal rank. Using the inclusion  $G_2 \subset SO_7 = SO(\text{Im}(\mathbb{O}))$  both of the following lemma describe the rank-7 vector bundle (and the non-degenerate trilinear form) associated to a principal  $G_2$ -bundle which admits either a  $SL_3$ -reduction or a  $SO_4$ -reduction.

Note that  $M_C(SO_4)$  has two connected components distinguished by the second Stiefel Whitney class. We only make here explicit computations with regards to the connected component  $M_C^+(SO_4)$  of  $M_C(SO_4)$  containing the trivial bundle.

**2.2.1. Principal  $G_2$ -bundles arising from rank-3 vector bundles.**

**Lemma 2.2.**

Let  $E$  be a rank-3 vector bundle with trivial determinant and let  $E(G_2)$  be his associated principal  $G_2$ -bundle and  $\mathcal{V}$  be his associated rank-7 vector bundle.

Then,  $\mathcal{V}$  has the following decomposition and the local sections basis  $\mathcal{B}_2$  is adapted to this decomposition:

$$\mathcal{V} = E \oplus E^* \oplus \mathcal{O}_C.$$

The non-degenerate trilinear form  $\omega$  is defined by the following local conditions:

- (1)  $\Lambda^3 E \simeq \Lambda^3 E^* \simeq \mathbb{C}$  and  $\omega(y_2, y_3, y_4) = \omega(y_5, y_6, y_1) = -\sqrt{2}$ .
- (2) On  $E \times E^* \times \mathcal{O}_C$ :

$$\omega(y_2, y_5, y_7) = \omega(y_3, y_6, y_7) = \omega(y_4, y_1, y_7) = i,$$

- (3) All other computation, not obtainable by permutation of the previous triplets, equals zero.

*Proof.* Under the action of  $SL_3$ ,  $\text{Im}(\mathbb{O})$  decomposes into  $SL_3$ -modules:

$$\begin{aligned} \text{Im}(\mathbb{O}) &= \langle y_2, y_3, y_4 \rangle \oplus \langle y_5, y_6, y_1 \rangle \oplus \langle y_7 \rangle \\ &= \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C} \end{aligned}$$

where  $\{y_2, y_3, y_4, y_5, y_6, y_1, y_7\}$  is the basis  $\mathcal{B}_2$  defined in Appendix A;  $\langle y_2, y_3, y_4 \rangle$  is an isotropic subspace of dimension 3, dual of  $\langle y_5, y_6, y_1 \rangle$ .

So, the 7-rank vector bundle  $\mathcal{V}$  associated to a rank-3 vector bundle  $E$  is:

$$\begin{aligned} \mathcal{V} &= E \times^{SL_3} (\mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C}), \\ \mathcal{V} &= E \oplus E^* \oplus \mathcal{O}_C. \end{aligned}$$

Evaluations given for  $\omega$  are deduced from Table 5 of Appendix A.  $\square$

**2.2.2. Principal  $G_2$ -bundles arising from two rank-2 vector bundles.** The following lemma makes explicit the vector bundle associated to a principal  $G_2$ -bundle extension of an element of  $M_C^+(SO_4)$ , using the surjective map  $M_C(SL_2) \times M_C(SL_2) \rightarrow M_C^+(SO_4)$ .

**Lemma 2.3.**

Let  $E, F$  be two rank-2 vector bundles of trivial determinant. Denote by  $(E, F)$  the associated principal  $SO_4$ -bundle,  $P$  the associated principal  $G_2$ -bundle and  $\mathcal{V}$  the associated rank-7 vector bundle. Then,  $\mathcal{V}$  has the following decomposition and the local sections basis  $\mathcal{B}_3$  is adapted to this decomposition:

$$\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).$$

The non-degenerate trilinear form  $\omega$  on  $\mathcal{V}$  is defined by the following local conditions:

- (1) On  $(E^* \otimes F)^3$  and on  $(E^* \otimes F) \times (\text{End}_0(F))^2$ ,  $\omega$  is identically zero,
- (2) On  $(E^* \otimes F)^2 \times \text{End}_0(F)$ :

$$\begin{aligned} \omega(y_2, y_4, y_3) &= \omega(y_5, y_1, y_6) = \sqrt{2}, \\ \omega(y_4, y_1, y_7) &= \omega(y_2, y_5, y_7) = i. \end{aligned}$$

- (3)  $\Lambda^3 \text{End}_0(F) \simeq \mathbb{C}$  and  $\omega(y_3, y_6, y_7) = i$ ,
- (4) All other computation, not obtainable by permutation of the previous triplets, equals zero.

*Proof.* The groups  $SO_4$  and  $SL_2 \times SL_2$  are isogenous. Under the action of  $SO_4$ ,  $\text{Im}(\mathbb{O})$  has the following decomposition:

$$\begin{aligned} \text{Im}(\mathbb{O}) &= \langle y_1, y_2, y_4, y_5 \rangle \oplus \langle y_3, y_6, y_7 \rangle, \\ &\simeq M^* \otimes N \oplus^\perp \text{End}_0(N) \end{aligned}$$

where  $M, N$  are 2-dimensional; an element  $\overline{(A, B)}$  of  $SO_4$  ( $A, B \in SL_2$ ) acts on  $M^* \otimes N$  by  $A \otimes B$  and by conjugation  $\text{End}_0(N)$ .

