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The space of generalized G_2 -theta functions

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Abstract. Let G_2 be the exceptional Lie group of automorphisms of the complex Cayley algebra and C be a smooth, connected, projective curve of genus at least 2. Using the map obtained from extension of structure groups, we prove explicit links between the space of generalized G_2 -theta functions over C and spaces of generalized theta functions associated to the classical Lie groups SL_2 and SL_3 .

Key words. Generalized theta functions, moduli stacks, principal bundles.

Mathematics Subject Classification. 14H60

1. INTRODUCTION

Throughout this paper we fix a smooth, connected, projective curve C of genus at least 2. For a complex Lie group G we denote by $\mathcal{M}_C(G)$ the moduli stack of principal G -bundles and by \mathcal{L} the ample line bundle that generates the Picard group $\text{Pic}(\mathcal{M}_C(G))$. The spaces $H^0(\mathcal{M}_C(G), \mathcal{L}^l)$ of generalized G -theta functions of level l are well-known for classical Lie groups but less understood for exceptional Lie groups. Let G_2 be the smallest exceptional Lie group: the group of automorphisms of the complex Cayley algebra. Our aim is to relate the space of generalized G_2 -theta functions $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ of level one to other spaces of generalized theta functions associated to classical Lie groups.

Using the Verlinde formula, which gives the dimension of the space of generalized G -theta functions for any simple and simply-connected Lie group G , we observe a numerical coincidence:

$$\dim H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \dim H^0(\mathcal{M}_C(G_2), \mathcal{L}).$$

In addition, we link together $\dim H^0(\mathcal{M}_C(G_2), \mathcal{L})$ and $\dim H^0(\mathcal{M}_C(SL_3), \mathcal{L})$. Our aim is to give a geometric interpretation of these dimension equalities.

According to the Borel-De Siebenthal classification [BDS49], the groups SL_3 and SO_4 appear as the two maximal subgroups of G_2 among the connected subgroups of G_2 of maximal rank. We define two linear maps by pull-back of the corresponding extension maps: on the one hand

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})$$

and on the other hand using the isogeny $SL_2 \times SL_2 \rightarrow SO_4$:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3).$$

These maps take values in the invariant part by the duality involution for the first map and by the action of 2-torsion elements of the Jacobian for the second one. We denote these invariant spaces by $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ and $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ respectively.

Using the natural isomorphism proved in [BNR89]:

$$H^0(\mathcal{M}_C(SL_3), \mathcal{L})^* \simeq H^0(\text{Pic}^{g-1}(C), 3\Theta)$$

where $\Theta = \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, L) > 0\}$, we prove the following theorem:

Theorem A.

The linear map

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$$

obtained by pull-back of the extension map $\mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$ is surjective for a general curve and is an isomorphism when the genus of the curve equals 2.

A curve is said satisfying the cubic normality when the multiplication map $\text{Sym}^3 H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$ is surjective. Using an explicit basis of $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)$ described in [Bea91], we prove the theorem:

Theorem B.

The linear map

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

obtained by pull-back of the extension map $\mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$ is an isomorphism for a general curve satisfying the cubic normality.

Notation. We use the following notations:

- C a smooth, connected, projective curve of genus g at least 2,
- G a connected and simply-connected simple complex Lie group,
- $\mathcal{M}_C(G)$ the moduli stack of principal G -bundles on C ,
- K_C the canonical bundle on C ,
- $\text{Pic}^{g-1}(C)$ the Picard group parametrizing line bundles on C of degree $g-1$,
- $h^0(X, \mathcal{L}) = \dim H^0(X, \mathcal{L})$.

2. PRINCIPAL G_2 -BUNDLES ARISING FROM VECTOR BUNDLES OF RANK TWO AND THREE

2.1. The octonions algebra. Let \mathbb{O} be the complex algebra of the octonions, $\text{Im}(\mathbb{O})$ the 7-dimensional subalgebra of the imaginary part of \mathbb{O} and $\mathcal{B}_0 = (e_1, \dots, e_7)$ the canonical basis of $\text{Im}(\mathbb{O})$ (see [Bae02]). The exceptional Lie group G_2 is the group of automorphisms of the octonions.

In Appendix A we give the multiplication table in $\text{Im}(\mathbb{O})$ by the Fano diagram and we introduce another basis $\mathcal{B}_1 = (y_1, \dots, y_7)$ of $\text{Im}(\mathbb{O})$ so

that $\langle y_1, y_2, y_3 \rangle$ and $\langle y_4, y_5, y_6 \rangle$ are isotropic and orthogonal and y_7 is orthogonal to both of these subspaces. This basis is defined in Appendix A, as well as two other basis obtained by permutation of elements of \mathcal{B}_1 : $\mathcal{B}_2 = (y_2, y_3, y_4, y_5, y_6, y_1, y_7)$ and $\mathcal{B}_3 = (y_1, y_2, y_4, y_5, y_3, y_6, y_7)$.

In the following paragraphs, we use local sections of rank-7 vector bundles satisfying multiplication rules of \mathcal{B}_2 or \mathcal{B}_3 .

2.2. Principal G_2 -bundles admitting a reduction. We introduce the notion of non-degenerated trilinear form on $\text{Im}(\mathbb{O})$ as Engel did it in [Enga] and [Engb]. A trilinear form ω on $\text{Im}(\mathbb{O})$ is said non-degenerated if the associated bilinear symmetric form B_ω is non-degenerated where $B_\omega(x, y) = \omega(x, \cdot, \cdot) \wedge \omega(y, \cdot, \cdot) \wedge \omega(\cdot, \cdot, \cdot) \forall x, y \in \text{Im}(\mathbb{O})$.

Lemma 2.1.

Giving a principal G_2 -bundle is equivalent to giving a rank-7 vector bundle with a non-degenerated trilinear form.

