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AN UNCONDITIONALLY STABLE UNCOUPLED SCHEME FOR THE APPROXIMATION OF A TRIPHASIC CAHN-HILLIARD/NAVIER-STOKES MODEL

SEBASTIAN MINJEAUD^{†‡}

Abstract. We propose an original scheme for the time discretization of a triphasic Cahn-Hilliard/Navier-Stokes model. This scheme allows an uncoupled resolution of the discrete Cahn-Hilliard and Navier-Stokes system, is unconditionally stable and preserves, at the discrete level, the main properties of the continuous model. The existence of discrete solution is proved and a convergence study is performed in the case where the densities of the three phases are the same.

Key words. Cahn-Hilliard/Navier-Stokes model; Time discretization; Unconditional stability.

AMS subject classifications. 35K55, 65M60, 65M12, 76T30.

1. Introduction. The complexity of multiphasic flows basically lies in the fact that the time evolution of interfaces, whose position is an unknown of the problem, may lead to their deformation, their break-up or coalescence. Moreover, interfaces obey to physical phenomena where capillar effects play a important role.

The various domains of application, where multiphasic flows are involved, are generally complex; the experimentations and measurements are quite difficult, onerous and most often not very accurate. For instance, in nuclear safety [18], the understanding of interaction between molten corium (lava-like molten mixture of portions of nuclear reactor core) and concrete (last confinement barrier) is a major issue. An approach using numerical direct simulations allows to access to instantaneous quantities at each points of the flows.

Because of their ability to capture interfaces implicitly, diffuse interfaces models are attractive for the numerical simulations of multiphase flows. In this article, we consider a model which couple the Cahn-Hilliard system and the Navier-Stokes equations.

1.1. The Cahn-Hilliard model. In diffuse interfaces theory, the interfaces are assumed to have a non-zero thickness ε (which here a constant parameter of the model). Interfaces are considered as mixing areas and the phase i can be represented by a smooth phase indicator c_i called *order parameter* (which may be understood here as the volumic fraction of the phase i). Thus, the system contains as many unknowns c_i as phases. These unknowns vary between 0 and 1 (values which correspond to pure phases by convention) and are linked by the relationship $\sum_i c_i = 1$.

A complete derivation of this kind of model for diphasic flows is presented in references [1], [2], [13] or [16]. Different extensions have been proposed for the simulations of three-phase flows in references [4], [10] or [15]. We consider in this paper the triphasic Cahn-Hilliard model taken from reference [4]:

$$\begin{cases} \frac{\partial c_i}{\partial t} = \operatorname{div} \left(\frac{M_0(\mathbf{c})}{\Sigma_i} \nabla \mu_i \right), & \text{for } i = 1, 2, 3, \\ \mu_i = f_i^F(\mathbf{c}) - \frac{3}{4} \varepsilon \Sigma_i \Delta c_i, & \text{for } i = 1, 2, 3, \end{cases} \quad (1.1)$$

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where $M_0(\mathbf{c})$ is a diffusion coefficient which is called *mobility* and can eventually depend on $\mathbf{c} = (c_1, c_2, c_3)$. The functions f_i^F are defined by:

$$f_i^F(\mathbf{c}) = \frac{4\Sigma_T}{\varepsilon} \sum_{j \neq i} \left(\frac{1}{\Sigma_j} (\partial_i F(\mathbf{c}) - \partial_j F(\mathbf{c})) \right), \quad (1.2)$$

where Σ_T is given by $\frac{3}{\Sigma_T} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}$. This system is a gradient flow for the following energy functional under the constraint of volume conservation:

$$\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(c_1, c_2, c_3) = \int_{\Omega} \frac{12}{\varepsilon} F(c_1, c_2, c_3) + \frac{3}{8} \varepsilon \sum_{i=1}^3 \Sigma_i |\nabla c_i|^2 dx, \quad (1.3)$$

where Ω denote an open, bounded, connected and smooth domain of \mathbb{R}^d ($d = 2$ or $d = 3$). The “intermediate” unknowns μ_i , called *chemical potentials*, are the functional derivatives of the triphasic Cahn-Hilliard energy (1.3). The rather intricate expression of f_i^F is due to the use of a Lagrange multiplier to ensure the constraint:

$$c_1 + c_2 + c_3 = 1. \quad (1.4)$$

We introduce the hyperplane $\mathcal{S} = \{(c_1, c_2, c_3) \in \mathbb{R}^3; c_1 + c_2 + c_3 = 1\}$ of \mathbb{R}^3 , to simplify notation in the sequel.

The expressions of the triphasic Cahn-Hilliard potential F and of the constant triplet $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3)$ was derived in [4], so that the model can correctly take into account the surface tensions values σ_{12} , σ_{13} and σ_{23} prescribed between the different pairs of phases and so that it is consistent with the two-phase situations: the triphasic model has to exactly reproduce diphasic situations when one of the three phases is not present. The coefficient Σ_i is given as a function of the surface tensions:

$$\Sigma_i = \sigma_{ij} + \sigma_{ik} - \sigma_{jk}, \quad \forall i \in \{1, 2, 3\}. \quad (1.5)$$

and the triphasic potential F is a polynomial function of \mathbf{c} :

$$F(\mathbf{c}) = \sigma_{12} c_1^2 c_2^2 + \sigma_{13} c_1^2 c_3^2 + \sigma_{23} c_2^2 c_3^2 + c_1 c_2 c_3 (\Sigma_1 c_1 + \Sigma_2 c_2 + \Sigma_3 c_3) + c_1^2 c_2^2 c_3^2 \Lambda(\mathbf{c}), \quad \forall \mathbf{c} \in \mathcal{S}, \quad (1.6)$$

where Λ is an arbitrary smooth function of \mathbf{c} .

Note that, in the sequel, we do not assume that the coefficients Σ_i are non negative, so that the model can handle some total spreading situations. However, as it is proved in [4], the following condition is necessary to ensure that the system is wellposed:

$$\Sigma_1 \Sigma_2 + \Sigma_1 \Sigma_3 + \Sigma_2 \Sigma_3 > 0. \quad (1.7)$$

This condition is equivalent to the coercivity of capillary terms and ensure that these terms brings a positive contribution to the free energy $\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}$. This is detailed in the following proposition:

PROPOSITION 1.1 ([4, Prop 2.1]). *Let $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}^3$. There exists $\underline{\Sigma} > 0$ such that, for all $n \geq 1$, for all $(\xi_1, \xi_2, \xi_3) \in (\mathbb{R}^n)^3$ such that $\xi_1 + \xi_2 + \xi_3 = 0$,*

$$\Sigma_1 |\xi_1|^2 + \Sigma_2 |\xi_2|^2 + \Sigma_3 |\xi_3|^2 \geq \underline{\Sigma} (|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2),$$

if and only if the two following conditions are satisfied:

$$\Sigma_1\Sigma_2 + \Sigma_1\Sigma_3 + \Sigma_2\Sigma_3 > 0 \quad \text{et} \quad \Sigma_i + \Sigma_j > 0, \quad \forall i \neq j. \quad (1.8)$$

This proposition (which is not obvious only when one of the coefficient Σ_i is negative) will be usefull in the sequel.

REMARK 1. *Owing to expression (1.5) of coefficients Σ_i , the second part of condition (1.8) is always satisfied and consequently it is sufficient to assume that the condition (1.7) holds, for application of lemma 1.1.*

The existence of weak solutions for problem (1.1) together with initial and Neumann boundary conditions (for order parameters c_i and chemical potentials μ_i) was proved in [4] (see [6] for an alternative proof based a numerical schemes) in 2D and 3D under the following general assumptions:

- the mobility M_0 is a bounded function of $\mathcal{C}^1(\mathbb{R}^3)$ class and there exists three positive constants M_1 , M_2 and M_3 such that:

$$\begin{aligned} \forall \mathbf{c} \in \mathcal{S}, \quad 0 < M_1 \leq M_0(\mathbf{c}) \leq M_2, \\ |DM_0(\mathbf{c})| \leq M_3. \end{aligned} \quad (1.9)$$

- the Cahn-Hilliard potential F is a positive function of $\mathcal{C}^2(\mathbb{R}^3)$ class which satisfies the following assumptions of polynomial growth: there exist a constant $B_1 > 0$ and a real p such that $2 \leq p < +\infty$ if $d = 2$ or $2 \leq p \leq 6$ if $d = 3$, and

$$\begin{aligned} \forall \mathbf{c} \in \mathcal{S}, \quad |F(\mathbf{c})| &\leq B_1 (1 + |\mathbf{c}|^p), \\ |DF(\mathbf{c})| &\leq B_1 (1 + |\mathbf{c}|^{p-1}), \\ |D^2F(\mathbf{c})| &\leq B_1 (1 + |\mathbf{c}|^{p-2}). \end{aligned} \quad (1.10)$$

1.2. Coupling with hydrodynamic. The coupling between the Cahn-Hilliard and Navier-Stokes systems is obtained by:

1. adding a transport term $\mathbf{u} \cdot \nabla c_i$ in the evolution equation of each order parameter c_i , ($i \in \{1, 2, 3\}$), that is the first equation of system (1.1).
2. defining the density and viscosity as smooth function of order parameters \mathbf{c} .
3. adding a capillary force term $\sum_{i=1}^3 \mu_i \nabla c_i$ in the right hand side of the momentum balance (in the Navier-Stokes equations).

Furthermore, we adopt the form of the Navier-Stokes equations, initially proposed in [12] (see also [5] and [16]), which ensures an energy balance without using the equation of mass conservation. It relies on the following inequality:

$$\frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \varrho |\mathbf{u}|^2 dx = \int_{\Omega_t} \left[\sqrt{\varrho} \frac{\partial}{\partial t} (\sqrt{\varrho} \mathbf{u}) + (\varrho \mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{2} \operatorname{div} (\varrho \mathbf{u}) \right] \cdot \mathbf{u} dx,$$

the domain Ω_t being an arbitrary bounded smooth domain moving at the fluid velocity \mathbf{u} [3].

Hence, the triphasic Cahn-Hilliard/Navier-Stokes, we study here, is constituted with following equations:

$$\left\{ \begin{array}{l} \frac{\partial c_i}{\partial t} + \mathbf{u} \cdot \nabla c_i = \operatorname{div} \left(\frac{M_0}{\Sigma_i} \nabla \mu_i \right), \quad \forall i = 1, 2, 3, \\ \mu_i = \frac{4\Sigma_T}{\varepsilon} \sum_{j \neq i} \left(\frac{1}{\Sigma_j} (\partial_i F(\mathbf{c}) - \partial_j F(\mathbf{c})) \right) - \frac{3}{4} \varepsilon \Sigma_i \Delta c_i, \quad \forall i = 1, 2, 3, \\ \sqrt{\varrho(\mathbf{c})} \frac{\partial}{\partial t} \left(\sqrt{\varrho(\mathbf{c})} \mathbf{u} \right) + \left(\varrho(\mathbf{c}) \mathbf{u} \cdot \nabla \right) \mathbf{u} + \frac{\mathbf{u}}{2} \operatorname{div} \left(\varrho(\mathbf{c}) \mathbf{u} \right) \\ \quad - \operatorname{div} \left(2\eta(\mathbf{c}) D(\mathbf{u}) \right) + \nabla p = \sum_{i=1}^3 \mu_i \nabla c_i + \varrho(\mathbf{c}) \mathbf{g}, \\ \operatorname{div} \mathbf{u} = 0, \end{array} \right. \quad (1.11)$$

where the vector \mathbf{g} stands for the gravity; the density and viscosity are defined by:

$$\varrho(\mathbf{c}) = \frac{\sum_{i=1}^3 \varrho_i h_\lambda(c_i - 0.5)}{\sum_{i=1}^3 h_\lambda(c_i - 0.5)} \quad \text{and} \quad \eta(\mathbf{c}) = \frac{\sum_{i=1}^3 \eta_i h_\lambda(c_i - 0.5)}{\sum_{i=1}^3 h_\lambda(c_i - 0.5)}, \quad (1.12)$$

where ϱ_1 (resp. ϱ_2 , resp. ϱ_3) and η_1 (resp. η_2 , resp. η_3) are the values (assumed to be constants) in phase 1 (resp. 2, resp. 3) and the function h_λ ($\lambda = 0.5$) is defined by:

$$h_\lambda(x) = \begin{cases} 0 & \text{if } x < -\lambda, \\ \frac{1}{2} \left(\frac{x}{\lambda} + \frac{1}{\pi} \sin \left(\pi \frac{x}{\lambda} \right) \right) & \text{if } -\lambda \leq x \leq \lambda, \\ 1 & \text{if } x > \lambda. \end{cases} \quad (1.13)$$

We supplement this system with Neumann boundary conditions for order parameters c_i and chemical potentials μ_i , *i.e.* for $i = 1, 2, 3$,

$$\nabla c_i \cdot \mathbf{n} = 0 \text{ and } M_0 \nabla \mu_i \cdot \mathbf{n} = 0, \text{ on } \Gamma, \quad (1.14)$$

and with a homogeneous Dirichlet boundary condition for the velocity, *i.e.*

$$\mathbf{u} = 0 \text{ sur } \Gamma. \quad (1.15)$$

Owing to these boundary conditions (1.14) and (1.15), we introduce the following functional spaces:

$$\begin{aligned} \mathcal{V}^c &= \mathcal{V}^\mu = H^1(\Omega), \\ \mathcal{V}_S^c &= \left\{ \mathbf{c} = (c_1, c_2, c_3) \in (H^1(\Omega))^3; \mathbf{c}(x) \in \mathcal{S} \text{ for almost every } x \in \Omega \right\}, \\ \mathcal{V}^u &= \left(H^1(\Omega) \right)^d, \\ \mathcal{V}_0^u &= \left(H_0^1(\Omega) \right)^d, \\ \mathcal{V}^p &= \{ p \in L^2(\Omega), \int_\Omega p \, dx = 0 \}. \end{aligned}$$

Finally, we assume that the following initial condition holds:

$$c_i(t=0) = c_i^0, \quad \text{and} \quad \mathbf{u}(t=0) = \mathbf{u}^0, \quad (1.16)$$

where $\mathbf{c}^0 = (c_1^0, c_2^0, c_3^0) \in \mathcal{V}_S^c$ and $\mathbf{u}^0 \in \mathcal{V}_0^u$ are given.