So, the rank-7 vector bundle  $\mathcal{V}$  associated to  $(E, F)(G_2)$ , when  $E, F$  are two rank-7 vector bundle of trivial determinant, is:

$$\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).$$

Evaluations given for  $\omega$  are deduced from Table 6 of Appendix A. □

### 3. EQUALITIES BETWEEN DIMENSIONS OF SPACES OF GENERALIZED THETA FUNCTIONS

Here are some dimension counts, using the Verlinde Formula, to calculate  $h^0(\mathcal{M}_C(G_2), \mathcal{L})$  and  $h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$ .

**Proposition 3.1.**

Dimensional equalities between the following spaces of generalized theta functions occur:

$$h^0(\mathcal{M}_C(G_2), \mathcal{L}) = \left( \frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{2} \right)^{g-1}, \quad (1)$$

$$h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \left[ \left( \frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{2} \right)^{g-1} \right], \quad (2)$$

$$\text{so, } h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}). \quad (3)$$

*Proof.* See Appendix B and C □

### 4. SURJECTIVITIES AND ISOMORPHISMS BETWEEN $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ AND $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$

To avoid confusion, we sometimes specify the group  $G$  in the notation of the generator of the Picard group  $\mathcal{L}$  writing  $\mathcal{L}_G$ .

Let  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$  be the invariant part of  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$  by the duality involution: the eigenspace of  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$  associated to the eigenvalue 1 under the natural involution  $E \mapsto \sigma(E) = E^*$ .

We consider the extension map  $i : \mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$  which associates to a rank-3 vector bundle of trivial determinant the associated principal  $G_2$ -bundle. The pull-back  $i^*(\mathcal{L}_{G_2})$  equals  $\mathcal{L}_{SL_3}$ .

**Theorem 4.1.**

The extension map  $i : \mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$  induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), i^*(\mathcal{L}_{G_2})).$$

This map takes values in  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ .

We denote by  $\Phi$  this map:

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+.$$

*Proof.* (1) The pull-back  $i^*(\mathcal{L}_{G_2})$  equals  $\mathcal{L}_{SL_3}$ . Indeed, let  $E$  be a rank-3 vector bundle. By Lemma 2.2, the rank-7 vector bundle associated to  $E$  is  $E \oplus E^* \oplus \mathcal{O}_C$ . We study the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow i & & \uparrow \rho_2 \\ \mathcal{M}_C(SL_3) & \xrightarrow{\rho_3} & \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3) \end{array}$$

where  $i$  and  $\rho_1$  are maps of extension of group of structure and  $\forall E \in \mathcal{M}_C(SL_3)$ ,  $\rho_3(E) = (E, E^*)$  and  $\forall (E, F) \in \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3)$ ,  $\rho_2(E, F) = E \oplus F \oplus \mathcal{O}_C$ . By Proposition 2.6 of [LS97], applied with  $SL_7$ ,  $G_2$  and the irreducible representation  $\rho_1$  of highest weight  $\varpi_1$ , the Dynkin index of  $d(\rho_1)$  equals 2. Therefore,  $\rho_1^*(\mathcal{D}_{SL_7}) = \mathcal{L}_{G_2}^2$  where  $\mathcal{D}_{SL_7}$  the determinant bundle generator of  $\text{Pic}(\mathcal{M}_C(SL_7))$  (see [KNR94] and [LS97]). In addition, by the same proposition,  $\rho_2^*(\mathcal{D}_{SL_7}) = \mathcal{L}_{SL_3} \boxtimes \mathcal{L}_{SL_3}$  and  $\rho_3^*(\rho_2^*(\mathcal{D}_{SL_7})) = \mathcal{L}_{SL_3}^2$ . So,  $i^*(\mathcal{L}_{G_2})^2 = \mathcal{L}_{SL_3}^2$  so that  $i^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_3}$  since the Picard group  $\text{Pic}(\mathcal{M}_C(SL_3))$  is isomorphic to  $\mathbb{Z}$ .

(2) We show that the image of the linear map  $\Phi$  is contained in  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ .

The morphism  $i$  is  $\sigma$ -invariant: for all  $E \in \mathcal{M}_C(SL_3)$ , the  $G_2$ -principal bundles  $E(G_2)$  and  $E^*(G_2)$  are isomorphic. Indeed, the Weyl group  $W(SL_3)$  is contained in  $W(G_2)$  since so are there normalizer; and  $W(SL_3)$  is a subgroup of  $W(G_2)$  of index 2. We consider  $\bar{g}$  in the Weyl group  $W(G_2) \setminus W(SL_3)$  and  $g \in G_2$  a representative of the equivalence class of  $\bar{g}$ . Then,  $g \notin SL_3$ . Let  $C_g$  be the inner automorphism of  $G_2$  induced by  $g$ . As the subalgebra  $\mathfrak{sl}_3$  of  $\mathfrak{g}_2$  corresponds to the long roots and as each element of the Weyl group  $W(G_2)$  respects the Killing form on  $\mathfrak{g}_2$ ,  $C_g(SL_3)$  is contained in  $SL_3$ . The restriction of  $C_g$  to  $SL_3$  is then an exterior automorphism of  $SL_3$  which we call  $\alpha : SL_3 \rightarrow SL_3$ . This automorphism exchanges the two fundamental representations of  $SL_3$ . So,  $\alpha$  induces an automorphism  $\tilde{\alpha}$  on  $\mathcal{M}_C(SL_3)$  such that,  $\forall E \in \mathcal{M}_C(SL_3)$ ,  $\tilde{\alpha}(E) = E^*$ . Consider the following commutative diagram, where  $\tilde{C}_g$  is the inner

automorphism given by  $g$ :

$$\begin{array}{ccc} \mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2) \\ \downarrow \tilde{\alpha} & & \downarrow \tilde{C}_g \\ \mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2). \end{array}$$

Then,  $\forall E \in \mathcal{M}_C(SL_3)$ ,

$$E^*(G_2) = \tilde{\alpha}(E)(G_2) = \tilde{C}_g(E(G_2)) \simeq E(G_2)$$

since  $\tilde{C}_g$  is an inner automorphism. Thus,  $i(E)$  and  $i(\sigma(E))$  are isomorphic.