Proof. Let P be a principal G_2 -bundle on C and V the associated rank-7 vector bundle and let ω be any non-degenerated trilinear form on $\text{Im}(\mathbb{O})$. By construction, V has a reduction to G_2 , *i.e.* it exists a section $\sigma : C \rightarrow GL_7/G_2$. The Lie group G_2 is the stabilizer $\text{Stab}_{SL_7}(\omega)$ under the action of SL_7 . Besides, under the action of GL_7 , $\text{Stab}_{GL_7}(\omega_0) \simeq G_2 \times \mathbb{Z}/3\mathbb{Z}$. Then

$$\sigma : C \xrightarrow{\sigma} GL_7/G_2 \twoheadrightarrow GL_7/(G_2 \times \mathbb{Z}/3\mathbb{Z}) \simeq GL_7/\text{Stab}_{GL_7}(\omega) \simeq \text{Orb}_{GL_7}(\omega).$$

In addition, the orbit $\text{Orb}_{GL_7}(\omega)$ is the set of all the non-degenerated trilinear form on $\text{Im}(\mathbb{O})$. So, V is fitted with a non-degenerated trilinear form. Reciprocally any rank-7 vector bundle fitted with a non-degenerated trilinear form defines a G_2 -vector bundle. \square

For a principal G_2 -bundle, we use the non-degenerate trilinear form ω on \mathcal{V} , locally defined by $\omega(x, y, z) = -\text{Re}[(xy)z]$.

According to the Borel-De Siebenthal classification (see [BDS49]), SL_3 and SO_4 are, up to conjugation, the two maximal subgroups of G_2 among the connected subgroups of G_2 of maximal rank. Using the inclusion $G_2 \subset SO_7 = SO(\text{Im}(\mathbb{O}))$ both of the following lemma describe the rank-7 vector bundle (and the non-degenerate trilinear form) associated to a principal G_2 -bundle which admits either a SL_3 -reduction or a SO_4 -reduction.

Note that $M_C(SO_4)$ has two connected components distinguished by the second Stiefel Whitney class. We only make here explicit computations with regards to the connected component $M_C^+(SO_4)$ of $M_C(SO_4)$ containing the trivial bundle.

2.2.1. Principal G_2 -bundles arising from rank-3 vector bundles.

Lemma 2.2.

Let E be a rank-3 vector bundle with trivial determinant and let $E(G_2)$ be his associated principal G_2 -bundle and \mathcal{V} be his associated rank-7 vector bundle.

Then, \mathcal{V} has the following decomposition and the local sections basis \mathcal{B}_2 is adapted to this decomposition:

$$\mathcal{V} = E \oplus E^* \oplus \mathcal{O}_C.$$

The non-degenerate trilinear form ω is defined by the following local conditions:

- (1) $\Lambda^3 E \simeq \Lambda^3 E^* \simeq \mathbb{C}$ and $\omega(y_2, y_3, y_4) = \omega(y_5, y_6, y_1) = -\sqrt{2}$.
- (2) On $E \times E^* \times \mathcal{O}_C$:

$$\omega(y_2, y_5, y_7) = \omega(y_3, y_6, y_7) = \omega(y_4, y_1, y_7) = i,$$

- (3) All other computation, not obtainable by permutation of the previous triplets, equals zero.

Proof. Under the action of SL_3 , $\text{Im}(\mathbb{O})$ decomposes into SL_3 -modules:

$$\begin{aligned} \text{Im}(\mathbb{O}) &= \langle y_2, y_3, y_4 \rangle \oplus \langle y_5, y_6, y_1 \rangle \oplus \langle y_7 \rangle \\ &= \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C} \end{aligned}$$

where $\{y_2, y_3, y_4, y_5, y_6, y_1, y_7\}$ is the basis \mathcal{B}_2 defined in Appendix A; $\langle y_2, y_3, y_4 \rangle$ is an isotropic subspace of dimension 3, dual of $\langle y_5, y_6, y_1 \rangle$.

So, the 7-rank vector bundle \mathcal{V} associated to a rank-3 vector bundle E is:

$$\begin{aligned} \mathcal{V} &= E \times^{SL_3} (\mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C}), \\ \mathcal{V} &= E \oplus E^* \oplus \mathcal{O}_C. \end{aligned}$$

Evaluations given for ω are deduced from Table 5 of Appendix A. \square

2.2.2. Principal G_2 -bundles arising from two rank-2 vector bundles. The following lemma makes explicit the vector bundle associated to a principal G_2 -bundle extension of an element of $M_C^+(SO_4)$, using the surjective map $M_C(SL_2) \times M_C(SL_2) \rightarrow M_C^+(SO_4)$.

Lemma 2.3.

Let E, F be two rank-2 vector bundles of trivial determinant. Denote by (E, F) the associated principal SO_4 -bundle, P the associated principal G_2 -bundle and \mathcal{V} the associated rank-7 vector bundle. Then, \mathcal{V} has the following decomposition and the local sections basis \mathcal{B}_3 is adapted to this decomposition:

$$\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).$$

The non-degenerate trilinear form ω on \mathcal{V} is defined by the following local conditions:

- (1) On $(E^* \otimes F)^3$ and on $(E^* \otimes F) \times (\text{End}_0(F))^2$, ω is identically zero,
- (2) On $(E^* \otimes F)^2 \times \text{End}_0(F)$:

$$\begin{aligned} \omega(y_2, y_4, y_3) &= \omega(y_5, y_1, y_6) = \sqrt{2}, \\ \omega(y_4, y_1, y_7) &= \omega(y_2, y_5, y_7) = i. \end{aligned}$$

- (3) $\Lambda^3 \text{End}_0(F) \simeq \mathbb{C}$ and $\omega(y_3, y_6, y_7) = i$,
- (4) All other computation, not obtainable by permutation of the previous triplets, equals zero.

Proof. The groups SO_4 and $SL_2 \times SL_2$ are isogenous. Under the action of SO_4 , $\text{Im}(\mathbb{O})$ has the following decomposition:

$$\begin{aligned} \text{Im}(\mathbb{O}) &= \langle y_1, y_2, y_4, y_5 \rangle \oplus \langle y_3, y_6, y_7 \rangle, \\ &\simeq M^* \otimes N \oplus^\perp \text{End}_0(N) \end{aligned}$$

where M, N are 2-dimensional; an element $\overline{(A, B)}$ of SO_4 ($A, B \in SL_2$) acts on $M^* \otimes N$ by $A \otimes B$ and by conjugation $\text{End}_0(N)$.