1.3. Outline of the article. In section 2, we describe the time and space discretization. The section 3 is then devoted to the two phase case. We then prove in sections 4, 5 and 6, the unconditional stability of the scheme, the existence of solutions and their convergence towards weak solutions of the Cahn-Hilliard/Navier-Stokes system. In particular, we prove the following existence theorem by passing to the limit in the numerical scheme in section 6:

THEOREM 1.2 (Existence of weak solution in the homogeneous case). *Assume the coefficients $(\Sigma_1, \Sigma_2, \Sigma_3)$ satisfy the condition (1.7), the mobility satisfy (1.9), and that the Cahn-Hilliard potential F satisfy the condition (1.10). Assume the densities of the three fluids are equal, i.e. $\varrho_1 = \varrho_2 = \varrho_3 = \varrho_0$, $\varrho_0 \in \mathbb{R}$. Consider the problem (1.11) together with initial condition (1.16) and boundary condition (1.14)-(1.15). Then, there exists a weak solution $(\mathbf{c}, \boldsymbol{\mu}, \mathbf{u}, p)$ on $[0, t_f]$ such that*

$$\mathbf{c} \in L^\infty(0, t_f; (H^1(\Omega))^3) \cap C^0([0, t_f]; (L^q(\Omega))^3), \text{ for all } q < 6,$$

$$\boldsymbol{\mu} \in L^2(0, t_f; (H^1(\Omega))^3),$$

$$\mathbf{u} \in L^\infty(0, t_f; (L^2(\Omega))^3) \cap L^2(0, t_f; (H^1(\Omega))^3),$$

$$\mathbf{c}(t, x) \in \mathcal{S}, \text{ for almost every } (t, x) \in [0, t_f] \times \Omega.$$

2. Discretization of the Cahn-Hilliard/Navier-Stokes model.

2.1. Time discretization. Let $N \in \mathbb{N}^*$ and $t_f \in]0, +\infty[$. The time domain $[0, t_f]$ is uniformly discretized with a fixed time step $\Delta t = \frac{t_f}{N}$; we define $t_n = n\Delta t$, for all $n \in \llbracket 0; N \rrbracket$.

We assume that the function $\mathbf{c}^n \in \mathcal{V}_S^c$ and $\mathbf{u}^n \in \mathcal{V}_0^u$ ($n \in \llbracket 0; N \rrbracket$) are given and we describe the system we have to solve to compute the unknowns $\mathbf{c}^{n+1} \in \mathcal{V}_S^c$ and $\mathbf{u}^{n+1} \in \mathcal{V}_0^u$ at time t^{n+1} .

We first describe, in two distinct paragraphs, the schemes we use to discretize the Cahn-Hilliard and Navier-Stokes equations without considering the coupling terms. For more details on the time discretizations of the triphasic Cahn-Hilliard model, the reader may refer to the article [6] (and reference therein). Several articles in the literature are devoted to the study of discretizations of Navier-Stokes equations: we refer in particular to the articles [12] and [17] which deal with variable density.

We then explain, in the two next paragraphs, the reasoning which led to the discretization of the coupling terms before writing the complete scheme in the last paragraph of this section.

2.1.1. Cahn-Hilliard system. We consider a time discretization of the Cahn-Hilliard system of the form: for $i = 1, 2, 3$,

$$\begin{cases} \frac{c_i^{n+1} - c_i^n}{\Delta t} + \frac{\text{transport}}{\text{term}} = \operatorname{div} \left(\frac{M_0^{n+\alpha}}{\Sigma_i} \nabla \mu_i^{n+1} \right), \\ \mu_i^{n+1} = D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1}) - \frac{3}{4} \varepsilon \Sigma_i \Delta c_i^{n+\beta}. \end{cases}$$

where $c_i^{n+\beta} = (1 - \beta)c_i^n + \beta c_i^{n+1}$, $\beta \in [0.5, 1]$, and $M_0^{n+\alpha} = M_0((1 - \alpha)\mathbf{c}^n + \alpha\mathbf{c}^{n+1})$; the discretization of transport term is postpone to paragraph 2.1.3.

This kind of discretizations was presented and studied in reference [6]. This is out the scope of the article. We assume that the discretization $D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1})$ of the term f_i^F is of the form:

$$D_i^F(\mathbf{a}^n, \mathbf{a}^{n+1}) = \frac{4\Sigma T}{\varepsilon} \sum_{j \neq i} \left(\frac{1}{\Sigma_j} \left(d_i^F(\mathbf{a}^n, \mathbf{a}^{n+1}) - d_j^F(\mathbf{a}^n, \mathbf{a}^{n+1}) \right) \right), \quad \forall (\mathbf{a}^n, \mathbf{a}^{n+1}) \in \mathcal{S}^2,$$

where d_i^F stands for a semi-implicit discretization of $\partial_i F$. We assume that the two following basic properties hold for all $i \in \{1, 2, 3\}$:

$$\forall \mathbf{c} \in \mathcal{S}, \quad D_i^F(\mathbf{c}, \mathbf{c}) = f_i^F(\mathbf{c}). \quad (2.1)$$

$$\begin{aligned} \forall (\mathbf{a}^n, \mathbf{a}^{n+1}) \in \mathcal{S}^2, \quad & |d_i^F(\mathbf{a}^n, \mathbf{a}^{n+1})| \leq B_1 \left(1 + |\mathbf{a}^n|^{p-1} + |\mathbf{a}^{n+1}|^{p-1} \right), \\ & |D(d_i^F(\mathbf{a}^n, \cdot))(\mathbf{a}^{n+1})| \leq B_1 \left(1 + |\mathbf{a}^n|^{p-2} + |\mathbf{a}^{n+1}|^{p-2} \right), \end{aligned} \quad (2.2)$$

the notation D means here the derivative of d_i^F with respect to the second variable. The assumption (2.1) is a consistency assumption and the assumption (2.2) is the counterpart of the polynomial growth assumption (1.10) on F . Many possible choices for the discretization of the term d_i^F was presented in [6]. We consider here the following expression:

$$\begin{aligned} d_i^{F_0}(\mathbf{a}^n, \mathbf{a}^{n+1}) &= \frac{\Sigma_i}{4} [a_i^{n+1} + a_i^n] [(a_j^{n+1} + a_k^{n+1})^2 + (a_j^n + a_k^n)^2] \\ &+ \frac{\Sigma_j}{4} [(a_j^{n+1})^2 + (a_j^n)^2] [a_i^{n+1} + a_k^{n+1} + a_i^n + a_k^n] \\ &+ \frac{\Sigma_k}{4} [(a_k^{n+1})^2 + (a_k^n)^2] [a_i^{n+1} + a_j^{n+1} + a_i^n + a_j^n] \\ &+ \Lambda [a_i^n + a_i^{n+1}] \left[(a_j^n)^2 (a_k^n)^2 + \frac{1}{2} (a_j^{n+1})^2 (a_k^n)^2 + \frac{1}{2} (a_j^n)^2 (a_k^{n+1})^2 + (a_j^{n+1})^2 (a_k^{n+1})^2 \right]. \end{aligned}$$

This scheme was built in order to ensure the following equality:

$$F(\mathbf{a}^{n+1}) - F(\mathbf{a}^n) = \sum_{i=1}^3 d_i^F(\mathbf{a}^n, \mathbf{a}^{n+1})(a_i^{n+1} - a_i^n), \quad \forall (\mathbf{a}^n, \mathbf{a}^{n+1}) \in \mathcal{S}^2,$$

and consequently a discrete energy equality is which obtained by multiplying the first equation of the Cahn-Hilliard system by μ_i^{n+1} , the second one by $c_i^{n+1} - c_i^n$, writing the equality of left hand sides and summing for $i = 1, 2, 3$.

2.1.2. Navier-Stokes system. We now present the time discretization of the momentum balance of the Navier-Stokes system:

$$\begin{aligned} & \underbrace{\sqrt{\varrho(\mathbf{c})} \frac{\partial}{\partial t} (\sqrt{\varrho(\mathbf{c})} \mathbf{u})}_{(1)} + \underbrace{(\varrho(\mathbf{c}) \mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{2} \operatorname{div} (\varrho(\mathbf{c}) \mathbf{u})}_{(2)} \\ & - \underbrace{\operatorname{div} (2\eta(\mathbf{c}) D(\mathbf{u}))}_{(3)} + \nabla p = \sum_{i=1}^3 \mu_i \nabla c_i + \varrho(\mathbf{c}) \mathbf{g}. \end{aligned}$$

We separatly present the discretization of the different terms (1), (2) and (3) involving in the above equation; for each of them, we give their contribution to the energy balance obtained at the discrete level by multiplying the equation by \mathbf{u}^{n+1} .

Term (1): Using the formal equality

$$\sqrt{\varrho} \frac{\partial}{\partial t} (\sqrt{\varrho} \mathbf{u}) = \varrho \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \frac{\partial \varrho}{\partial t} \mathbf{u}, \quad (2.3)$$

we give two possible discretizations of term (1). The first (taken from [12]), which takes advantage of the expression given in the left hand side of 2.3, reads:

$$\frac{\sqrt{\varrho^{n+1}} \sqrt{\varrho^{n+1}} \mathbf{u}^{n+1} - \sqrt{\varrho^n} \mathbf{u}^n}{\Delta t} = \frac{\varrho^{n+1} \mathbf{u}^{n+1} - \sqrt{\varrho^n \varrho^{n+1}} \mathbf{u}^n}{\Delta t}.$$

Its contribution to the energy balance is:

$$\begin{aligned} \int_{\Omega} \sqrt{\varrho^{n+1}} \frac{\sqrt{\varrho^{n+1}} \mathbf{u}^{n+1} - \sqrt{\varrho^n} \mathbf{u}^n}{\Delta t} \cdot \mathbf{u}^{n+1} dx \\ = \frac{1}{2\Delta t} \left[\left| \sqrt{\varrho^{n+1}} \mathbf{u}^{n+1} \right|_{L^2(\Omega)}^2 - \left| \sqrt{\varrho^n} \mathbf{u}^n \right|_{L^2(\Omega)}^2 \right. \\ \left. + \left| \sqrt{\varrho^{n+1}} \mathbf{u}^{n+1} - \sqrt{\varrho^n} \mathbf{u}^n \right|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

The second (taken from [17]), which takes advantage of the expression given in the right hand side of 2.3, reads:

$$\varrho^n \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{1}{2} \frac{\varrho^{n+1} - \varrho^n}{\Delta t} \mathbf{u}^{n+1} = \frac{\frac{\varrho^n + \varrho^{n+1}}{2} \mathbf{u}^{n+1} - \varrho^n \mathbf{u}^n}{\Delta t}.$$

Its contribution to the energy balance is:

$$\begin{aligned} \int_{\Omega} \varrho^n \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \mathbf{u}^{n+1} dx + \int_{\Omega} \frac{1}{2} \frac{\varrho^{n+1} - \varrho^n}{\Delta t} \mathbf{u}^{n+1} \cdot \mathbf{u}^{n+1} dx \\ = \frac{1}{2\Delta t} \left[\left| \sqrt{\varrho^{n+1}} \mathbf{u}^{n+1} \right|_{L^2(\Omega)}^2 - \left| \sqrt{\varrho^n} \mathbf{u}^n \right|_{L^2(\Omega)}^2 \right. \\ \left. + \left| \sqrt{\varrho^n} (\mathbf{u}^{n+1} - \mathbf{u}^n) \right|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

These two discretizations are of the form $\frac{\tilde{\varrho}^{n+1} \mathbf{u}^{n+1} - \tilde{\varrho}^n \mathbf{u}^n}{\Delta t}$ where $\tilde{\varrho}^\ell$ ($\ell = n$ or $n+1$) is either ϱ^ℓ , either some mean of ϱ^n and ϱ^{n+1} . Indeed, the first form corresponds to $\tilde{\varrho}^{n+1} = \varrho^{n+1}$ and $\tilde{\varrho}^n = \sqrt{\varrho^n \varrho^{n+1}}$ (geometric mean of ϱ^n and ϱ^{n+1}); the second one corresponds to $\tilde{\varrho}^{n+1} = \frac{\varrho^n + \varrho^{n+1}}{2}$ (arithmetic mean of ϱ^n and ϱ^{n+1}) and $\tilde{\varrho}^n = \varrho^n$. The discretization of coupling term that we present in the sequel involves the coefficient $\tilde{\varrho}^n$ in the Cahn-Hilliard system. Hence, we use the second form to avoid the presence of a supplementary non-linearity through ϱ^{n+1} (which is equal to $\varrho(\mathbf{c}^{n+1})$).