The  $\sigma$ -invariance of  $i$  implies that the image of  $\Phi$  is contained in one of the two eigenspaces of  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ . As  $\sigma^*(\mathcal{L}_{SL_3}) \simeq \mathcal{L}_{SL_3}$ , which is the isomorphism which implies identity over the trivial bundle, we get  $\sigma(i^*(\mathcal{L}_{G_2})) = \sigma^*(\mathcal{L}_{SL_3}) = \mathcal{L}_{SL_3} = i^*(\mathcal{L}_{G_2})$ . Thus, the image of  $\Phi$  is contained in  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ , the eigenspace relative to the eigenvalue 1.  $\square$

We remind two points of vocabulary: an *even theta-characteristic*  $\kappa$  on a curve  $C$  is an element  $\kappa$  of  $\text{Pic}^{g-1}(C)$  such that  $\kappa \otimes \kappa = K_C$  and  $h^0(C, \kappa)$  is even ; a curve  $C$  is said *without effective theta-constant* if  $h^0(C, \kappa) = 0$  for all even theta-characteristic  $\kappa$ . The set of all even theta-characteristics is named  $\Theta^{\text{even}}(C)$ .

**Theorem 4.2.**

The linear map  $\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$

- (1) is surjective when the curve  $C$  is without effective theta-constant.
- (2) is an isomorphism if the genus of  $C$  equals 2.

*Proof.* (1) Let  $C$  be a curve without effective theta-constant and consider the following diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow i & \nearrow \varphi & \\ \mathcal{M}_C(SL_3) & & \end{array}$$

We introduce the element  $\Delta_\kappa$  defined for each even theta-characteristic  $\kappa$ :

$$\Delta_\kappa = \{P \in \mathcal{M}_C(SL_7) \mid h^0(C, P(\mathbb{C}^7) \otimes \kappa) > 0\}.$$



These  $\Delta_\kappa$  are Cartier divisors, so they define, up to a scalar, an element of  $H^0(\mathcal{M}_C(SL_7), \mathcal{L})$ . The image  $\Phi(\rho_1^*(\Delta_\kappa)) = \rho^*(\Delta_\kappa)$  is

$$\begin{aligned} \rho^*(\Delta_\kappa) &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \oplus E^* \oplus \mathcal{O}_C) \otimes \kappa > 0\}, \\ &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) + h^0(C, \kappa) > 0\}, \\ &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) > 0\}, \\ &\quad \text{because } C \text{ is without effective theta-constant,} \\ &= \{E \in \mathcal{M}_C(SL_3) \mid 2h^0(C, E \otimes \kappa) > 0\}, \text{ by Serre duality.} \end{aligned}$$

Thus,  $\rho^*(\Delta_\kappa) = 2H_\kappa$  where  $H_\kappa := \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) > 0\}$ . Therefore, to show the surjectivity of  $\Phi$ , it suffices to show that  $\{H_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ . We consider

$$\Theta = \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, L) > 0\}$$

and the natural map between the spaces  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})^*$  and  $H^0(\text{Pic}^{g-1}(C), 3\Theta)$ . By Theorem 3 of [BNR89], this map is an isomorphism and, besides, it is equivariant for the two involutions on  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})^*$  and  $H^0(\text{Pic}^{g-1}(C), 3\Theta)$  (respectively  $E \mapsto E^*$  and  $L \mapsto K_C \otimes L^{-1}$ ). So, the components (+) and (-) of each part are in correspondence:  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^*$  is isomorphic to  $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$ . Denote by  $\varphi$  this isomorphism  $\varphi : \mathbb{P}H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^* \xrightarrow{\sim} \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)_+^*$ . For all even theta-characteristic  $\kappa$ , the image  $\varphi(H_\kappa)$  is  $\varphi_{3\Theta}(\kappa)$  where  $\varphi_{3\Theta}$  is the following map:

$$\varphi_{3\Theta} : \text{Pic}^{g-1}(C) \rightarrow \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)_+^* = |3\Theta|_+^*.$$

The set  $\{\varphi_{3\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates  $|3\Theta|_+^*$ . Indeed, in the following commutative diagram

$$\begin{array}{ccc} & & |4\Theta|_+^* \\ & \nearrow \varphi_{4\Theta} & \vdots \\ \text{Pic}^{g-1}(C) & \xrightarrow{\varphi_{3\Theta}} & |3\Theta|_+^* \end{array}$$

the map  $|4\Theta|_+^* \dashrightarrow |3\Theta|_+^*$  is surjective because it is induced by the inclusion  $D \in H^0(C, 3\Theta)_+ \mapsto D + \Theta \in H^0(C, 4\Theta)_+$ . In addition, by [KPS09], when  $C$  is without effective theta-constant,  $\{\varphi_{4\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$  is a base of  $|4\Theta|_+^*$  (the number of even theta-characteristics equals  $2^{g-1}(2^g + 1)$  which equals the linear dimension of  $|4\Theta|_+^*$ ). Thus,  $\{\varphi_{3\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates  $|3\Theta|_+^*$  and  $\{H_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates the space  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ . As we have shown that  $H_\kappa$  equals  $\Phi(\rho_1(\Delta_\kappa))$  for all  $\kappa$  even theta-characteristic, the map  $\Phi$  is surjective.