So, the rank-7 vector bundle \mathcal{V} associated to $(E, F)(G_2)$, when E, F are two rank-7 vector bundle of trivial determinant, is:

$$\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).$$

Evaluations given for ω are deduced from Table 6 of Appendix A. □

3. EQUALITIES BETWEEN DIMENSIONS OF SPACES OF GENERALIZED THETA FUNCTIONS

Here are some dimension counts, using the Verlinde Formula, to calculate $h^0(\mathcal{M}_C(G_2), \mathcal{L})$ and $h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$.

Proposition 3.1.

Dimensional equalities between the following spaces of generalized theta functions occur:

$$h^0(\mathcal{M}_C(G_2), \mathcal{L}) = \left(\frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left(\frac{5 - \sqrt{5}}{2} \right)^{g-1}, \quad (1)$$

$$h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \left[\left(\frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left(\frac{5 - \sqrt{5}}{2} \right)^{g-1} \right], \quad (2)$$

$$\text{so, } h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}). \quad (3)$$

Proof. See Appendix B and C □

4. SURJECTIVITIES AND ISOMORPHISMS BETWEEN $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ AND $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$

To avoid confusion, we sometimes specify the group G in the notation of the generator of the Picard group \mathcal{L} writing \mathcal{L}_G .

Let $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ be the invariant part of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ by the duality involution: the eigenspace of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ associated to the eigenvalue 1 under the natural involution $E \mapsto \sigma(E) = E^*$.

We consider the extension map $i : \mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$ which associates to a rank-3 vector bundle of trivial determinant the associated principal G_2 -bundle. The pull-back $i^*(\mathcal{L}_{G_2})$ equals \mathcal{L}_{SL_3} .

Theorem 4.1.

The extension map $i : \mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$ induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), i^*(\mathcal{L}_{G_2})).$$

This map takes values in $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$.

We denote by Φ this map:

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+.$$

Proof. (1) The pull-back $i^*(\mathcal{L}_{G_2})$ equals \mathcal{L}_{SL_3} . Indeed, let E be a rank-3 vector bundle. By Lemma 2.2, the rank-7 vector bundle associated to E is $E \oplus E^* \oplus \mathcal{O}_C$. We study the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow i & & \uparrow \rho_2 \\ \mathcal{M}_C(SL_3) & \xrightarrow{\rho_3} & \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3) \end{array}$$

where i and ρ_1 are maps of extension of group of structure and $\forall E \in \mathcal{M}_C(SL_3)$, $\rho_3(E) = (E, E^*)$ and $\forall (E, F) \in \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3)$, $\rho_2(E, F) = E \oplus F \oplus \mathcal{O}_C$. By Proposition 2.6 of [LS97], applied with SL_7 , G_2 and the irreducible representation ρ_1 of highest weight ϖ_1 , the Dynkin index of $d(\rho_1)$ equals 2. Therefore, $\rho_1^*(\mathcal{D}_{SL_7}) = \mathcal{L}_{G_2}^2$ where \mathcal{D}_{SL_7} the determinant bundle generator of $\text{Pic}(\mathcal{M}_C(SL_7))$ (see [KNR94] and [LS97]). In addition, by the same proposition, $\rho_2^*(\mathcal{D}_{SL_7}) = \mathcal{L}_{SL_3} \boxtimes \mathcal{L}_{SL_3}$ and $\rho_3^*(\rho_2^*(\mathcal{D}_{SL_7})) = \mathcal{L}_{SL_3}^2$. So, $i^*(\mathcal{L}_{G_2})^2 = \mathcal{L}_{SL_3}^2$ so that $i^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_3}$ since the Picard group $\text{Pic}(\mathcal{M}_C(SL_3))$ is isomorphic to \mathbb{Z} .

(2) We show that the image of the linear map Φ is contained in $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$.

The morphism i is σ -invariant: for all $E \in \mathcal{M}_C(SL_3)$, the G_2 -principal bundles $E(G_2)$ and $E^*(G_2)$ are isomorphic. Indeed, the Weyl group $W(SL_3)$ is contained in $W(G_2)$ since so are there normalizer; and $W(SL_3)$ is a subgroup of $W(G_2)$ of index 2. We consider \bar{g} in the Weyl group $W(G_2) \setminus W(SL_3)$ and $g \in G_2$ a representative of the equivalence class of \bar{g} . Then, $g \notin SL_3$. Let C_g be the inner automorphism of G_2 induced by g . As the subalgebra \mathfrak{sl}_3 of \mathfrak{g}_2 corresponds to the long roots and as each element of the Weyl group $W(G_2)$ respects the Killing form on \mathfrak{g}_2 , $C_g(SL_3)$ is contained in SL_3 . The restriction of C_g to SL_3 is then an exterior automorphism of SL_3 which we call $\alpha : SL_3 \rightarrow SL_3$. This automorphism exchanges the two fundamental representations of SL_3 . So, α induces an automorphism $\tilde{\alpha}$ on $\mathcal{M}_C(SL_3)$ such that, $\forall E \in \mathcal{M}_C(SL_3)$, $\tilde{\alpha}(E) = E^*$. Consider the following commutative diagram, where \tilde{C}_g is the inner

automorphism given by g :

$$\begin{array}{ccc} \mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2) \\ \downarrow \tilde{\alpha} & & \downarrow \tilde{C}_g \\ \mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2). \end{array}$$

Then, $\forall E \in \mathcal{M}_C(SL_3)$,

$$E^*(G_2) = \tilde{\alpha}(E)(G_2) = \tilde{C}_g(E(G_2)) \simeq E(G_2)$$

since \tilde{C}_g is an inner automorphism. Thus, $i(E)$ and $i(\sigma(E))$ are isomorphic.

The σ -invariance of i implies that the image of Φ is contained in one of the two eigenspaces of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$. As $\sigma^*(\mathcal{L}_{SL_3}) \simeq \mathcal{L}_{SL_3}$, which is the isomorphism which implies identity over the trivial bundle, we get $\sigma(i^*(\mathcal{L}_{G_2})) = \sigma^*(\mathcal{L}_{SL_3}) = \mathcal{L}_{SL_3} = i^*(\mathcal{L}_{G_2})$. Thus, the image of Φ is contained in $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$, the eigenspace relative to the eigenvalue 1. \square

We remind two points of vocabulary: an *even theta-characteristic* κ on a curve C is an element κ of $\text{Pic}^{g-1}(C)$ such that $\kappa \otimes \kappa = K_C$ and $h^0(C, \kappa)$ is even ; a curve C is said *without effective theta-constant* if $h^0(C, \kappa) = 0$ for all even theta-characteristic κ . The set of all even theta-characteristics is named $\Theta^{\text{even}}(C)$.