Term (2): The term (2) is linearized by using an explicit velocity for the transport:

$$(\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} + \frac{\mathbf{u}^{n+1}}{2} \operatorname{div} (\varrho^{n+1} \mathbf{u}^n).$$

Its contribution to the energy balance vanishes. Indeed, for all $\boldsymbol{\nu}^{\mathbf{u}} \in \mathcal{V}_0^{\mathbf{u}}$, we have:

$$\begin{aligned} \int_{\Omega} (\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx + \int_{\Omega} \frac{1}{2} \operatorname{div} (\varrho^{n+1} \mathbf{u}^n) \mathbf{u}^{n+1} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \\ = \frac{1}{2} \left[\int_{\Omega} (\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx - \int_{\Omega} (\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \boldsymbol{\nu}^{\mathbf{u}} \cdot \mathbf{u}^{n+1} dx \right]. \end{aligned}$$

In particular, when we take $\boldsymbol{\nu}^{\mathbf{u}} = \mathbf{u}^{n+1}$, the above term vanishes.

Term (3): We discretize the term (3) with an implicit scheme:

$$-\operatorname{div} (2\eta^{n+1} D\mathbf{u}^{n+1}).$$

Its contribution to the energy balance is:

$$\int_{\Omega} \eta^{n+1} |D\mathbf{u}^{n+1}|^2 dx.$$

Thus, we adopt the following discretization of the Navier-Stokes equations:

$$\begin{cases} \varrho^n \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{1}{2} \frac{\varrho^{n+1} - \varrho^n}{\Delta t} \mathbf{u}^{n+1} + \left(\varrho^{n+1} \mathbf{u}^n \cdot \nabla \right) \mathbf{u}^{n+1} + \frac{\mathbf{u}^{n+1}}{2} \operatorname{div} \left(\varrho^{n+1} \mathbf{u}^n \right) \\ \quad - \operatorname{div} \left(\eta^{n+1} D\mathbf{u}^{n+1} \right) + \nabla p^{n+1} = \underset{\text{capillaire}}{\text{terme de force}} + \varrho^{n+1} \mathbf{g}, \\ \operatorname{div} (\mathbf{u}^{n+1}) = 0, \end{cases}$$

the discretization of the capillary force term is described in the next paragraph.

2.1.3. Coupling term. We give in this paragraph the discretization of coupling terms. That is the transport terms $\mathbf{u} \cdot \nabla c_i$ in the Cahn-Hilliard equations, and the capillary force term in the momentum balance (Navier-Stokes equation). At the continuous level, when writing the energy balance, the contribution of these two terms counterbalance each other. At the discrete level, we saw that the energy balance is obtained, for Cahn-Hilliard system, by multiplying the the transport terms by μ_i^{n+1} before summing up for $i = 1, 2, 3$ and, for the Navier-Stokes equations, by multiplying the capillary force term by \mathbf{u}^{n+1} .

Consequently, it is easy to see that when all the terms mentionned above are discretized with an implicit scheme (*cf* [11] for the diphasic case), *i.e.* $\mathbf{u}^{n+1} \cdot \nabla c_i^{n+1}$ and $\sum_{i=1}^3 \mu_i^{n+1} \nabla c_i^{n+1}$, the balance is also true at the discrete level. However, this discretization introduces a strong coupling between Cahn-Hilliard and Navier-Stokes systems. The discrete system is difficult to solve in practice.

It is possible to uncouple the system (*cf* [14] for the diphasic case, [5] for the triphasic case) by using an explicit velocity (*i.e.* the velocity at time t^n) in the Cahn-Hilliard equation: $\mathbf{u}^n \cdot \nabla c_i^{n+1}$. However, the contributions of the transport terms in the Cahn-Hilliard system and the contribution of the capillary forces in the Navier-Stokes do not counterbalance when writing the discrete energy balance which contains the additional term: $(\mathbf{u}^{n+1} - \mathbf{u}^n) \cdot \sum_{i=1}^3 \mu_i^{n+1} \nabla c_i^{n+1}$. It is difficult to attribute a sign to this term and the scheme stability is obtained only conditionnaly (*cf* [14], assuming for instance that the ratio between the time step and the mesh size is bounded).

We first observe that it is possible to uncouple the resolution of Navier-Stokes system and the taking into account of capillary forces. The taking into account of the capillary forces is performed during a first step which provide an intermediate velocity \mathbf{u}^* used in the Cahn-Hilliard system. The Navier-Stokes system is then solved in a second step. The scheme reads:

(i) Taking into account of capillary forces:

$$\begin{cases} \varrho^n \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + \nabla p^* = \sum_{i=1}^3 \mu_i^{n+1} \nabla c_i^{n+1}, \\ \operatorname{div} (\mathbf{u}^*) = 0. \end{cases}$$

(ii) Cahn-Hilliard system:

$$\begin{cases} \frac{c_i^{n+1} - c_i^n}{\Delta t} + \mathbf{u}^* \cdot \nabla c_i^{n+1} = \operatorname{div} \left(\frac{M_0^{n+\alpha}}{\Sigma_i} \nabla \mu_i^{n+1} \right), \\ \mu_i^{n+1} = D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1}) - \frac{3}{4} \varepsilon \Sigma_i \Delta c_i^{n+\beta}. \end{cases}$$

(iii) Navier-Stokes system:

$$\begin{cases} \varrho^n \frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} + \frac{1}{2} \frac{\varrho^{n+1} - \varrho^n}{\Delta t} \mathbf{u}^{n+1} + (\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} + \frac{\mathbf{u}^{n+1}}{2} \operatorname{div} (\varrho^{n+1} \mathbf{u}^n) \\ \quad - \operatorname{div} (\eta^{n+1} D \mathbf{u}^{n+1}) + \nabla (p^{n+1} - p^*) = \varrho^{n+1} \mathbf{g}, \\ \operatorname{div} (\mathbf{u}^{n+1}) = 0. \end{cases}$$

This discretization is inconditionnaly stable but the systems of step (i) (Darcy problem) is still coupled with the Cahn-Hilliard equations (system (ii)).

We propose to forget for a moment the divergence free constraint imposed to \mathbf{u}^* (and consequently the associated pressure term ∇p^*) in the system of the step (i). This leads to define:

$$\varrho^n \frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \sum_{i=1}^3 c_i^{n+1} \nabla \mu_i^{n+1}.$$

This definition of \mathbf{u}^* is explicit and \mathbf{u}^* can be replaced by its expression in the Cahn-Hilliard system eliminating the coupling with Navier-Stokes equations.

The problem is that \mathbf{u}^* is not divergence free. Nevertheless, note that the property $\mathbf{u}^* \cdot \mathbf{n} = 0$ is still satisfied on Γ . Now, the question is : is it possible to discretize the transport term in Cahn-Hilliard equation in order to preserve its fundamental properties (volume conservation and the fact that the sum of the three order parameters is equal to 1) ? The answer is given in the next paragraph.

2.1.4. Transport term in the Cahn-Hilliard system when the velocity is not divergence free. In this paragraph, we are interesting in the form of the transport term in the Cahn-Hilliard equation when the advection velocity, denoted by \mathbf{u}^* is not divergence free but satisfy the boundary condition $\mathbf{u}^* \cdot \mathbf{n} = 0$ on Γ .

Preserving properties of the Cahn-Hilliard when the advective velocity is not divergence free may be usefull in other contexts. For instance, when using a incremental projection method (cf [8], [20]), the end step velocity is not divergence free.

The transport term may be written in conservative or non conservative form (these two forms are not equivalent since *a priori* $\operatorname{div} (\mathbf{u}^*) \neq 0$):

- non conservative form: $\mathbf{u}^* \cdot \nabla c_i$,
- conservative form: $\operatorname{div} (c_i \mathbf{u}^*)$.

The conservative form ensures volume conservation (since $\mathbf{u}^* \cdot \mathbf{n} = 0$ on Γ). This is not the case for the non conservative form since *a priori* $\int_{\Omega} \mathbf{u}^* \cdot \nabla c_i \, dx \neq 0$. Conversely, when using the conservative form, a necessary condition to ensure that the sum of the three order parameters c_i remains constant equal to 1, is that $\operatorname{div} (\mathbf{u}^*) = 0$. Neither the conservative form nor the non conservative form ensure both volume conservation and the fact that the sum of the three order parameters remains equal to 1.

We propose to use the following formulation:

$$\operatorname{div}((c_i - \alpha_i)\mathbf{u}^*),$$

where α_i is a constant. This formulation allows to ensure the two desired properties if $\sum_{i=1}^3 \alpha_i = 1$. To guarantee the consistency with diphasic model, the constant α_i may be zero when the phase i is not present. In the sequel, we propose to choose:

$$\alpha_i = \int_{\Omega} c_i^0 dx.$$

This formulation allows to use an advective velocity which is not divergence free. The term $-\alpha_i \operatorname{div}(\mathbf{u}^*)$ is added in the Cahn-Hilliard system, its role is to re-equilibrate the values of each order parameters to ensure the fact that their sum remains equal to 1. We prove in section 6 that this term is of order $O(h + \Delta t)$ and so it does not disturb the consistency of the scheme.

Owing to this formulation of the transport term, it seems natural to adopt the following definition for the capillary forces term in the Navier-Stokes equations:

$$-\sum_{i=1}^3 (c_i - \alpha_i) \nabla \mu_i.$$

This is equivalent to changed the definition of the pressure by adding the term $\sum_{i=1}^3 (c_i - \alpha_i) \mu_i$.

2.1.5. Time discretization of the Cahn-Hilliard/Navier-Stokes system.

Finally, the different considerations presented in previous paragraphs lead to propose the following scheme:

PROBLEM 1.

- *Step 1 : resolution of Cahn-Hilliard system*
Find $(\mathbf{c}^{n+1}, \boldsymbol{\mu}^{n+1}) \in (\mathcal{V}^c)^3 \times (\mathcal{V}^\mu)^3$ such that, for $i = 1, 2$ and 3 ,

$$\left\{ \begin{array}{l} \frac{c_i^{n+1} - c_i^n}{\Delta t} + \operatorname{div} \left([c_i^n - \alpha_i] \left[\mathbf{u}^n - \frac{\Delta t}{\varrho^n} \sum_{j=1}^3 (c_j^n - \alpha_j) \nabla \mu_j^{n+1} \right] \right) \\ \hspace{15em} = \operatorname{div} \left(\frac{M_0^{n+\alpha}}{\Sigma_i} \nabla \mu_i^{n+1} \right), \\ \mu_i^{n+1} = D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1}) - \frac{3}{4} \Sigma_i \varepsilon \Delta c_i^{n+\beta}, \end{array} \right. \quad (2.4)$$

with α_j a constant : $\alpha_j = \int_{\Omega} c_j^0 dx$.

- *Step 2 : resolution of Navier-Stokes system*
Find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathcal{V}_0^{\mathbf{u}} \times \mathcal{V}^p$ such that,

$$\left\{ \begin{array}{l} \varrho^n \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{1}{2} \frac{\varrho^{n+1} - \varrho^n}{\Delta t} \mathbf{u}^{n+1} + (\varrho^{n+1} \mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} \\ \hspace{1.5em} + \frac{\mathbf{u}^{n+1}}{2} \operatorname{div}(\varrho^{n+1} \mathbf{u}^n) + \operatorname{div}(2\eta^{n+1} D\mathbf{u}^{n+1}) \\ \hspace{15em} + \nabla p^{n+1} = \varrho^{n+1} \mathbf{g} + \sum_{j=1}^3 (c_j^n - \alpha_j) \nabla \mu_j^{n+1}, \\ \operatorname{div}(\mathbf{u}^{n+1}) = 0, \end{array} \right. \quad (2.5)$$

where $\eta^{n+1} = \eta(\mathbf{c}_h^{n+1})$ and $\varrho^\ell = \varrho(\mathbf{c}^\ell)$, for $\ell = n$ and $\ell = n+1$.

In the above scheme, the discretization of the order parameters is explicit both in the transport term of Cahn-Hilliard equation: $\operatorname{div} \left([c_i^n - \alpha_i] [\mathbf{u}^n - \frac{\Delta t}{\varrho^n} \sum_{j=1}^3 (c_j^n - \alpha_j) \nabla \mu_j^{n+1}] \right)$ and in the capillary forces term of Navier-Stokes equation: $\sum_{j=1}^3 (c_j^n - \alpha_j) \nabla \mu_j^{n+1}$. It is also possible to use an implicit version *i.e.* $\operatorname{div} \left([c_i^{n+1} - \alpha_i] [\mathbf{u}^n - \frac{\Delta t}{\varrho^n} \sum_{j=1}^3 (c_j^{n+1} - \alpha_j) \nabla \mu_j^{n+1}] \right)$ and $\sum_{j=1}^3 (c_j^{n+1} - \alpha_j) \nabla \mu_j^{n+1}$, but it introduces an additional non linearity in the Cahn-Hilliard system.