- (2) By [BNR89], the dimension of  $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^*$  equals the dimension of  $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$ , that is  $\frac{3g+1}{2}$ . When the genus of  $C$  is 2,

the dimension of  $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$  equals 5 which is also the dimension of  $H^0(\mathcal{M}_C(G_2), \mathcal{L})$  by Proposition 3.1.(1). A curve of genus 2 is without effective theta-constant. So, by this dimension equality and by the point (1) of the theorem,  $\Phi$  is an isomorphism when the genus of  $C$  equals 2.  $\square$

## 5. ISOMORPHISMS BETWEEN SPACES OF GENERALIZED $G_2$ -THETA FUNCTIONS AND GENERALIZED $SL_2$ -THETA FUNCTIONS

Let  $\text{JC}[2]$  be the group of 2-torsion elements of the Jacobian:  $\text{JC}[2] = \{\alpha \in \text{Pic}^0(C) \mid \alpha \otimes \alpha = \mathcal{O}_C\}$ . This group acts on  $\mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$ : for  $\alpha \in \text{JC}[2]$  and  $(E, F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$ , we associate  $(E \otimes \alpha, F \otimes \alpha)$ . Let  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$  be the invariant part of  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]$  under the action of the element of the Jacobian group  $\text{JC}[2]$ .

We study the subgroup  $SO_4$  of  $G_2$ , which is isogenous to  $SL_2 \times SL_2$ , and the linear map induced by pull-back by the extension map  $j : \mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$  which associates to two rank-2 vector bundle of trivial determinant the associated principal  $G_2$ -bundle. The pull-back  $j^*(\mathcal{L}_{G_2})$  equals  $\mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3$ .

### Theorem 5.1.

The extension map  $j : \mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$  induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SO_4), j^*(\mathcal{L}_{G_2})).$$

This map takes values in  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ .

We denote by  $\Psi$  this map:

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

*Proof.* (1) Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow j & & \uparrow \rho_4 \\ \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) & \xrightarrow{(f_1, f_2)} & \mathcal{M}_C(SL_4) \times \mathcal{M}_C(SL_3) \end{array}$$

where  $j$  and  $\rho_1$  are the extension maps,  $f_1(M, N) = M^* \otimes N$ ,  $f_2(M, N) = \text{End}_0(N)$  and  $\rho_4(A, B) = A \oplus B$ .

As in the previous section, we calculate explicitly  $j^*(\mathcal{L}_{G_2})$ .

Let  $\mathcal{D}_{SL_7}$  be the determinant bundle of  $\mathcal{M}_C(SL_7)$  and  $\text{pr}_1$  and  $\text{pr}_2$  the canonical projections of  $SL_2 \times SL_2$ . We get

$$f_1^*(\mathcal{L}_{SL_4}) = \text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^2 = \mathcal{L}_{SL_2}^2 \boxtimes \mathcal{L}_{SL_2}^2$$

and according to Table  $B$  of [Sor00]:

$$f_2^*(\mathcal{L}_{SL_3}) = \text{pr}_2^*(\mathcal{L}_{SL_2})^4,$$

since  $f_2$  is associated to the adjoint representation of  $SL_2$ , which has Dynkin index 4.

$$\begin{aligned} \rho_4^*(\mathcal{D}) &= \mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3}, \\ \text{so } j^*(\mathcal{L}_{G_2}^2) &= (f_1, f_2)^*(\mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3}), \\ &= f_1^*(\mathcal{L}_{SL_4}) \otimes f_2^*(\mathcal{L}_{SL_3}), \\ &= [\text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^2] \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^4, \\ j^*(\mathcal{L}_{G_2}^2) &= \text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^6, \\ \text{so } j^*(\mathcal{L}_{G_2}) &= \text{pr}_1^*(\mathcal{L}_{SL_2}) \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^3. \end{aligned}$$

We get

$$j^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3.$$

- (2) The morphism  $j : \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2} \rightarrow \mathcal{M}_{G_2}$  is invariant under the action of  $JC[2]$ :  $(E, F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$  and  $\alpha \in JC[2]$  then the rank-7 vector bundle associated by  $j$  to  $(E \otimes \alpha, F \otimes \alpha)$  is

$$\begin{aligned} &(E \otimes \alpha)^* \otimes (F \otimes \alpha) \oplus \text{End}_0(F \otimes \alpha) \\ &= E^* \otimes F \otimes \alpha^* \otimes \alpha \oplus \text{End}_0(F \otimes \alpha), \\ &= E \otimes F \oplus \text{End}_0(F) = j(E, F). \end{aligned}$$

Therefore, the image of  $\Psi$  is contained in the expected vector space  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ .  $\square$

Before going further in the study of the morphism  $j$ , we compare the dimensions of the involved sets.