Theorem 4.2.

The linear map $\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$

- (1) is surjective when the curve C is without effective theta-constant.
- (2) is an isomorphism if the genus of C equals 2.

Proof. (1) Let C be a curve without effective theta-constant and consider the following diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow i & \nearrow \varphi & \\ \mathcal{M}_C(SL_3) & & \end{array}$$

We introduce the element Δ_κ defined for each even theta-characteristic κ :

$$\Delta_\kappa = \{P \in \mathcal{M}_C(SL_7) \mid h^0(C, P(\mathbb{C}^7) \otimes \kappa) > 0\}.$$

These Δ_κ are Cartier divisors, so they define, up to a scalar, an element of $H^0(\mathcal{M}_C(SL_7), \mathcal{L})$. The image $\Phi(\rho_1^*(\Delta_\kappa)) = \rho^*(\Delta_\kappa)$ is

$$\begin{aligned} \rho^*(\Delta_\kappa) &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \oplus E^* \oplus \mathcal{O}_C) \otimes \kappa > 0\}, \\ &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) + h^0(C, \kappa) > 0\}, \\ &= \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) > 0\}, \\ &\quad \text{because } C \text{ is without effective theta-constant,} \\ &= \{E \in \mathcal{M}_C(SL_3) \mid 2h^0(C, E \otimes \kappa) > 0\}, \text{ by Serre duality.} \end{aligned}$$

Thus, $\rho^*(\Delta_\kappa) = 2H_\kappa$ where $H_\kappa := \{E \in \mathcal{M}_C(SL_3) \mid h^0(C, E \otimes \kappa) > 0\}$. Therefore, to show the surjectivity of Φ , it suffices to show that $\{H_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$. We consider

$$\Theta = \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, L) > 0\}$$

and the natural map between the spaces $H^0(\mathcal{M}_C(SL_3), \mathcal{L})^*$ and $H^0(\text{Pic}^{g-1}(C), 3\Theta)$. By Theorem 3 of [BNR89], this map is an isomorphism and, besides, it is equivariant for the two involutions on $H^0(\mathcal{M}_C(SL_3), \mathcal{L})^*$ and $H^0(\text{Pic}^{g-1}(C), 3\Theta)$ (respectively $E \mapsto E^*$ and $L \mapsto K_C \otimes L^{-1}$). So, the components (+) and (-) of each part are in correspondence: $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^*$ is isomorphic to $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$. Denote by φ this isomorphism $\varphi : \mathbb{P}H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^* \xrightarrow{\sim} \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)_+^*$. For all even theta-characteristic κ , the image $\varphi(H_\kappa)$ is $\varphi_{3\Theta}(\kappa)$ where $\varphi_{3\Theta}$ is the following map:

$$\varphi_{3\Theta} : \text{Pic}^{g-1}(C) \rightarrow \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)_+^* = |3\Theta|_+^*.$$

The set $\{\varphi_{3\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $|3\Theta|_+^*$. Indeed, in the following commutative diagram

$$\begin{array}{ccc} & & |4\Theta|_+^* \\ & \nearrow \varphi_{4\Theta} & \vdots \\ \text{Pic}^{g-1}(C) & \xrightarrow{\varphi_{3\Theta}} & |3\Theta|_+^* \end{array}$$

the map $|4\Theta|_+^* \dashrightarrow |3\Theta|_+^*$ is surjective because it is induced by the inclusion $D \in H^0(C, 3\Theta)_+ \mapsto D + \Theta \in H^0(C, 4\Theta)_+$. In addition, by [KPS09], when C is without effective theta-constant, $\{\varphi_{4\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$ is a base of $|4\Theta|_+^*$ (the number of even theta-characteristics equals $2^{g-1}(2^g + 1)$ which equals the linear dimension of $|4\Theta|_+^*$). Thus, $\{\varphi_{3\Theta}(\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $|3\Theta|_+^*$ and $\{H_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates the space $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$. As we have shown that H_κ equals $\Phi(\rho_1(\Delta_\kappa))$ for all κ even theta-characteristic, the map Φ is surjective.

- (2) By [BNR89], the dimension of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+^*$ equals the dimension of $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$, that is $\frac{3g+1}{2}$. When the genus of C is 2,

the dimension of $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$ equals 5 which is also the dimension of $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ by Proposition 3.1.(1). A curve of genus 2 is without effective theta-constant. So, by this dimension equality and by the point (1) of the theorem, Φ is an isomorphism when the genus of C equals 2. \square

5. ISOMORPHISMS BETWEEN SPACES OF GENERALIZED G_2 -THETA FUNCTIONS AND GENERALIZED SL_2 -THETA FUNCTIONS

Let $\text{JC}[2]$ be the group of 2-torsion elements of the Jacobian: $\text{JC}[2] = \{\alpha \in \text{Pic}^0(C) \mid \alpha \otimes \alpha = \mathcal{O}_C\}$. This group acts on $\mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$: for $\alpha \in \text{JC}[2]$ and $(E, F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$, we associate $(E \otimes \alpha, F \otimes \alpha)$. Let $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ be the invariant part of $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]$ under the action of the element of the Jacobian group $\text{JC}[2]$.

We study the subgroup SO_4 of G_2 , which is isogenous to $SL_2 \times SL_2$, and the linear map induced by pull-back by the extension map $j : \mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$ which associates to two rank-2 vector bundle of trivial determinant the associated principal G_2 -bundle. The pull-back $j^*(\mathcal{L}_{G_2})$ equals $\mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3$.

Theorem 5.1.

The extension map $j : \mathcal{M}_C(SO_4) \rightarrow \mathcal{M}_C(G_2)$ induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \rightarrow H^0(\mathcal{M}_C(SO_4), j^*(\mathcal{L}_{G_2})).$$

This map takes values in $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$.

We denote by Ψ this map:

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

Proof. (1) Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_7) \\ \uparrow j & & \uparrow \rho_4 \\ \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) & \xrightarrow{(f_1, f_2)} & \mathcal{M}_C(SL_4) \times \mathcal{M}_C(SL_3) \end{array}$$

where j and ρ_1 are the extension maps, $f_1(M, N) = M^* \otimes N$, $f_2(M, N) = \text{End}_0(N)$ and $\rho_4(A, B) = A \oplus B$.