To finish this section, it is interesting to examine the scheme obtained for the resolution of Cahn-Hilliard system when the velocity \mathbf{u}^n vanishes:

$$\begin{cases} \frac{c_i^{n+1} - c_i^n}{\Delta t} = \operatorname{div} \left(\frac{M_0^{n+\alpha}}{\Sigma_i} \nabla \mu_i^{n+1} \right) \\ \quad + \operatorname{div} \left([c_i^n - \alpha_i] \left[\frac{\Delta t}{\varrho^n} \sum_{j=1}^3 (c_j^n - \alpha_j) \nabla \mu_j^{n+1} \right] \right), \\ \mu_i^{n+1} = D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1}) - \frac{3}{4} \Sigma_i \varepsilon \Delta c_i^{n+\beta}. \end{cases} \quad (2.6)$$

This scheme is different from the one presented in reference [6]: an additional diffusion term (with coefficient of order Δt) appears. It remains possible to prove an energy equality:

$$\begin{aligned} \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}^{n+1}) - \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}^n) + \Delta t \sum_{i=1}^3 \int_{\Omega} \frac{M_0^{n+\alpha}}{\Sigma_i} |\nabla \mu_i^{n+1}|^2 dx \\ + \Delta t^2 \int_{\Omega} \frac{1}{\varrho^n} \left| \sum_{i=1}^3 (c_i^n - \alpha_i) \nabla \mu_i^{n+1} \right|^2 dx + \frac{3}{8} (2\beta - 1) \varepsilon \int_{\Omega} \sum_{i=1}^3 \Sigma_i |\nabla c_i^{n+1} - \nabla c_i^n|^2 dx \\ = \frac{12}{\varepsilon} \int_{\Omega} [F(\mathbf{c}^{n+1}) - F(\mathbf{c}^n) - \mathbf{d}^F(\mathbf{c}^n, \mathbf{c}^{n+1}) \cdot (\mathbf{c}^{n+1} - \mathbf{c}^n)] dx. \end{aligned}$$

Thus, the additional term contributes to the energy decrease. In particular, the scheme we propose allows to correctly compute equilibrium states.

2.2. Space discretization. For the space discretization, we use the Galerkin method and the finite elements method. Let \mathcal{V}_h^c , \mathcal{V}_h^μ , \mathcal{V}_h^u and \mathcal{V}_h^p be finite elements approximation spaces of \mathcal{V}^c , \mathcal{V}^μ , \mathcal{V}^u and \mathcal{V}^p respectively. Since the velocity satisfies homogeneous Dirichlet boundary conditions on Γ , we define the following approximation space:

$$\mathcal{V}_{h,0}^u = \left\{ \nu_h^u \in \mathcal{V}_h^u; \nu_h^u = 0 \text{ sur } \Gamma \right\}.$$

To simplify the notation, we introduce also the following space:

$$\mathcal{V}_{h,\mathcal{S}}^c = \left\{ \mathbf{c}_h = (c_{1h}, c_{2h}, c_{3h}) \in (\mathcal{V}_h^c)^3; \mathbf{c}_h(x) \in \mathcal{S} \text{ for almost every } x \in \Omega \right\}.$$

The general assumptions required on approximation spaces are:

$$\bullet 1 \in \mathcal{V}_h^c \quad \text{et} \quad 1 \in \mathcal{V}_h^\mu, \quad (2.7)$$

$$\bullet \forall \nu^c \in \mathcal{V}^c, \quad \inf_{\nu_h^c \in \mathcal{V}_h^c} |\nu^c - \nu_h^c|_{H^1(\Omega)} \xrightarrow{h \rightarrow 0} 0, \quad (2.8)$$

$$\bullet \forall \nu^\mu \in \mathcal{V}^\mu, \quad \inf_{\nu_h^\mu \in \mathcal{V}_h^\mu} |\nu^\mu - \nu_h^\mu|_{H^1(\Omega)} \xrightarrow{h \rightarrow 0} 0, \quad (2.9)$$

$$\bullet \forall \nu^\mathbf{u} \in \mathcal{V}_0^\mathbf{u}, \quad \inf_{\nu_h^\mathbf{u} \in \mathcal{V}_{h,0}^\mathbf{u}} |\nu^\mathbf{u} - \nu_h^\mathbf{u}|_{(H^1(\Omega))^d} \xrightarrow{h \rightarrow 0} 0, \quad (2.10)$$

$$\bullet \forall \nu^p \in \mathcal{V}^p, \quad \inf_{\nu_h^p \in \mathcal{V}_h^p} |\nu^p - \nu_h^p|_{(L^2(\Omega))^d} \xrightarrow{h \rightarrow 0} 0, \quad (2.11)$$

• There exists a positive constant β (independent of h) such that

$$\inf_{\nu_h^p \in \mathcal{V}_h^p} \sup_{\nu_h^\mathbf{u} \in \mathcal{V}_{h,0}^\mathbf{u}} \frac{\int_{\Omega} \nu_h^p \operatorname{div} \nu_h^\mathbf{u} dx}{|\nu_h^p|_{L^2(\Omega)} |\nu_h^\mathbf{u}|_{(H^1(\Omega))^d}} \geq \beta, \quad (2.12)$$

• there exists a positive constant C independent of h such that

$$\forall \nu^c \in \mathcal{V}^c, \quad \left| \Pi_0^{\nu^c}(\nu^c) \right|_{H^1(\Omega)} \leq |\nu^c|_{H^1(\Omega)}, \quad (2.13)$$

$$\forall \nu^\mu \in \mathcal{V}^\mu, \quad \left| \Pi_0^{\nu^\mu}(\nu^\mu) \right|_{H^1(\Omega)} \leq C |\nu^\mu|_{H^1(\Omega)}, \quad (2.14)$$

where $\Pi_0^{\nu_h}$ is the $L^2(\Omega)$ projection on \mathcal{V}_h ,

• there exists a function C_{inv} of h such that

$$\forall \nu_h^c \in \mathcal{V}_h^c, \quad |\nu_h^c|_{L^\infty(\Omega)}^2 \leq C_{\text{inv}}(h) |\nu_h^c|_{H^1(\Omega)}^2, \quad (2.15)$$

$$\bullet \mathcal{V}_h^c \subset \mathcal{V}_h^\mu. \quad (2.16)$$

REMARK 2. In addition, to standard assumptions on the approximation spaces, we assume that the approximation space for order parameters satisfies the inverse inequality (2.15). This property is (for instance) satisfied when the mesh family is quasi-uniform and the approximation spaces are associated to corresponding Lagrange finite elements; in this case we can choose $C_{\text{inv}}(h) = C(1 + \ln(h))$ if $d = 2$ and $C_{\text{inv}}(h) = Ch^{-1}$ if $d = 3$ where C is a constant which only depends on the mesh regularity (cf [7, 4.5.11 (p. 112) et 4.9.2 (p. 123)]). Furthermore, it is necessary that approximation spaces for velocity and pressure satisfy the so-called inf-sup condition.

We begin with the definition of discrete functions $\mathbf{c}_h^0 \in \mathcal{V}_S^c$ and $\mathbf{u}_h^0 \in \mathcal{V}_{h,0}^\mathbf{u}$ at the initial time:

$$\mathbf{c}_h^0(x) \in \mathcal{S}, \quad \forall h > 0, \quad \text{for almost every } x \in \Omega \quad \text{and} \quad |\mathbf{c}_h^0 - \mathbf{c}^0|_{(H^1(\Omega))^3} \xrightarrow{h \rightarrow 0} 0, \quad (2.17)$$

$$|\mathbf{u}_h^0 - \mathbf{u}^0|_{(H^1(\Omega))^d} \xrightarrow{h \rightarrow 0} 0. \quad (2.18)$$

These discrete functions \mathbf{c}_h^0 and \mathbf{u}_h^0 can be obtained from initial conditions \mathbf{c}^0 and \mathbf{u}^0 by $H^1(\Omega)$ projection, or as it is the case in practice, by finite elements interpolation provided that \mathbf{c}_i^0 and \mathbf{u}^0 are smooth enough.

Assume now that $\mathbf{c}_h^n \in \mathcal{V}_S^c$ and $\mathbf{u}_h^n \in \mathcal{V}_{h,0}^\mathbf{u}$ are given, the Galerkin approximation of problem 1 reads:

PROBLEM 2 (Formulation with three order parameters).

• Step 1: resolution of Cahn-Hilliard system

Find $(\mathbf{c}_h^{n+1}, \boldsymbol{\mu}_h^{n+1}) \in (\mathcal{V}_{h,S}^c)^3 \times (\mathcal{V}_h^\mu)^3$ such that $\forall \nu_h^c \in \mathcal{V}_h^c, \forall \nu_h^\mu \in \mathcal{V}_h^\mu$, we have, for $i = 1, 2$ and 3 ,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx \\ - \int_{\Omega} [c_{ih}^n - \alpha_{ih}] \left[\mathbf{u}_h^n - \frac{\Delta t}{\varrho_h^n} \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1} \right] \cdot \nabla \nu_h^\mu dx \\ \hspace{15em} = - \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx, \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} D_i^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_h^c dx + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \nabla \nu_h^c dx, \end{array} \right. \quad (2.19)$$

where α_{jh} is the constant defined by $\alpha_{jh} = \int_{\Omega} c_{jh}^0 dx$.

- *Step 2: resolution of Navier-Stokes equations*

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathcal{V}_{h,0}^u \times \mathcal{V}_h^p$ such that $\forall \boldsymbol{\nu}_h^u \in \mathcal{V}_{h,0}^u, \forall \nu_h^p \in \mathcal{V}_h^p$,

$$\left\{ \begin{array}{l} \int_{\Omega} \varrho_h^n \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \cdot \boldsymbol{\nu}_h^u dx + \frac{1}{2} \int_{\Omega} \frac{\varrho_h^{n+1} - \varrho_h^n}{\Delta t} \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^u dx \\ + \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^u dx - \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \boldsymbol{\nu}_h^u \cdot \mathbf{u}_h^{n+1} dx \\ + \int_{\Omega} 2\eta_h^{n+1} D\mathbf{u}_h^{n+1} : D\boldsymbol{\nu}_h^u dx - \int_{\Omega} p_h^{n+1} \operatorname{div}(\boldsymbol{\nu}_h^u) dx \\ \hspace{10em} = \int_{\Omega} \varrho_h^{n+1} \mathbf{g} \cdot \boldsymbol{\nu}_h^u dx - \int_{\Omega} \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1} \cdot \boldsymbol{\nu}_h^u dx, \\ \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0, \end{array} \right. \quad (2.20)$$

where $\eta_h^{n+1} = \eta(\mathbf{c}_h^{n+1})$ and $\varrho_h^\ell = \varrho(c_h^\ell)$, for $\ell = n$ and $\ell = n+1$.

2.3. Equivalence with Cahn-Hilliard system with two equations. In practice, for the resolution of Cahn-Hilliard system, we only solve the equations satisfied by (c_1, c_2, μ_1, μ_2) . Indeed, problem 2 is equivalent to the following one:

PROBLEM 3 (Formulation with two order parameters).

- *Step 1: resolution of Cahn-Hilliard system*

Find $(c_{1h}^{n+1}, c_{2h}^{n+1}, \mu_{1h}^{n+1}, \mu_{2h}^{n+1}) \in (\mathcal{V}_h^c)^2 \times (\mathcal{V}_h^\mu)^2$ such that $\forall \nu_h^c \in \mathcal{V}_h^c, \forall \nu_h^\mu \in \mathcal{V}_h^\mu$, for $i = 1$ and 2 ,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx \\ - \int_{\Omega} [c_{ih}^n - \alpha_i] \left[\mathbf{u}_h^n - \frac{\Delta t}{\varrho_h^n} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1} \right] \cdot \nabla \nu_h^\mu dx \\ \hspace{15em} = - \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx, \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} D_i^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_h^c dx + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \nabla \nu_h^c dx. \end{array} \right. \quad (2.21)$$

with $\mathbf{c}_h^{n+1} = (c_{1h}^{n+1}, c_{2h}^{n+1}, 1 - c_{1h}^{n+1} - c_{2h}^{n+1})$.

- We then define:

$$c_{3h}^{n+1} = 1 - c_{1h}^{n+1} - c_{2h}^{n+1} \quad \text{and} \quad \mu_{3h}^{n+1} = - \left(\frac{\Sigma_3}{\Sigma_1} \mu_{1h}^{n+1} + \frac{\Sigma_3}{\Sigma_2} \mu_{2h}^{n+1} \right). \quad (2.22)$$

- Step 2: the resolution of Navier-Stokes system remains unchanged; (cf problem 2).

REMARK 3. In system where only the unknowns $(c_{1h}^{n+1}, \mu_{1h}^{n+1}, c_{2h}^{n+1}, \mu_{2h}^{n+1})$ are present, the notation \mathbf{c}_h^{n+1} stands for the vector $(c_{1h}^{n+1}, c_{2h}^{n+1}, 1 - c_{1h}^{n+1} - c_{2h}^{n+1})$.

The resolution of problems 2 and 3 are equivalent. The proof of this result which is stated in theorem 2.1 below is very closed of the proof theorem 2.6 in the article [6]. For the sake of brevity, we skip this proof.