**Lemma 5.2.**

The dimension of the space  $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$  equals the dimension of  $H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2})$ .

*Proof.* By Proposition 3.1,(3), we notice the remarkable following relation:

$$2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}) = h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3). \quad (4)$$

So, as  $h^0(\mathcal{M}_C(SL_2), \mathcal{L}) = 2^g$  (see [Bea88]), we get

$$\begin{aligned} &\dim \left( [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0 \right) \\ &= \frac{1}{2^{2g}} \times h^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \times h^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3) \quad \text{because } |JC[2]| = 2^{2g}, \\ &= \frac{1}{2^{2g}} \times 2^g \times 2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \quad \text{by (4),} \\ &= h^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}). \end{aligned}$$

$\square$

All the following results are based on the *cubic normality conjecture*. Its statement is:

**Conjecture 5.3.**

For a general curve  $C$ , the multiplication map

$$\eta : \text{Sym}^3 H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)$$

is surjective.

When the previous map  $\eta$  is surjective, we say that the curve  $C$  satisfies *cubic normality*.

**Proposition 5.4.**

cubic normality holds for all curve of genus 2, all non hyper-elliptic curve of genus 3 and all curve of genus 4 without effective theta-constant.

*Proof.* For a curve of genus 2,  $\mathcal{M}_C(SL_2)$  is isomorphic to  $\mathbb{P}^3$  and  $\mathcal{L}_{SL_2}$  to  $\mathcal{O}(1)$  (see [NR75]). A non-hyper-elliptic curve of genus 3 is a Coble quartic (see [NR87]). The cubic normality is true in both of these cases. For a general curve of genus 4 without effective theta-constant, cubic normality is proved in Theorem 4.1 of [OP99].  $\square$

When this conjecture is true, we get this theorem:

**Theorem 5.5.**

Let  $C$  be a curve of genus at least 2 without effective theta-constant and satisfying the cubic normality and let  $\Psi$  be the map defined in Theorem 5.1:

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0.$$

- (1) The map  $\Psi$  is an isomorphism,
- (2) The space of generalized  $G_2$ -theta functions  $H^0(\mathcal{M}_C(G_2), \mathcal{L})$  is linearly generated by the divisors  $\rho_0^*(\Delta_\kappa)$  for  $\kappa$  even theta-characteristic, where  $\rho_0$  is the extension morphism

$$\rho_0 : \mathcal{M}_C(G_2) \rightarrow \mathcal{M}_C(SO_7).$$

*Proof.* (1) According to the dimension equality proved in Lemma 5.2, it suffices to prove the surjectivity of  $\Psi$ .

Denote by  $V$  the vector space  $H^0(\mathcal{M}_C(SL_2), \mathcal{L})$ .

Using the notations of [Bea91], we associate to each even theta-characteristic  $\kappa$  an element  $d_\kappa$  of  $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 2})$  and an element  $\xi_\kappa$  of  $V \otimes V$ . For each even theta-characteristic  $\kappa$ ,  $d_\kappa$  is the section of  $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 2})$  such that  $D_\kappa$  is the divisor of the zeros of  $d_\kappa$ , where  $D_\kappa = \{S \in \mathcal{M}_C(SL_2) \mid h^0(C, \text{End}_0(S) \otimes \kappa) > 0\}$ . Consider the following maps:

$$\begin{aligned} \rho_0^* : & H^0(\mathcal{M}_C^+(SO_7), \mathcal{L}) & \longrightarrow & H^0(\mathcal{M}_C(G_2), \mathcal{L}) \\ \text{and } \beta : & [V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]_0 & \longrightarrow & [V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)]_0 \end{aligned}$$

where  $\beta(A, B, D) = (A, BD)$ . For any even theta-characteristic  $\kappa$ , the image  $\Psi(\rho_0^*(\Delta_\kappa))$  equals  $\beta(\xi_\kappa \otimes d_\kappa)$ . Indeed,  $\Psi$  is induced by:

$$\begin{aligned} j: \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) &\rightarrow \mathcal{M}_C(G_2) \\ (E, F) &\mapsto \text{Hom}(E, F) \oplus \text{End}_0(F); \end{aligned}$$

the pull-back  $\Psi(\rho_0^*(\Delta_\kappa))$  is the sum of two divisors:

$$\begin{aligned} \Delta_1 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{End}_0(F) \otimes \kappa) > 0\}, \\ \Delta_2 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{Hom}(E, F) \otimes \kappa) > 0\}. \end{aligned}$$

In addition,  $\mathcal{O}(\Delta_1) = \mathcal{O}_C \boxtimes \mathcal{L}^2$  and  $\mathcal{O}(\Delta_2) = \mathcal{L} \boxtimes \mathcal{L}$  (see [Bea91]) and more precisely:

$$\Delta_1 = \text{Zeros}(d_\kappa) \text{ et } \Delta_2 = \text{Zeros}(\xi_\kappa).$$