As in the previous section, we calculate explicitly $j^*(\mathcal{L}_{G_2})$.

Let \mathcal{D}_{SL_7} be the determinant bundle of $\mathcal{M}_C(SL_7)$ and pr_1 and pr_2 the canonical projections of $SL_2 \times SL_2$. We get

$$f_1^*(\mathcal{L}_{SL_4}) = \text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^2 = \mathcal{L}_{SL_2}^2 \boxtimes \mathcal{L}_{SL_2}^2$$

and according to Table B of [Sor00]:

$$f_2^*(\mathcal{L}_{SL_3}) = \text{pr}_2^*(\mathcal{L}_{SL_2})^4,$$

since f_2 is associated to the adjoint representation of SL_2 , which has Dynkin index 4.

$$\begin{aligned} \rho_4^*(\mathcal{D}) &= \mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3}, \\ \text{so } j^*(\mathcal{L}_{G_2}^2) &= (f_1, f_2)^*(\mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3}), \\ &= f_1^*(\mathcal{L}_{SL_4}) \otimes f_2^*(\mathcal{L}_{SL_3}), \\ &= [\text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^2] \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^4, \\ j^*(\mathcal{L}_{G_2}^2) &= \text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^6, \\ \text{so } j^*(\mathcal{L}_{G_2}) &= \text{pr}_1^*(\mathcal{L}_{SL_2}) \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^3. \end{aligned}$$

We get

$$j^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3.$$

- (2) The morphism $j : \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2} \rightarrow \mathcal{M}_{G_2}$ is invariant under the action of $JC[2]$: $(E, F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$ and $\alpha \in JC[2]$ then the rank-7 vector bundle associated by j to $(E \otimes \alpha, F \otimes \alpha)$ is

$$\begin{aligned} &(E \otimes \alpha)^* \otimes (F \otimes \alpha) \oplus \text{End}_0(F \otimes \alpha) \\ &= E^* \otimes F \otimes \alpha^* \otimes \alpha \oplus \text{End}_0(F \otimes \alpha), \\ &= E \otimes F \oplus \text{End}_0(F) = j(E, F). \end{aligned}$$

Therefore, the image of Ψ is contained in the expected vector space $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$. \square

Before going further in the study of the morphism j , we compare the dimensions of the involved sets.

Lemma 5.2.

The dimension of the space $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ equals the dimension of $H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2})$.

Proof. By Proposition 3.1,(3), we notice the remarkable following relation:

$$2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}) = h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3). \quad (4)$$

So, as $h^0(\mathcal{M}_C(SL_2), \mathcal{L}) = 2^g$ (see [Bea88]), we get

$$\begin{aligned} &\dim \left([H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0 \right) \\ &= \frac{1}{2^{2g}} \times h^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \times h^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3) \quad \text{because } |JC[2]| = 2^{2g}, \\ &= \frac{1}{2^{2g}} \times 2^g \times 2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \quad \text{by (4),} \\ &= h^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}). \end{aligned}$$

\square

All the following results are based on the *cubic normality conjecture*. Its statement is:

Conjecture 5.3.

For a general curve C , the multiplication map

$$\eta : \text{Sym}^3 H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)$$

is surjective.

When the previous map η is surjective, we say that the curve C satisfies *cubic normality*.

Proposition 5.4.

Cubic normality holds for all curve of genus 2, all non hyper-elliptic curve of genus 3 and all curve of genus 4 without effective theta-constant.

Proof. For a curve of genus 2, $\mathcal{M}_C(SL_2)$ is isomorphic to \mathbb{P}^3 and \mathcal{L}_{SL_2} to $\mathcal{O}(1)$ (see [NR75]). A non-hyper-elliptic curve of genus 3 is a Coble quartic (see [NR87]). The cubic normality is true in both of these cases. For a general curve of genus 4 without effective theta-constant, cubic normality is proved in Theorem 4.1 of [OP99]. \square

When this conjecture is true, we get this theorem:

Theorem 5.5.

Let C be a curve of genus at least 2 without effective theta-constant and satisfying the cubic normality and let Ψ be the map defined in Theorem 5.1:

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \rightarrow [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0.$$

- (1) The map Ψ is an isomorphism,
- (2) The space of generalized G_2 -theta functions $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ is linearly generated by the divisors $\rho_0^*(\Delta_\kappa)$ for κ even theta-characteristic, where ρ_0 is the extension morphism

$$\rho_0 : \mathcal{M}_C(G_2) \rightarrow \mathcal{M}_C(SO_7).$$

Proof. (1) According to the dimension equality proved in Lemma 5.2, it suffices to prove the surjectivity of Ψ .

Denote by V the vector space $H^0(\mathcal{M}_C(SL_2), \mathcal{L})$.

Using the notations of [Bea91], we associate to each even theta-characteristic κ an element d_κ of $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 2})$ and an element ξ_κ of $V \otimes V$. For each even theta-characteristic κ , d_κ is the section of $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 2})$ such that D_κ is the divisor of the zeros of d_κ , where $D_\kappa = \{S \in \mathcal{M}_C(SL_2) \mid h^0(C, \text{End}_0(S) \otimes \kappa) > 0\}$. Consider the following maps:

$$\begin{aligned} \rho_0^* : & H^0(\mathcal{M}_C^+(SO_7), \mathcal{L}) & \longrightarrow & H^0(\mathcal{M}_C(G_2), \mathcal{L}) \\ \text{and } \beta : & [V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]_0 & \longrightarrow & [V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)]_0 \end{aligned}$$

where $\beta(A, B, D) = (A, BD)$. For any even theta-characteristic κ , the image $\Psi(\rho_0^*(\Delta_\kappa))$ equals $\beta(\xi_\kappa \otimes d_\kappa)$. Indeed, Ψ is induced by:

$$\begin{aligned} j: \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) &\rightarrow \mathcal{M}_C(G_2) \\ (E, F) &\mapsto \text{Hom}(E, F) \oplus \text{End}_0(F); \end{aligned}$$

the pull-back $\Psi(\rho_0^*(\Delta_\kappa))$ is the sum of two divisors:

$$\begin{aligned} \Delta_1 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{End}_0(F) \otimes \kappa) > 0\}, \\ \Delta_2 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{Hom}(E, F) \otimes \kappa) > 0\}. \end{aligned}$$