THEOREM 2.1. Problem 2 is equivalent to problem 3. In particular, each solution $(\mathbf{c}_h^{n+1}, \boldsymbol{\mu}_h^{n+1}, \mathbf{u}^{n+1}, p^{n+1})$ of problem 2 satisfies

$$\sum_{i=1}^3 c_{ih}^{n+1} = 1 \quad \text{and} \quad \sum_{i=1}^3 \frac{\mu_{ih}^{n+1}}{\Sigma_i} = 0. \quad (2.23)$$

3. Corresponding scheme in the diphasic case. Consider a system with two components (denoted below with indexes 1 and 2) and assume that the evolution of order parameters c_i , ($i = 1, 2$) and chemical potential $\tilde{\mu}_i$, ($i = 1, 2$) associated to these two phases are governed by the diphasic Cahn-Hilliard model:

$$\begin{cases} \frac{\partial c_i}{\partial t} = \operatorname{div} (M(c_1, c_2) \nabla \tilde{\mu}_i), & \text{for } i = 1, 2, \\ \tilde{\mu}_i = \frac{12}{\varepsilon} \sigma_{12} f'(c_i) - \frac{3}{2} \varepsilon \sigma_{12} \Delta c_i & \text{for } i = 1, 2, \end{cases} \quad (3.1)$$

where ε is the interface thickness, $M(c_1, c_2)$ the mobility and σ_{12} the surface tension between the two phases. The unknowns are linked by the following relationship: $c_1 + c_2 = 1$ and $\tilde{\mu}_1 + \tilde{\mu}_2 = 0$.

The algebraic consistency of the triphasic model ensures that the triplet $(c_1, c_2 = 1 - c_1, c_3 = 0)$ is a particular solution of the triphasic Cahn-Hilliard model (1.1) (with $M_0(\mathbf{c}) = 2\sigma_{12}M(c_1, c_2)$) and for all choices of the values of the surface tensions σ_{13} and σ_{23} involving the third component. In this case, the triphasic chemical potentials are given by $\mu_i = \frac{\Sigma_i}{2\sigma_{12}} \tilde{\mu}_i$ for $i = 1, 2$ and $\mu_3 = 0$.

We give here the equivalent of this result at the discrete level. Let us consider the following discretization of the diphasic model:

PROBLEM 4.

- Step 1: resolution of Cahn-Hilliard system

Find $(c_{1h}^{n+1}, c_{2h}^{n+1}, \tilde{\mu}_{1h}^{n+1}, \tilde{\mu}_{2h}^{n+1}) \in (\mathcal{V}_h^c)^2 \times (\mathcal{V}_h^\mu)^2$ such that $\forall \nu_h^c \in \mathcal{V}_h^c, \forall \nu_h^\mu \in \mathcal{V}_h^\mu$, for $i = 1$ and 2, we have,

$$\begin{cases} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx - \int_{\Omega} (c_{ih}^n - \alpha_i) \mathbf{u}_h^n \cdot \nabla \nu_h^\mu dx \\ \quad = - \int_{\Omega} \left[M + \frac{\Delta t}{\varrho_h^n} (c_{ih}^n - \alpha_i)^2 \right] \nabla \tilde{\mu}_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx, \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} D_i^F \left((c_{1h}^n, c_{2h}^n, 0), (c_{1h}^{n+1}, c_{2h}^{n+1}, 0) \right) \nu_h^c dx \\ \quad + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \nabla \nu_h^c dx. \end{cases} \quad (3.2)$$

- *Step 2: resolution of Navier-Stokes system*

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathcal{V}_{h,0}^{\mathbf{u}} \times \mathcal{V}_h^p$ such that $\forall \boldsymbol{\nu}_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}, \forall \nu_h^p \in \mathcal{V}_h^p$,

$$\left\{ \begin{aligned} & \int_{\Omega} \varrho_h^n \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx + \frac{1}{2} \int_{\Omega} \frac{\varrho_h^{n+1} - \varrho_h^n}{\Delta t} \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \\ & + \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx - \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \boldsymbol{\nu}_h^{\mathbf{u}} \cdot \mathbf{u}_h^{n+1} dx \\ & + \int_{\Omega} 2\eta_h^{n+1} D\mathbf{u}_h^{n+1} : D\boldsymbol{\nu}_h^{\mathbf{u}} dx - \int_{\Omega} p_h^{n+1} \operatorname{div}(\boldsymbol{\nu}_h^{\mathbf{u}}) dx \\ & = \int_{\Omega} \varrho_h^{n+1} \mathbf{g} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx - \frac{1}{2} \int_{\Omega} \sum_{j=1}^2 (c_{jh}^n - \alpha_{jh}) \nabla \tilde{\mu}_{jh}^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx, \\ & \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0, \end{aligned} \right. \quad (3.3)$$

where $\eta_h^{n+1} = \eta(\mathbf{c}_h^{n+1})$ and $\varrho_h^\ell = \varrho(\mathbf{c}_h^\ell)$, for $\ell = n$ and $\ell = n+1$.

In diphasic case, the essential difference between the scheme we propose and the more standard scheme with an explicit velocity in Cahn-Hilliard systems (cf [14]) is the additional term $\frac{\Delta t}{\varrho_h^n} (c_{ih}^n - \alpha_i)^2$ in the mobility coefficient. This term can be interpreted as an additional diffusion (proportional to Δt) which aimed at stabilizing the scheme proposed in [14].

PROPOSITION 3.1. Defining $M_0 = 2\sigma_{12}M$, $\mu_{ih}^{n+1} = \frac{\Sigma_i}{2\sigma_{12}} \tilde{\mu}_{ih}^{n+1}$ for $i = 1, 2$ and $\mu_{3h}^{n+1} = 0$, we have the following results:

- if $((c_{1h}^{n+1}, \tilde{\mu}_{1h}^{n+1}), (c_{2h}^{n+1}, \tilde{\mu}_{2h}^{n+1}))$ is a solution of the diphasic problem 4 then $((c_{1h}^{n+1}, \mu_{1h}^{n+1}), (c_{2h}^{n+1}, \mu_{2h}^{n+1}), (0, 0))$ is the solution of the triphasic problem 2.
- Conversely, if $((c_{1h}^{n+1}, \mu_{1h}^{n+1}), (c_{2h}^{n+1}, \mu_{2h}^{n+1}), (0, 0))$ is a solution of triphasic problem 2 then $((c_{1h}^{n+1}, \tilde{\mu}_{1h}^{n+1}), (c_{2h}^{n+1}, \tilde{\mu}_{2h}^{n+1}))$ is a solution of diphasic problem 4.

4. Inconditionnal stability of the scheme. We prove in this section the energy equality which ensures the unconditional stability of the scheme.

PROPOSITION 4.1 (Discrete energy equality). Let $\mathbf{c}_h^n \in \mathcal{V}_{h,S}^c$ and $\mathbf{u}_h^n \in \mathcal{V}_{h,0}^{\mathbf{u}}$. Assume that there exists a solution $(\mathbf{c}_h^{n+1}, \boldsymbol{\mu}_h^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1})$ of problem 2. Then, we have the following equality:

$$\begin{aligned} & \left[\mathcal{F}_{\Sigma,\varepsilon}^{triph}(\mathbf{c}_h^{n+1}) + \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} |\mathbf{u}_h^{n+1}|^2 dx \right] - \left[\mathcal{F}_{\Sigma,\varepsilon}^{triph}(\mathbf{c}_h^n) + \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}_h^n|^2 dx \right] \\ & + \Delta t \sum_{i=1}^3 \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} |\nabla \mu_{ih}^{n+1}|^2 dx + \Delta t \int_{\Omega} 2\eta_h^{n+1} |D\mathbf{u}_h^{n+1}|^2 dx \\ & + \frac{3}{8}(2\beta - 1)\varepsilon \int_{\Omega} \sum_{i=1}^3 \Sigma_i |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx + \frac{1}{2} \int_{\Omega} \varrho_h^n [|\mathbf{u}_h^{n+1} - \mathbf{u}^*|^2 + |\mathbf{u}^* - \mathbf{u}_h^n|^2] dx \\ & = \frac{12}{\varepsilon} \int_{\Omega} [F(\mathbf{c}_h^{n+1}) - F(\mathbf{c}_h^n) - \mathbf{d}^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \cdot (\mathbf{c}_h^{n+1} - \mathbf{c}_h^n)] dx + \Delta t \int_{\Omega} \varrho_h^{n+1} \mathbf{g} \cdot \mathbf{u}_h^{n+1} dx, \end{aligned} \quad (4.1)$$

where $\mathbf{d}^F(\cdot, \cdot)$ is the vector $(d_i^F(\cdot, \cdot))_{i=1,2,3}$ and

$$\mathbf{u}^* = \mathbf{u}_h^n - \frac{\Delta t}{\varrho_h^n} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1}. \quad (4.2)$$

Proof. The key point of the proof is the following observation: the Cahn-Hilliard and Navier-Stokes system can be re-written using the function \mathbf{u}^* defined by (4.2). Then, standard estimations for Cahn-Hilliard and Navier-Stokes systems are done (step 1 and 3) and an estimation on the L^2 norm of \mathbf{u}^* gives the conclusion (step 2).

Step 1: Owing to the definition (4.2) of the function \mathbf{u}^* , we observe that the system 2.4 can be rewritten as follows:

$$\begin{cases} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx - \int_{\Omega} [c_{ih}^n - \alpha_i] \mathbf{u}^* \cdot \nabla \nu_h^\mu dx = - \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx, \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} D_i^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_h^c dx + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \nabla \nu_h^c dx. \end{cases}$$

We take $\nu_h^\mu = \mu_{ih}^{n+1}$ and $\nu_h^c = \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t}$ as test functions in this system. After some standard calculation (see [6]), this yields:

$$\begin{aligned} & \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^{n+1}) - \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^n) + \Delta t \sum_{i=1}^3 \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} |\nabla \mu_{ih}^{n+1}|^2 dx \\ & + \frac{3}{8} (2\beta - 1) \varepsilon \int_{\Omega} \sum_{i=1}^3 \Sigma_i |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx = \Delta t \int_{\Omega} \mathbf{u}^* \cdot \sum_{i=1}^3 (c_{ih}^n - \alpha_i) \nabla \mu_{ih}^{n+1} dx \\ & + \frac{12}{\varepsilon} \int_{\Omega} \left[F(\mathbf{c}_h^{n+1}) - F(\mathbf{c}_h^n) - \mathbf{d}^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \cdot (\mathbf{c}_h^{n+1} - \mathbf{c}_h^n) \right] dx. \end{aligned} \quad (4.3)$$

Step 2: It is possible to obtain an estimation of the first term of the right hand side of the previous equality. By definition of \mathbf{u}^* , we have:

$$\sqrt{\varrho_h^n} \mathbf{u}^* = \sqrt{\varrho_h^n} \mathbf{u}_h^n - \frac{\Delta t}{\sqrt{\varrho_h^n}} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1}.$$

Multiplying by $\sqrt{\varrho_h^n} \mathbf{u}^*$, and integrating on Ω , yields:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}^*|^2 dx - \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}_h^n|^2 dx + \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}^* - \mathbf{u}_h^n|^2 dx \\ & = -\Delta t \int_{\Omega} \mathbf{u}^* \cdot \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1} dx. \end{aligned} \quad (4.4)$$

Step 3: The system (2.5) can also be re-written using the function \mathbf{u}^* :

$$\begin{cases} \int_{\Omega} \varrho_h^n \frac{\mathbf{u}_h^{n+1} - \mathbf{u}^*}{\Delta t} \cdot \nu_h^u dx + \frac{1}{2} \int_{\Omega} \mathbf{u}_h^{n+1} \frac{\varrho_h^{n+1} - \varrho_h^n}{\Delta t} \cdot \nu_h^u dx \\ \quad + \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \nu_h^u dx - \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \nu_h^u \cdot \mathbf{u}_h^{n+1} dx \\ \quad + \int_{\Omega} 2\eta_h^{n+1} D\mathbf{u}_h^{n+1} : D\nu_h^u dx - \int_{\Omega} p_h^{n+1} \operatorname{div}(\nu_h^u) dx = \int_{\Omega} \varrho_h^{n+1} \mathbf{g} \cdot \nu_h^u dx, \\ \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0. \end{cases}$$

We take $\nu_h^{\mathbf{u}} = \mathbf{u}_h^{n+1}$ et $\nu_h^p = p_h^{n+1}$ as test functions in this system. This yields:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} |\mathbf{u}_h^{n+1}|^2 dx - \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}^*|^2 dx + \frac{1}{2} \int_{\Omega} \varrho_h^n |\mathbf{u}_h^{n+1} - \mathbf{u}^*|^2 dx \\ & + \Delta t \int_{\Omega} 2\eta_h^{n+1} |D\mathbf{u}_h^{n+1}|^2 dx = \Delta t \int_{\Omega} \varrho_h^{n+1} \mathbf{g} \cdot \mathbf{u}_h^{n+1} dx. \end{aligned} \quad (4.5)$$

The conclusion is obtained by summing up the equations (4.3), (4.4) and (4.5). \square

REMARK 4. *An important difference with the work presented in [14] in the case of homogeneous diphasic Cahn-Hilliard/Navier-Stokes model is that no condition is required on the time step to ensure the stability.*

5. Existence of solution for discrete problem. We prove in this section the existence of solutions for the non linear discrete problem 2.