When the curve  $C$  is of genus at least 2 without effective theta-constant, it is proved in [BNR89] that the map

$$\begin{aligned} \varphi_0^*: \text{Sym}^2 V &\longrightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) \\ \xi_\kappa &\mapsto d_\kappa \end{aligned}$$

is an isomorphism. We identify  $\text{Sym}^2 V$  with the invariant space of  $V \otimes V$  under the involution  $a \otimes b \mapsto b \otimes a$ . By Theorem 1.2 and Proposition A.5 of [Bea91], the set  $\{d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$  is a basis of  $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)$  and  $\{\xi_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$  is a basis of  $\text{Sym}^2 V$ . Then, the vector space  $[V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]_0$  is generated by  $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ . Thus, to prove the surjectivity of the map  $\Psi$ , it is sufficient to show the surjectivity of the map  $\beta$ . Consider the following diagram:

$$\begin{array}{ccc} V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) & \longrightarrow & H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) \\ \uparrow & & \uparrow \eta \\ V \otimes V \otimes V & \longrightarrow & \text{Sym}^3 V. \end{array}$$

With the hypothesis of cubic normality, the map  $\eta$  is surjective. Therefore the map  $V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$  is also surjective. By restriction to invariant sections under the action of  $J\mathcal{C}[2]$ ,  $\beta$  is surjective.

The map  $\Psi$  is thus an isomorphism.

- (2) The point (2) is a consequence of the previous facts: for each element  $\kappa \in \Theta^{\text{even}}(C)$ , the image of  $\rho_0^*(\Delta_\kappa)$  by  $\Psi$  is  $\xi_\kappa \otimes d_\kappa$ . As  $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates  $[V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]$ , the set  $\{\rho_0^*(\Delta_\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$  generates  $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ . □

**Remark 5.6.** By Proposition 5.4 the linear map  $\Psi$  is an isomorphism for each curve of genus 2, each non hyperelliptic curve of genus 3 and each curve of genus 4 without effective theta-constant.

#### APPENDIX A. THE OCTONION ALGEBRA

Let  $\mathcal{B}_0 = \{e_1, \dots, e_7\}$  be the canonical basis of the subalgebra of the imaginary part of octonions. The multiplication rules are:

- $\forall i \in \{1, \dots, 7\}, \quad e_i^2 = -1,$
- $e_i e_j = -e_j e_i = e_k$  when  $(e_i, e_j, e_k)$  are three points on the same edge of on the oriented Fano diagram.

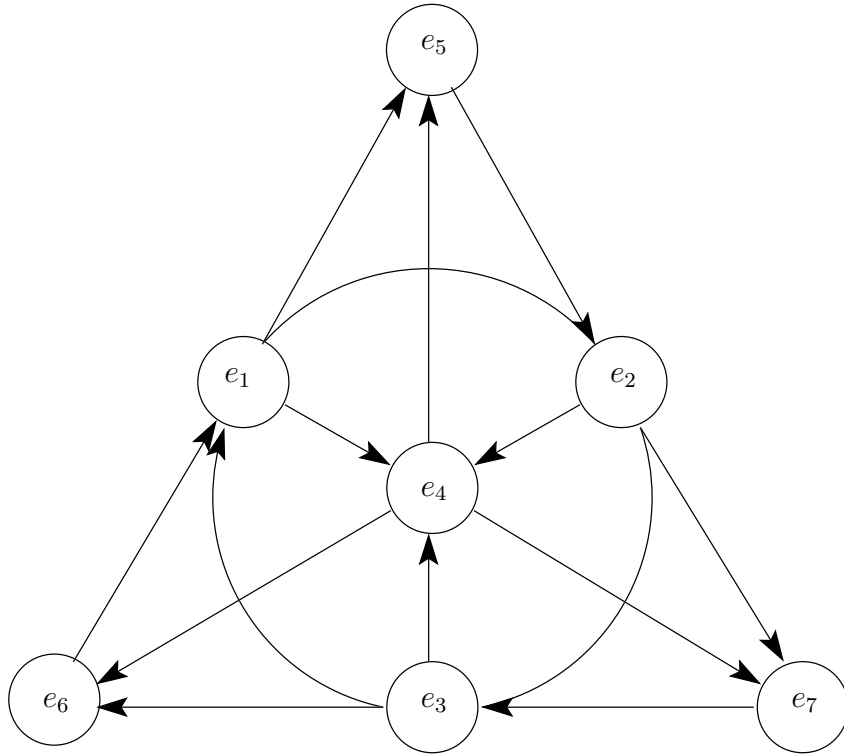


FIGURE 1. Fano Diagram

We introduced the basis  $\mathcal{B}_1 = \{y_1, \dots, y_7\}$  obtained by basis change by

the change of basis matrix  $P = \frac{\sqrt{2}}{2} \left( \begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & -i & 0 & 0 & i & 0 & 0 \\ -i & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & i & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{array} \right)$ . The

canonical quadratic form on  $\text{Im}\mathbb{O}$  expressed in the basis  $\mathcal{B}_1$  is  $Q = \left( \begin{array}{c|c|c} 0 & I_3 & 0 \\ \hline I_3 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$ .