In addition, $\mathcal{O}(\Delta_1) = \mathcal{O}_C \boxtimes \mathcal{L}^2$ and $\mathcal{O}(\Delta_2) = \mathcal{L} \boxtimes \mathcal{L}$ (see [Bea91]) and more precisely:

$$\Delta_1 = \text{Zeros}(d_\kappa) \text{ et } \Delta_2 = \text{Zeros}(\xi_\kappa).$$

When the curve C is of genus at least 2 without effective theta-constant, it is proved in [BNR89] that the map

$$\begin{aligned} \varphi_0^*: \text{Sym}^2 V &\longrightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) \\ \xi_\kappa &\mapsto d_\kappa \end{aligned}$$

is an isomorphism. We identify $\text{Sym}^2 V$ with the invariant space of $V \otimes V$ under the involution $a \otimes b \mapsto b \otimes a$. By Theorem 1.2 and Proposition A.5 of [Bea91], the set $\{d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ is a basis of $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)$ and $\{\xi_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ is a basis of $\text{Sym}^2 V$. Then, the vector space $[V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]_0$ is generated by $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$. Thus, to prove the surjectivity of the map Ψ , it is sufficient to show the surjectivity of the map β . Consider the following diagram:

$$\begin{array}{ccc} V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) & \longrightarrow & H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) \\ \uparrow & & \uparrow \eta \\ V \otimes V \otimes V & \longrightarrow & \text{Sym}^3 V. \end{array}$$

With the hypothesis of cubic normality, the map η is surjective. Therefore the map $V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2) \rightarrow H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$ is also surjective. By restriction to invariant sections under the action of $J\mathcal{C}[2]$, β is surjective.

The map Ψ is thus an isomorphism.

- (2) The point (2) is a consequence of the previous facts: for each element $\kappa \in \Theta^{\text{even}}(C)$, the image of $\rho_0^*(\Delta_\kappa)$ by Ψ is $\xi_\kappa \otimes d_\kappa$. As $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $[V \otimes V \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)]$, the set $\{\rho_0^*(\Delta_\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $H^0(\mathcal{M}_C(G_2), \mathcal{L})$. □

Remark 5.6. By Proposition 5.4 the linear map Ψ is an isomorphism for each curve of genus 2, each non hyperelliptic curve of genus 3 and each curve of genus 4 without effective theta-constant.

APPENDIX A. THE OCTONION ALGEBRA

Let $\mathcal{B}_0 = \{e_1, \dots, e_7\}$ be the canonical basis of the subalgebra of the imaginary part of octonions. The multiplication rules are:

- $\forall i \in \{1, \dots, 7\}, e_i^2 = -1,$
- $e_i e_j = -e_j e_i = e_k$ when (e_i, e_j, e_k) are three points on the same edge of on the oriented Fano diagram.

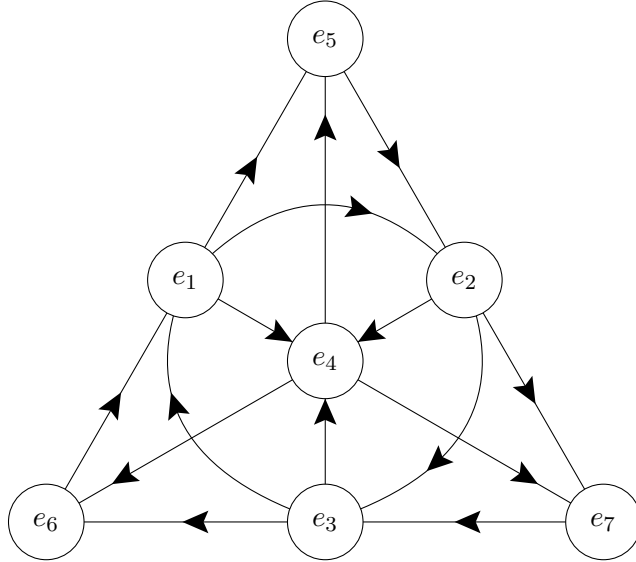


FIGURE 1. Fano Diagram

We introduced the basis $\mathcal{B}_1 = \{y_1, \dots, y_7\}$ obtained by basis change by

the change of basis matrix $P = \frac{\sqrt{2}}{2} \left(\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & -i & 0 & 0 & i & 0 & 0 \\ -i & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & i & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{array} \right).$ The

canonical quadratic form on $\text{Im}\mathbb{O}$ expressed in the basis \mathcal{B}_1 is $Q = \left(\begin{array}{c|c|c} 0 & I_3 & 0 \\ \hline I_3 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$.

In the basis $\mathcal{B}_2 = \{y_2, y_3, y_4, y_5, y_6, y_1, y_7\}$, the multiplication table is

\nearrow	y_2	y_3	y_4	y_5	y_6	y_1	y_7
y_2	0	$-\sqrt{2}y_1$	$\sqrt{2}y_6$	$-1 + iy_7$	0	0	$-iy_2$
y_3	$\sqrt{2}y_1$	0	$-\sqrt{2}y_5$	0	$-1 + iy_7$	0	$-iy_3$
y_4	$-\sqrt{2}y_6$	$\sqrt{2}y_5$	0	0	0	$-1 + iy_7$	$-iy_4$
y_5	$-1 - iy_7$	0	0	0	$-\sqrt{2}y_4$	$\sqrt{2}y_3$	iy_5
y_6	0	$-1 - iy_7$	0	$\sqrt{2}y_4$	0	$-\sqrt{2}y_2$	iy_6
y_1	0	0	$-1 - iy_7$	$-\sqrt{2}y_3$	$\sqrt{2}y_2$	0	iy_1
y_7	iy_2	iy_3	iy_4	$-iy_5$	$-iy_6$	$-iy_1$	-1

(5)

In the basis $\mathcal{B}_3 = \{y_1, y_2, y_4, y_5, y_3, y_6, y_7\}$, the multiplication table is