THEOREM 5.1. *Given $\mathbf{c}_h^n \in \mathcal{V}_S^c$, $\mathbf{u}_h^n \in \mathcal{V}_0^{\mathbf{u}}$, we assume that*

- *the coefficients $(\Sigma_1, \Sigma_2, \Sigma_3)$ satisfy (1.7), the mobility satisfy (1.9), and the Cahn-Hilliard potential F satisfy (1.10),*
- *the discretization of non linear terms \mathbf{d}^F satisfy (2.2) and the following property: there exists $K_1^{\mathbf{c}_h^n} > 0$ (eventually depending on \mathbf{c}_h^n) such that*

$$\int_{\Omega} [F(\mathbf{a}_h^{n+1}) - F(\mathbf{c}_h^n) - \mathbf{d}^F(\mathbf{c}_h^n, \mathbf{a}_h^{n+1}) \cdot (\mathbf{a}_h^{n+1} - \mathbf{c}_h^n)] dx \leq K_1^{\mathbf{c}_h^n}, \quad \forall \mathbf{a}_h^{n+1} \in \mathcal{V}_S^c. \quad (5.1)$$

Then, there exists at least one solution $(\mathbf{c}_h^{n+1}, \boldsymbol{\mu}_h^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1})$ to the problem 2.

The proof relies on the following lemma from the topological degree theory [9].

LEMMA 5.2 (Topological degree). *Let W be a finite dimensional vector space, G be a continuous function from W to W . Assume that there exists a continuous function H from $W \times [0; 1]$ to W satisfying*

- (i) *$H(\cdot, 1) = G$ and $H(\cdot, 0)$ is affine,*
- (ii) *$\exists R > 0$ s.t. $\forall (w, \delta) \in W \times [0; 1]$, if $H(w, \delta) = 0$ then $|w|_W < R$,*
- (iii) *the equation $H(w, 0) = 0$ has a solution $w \in W$,*

Then there exists at least one solution $w \in W$ such that $G(w) = 0$ and $|w|_W < R$.

The idea is to link the non linear discrete problem to a more simple (linear) problem (using an homotopy, function H of lemma 5.2) for which we are able to prove existence of solutions (assumption (ii) of lemma 5.2). The topological degree theory allows to deduce the existence of solutions for the non linear problem from *a priori* estimates which are in our case deduced from the energy equality (4.1) proved in proposition 4.1.

Proof. Proof of theorem 5.1. We first reformulate the problem 2 to enter in the framework of lemma 5.2, before validating the assumptions (i), (ii) and (iii) of this lemma.

Reformulation of the problem

Let W be a finite dimensional vector space $(\mathcal{V}^c)^2 \times (\mathcal{V}_h^{\mu})^2 \times \mathcal{V}_{h,0}^{\mathbf{u}} \times \mathcal{V}^p$. We define the following norm on W : for all $w = (c_{1h}, c_{2h}, \mu_{1h}, \mu_{2h}, \mathbf{u}_h, p_h) \in W$,

$$|w|_W^2 = |c_{1h}|_{H^1(\Omega)}^2 + |c_{2h}|_{H^1(\Omega)}^2 + |\mu_{1h}|_{H^1(\Omega)}^2 + |\mu_{2h}|_{H^1(\Omega)}^2 + |\mathbf{u}_h|_{(H^1(\Omega))^d}^2 + |p_h|_{L^2(\Omega)}^2,$$

and we introduce the function H such that

$$H : W \times [0; 1] \rightarrow W$$

$$(w^{n+1}, \delta) = (c_{1h}^{n+1}, c_{2h}^{n+1}, \mu_{1h}^{n+1}, \mu_{2h}^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1}, \delta) \mapsto (\mathcal{R}_{\delta}^{\mu_1}, \mathcal{R}_{\delta}^{c_1}, \mathcal{R}_{\delta}^{\mu_2}, \mathcal{R}_{\delta}^{c_2}, \mathcal{R}_{\delta}^{\mathbf{u}}, \mathcal{R}_{\delta}^p)$$

where $\mathcal{R}_\delta^{c_1}$ and $\mathcal{R}_\delta^{c_2}$, (resp. $\mathcal{R}_\delta^{\mu_1}$ and $\mathcal{R}_\delta^{\mu_2}$, resp. $\mathcal{R}_\delta^{\mathbf{u}}$, resp. \mathcal{R}_δ^p) are defined with their coordinates in the finite elements basis $(\nu_I^c)_{I \in \llbracket 1; \dim(\mathcal{V}_h^c) \rrbracket}$ (resp. $(\nu_I^\mu)_{I \in \llbracket 1; \dim(\mathcal{V}_h^\mu) \rrbracket}$, resp. $(\nu_I^{\mathbf{u}})_{I \in \llbracket 1; \dim(\mathcal{V}_{h,0}^{\mathbf{u}}) \rrbracket}$, resp. $(\nu_I^p)_{I \in \llbracket 1; \dim(\mathcal{V}_h^p) \rrbracket}$) of \mathcal{V}_h^c (resp. \mathcal{V}_h^μ , resp. $\mathcal{V}_{h,0}^{\mathbf{u}}$, resp. \mathcal{V}_h^p):

$$\forall I \in \llbracket 1; \dim(\mathcal{V}_h^\mu) \rrbracket,$$

$$\begin{aligned} (\mathcal{R}_\delta^{\mu_i})_I &= \int_\Omega \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_I^\mu dx + \int_\Omega \frac{M_{0h\delta}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_I^\mu dx \\ &\quad - \delta \int_\Omega \left[c_{ih}^n - \alpha_{ih} \right] \left[\mathbf{u}_h^n - \frac{\Delta t}{\varrho_{h\delta}^n} \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1} \right] \cdot \nabla \nu_I^\mu dx, \end{aligned}$$

$$\forall I \in \llbracket 1; \dim(\mathcal{V}_h^c) \rrbracket,$$

$$(\mathcal{R}_\delta^{c_i})_I = \int_\Omega \mu_{ih}^{n+1} \nu_I^c dx - \int_\Omega \delta D_i(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_I^c dx - \int_\Omega \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \cdot \nabla \nu_I^c dx,$$

$$\forall I \in \llbracket 1; \dim(\mathcal{V}_{h,0}^{\mathbf{u}}) \rrbracket,$$

$$\begin{aligned} (\mathcal{R}_\delta^{\mathbf{u}})_I &= \int_\Omega \varrho_{h\delta}^n \frac{\mathbf{u}_h^{n+1} - \delta \mathbf{u}_h^n}{\Delta t} \cdot \nu_I^{\mathbf{u}} dx + \frac{1}{2} \int_\Omega \frac{\varrho_{h\delta}^{n+1} - \varrho_{h\delta}^n}{\Delta t} \mathbf{u}_h^{n+1} \cdot \nu_I^{\mathbf{u}} dx \\ &\quad + \frac{\delta}{2} \int_\Omega \varrho_{h\delta}^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \nu_I^{\mathbf{u}} dx - \frac{\delta}{2} \int_\Omega \varrho_{h\delta}^{n+1} (\mathbf{u}_h^n \cdot \nabla) \nu_I^{\mathbf{u}} \cdot \mathbf{u}_h^{n+1} dx \\ &\quad + \int_\Omega 2\eta_{h\delta}^{n+1} D\mathbf{u}_h^{n+1} : D\nu_I^{\mathbf{u}} dx - \int_\Omega p_h^{n+1} \operatorname{div}(\nu_I^{\mathbf{u}}) dx \\ &\quad - \int_\Omega \varrho_{h\delta}^{n+1} \mathbf{g} \cdot \nu_I^{\mathbf{u}} dx + \delta \int_\Omega \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1} \cdot \nu_I^{\mathbf{u}} dx, \end{aligned}$$

$$\forall I \in \llbracket 1; \dim(\mathcal{V}_h^p) \rrbracket, \quad (\mathcal{R}_\delta^p)_I = \int_\Omega \nu_I^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx,$$

with $M_{0h\delta}^{n+\alpha} = M_0((1 - \delta\alpha)\mathbf{c}_h^n + \delta\alpha\mathbf{c}_h^{n+1})$, $\varrho_{h\delta}^\ell = \varrho((1 - \delta)\mathbf{c}_h^{\ell-1} + \delta\mathbf{c}_h^\ell)$ for $\ell = n$ or $\ell = n + 1$ and $\eta_{h\delta}^{n+1} = \eta((1 - \delta)\mathbf{c}_h^n + \delta\mathbf{c}_h^{n+1})$. The function G is defined by:

$$\begin{aligned} G : W &\rightarrow W \\ w &\mapsto H(w, 1) \end{aligned}$$

The problem “Find w^{n+1} such that $G(w^{n+1}) = 0$ ” is equivalent (by definition of the function H) to the problem 2. To prove the theorem, we are going to prove that the functions H et G satisfy the assumptions of lemma 5.2. The continuity of the function H is obtained using the continuity of the different non linear functions (D_i^F , ϱ and η) and the Lebesgue’s theorem. The function $H(\cdot, 0)$ is clearly affine by construction.

Validation of assumption (ii) of lemma 5.2

Let $(w^{n+1}, \delta) \in W \times [0; 1]$ such that $H(w^{n+1}, \delta) = 0$. Note that $H(w^{n+1}, \delta) = 0$ is equivalent to say that $w^{n+1} = (c_{1h}^{n+1}, c_{2h}^{n+1}, \mu_{1h}^{n+1}, \mu_{2h}^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1})$ is a solution of a problem closely related to the problem 2. Thus, we can perform the same calculations as in the proof of proposition 4.1. Indeed, defining

$$\mathbf{u}_\delta^* = \delta \mathbf{u}_h^n - \delta \frac{\Delta t}{\varrho_{h\delta}^n} \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1},$$

the equality $H(w^{n+1}, \delta) = 0$ exactly means that we have: $\forall \nu_h^\mu \in \mathcal{V}_h^\mu, \forall \nu_h^c \in \mathcal{V}_h^c, \forall \nu_h^u \in \mathcal{V}_{h,0}^u, \forall \nu_h^p \in \mathcal{V}_h^p$,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx - \delta \int_{\Omega} (c_{ih}^n - \alpha_{ih}) \mathbf{u}_\delta^* \cdot \nabla \nu_h^\mu dx = - \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx, \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} \delta D_i(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_h^c dx + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \cdot \nabla \nu_h^c dx, \\ \int_{\Omega} \varrho_{h\delta}^n \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_\delta^*}{\Delta t} \cdot \nu_h^u dx + \frac{1}{2} \int_{\Omega} \frac{\varrho_{h\delta}^{n+1} - \varrho_{h\delta}^n}{\Delta t} \mathbf{u}_h^{n+1} \cdot \nu_h^u dx \\ \quad + \frac{\delta}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \nu_h^u dx - \frac{\delta}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \nu_h^u \cdot \mathbf{u}_h^{n+1} dx \\ \quad + \int_{\Omega} 2\eta_h^{n+1} D\mathbf{u}_h^{n+1} : D\nu_h^u dx - \int_{\Omega} p_h^{n+1} \operatorname{div}(\nu_h^u) dx = \int_{\Omega} \varrho_{h\delta}^{n+1} \mathbf{g} \cdot \nu_h^u dx, \\ \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0. \end{array} \right. \quad (5.2)$$

We take $\nu_h^\mu = \mu_{ih}^{n+1}$, $\nu_h^c = \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t}$, $\nu_h^u = \mathbf{u}_h^{n+1}$ and $\nu_h^p = p^{n+1}$ as test functions in this system to obtain:

$$\begin{aligned} & \left[\mathcal{F}_{\Sigma, \varepsilon, \delta}^{\text{triph}}(\mathbf{c}_h^{n+1}) + \frac{1}{2} \int_{\Omega} \varrho_h^{n+1} |\mathbf{u}_h^{n+1}|^2 dx \right] - \left[\mathcal{F}_{\Sigma, \varepsilon, \delta}^{\text{triph}}(\mathbf{c}_h^n) + \frac{1}{2} \int_{\Omega} \varrho_h^n |\delta \mathbf{u}_h^n|^2 dx \right] \\ & + \Delta t \sum_{i=1}^3 \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} |\nabla \mu_{ih}^{n+1}|^2 dx + \Delta t \int_{\Omega} 2\eta_h^{n+1} |D\mathbf{u}_h^{n+1}|^2 dx \\ & + \frac{3}{8} (2\beta - 1) \varepsilon \int_{\Omega} \sum_{i=1}^3 \Sigma_i |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx \\ & + \frac{1}{2} \int_{\Omega} \varrho_{h\delta}^n \left[|\mathbf{u}_h^{n+1} - \mathbf{u}_\delta^*|_{L^2(\Omega)}^2 + |\mathbf{u}_\delta^* - \delta \mathbf{u}_h^n|_{L^2(\Omega)}^2 \right] dx \\ & = \Delta t \int_{\Omega} \varrho_{h\delta}^{n+1} \mathbf{g} \cdot \mathbf{u}_h^{n+1} dx + \frac{12}{\varepsilon} \delta \int_{\Omega} \left[F(\mathbf{c}_h^{n+1}) - F(\mathbf{c}_h^n) - \mathbf{d}^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \cdot (\mathbf{c}_h^{n+1} - \mathbf{c}_h^n) \right] dx, \end{aligned}$$

where $\mathcal{F}_{\Sigma, \varepsilon, \delta}^{\text{triph}}(\mathbf{c}_h^\ell) = \int_{\Omega} \delta \frac{12}{\varepsilon} F(\mathbf{c}_h^\ell) + \sum_{i=1}^3 \frac{3}{8} \varepsilon \Sigma_i |\nabla c_{ih}^\ell|^2 dx$. Using proposition 1.1, the

fact that F is non negative, the positive lower bounds ϱ_{\min} and η_{\min} for the density and viscosity, the fact that the mobility is bounded from below, the Korn lemma (cf [3, lemme VII.3.5]) and assumption (5.1), we obtain:

$$\begin{aligned} & \frac{3}{8} \varepsilon \sum_{i=1}^3 |\nabla c_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 + \frac{\varrho_{\min}}{2} |\mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d}^2 \\ & + \frac{M_1 \Sigma \Delta t}{\max_{i=1,2,3} (|\Sigma_i|)} \sum_{i=1}^3 |\nabla \mu_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 + 2\Delta t \eta_{\min} C_k |\nabla \mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d}^2 \\ & \leq \mathcal{F}_{\Sigma, \varepsilon, \delta}^{\text{triph}}(\mathbf{c}_h^n) + \frac{\varrho_{\max}}{2} |\delta \mathbf{u}_h^n|_{L^2(\Omega)}^2 + \Delta t \varrho_{\max} |g|_2 |\Omega|^{\frac{1}{2}} |\mathbf{u}_h^{n+1}|_{L^2(\Omega)^2} + \delta \frac{12}{\varepsilon} K_1^{\mathbf{c}_h^n}. \end{aligned} \quad (5.3)$$