In the basis  $\mathcal{B}_2 = \{y_2, y_3, y_4, y_5, y_6, y_1, y_7\}$ , the multiplication table is

$\nearrow$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_1$	$y_7$
$y_2$	0	$-\sqrt{2}y_1$	$\sqrt{2}y_6$	$-1 + iy_7$	0	0	$-iy_2$
$y_3$	$\sqrt{2}y_1$	0	$-\sqrt{2}y_5$	0	$-1 + iy_7$	0	$-iy_3$
$y_4$	$-\sqrt{2}y_6$	$\sqrt{2}y_5$	0	0	0	$-1 + iy_7$	$-iy_4$
$y_5$	$-1 - iy_7$	0	0	0	$-\sqrt{2}y_4$	$\sqrt{2}y_3$	$iy_5$
$y_6$	0	$-1 - iy_7$	0	$\sqrt{2}y_4$	0	$-\sqrt{2}y_2$	$iy_6$
$y_1$	0	0	$-1 - iy_7$	$-\sqrt{2}y_3$	$\sqrt{2}y_2$	0	$iy_1$
$y_7$	$iy_2$	$iy_3$	$iy_4$	$-iy_5$	$-iy_6$	$-iy_1$	-1

(5)

In the basis  $\mathcal{B}_3 = \{y_1, y_2, y_4, y_5, y_3, y_6, y_7\}$ , the multiplication table is

$\nearrow$	$y_1$	$y_2$	$y_4$	$y_5$	$y_3$	$y_6$	$y_7$
$y_1$	0	0	$-1 - iy_7$	$-\sqrt{2}y_3$	0	$\sqrt{2}y_2$	$iy_1$
$y_2$	0	0	$\sqrt{2}y_6$	$-1 + iy_7$	$-\sqrt{2}y_1$	0	$-iy_2$
$y_4$	$-1 + iy_7$	$-\sqrt{2}y_6$	0	0	$\sqrt{2}y_5$	0	$-iy_4$
$y_5$	$\sqrt{2}y_3$	$-1 - iy_7$	0	0	0	$-\sqrt{2}y_4$	$iy_5$
$y_3$	0	$\sqrt{2}y_1$	$-\sqrt{2}y_5$	0	0	$-1 + iy_7$	$-iy_3$
$y_6$	$-\sqrt{2}y_2$	0	0	$\sqrt{2}y_4$	$-1 - iy_7$	0	$iy_6$
$y_7$	$-iy_1$	$iy_2$	$iy_4$	$-iy_5$	$iy_3$	$-iy_6$	-1

(6)

## APPENDIX B. COMPUTATION OF $h^0(\mathcal{M}_C(G_2), \mathcal{L})$

First, we recall the Verlinde Formula.

**Proposition B.1** (Verlinde Formula).

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  of classical type or type  $\mathfrak{g}_2$ ,  $\mathcal{L}$  be the ample canonical line bundle on  $\mathcal{M}_C(G)$  and  $i$  be a positive integer. The integer  $h^0(\mathcal{M}_C(G), \mathcal{L}^i)$  is given by the following relation:

$$h^0(\mathcal{M}_C(G), \mathcal{L}^i) = (\#T_i)^{g-1} \sum_{\mu \in \mathcal{P}_i} \prod_{\alpha \in \Delta_+} \left[ 2 \sin \left( \frac{\pi \langle \alpha, \mu + \rho \rangle}{i + g^*} \right) \right]^{2-2g}.$$

where

$$\begin{aligned}
\#T_i &= (i + g^*)^{\text{rk}(\mathfrak{g})} \#(\mathcal{P}/Q) \#(Q/Q_{\text{lg}}), & \langle \cdot, \cdot \rangle & \text{ is the Killing form,} \\
& \text{where } \text{rk}(\mathfrak{g}) \text{ is the rank of } \mathfrak{g}, & \mathcal{P}_i &= \{\text{dominant weights } \mu \mid \langle \mu, \theta \rangle \leq i\}, \\
\mathcal{P} & \text{ is the weight lattice,} & \theta & \text{ is the maximal positive root,} \\
Q & \text{ is the root lattice,} & \Delta_+ &= \{\text{positive roots of } \mathfrak{g}\} \\
Q_{\text{lg}} & \text{ is the long root lattice,} & g^* & \text{ is the dual Coxeter number of the group } G. \\
\rho &= \frac{1}{2} \sum_{\alpha_j \in \Delta_+} \alpha_j,
\end{aligned}$$

*Proof.* See the survey [Sor96].  $\square$

The root system of  $G_2$  has six positive roots  $\alpha_1, \dots, \alpha_6$ , with two simple roots, called  $\alpha_1$  and  $\alpha_2$ .

For  $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ , the data used in the Verlinde Formula are  $g^* = 4$ ,  $\text{rk}(\mathfrak{g}_2) = 2$ ,  $\#T_1 = (1 + 4)^2 \times 1 \times 3 = 75$ ,  $\rho = 5\alpha_1 + 3\alpha_2$ ,  $\theta = \alpha_6 = 3\alpha_1 + 2\alpha_2$  and the fundamental weights are  $\varpi_1 = \alpha_5$  and  $\varpi_2 = \alpha_6$ .

To describe  $\mathcal{P}_1$ , we describe the Killing form. The angle formed between  $\alpha_1$  and  $\alpha_2$  is  $5\pi/6$  and the ratio  $\|\alpha_2\|/\|\alpha_1\| = \sqrt{3}$ . In order to normalize the Killing form, we impose the norm of maximal positive root  $\|\theta\|$  equals 2. As  $\theta = \alpha_6$ ,  $\|\alpha_2\| = \|\theta\| = 2$  and then  $\|\alpha_1\| = 2/3$  and  $\langle \alpha_1, \alpha_2 \rangle = -1$ . The evaluations of the Killing form on  $\varpi_i$  and  $\theta$  are  $\langle \varpi_1, \theta \rangle = 1$  and  $\langle \varpi_2, \theta \rangle = 2$ ; so  $\mathcal{P}_1 = \{0, \varpi_1\}$ .