\nearrow	y_1	y_2	y_4	y_5	y_3	y_6	y_7
y_1	0	0	$-1 - iy_7$	$-\sqrt{2}y_3$	0	$\sqrt{2}y_2$	iy_1
y_2	0	0	$\sqrt{2}y_6$	$-1 + iy_7$	$-\sqrt{2}y_1$	0	$-iy_2$
y_4	$-1 + iy_7$	$-\sqrt{2}y_6$	0	0	$\sqrt{2}y_5$	0	$-iy_4$
y_5	$\sqrt{2}y_3$	$-1 - iy_7$	0	0	0	$-\sqrt{2}y_4$	iy_5
y_3	0	$\sqrt{2}y_1$	$-\sqrt{2}y_5$	0	0	$-1 + iy_7$	$-iy_3$
y_6	$-\sqrt{2}y_2$	0	0	$\sqrt{2}y_4$	$-1 - iy_7$	0	iy_6
y_7	$-iy_1$	iy_2	iy_4	$-iy_5$	iy_3	$-iy_6$	-1

(6)

APPENDIX B. COMPUTATION OF $h^0(\mathcal{M}_C(G_2), \mathcal{L})$

First, we recall the Verlinde Formula.

Proposition B.1 (Verlinde Formula).

Let G be a Lie group with Lie algebra \mathfrak{g} of classical type or type \mathfrak{g}_2 , \mathcal{L} be the ample canonical line bundle on $\mathcal{M}_C(G)$ and i be a positive integer. The integer $h^0(\mathcal{M}_C(G), \mathcal{L}^i)$ is given by the following relation:

$$h^0(\mathcal{M}_C(G), \mathcal{L}^i) = (\#T_i)^{g-1} \sum_{\mu \in \mathcal{P}_i} \prod_{\alpha \in \Delta_+} \left[2 \sin \left(\frac{\pi \langle \alpha, \mu + \rho \rangle}{i + g^*} \right) \right]^{2-2g}.$$

where

$$\begin{aligned}
\#T_i &= (i + g^*)^{\text{rk}(\mathfrak{g})} \#(\mathcal{P}/Q) \#(Q/Q_{\text{lg}}), & \langle \cdot, \cdot \rangle & \text{ is the Killing form,} \\
& \text{where } \text{rk}(\mathfrak{g}) \text{ is the rank of } \mathfrak{g}, & \mathcal{P}_i &= \{\text{dominant weights } \mu \mid \langle \mu, \theta \rangle \leq i\}, \\
\mathcal{P} & \text{ is the weight lattice,} & \theta & \text{ is the maximal positive root,} \\
Q & \text{ is the root lattice,} & \Delta_+ &= \{\text{positive roots of } \mathfrak{g}\} \\
Q_{\text{lg}} & \text{ is the long root lattice,} & g^* & \text{ is the dual Coxeter number of the group } G. \\
\rho &= \frac{1}{2} \sum_{\alpha_j \in \Delta_+} \alpha_j,
\end{aligned}$$

Proof. See the survey [Sor96]. \square

The root system of G_2 has six positive roots $\alpha_1, \dots, \alpha_6$, with two simple roots, called α_1 and α_2 .

For $H^0(\mathcal{M}_C(G_2), \mathcal{L})$, the data used in the Verlinde Formula are $g^* = 4$, $\text{rk}(\mathfrak{g}_2) = 2$, $\#T_1 = (1 + 4)^2 \times 1 \times 3 = 75$, $\rho = 5\alpha_1 + 3\alpha_2$, $\theta = \alpha_6 = 3\alpha_1 + 2\alpha_2$ and the fundamental weights are $\varpi_1 = \alpha_5$ and $\varpi_2 = \alpha_6$.

To describe \mathcal{P}_1 , we describe the Killing form. The angle formed between α_1 and α_2 is $5\pi/6$ and the ratio $\|\alpha_2\|/\|\alpha_1\| = \sqrt{3}$. In order to normalize the Killing form, we impose the norm of maximal positive root $\|\theta\|$ equals 2. As $\theta = \alpha_6$, $\|\alpha_2\| = \|\theta\| = 2$ and then $\|\alpha_1\| = 2/3$ and $\langle \alpha_1, \alpha_2 \rangle = -1$. The evaluations of the Killing form on ϖ_i and θ are $\langle \varpi_1, \theta \rangle = 1$ and $\langle \varpi_2, \theta \rangle = 2$; so $\mathcal{P}_1 = \{0, \varpi_1\}$.

The evaluation of the Killing form on each positive root and ρ added to each value of \mathcal{P}_1 are the following:

$$\begin{aligned}
\langle \alpha_1, \rho \rangle &= 1/3 & \text{and} & & \langle \alpha_1, \varpi_1 + \rho \rangle &= 2/3, \\
\langle \alpha_2, \rho \rangle &= 1, & & & \langle \alpha_2, \varpi_1 + \rho \rangle &= 1, \\
\langle \alpha_3, \rho \rangle &= 4/3, & & & \langle \alpha_3, \varpi_1 + \rho \rangle &= 5/3, \\
\langle \alpha_4, \rho \rangle &= 5/3, & & & \langle \alpha_4, \varpi_1 + \rho \rangle &= 7/3, \\
\langle \alpha_5, \rho \rangle &= 2, & & & \langle \alpha_5, \varpi_1 + \rho \rangle &= 3, \\
\langle \alpha_6, \rho \rangle &= 3, & & & \langle \alpha_6, \varpi_1 + \rho \rangle &= 4.
\end{aligned}$$

So, by the Verlinde formula, the dimension $h^0(\mathcal{M}_C(G_2), \mathcal{L})$ is

$$\begin{aligned}
h^0(\mathcal{M}_C(G_2), \mathcal{L}) &= \left(\frac{2^{10}}{25}\right)^{1-g} \left[\left[\sin^2\left(\frac{\pi}{15}\right) \sin^2\left(\frac{4\pi}{15}\right) \sin^2\left(\frac{\pi}{5}\right) \sin^4\left(\frac{2\pi}{5}\right) \right]^{1-g} \right. \\
& \quad \left. + \left[\sin^2\left(\frac{2\pi}{15}\right) \sin^2\left(\frac{7\pi}{5}\right) \sin^2\left(\frac{2\pi}{5}\right) \sin^4\left(\frac{\pi}{5}\right) \right]^{1-g} \right].
\end{aligned}$$