Using the Poincaré and Young inequalities, since $\delta \leq 1$, yields:

$$\begin{aligned} & \frac{3}{8}\varepsilon \sum_{i=1}^3 |\nabla c_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 + \frac{\varrho_{\min}}{2} |\mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d}^2 \\ & + \frac{M_1 \underline{\Sigma} \Delta t}{\max_{i=1,2,3} (|\Sigma_i|)} \sum_{i=1}^3 |\nabla \mu_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 + \Delta t \eta_{\min} C_k |\nabla \mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d}^2 \\ & \leq \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^n) + \frac{\varrho_{\max}}{2} |\mathbf{u}_h^n|_{L^2(\Omega)}^2 + \frac{C_p^2 \Delta t \varrho_{\max}^2 |g|_2^2 |\Omega|}{4 \eta_{\min} C_k} + \frac{12}{\varepsilon} K_1^{\mathbf{c}_h^n}. \end{aligned}$$

The constant $K_2^{\mathbf{c}_h^n} = \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^n) + \frac{\varrho_{\max}}{2} |\mathbf{u}_h^n|_{L^2(\Omega)}^2 + \frac{C_p^2 \Delta t \varrho_{\max}^2 |g|_2^2 |\Omega|}{4 \eta_{\min} C_k} + \frac{12}{\varepsilon} K_1^{\mathbf{c}_h^n}$ is independent of δ and w^{n+1} and we have

$$\sum_{i=1}^3 |\nabla c_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 \leq K_3^{\mathbf{c}_h^n}, \quad (5.4)$$

$$\sum_{i=1}^3 |\nabla \mu_{ih}^{n+1}|_{(L^2(\Omega))^d}^2 \leq K_4^{\mathbf{c}_h^n}, \quad (5.5)$$

$$|\mathbf{u}_h^{n+1}|_{(H^1(\Omega))^d} \leq K_5^{\mathbf{c}_h^n}, \quad (5.6)$$

$$\text{with } K_3^{\mathbf{c}_h^n} = \frac{8K_2^{\mathbf{c}_h^n}}{3\varepsilon \underline{\Sigma}}, K_4^{\mathbf{c}_h^n} = \frac{\max_{i=1,2,3} (|\Sigma_i|) K_2^{\mathbf{c}_h^n}}{M_1 \underline{\Sigma}}, K_5^{\mathbf{c}_h^n} = \max \left(\frac{2K_2^{\mathbf{c}_h^n}}{\varrho_{\min}}, \frac{K_2^{\mathbf{c}_h^n}}{\Delta t \eta_{\min} C_k} \right)^{\frac{1}{2}}.$$

We now use the discrete form of the volume conservation: $m(c_{ih}^{n+1}) = m(c_{ih}^n)$ directly obtained by choosing $\nu_h^\mu \equiv 1$ in the system (5.2). Thus, thanks to Poincaré inequality, there exists a positive constant $K_6^{\mathbf{c}_h^n} = C_p \left(K_3^{\mathbf{c}_h^n} + m(c_{ih}^n) \right)$ independent of δ and w^{n+1} such that

$$|c_{ih}^{n+1}|_{H^1(\Omega)} \leq K_6^{\mathbf{c}_h^n}. \quad (5.7)$$

To estimate the mean $m(\mu_{ih}^{n+1})$, we take $\nu_h^c \equiv 1$ in the system (5.2). This yields:

$$m(\mu_{ih}^{n+1}) = \int_{\Omega} \delta D_i^F(\mathbf{c}_h^{n+1}, \mathbf{c}_h^n) dx.$$

This can be bounded using $|\mathbf{c}_h^{n+1}|_{H^1(\Omega)}$ and $|\mathbf{c}_h^n|_{H^1(\Omega)}$ under the assumption (2.2). Indeed, the polynomial growth (2.2) of d_i^F implies that there exists a positive constant $C_1 = \frac{16\Sigma_T}{3\Sigma_m} B_1$ such that

$$|D_i^F(\mathbf{c}_h^{n+1}, \mathbf{c}_h^n)| \leq C_1 \left(1 + |\mathbf{c}_h^{n+1}|^{p-1} + |\mathbf{c}_h^n|^{p-1} \right).$$

Thus, since $\delta \leq 1$, and by using (5.7), we obtain

$$\begin{aligned} m(\mu_{ih}^{n+1}) & \leq C_1 \left(|\Omega| + |\mathbf{c}_h^{n+1}|_{L^{p-1}}^{p-1} + |\mathbf{c}_h^n|_{L^{p-1}}^{p-1} \right) \\ & \leq C_1 \left(|\Omega| + \left(K_6^{\mathbf{c}_h^n} \right)^{p-1} + |\mathbf{c}_h^n|_{H^1}^{p-1} \right) := K_7^{h, \mathbf{c}_h^n}. \end{aligned}$$

Thanks to the Poincaré inequality, we have

$$|\mu_{ih}^{n+1}|_{H^1(\Omega)} \leq C_p \left(K_4^{\mathbf{c}_h^n} + K_7^{h, \mathbf{c}_h^n} \right) := K_8^{\mathbf{c}_h^n}. \quad (5.8)$$

The bound on pressure is obtained using the bound on the velocity (5.6) and the inf-sup condition (2.12) which ensures (cf [7, 21.5.10, p. 344]) that there exists $\mathbf{v}_h \in \mathcal{V}_{h,0}^{\mathbf{u}}$ such that

$$\forall \nu_h^p \in \mathcal{V}^p, \quad \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{v}_h) dx = \int_{\Omega} \nu_h^p p_h^{n+1} dx \quad \text{and} \quad |\mathbf{v}_h|_{(H^1(\Omega))^d} \leq \frac{1}{\beta} |p_h^{n+1}|_{L^2(\Omega)}. \quad (5.9)$$

Thus, taking $\boldsymbol{\nu}_h^{\mathbf{u}} = \mathbf{v}_h$ in the system (5.2) yields:

$$\begin{aligned} |p_h^{n+1}|_{L^2(\Omega)}^2 &= \int_{\Omega} \frac{\frac{\varrho_{h\delta}^n + \varrho_{h\delta}^{n+1}}{2} \mathbf{u}_h^{n+1} - \varrho_{h\delta}^n \delta \mathbf{u}_h^n}{\Delta t} \cdot \mathbf{v}_h dx + \int_{\Omega} 2\eta_{h\delta}^{n+1} D\mathbf{u}_h^{n+1} : D\mathbf{v}_h dx \\ &\quad + \frac{\delta}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \mathbf{v}_h dx - \frac{\delta}{2} \int_{\Omega} \varrho_h^{n+1} (\mathbf{u}_h^n \cdot \nabla) \mathbf{v}_h \cdot \mathbf{u}_h^{n+1} dx \\ &\quad - \int_{\Omega} \varrho_{h\delta}^{n+1} \mathbf{g} \cdot \mathbf{v}_h dx + \delta \int_{\Omega} \sum_{j=1}^3 (c_{jh}^n - \alpha_{jh}) \nabla \mu_{jh}^{n+1} \cdot \mathbf{v}_h dx. \end{aligned}$$

We use the Cauchy-Schwarz inequality, the upper bounds ϱ_{\max} and η_{\max} for the density and the viscosity and the estimates (5.6), (5.7), (5.8) and (5.9) to obtain:

$$\begin{aligned} |p_h^{n+1}|_{L^2(\Omega)}^2 &\leq \frac{\varrho_{\max}}{\Delta t} \left[|\mathbf{u}_h^n|_{(L^2(\Omega))^d} + |\mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d} \right] |\mathbf{v}_h|_{(L^2(\Omega))^d} \\ &\quad + 2\eta_{\max} |\nabla \mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d} |\nabla \mathbf{v}_h|_{(L^2(\Omega))^d} \\ &\quad + \frac{\delta}{2} \varrho_{\max} |\mathbf{u}_h^n|_{(L^4(\Omega))^d} |\nabla \mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d} |\mathbf{v}_h|_{(L^4(\Omega))^d} \\ &\quad + \frac{\delta}{2} \varrho_{\max} |\mathbf{u}_h^n|_{(L^4(\Omega))^d} |\nabla \mathbf{v}_h|_{(L^2(\Omega))^d} |\mathbf{u}_h^{n+1}|_{(L^4(\Omega))^d} \\ &\quad + \varrho_{\max} |\Omega|^{\frac{1}{2}} |\mathbf{g}|_2 |\mathbf{v}|_{(L^2(\Omega))^d} \\ &\quad + \delta \sum_{j=1}^3 |c_{jh}^n - \alpha_{jh}|_{(L^4(\Omega))^d} |\nabla \mu_{jh}^{n+1}|_{(L^2(\Omega))^d} |\mathbf{v}_h|_{(L^4(\Omega))^d} \\ &\leq \frac{1}{\beta} \left[\frac{\varrho_{\max}}{\Delta t} \left[|\mathbf{u}_h^n|_{(L^2(\Omega))^d} + K_5^{\mathbf{c}_h^n} \right] + 2\eta_{\max} K_5^{\mathbf{c}_h^n} \right. \\ &\quad \left. + \varrho_{\max} |\mathbf{u}_h^n|_{(L^4(\Omega))^d} K_5^{\mathbf{c}_h^n} + \varrho_{\max} |\Omega|^{\frac{1}{2}} |\mathbf{g}|_2 + 3K_6^{\mathbf{c}_h^n} K_8^{\mathbf{c}_h^n} \right] |p_h^{n+1}|_{L^2(\Omega)}. \end{aligned}$$

In conclusion, we have:

$$|p_h^{n+1}|_{L^2(\Omega)} \leq K_9^{\mathbf{c}_h^n}, \quad (5.10)$$

with $K_9^{\mathbf{c}_h^n} = \frac{\varrho_{\max}}{\Delta t} \left[|\mathbf{u}_h^n|_{(L^2(\Omega))^d} + K_5^{\mathbf{c}_h^n} \right] + 2\eta_{\max} K_5^{\mathbf{c}_h^n} + \varrho_{\max} |\mathbf{u}_h^n|_{(L^4(\Omega))^d} K_5^{\mathbf{c}_h^n} + \varrho_{\max} |\Omega|^{\frac{1}{2}} |\mathbf{g}|_2 + 3K_6^{\mathbf{c}_h^n} K_8^{\mathbf{c}_h^n}$.

Thus, by combining (5.7), (5.8), (5.6) and (5.10), we obtain a positive constant $K^{\mathbf{c}_h^n}$ independent of δ and \mathbf{c}_h^{n+1} such that

$$|w^{n+1}|_W \leq K^{\mathbf{c}_h^n}.$$

Hence, taking $R > K^{\mathbf{c}_h^n} \geq 0$ guarantees that for all $(w, \delta) \in W \times [0; 1]$, $H(w, \delta) = 0 \implies |w|_W < R$.

Validation of assumption (iii) of lemma 5.2

We have to show the existence of a solution to the linear problem $H(w^{n+1}, 0) = 0$. This problem can be written as three problems which are totally uncoupled:

- (1-2) Find $(c_{ih}^{n+1}, c_{2h}^{n+1}, \mu_{1h}^{n+1}, \mu_{2h}^{n+1}) \in (\mathcal{V}_h^c)^2 \times (\mathcal{V}_h^\mu)^2$ such that $\forall i = 1, 2, \forall \nu_h^\mu \in \mathcal{V}_h^\mu, \forall \nu_h^c \in \mathcal{V}_h^c$,

$$a_i((c_{ih}^{n+1}, \mu_{ih}^{n+1}), (\nu_h^c, \nu_h^\mu)) = \int_{\Omega} c_{ih}^n \nu_h^\mu dx,$$

where

$$\begin{aligned} a_i((c_{ih}^{n+1}, \mu_{ih}^{n+1}), (\nu_h^c, \nu_h^\mu)) &= \int_{\Omega} \left[c_{ih}^{n+1} \nu_h^\mu + \frac{M_{0h}^n}{\Sigma_i} \Delta t \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu \right] dx \\ &\quad + \int_{\Omega} \left[\frac{3}{4} \Sigma_i \varepsilon \beta \nabla c_{ih}^{n+1} \cdot \nabla \nu_h^c - \mu_{ih}^{n+1} \nu_h^c \right] dx, \end{aligned}$$

with $M_{0h}^n = M_0(\mathbf{c}_h^n)$.