The evaluation of the Killing form on each positive root and  $\rho$  added to each value of  $\mathcal{P}_1$  are the following:

$$\begin{aligned}
\langle \alpha_1, \rho \rangle &= 1/3 & \text{and} & & \langle \alpha_1, \varpi_1 + \rho \rangle &= 2/3, \\
\langle \alpha_2, \rho \rangle &= 1, & & & \langle \alpha_2, \varpi_1 + \rho \rangle &= 1, \\
\langle \alpha_3, \rho \rangle &= 4/3, & & & \langle \alpha_3, \varpi_1 + \rho \rangle &= 5/3, \\
\langle \alpha_4, \rho \rangle &= 5/3, & & & \langle \alpha_4, \varpi_1 + \rho \rangle &= 7/3, \\
\langle \alpha_5, \rho \rangle &= 2, & & & \langle \alpha_5, \varpi_1 + \rho \rangle &= 3, \\
\langle \alpha_6, \rho \rangle &= 3, & & & \langle \alpha_6, \varpi_1 + \rho \rangle &= 4.
\end{aligned}$$

So, by the Verlinde formula, the dimension  $h^0(\mathcal{M}_C(G_2), \mathcal{L})$  is

$$\begin{aligned}
h^0(\mathcal{M}_C(G_2), \mathcal{L}) &= \left(\frac{2^{10}}{25}\right)^{1-g} \left[ \left[ \sin^2\left(\frac{\pi}{15}\right) \sin^2\left(\frac{4\pi}{15}\right) \sin^2\left(\frac{\pi}{5}\right) \sin^4\left(\frac{2\pi}{5}\right) \right]^{1-g} \right. \\
& \quad \left. + \left[ \sin^2\left(\frac{2\pi}{15}\right) \sin^2\left(\frac{7\pi}{5}\right) \sin^2\left(\frac{2\pi}{5}\right) \sin^4\left(\frac{\pi}{5}\right) \right]^{1-g} \right].
\end{aligned}$$

To obtain a compact formula, we express these trigonometric products in  $\mathbb{Q}(\sqrt{5})$ :

$$\begin{aligned}
\sin^2\left(\frac{\pi}{5}\right) &= \frac{1}{8}(5 - \sqrt{5}), & \sin^2\left(\frac{\pi}{15}\right) \sin^2\left(\frac{4\pi}{15}\right) &= \frac{1}{25}(3 - \sqrt{5}), \\
\sin^2\left(\frac{2\pi}{5}\right) &= \frac{1}{8}(5 + \sqrt{5}), & \sin^2\left(\frac{2\pi}{15}\right) \sin^2\left(\frac{7\pi}{15}\right) &= \frac{1}{25}(3 + \sqrt{5}), \\
\sin^4\left(\frac{\pi}{5}\right) &= \frac{5}{2^5}(3 - \sqrt{5}), & \sin^4\left(\frac{2\pi}{5}\right) &= \frac{5}{2^5}(3 + \sqrt{5}).
\end{aligned}$$



So,

$$\begin{aligned}
& h^0(\mathcal{M}_C(G_2), \mathcal{L}) \\
&= \left(\frac{2^{10}}{25}\right)^{1-g} \left[ \left[\left(\frac{1}{2^3}\right) \left(\frac{1}{2^3}\right) \left(\frac{5}{2^5}\right) (3 - \sqrt{5}) (5 - \sqrt{5}) (3 + \sqrt{5})\right]^{1-g} \right. \\
&\quad \left. + \left[\left(\frac{1}{2^5}\right) \left(\frac{1}{2^3}\right) \left(\frac{5}{2^5}\right) (3 + \sqrt{5}) (5 + \sqrt{5}) (3 - \sqrt{5})\right]^{1-g} \right], \\
&= \left(\frac{5-\sqrt{5}}{10}\right)^{1-g} + \left(\frac{5+\sqrt{5}}{10}\right)^{1-g} = \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}.
\end{aligned}$$

### APPENDIX C. COMPUTATION OF $h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$

Using the previous notations, the evaluation on  $\alpha$  and each element of  $\mu + \rho$ , where  $\mu \in \mathcal{P}_3$ , are the following:

$$\begin{aligned}
\langle \alpha, \rho \rangle &= 1, & \text{and} & \quad \langle \alpha, 2\varpi_1 + \rho \rangle = 3, \\
\langle \alpha, \varpi_1 + \rho \rangle &= 2, & \langle \alpha, 3\varpi_1 + \rho \rangle &= 4.
\end{aligned}$$

By the Verlinde Formula,

$$\begin{aligned}
& h^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 3}) \\
&= (10)^{g-1} (2)^{2-2g} \left[ \left[\sin\left(\frac{\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{2\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{3\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{4\pi}{5}\right)\right]^{2-2g} \right], \\
&= \left(\frac{5}{2}\right)^{g-1} \left[ 2 \left[\sin^2\left(\frac{\pi}{5}\right)\right]^{1-g} + 2 \left[\sin^2\left(\frac{2\pi}{5}\right)\right]^{1-g} \right], \\
&= 2 \left(\frac{5}{2}\right)^{g-1} \left( \left(\frac{8}{5-\sqrt{5}}\right)^{g-1} + \left(\frac{8}{5+\sqrt{5}}\right)^{g-1} \right), \\
&= 2^g \left[ \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1} \right].
\end{aligned}$$

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