To obtain a compact formula, we express these trigonometric products in $\mathbb{Q}(\sqrt{5})$:

$$\begin{aligned}
\sin^2\left(\frac{\pi}{5}\right) &= \frac{1}{8}(5 - \sqrt{5}), & \sin^2\left(\frac{\pi}{15}\right) \sin^2\left(\frac{4\pi}{15}\right) &= \frac{1}{25}(3 - \sqrt{5}), \\
\sin^2\left(\frac{2\pi}{5}\right) &= \frac{1}{8}(5 + \sqrt{5}), & \sin^2\left(\frac{2\pi}{15}\right) \sin^2\left(\frac{7\pi}{15}\right) &= \frac{1}{25}(3 + \sqrt{5}), \\
\sin^4\left(\frac{\pi}{5}\right) &= \frac{5}{2^5}(3 - \sqrt{5}), & \sin^4\left(\frac{2\pi}{5}\right) &= \frac{5}{2^5}(3 + \sqrt{5}).
\end{aligned}$$

So,

$$\begin{aligned}
& h^0(\mathcal{M}_C(G_2), \mathcal{L}) \\
&= \left(\frac{2^{10}}{25}\right)^{1-g} \left[\left[\left(\frac{1}{2^3}\right) \left(\frac{1}{2^3}\right) \left(\frac{5}{2^5}\right) (3 - \sqrt{5}) (5 - \sqrt{5}) (3 + \sqrt{5})\right]^{1-g} \right. \\
&\quad \left. + \left[\left(\frac{1}{2^5}\right) \left(\frac{1}{2^3}\right) \left(\frac{5}{2^5}\right) (3 + \sqrt{5}) (5 + \sqrt{5}) (3 - \sqrt{5})\right]^{1-g} \right], \\
&= \left(\frac{5-\sqrt{5}}{10}\right)^{1-g} + \left(\frac{5+\sqrt{5}}{10}\right)^{1-g} = \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}.
\end{aligned}$$

APPENDIX C. COMPUTATION OF $h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$

Using the previous notations, the evaluation on α and each element of $\mu + \rho$, where $\mu \in \mathcal{P}_3$, are the following:

$$\begin{aligned}
\langle \alpha, \rho \rangle &= 1, & \text{and } \langle \alpha, 2\varpi_1 + \rho \rangle &= 3, \\
\langle \alpha, \varpi_1 + \rho \rangle &= 2, & \langle \alpha, 3\varpi_1 + \rho \rangle &= 4.
\end{aligned}$$

By the Verlinde Formula,

$$\begin{aligned}
& h^0(\mathcal{M}_C(SL_2), \mathcal{L}^{\otimes 3}) \\
&= (10)^{g-1} (2)^{2-2g} \left[\left[\sin\left(\frac{\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{2\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{3\pi}{5}\right)\right]^{2-2g} + \left[\sin\left(\frac{4\pi}{5}\right)\right]^{2-2g} \right], \\
&= \left(\frac{5}{2}\right)^{g-1} \left[2 \left[\sin^2\left(\frac{\pi}{5}\right)\right]^{1-g} + 2 \left[\sin^2\left(\frac{2\pi}{5}\right)\right]^{1-g} \right], \\
&= 2 \left(\frac{5}{2}\right)^{g-1} \left(\left(\frac{8}{5-\sqrt{5}}\right)^{g-1} + \left(\frac{8}{5+\sqrt{5}}\right)^{g-1} \right), \\
&= 2^g \left[\left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1} \right].
\end{aligned}$$

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REFERENCES

- [Bae02] J. C. BAEZ – “The octonions”, *Bull. Amer. Math. Soc. (N.S.)* **39** (2002), no. 2, p. 145–205 (electronic).
- [BDS49] A. BOREL & J. DE SIEBENTHAL – “Les sous-groupes fermés de rang maximum des groupes de Lie clos”, *Comment. Math. Helv.* **23** (1949), p. 200–221.
- [Bea88] A. BEAUVILLE – “Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta”, *Bull. Soc. Math. France* **116** (1988), no. 4, p. 431–448 (1989).
- [Bea91] ———, “Fibrés de rang deux sur une courbe, fibré déterminant et fonctions thêta. II”, *Bull. Soc. Math. France* **119** (1991), no. 3, p. 259–291.
- [BNR89] A. BEAUVILLE, M. S. NARASIMHAN & S. RAMANAN – “Spectral curves and the generalised theta divisor”, *J. Reine Angew. Math.* **398** (1989), p. 169–179.
- [Enga] F. ENGEL – “Sur un groupe simple à quatorze paramètres”.
- [Engb] F. ENGEL – “Ein neues, dem linearen complexe analoges gebilde”.
- [KNR94] S. KUMAR, M. S. NARASIMHAN & A. RAMANATHAN – “Infinite Grassmannians and moduli spaces of G -bundles”, *Math. Ann.* **300** (1994), no. 1, p. 41–75.
- [KPS09] Y. KOPELIOVICH, C. PAULY & O. SERMAN – “On theta functions of order 4”, *Bull. Lond. Math. Soc.* **41** (2009), no. 3, p. 423–428.

- [LS97] Y. LASZLO & C. SORGER – “The line bundles on the moduli of parabolic G -bundles over curves and their sections”, *Ann. Sci. École Norm. Sup. (4)* **30** (1997), no. 4, p. 499–525.
- [NR75] M. S. NARASIMHAN & S. RAMANAN – “Generalised Prym varieties as fixed points”, *J. Indian Math. Soc. (N.S.)* **39** (1975), p. 1–19 (1976).
- [NR87] ———, “ 2θ -linear systems on abelian varieties”, in *Vector bundles on algebraic varieties (Bombay, 1984)*, Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, p. 415–427.
- [OP99] W. OXBURY & C. PAULY – “Heisenberg invariant quartics and $SU_C(2)$ for a curve of genus four”, *Math. Proc. Cambridge Philos. Soc.* **125** (1999), no. 2, p. 295–319.
- [Sor96] C. SORGER – “La formule de Verlinde”, *Astérisque* (1996), no. 237, p. Exp. No. 794, 3, 87–114, Séminaire Bourbaki, Vol. 1994/95.
- [Sor00] ———, “Lectures on moduli of principal G -bundles over algebraic curves”, in *School on Algebraic Geometry (Trieste, 1999)*, ICTP Lect. Notes, vol. 1, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000, p. 1–57.