- (3) Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathcal{V}_{h,0}^{\mathbf{u}} \times \mathcal{V}_h^p$ such that $\forall \nu_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}, \forall \nu_h^p \in \mathcal{V}_h^p$,

$$\begin{cases} \int_{\Omega} \frac{\varrho_h^{n-1} + \varrho_h^n}{2\Delta t} \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx + \int_{\Omega} 2\eta_h^n D\mathbf{u}_h^{n+1} : D\boldsymbol{\nu}_h^{\mathbf{u}} dx \\ \quad - \int_{\Omega} p_h^{n+1} \operatorname{div}(\boldsymbol{\nu}_h^{\mathbf{u}}) dx = \int_{\Omega} \varrho_h^n \mathbf{g} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx, \\ \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0. \end{cases}$$

Since the linear problems (1-2) are posed in finite dimension, it is sufficient to prove that for all $(c_{ih}^{n+1}, \mu_{ih}^{n+1}) \in \mathcal{V}_h^c \times \mathcal{V}_h^\mu$:

$$(a_i((c_{ih}^{n+1}, \mu_{ih}^{n+1}), (\nu_h^c, \nu_h^\mu)) = 0, \quad \forall (\nu_h^c, \nu_h^\mu) \in \mathcal{V}_h^c \times \mathcal{V}_h^\mu) \implies (c_{ih}^{n+1}, \mu_{ih}^{n+1}) = (0, 0).$$

Let $(c_{ih}^{n+1}, \mu_{ih}^{n+1}) \in \mathcal{V}_h^c \times \mathcal{V}_h^\mu$ such that

$$(a_i((c_{ih}^{n+1}, \mu_{ih}^{n+1}), (\nu_h^c, \nu_h^\mu)) = 0, \quad \forall (\nu_h^c, \nu_h^\mu) \in \mathcal{V}_h^c \times \mathcal{V}_h^\mu), \quad (5.11)$$

Taking $(\nu_h^c, \nu_h^\mu) = (c_{ih}^{n+1}, \mu_{ih}^{n+1})$ in (5.11) yields:

$$\int_{\Omega} M_{0h}^n \Delta t |\nabla \mu_{ih}^{n+1}|^2 dx + \frac{3}{4} \Sigma_i \varepsilon \beta \int_{\Omega} |\nabla c_{ih}^{n+1}|^2 dx = 0.$$

Since the mobility satisfy (1.9), we obtain: $\nabla \mu_{ih}^{n+1} = \nabla c_{ih}^{n+1} = 0$. Hence, c_{ih}^{n+1} and μ_{ih}^{n+1} are constant. Using (5.11) then yields

$$(c_{ih}^{n+1}, \mu_{ih}^{n+1}) = (0, 0).$$

The problem (3) has a unique solution. Indeed, the lower bounds on density and viscosity, the Korn lemma (cf [3, lemme VII.3.5]) allows to prove that the continuous bilinear form

$$(\mathbf{u}, \mathbf{v}) \in \mathcal{V}_{h,0}^{\mathbf{u}} \rightarrow \int_{\Omega} \frac{\varrho_h^{n-1} + \varrho_h^n}{2\Delta t} \mathbf{u} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx + \int_{\Omega} 2\eta_h^n D\mathbf{u}_h^{n+1} : D\boldsymbol{\nu}_h^{\mathbf{u}} dx$$

is coercive. The inf-sup condition (2.12) allows to conclude. \square

6. Convergence of discrete solutions in the homogeneous case. In this section, we assume that $\varrho_1 = \varrho_2 = \varrho_3 = \varrho_0 > 0$. This implies that the function $\varrho(\mathbf{c})$ is a constant function:

$$\varrho(\mathbf{c}) = \varrho_0, \quad \forall \mathbf{c} \in \mathcal{S}.$$

In this particular case, the problem 2 reads:

PROBLEM 5.

- *Step 1: resolution of Cahn-Hilliard system*

Find $(\mathbf{c}_h^{n+1}, \boldsymbol{\mu}_h^{n+1}) \in \mathcal{V}_{h,\mathcal{S}}^{\mathbf{c}} \times (\mathcal{V}_h^{\boldsymbol{\mu}})^3$ such that $\forall \nu_h^c \in \mathcal{V}_{h,\mathcal{S}}^c, \forall \nu_h^\mu \in \mathcal{V}_h^\mu$, we have, for $i = 1, 2$ and 3 ,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx \\ - \int_{\Omega} [c_{ih}^n - \alpha_i] \left[\mathbf{u}_h^n - \frac{\Delta t}{\varrho_0} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1} \right] \cdot \nabla \nu_h^\mu dx \\ \\ \\ \int_{\Omega} \mu_{ih}^{n+1} \nu_h^c dx = \int_{\Omega} D_i^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \nu_h^c dx + \int_{\Omega} \frac{3}{4} \Sigma_i \varepsilon \nabla c_{ih}^{n+\beta} \nabla \nu_h^c dx, \end{array} \right. \quad (6.1)$$

with α_j defined by $\alpha_j = \int_{\Omega} c_{jh}^0 dx$.

- *Step 2: resolution of Navier-Stokes equations*

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathcal{V}_{h,0}^{\mathbf{u}} \times \mathcal{V}_h^p$ such that $\forall \boldsymbol{\nu}_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}, \forall \nu_h^p \in \mathcal{V}_h^p$,

$$\left\{ \begin{array}{l} \int_{\Omega} \varrho_0 \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx + \frac{1}{2} \int_{\Omega} \varrho_0 (\mathbf{u}_h^n \cdot \nabla) \mathbf{u}_h^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \\ - \frac{1}{2} \int_{\Omega} \varrho_0 (\mathbf{u}_h^n \cdot \nabla) \boldsymbol{\nu}_h^{\mathbf{u}} \cdot \mathbf{u}_h^{n+1} dx \\ + \int_{\Omega} 2\eta_h^{n+1} D\mathbf{u}_h^{n+1} : D\boldsymbol{\nu}_h^{\mathbf{u}} dx - \int_{\Omega} p_h^{n+1} \operatorname{div}(\boldsymbol{\nu}_h^{\mathbf{u}}) dx \\ \\ = \int_{\Omega} \varrho_0 \mathbf{g} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx - \int_{\Omega} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx, \\ \\ \int_{\Omega} \nu_h^p \operatorname{div}(\mathbf{u}_h^{n+1}) dx = 0, \end{array} \right. \quad (6.2)$$

where $\eta_h^{n+1} = \eta(\mathbf{c}_h^{n+1})$.

The existence of solutions is given by theorem 5.1. For all $N \in \mathbb{N}$, we can introduce the following functions of time $t \in [0, t_f]$:

$$\underline{c}_{ih}^N(t, \cdot) = c_{ih}^n(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[, \quad (6.3)$$

$$\bar{c}_{ih}^N(t, \cdot) = c_{ih}^{n+1}(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[, \quad (6.4)$$

$$c_{ih}^N(t, \cdot) = \frac{t_{n+1} - t}{\Delta t} c_{ih}^n(\cdot) + \frac{t - t_n}{\Delta t} c_{ih}^{n+1}(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[. \quad (6.5)$$

For chemical potentials, for all $N \in \mathbb{N}$, we introduce the following piecewise (in time) constant functions:

$$\mu_{ih}^N(t, \cdot) = \mu_{ih}^{n+1}(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[. \quad (6.6)$$

And finally for the velocity, we introduce the following function of time $t \in [0, t_f]$, for all $N \in \mathbb{N}$:

$$\underline{\mathbf{u}}_h^N(t, \cdot) = \mathbf{u}_h^n(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[, \quad (6.7)$$

$$\overline{\mathbf{u}}_h^N(t, \cdot) = \mathbf{u}_h^{n+1}(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[, \quad (6.8)$$

$$\mathbf{u}_h^N(t, \cdot) = \frac{t_{n+1} - t}{\Delta t} \mathbf{u}_h^n(\cdot) + \frac{t - t_n}{\Delta t} \mathbf{u}_h^{n+1}(\cdot), \quad \text{if } t \in]t_n, t_{n+1}[. \quad (6.9)$$

The convergence result is the following:

THEOREM 6.1 (Convergence theorem). *We assume that assumptions of theorem 5.1 are satisfied, so that a solution $(\mathbf{c}_h^N, \boldsymbol{\mu}_h^N, \mathbf{u}_h^N, p_h^N)$ of problem 2 exist for all $N \in \mathbb{N}^*$ and for all $h > 0$. We assume that $\beta \in]\frac{1}{2}, 1]$, that the consistency property (2.1) is satisfied and that there exists two constants $C > 0$ and $\Delta t_0 > 0$ such that for all $\Delta t \leq \Delta t_0$ and for all $n \in \llbracket 0; N-1 \rrbracket$,*

$$\begin{aligned} & \left[\mathcal{F}_{\Sigma, \varepsilon}^{triph}(\mathbf{c}_h^{n+1}) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^{n+1}|^2 dx \right] - \left[\mathcal{F}_{\Sigma, \varepsilon}^{triph}(\mathbf{c}_h^n) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^n|^2 dx \right] \\ & + C \left[\Delta t \sum_{i=1}^3 \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} |\nabla \mu_{ih}^{n+1}|^2 dx + \frac{3}{8} (2\beta - 1) \varepsilon \int_{\Omega} \sum_{i=1}^3 \Sigma_i |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx \right] \\ & + \Delta t \int_{\Omega} 2\eta_h^{n+1} |D\mathbf{u}_h^{n+1}|^2 dx + \frac{1}{4} \varrho_0 \int_{\Omega} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 dx \leq \Delta t \varrho_0 \int_{\Omega} \mathbf{g} \cdot \mathbf{u}_h^{n+1} dx. \end{aligned} \quad (6.10)$$

Consider the problem (1.11), the initial conditions (1.16) and the boundary conditions (1.14)-(1.15). Then, there exists a weak solution $(\mathbf{c}, \boldsymbol{\mu}, \mathbf{u}, p)$ defined on $[0, t_f[$ such that

$$\mathbf{c} \in L^\infty(0, t_f; (\mathbf{H}^1(\Omega))^3) \cap \mathcal{C}^0([0, t_f[; (\mathbf{L}^q(\Omega))^3), \text{ for all } q < 6$$

$$\boldsymbol{\mu} \in L^2(0, t_f; (\mathbf{H}^1(\Omega))^3),$$

$$\mathbf{u} \in L^\infty(0, t_f; (\mathbf{L}^2(\Omega))^d) \cap L^2(0, t_f; (\mathbf{H}^1(\Omega))^d),$$

$$\mathbf{c}(t, x) \in \mathcal{S}, \text{ for almost every } (t, x) \in [0, t_f[\times \Omega.$$

Moreover, for all sequences $(h_K)_{K \in \mathbb{N}^*}$ and $(N_K)_{K \in \mathbb{N}^*}$ satisfying the following properties:

- $h_K \xrightarrow{K \rightarrow +\infty} 0$ and $N_K \xrightarrow{K \rightarrow +\infty} +\infty$,
- there exists a constant A (independent of K) such that: (recall that the function C_{inv} is defined by (2.15))

$$\forall K \in \mathbb{N}^*, C_{\text{inv}}(h_K) \leq AN_K, \quad (6.11)$$

the sequences $(\mathbf{c}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$, $(\boldsymbol{\mu}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$ and $(\mathbf{u}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$ satisfy, up to a subsequence, the following convergence when $K \rightarrow +\infty$:

$$\mathbf{c}_{h_K}^{N_K} \rightarrow \mathbf{c} \text{ in } \mathcal{C}^0(0, t_f, (\mathbf{L}^q)^3) \text{ strong, for all } q < 6, \quad (6.12)$$

$$\mathbf{u}_{h_K}^{N_K} \rightarrow \mathbf{u} \text{ in } L^2(0, t_f, (\mathbf{L}^2)^d) \text{ strong,} \quad (6.13)$$

$$\boldsymbol{\mu}_{h_K}^{N_K} \rightharpoonup \boldsymbol{\mu} \text{ in } L^2(0, t_f, (\mathbf{H}^1)^3) \text{ weak.} \quad (6.14)$$

REMARK 5. The assumption (6.10) is obtained in practice by applying the proposition 4.1 and bounding the term:

$$\int_{\Omega} [F(\mathbf{c}_h^{n+1}) - F(\mathbf{c}_h^n) - \mathbf{d}^F(\mathbf{c}_h^n, \mathbf{c}_h^{n+1}) \cdot (\mathbf{c}_h^{n+1} - \mathbf{c}_h^n)] dx,$$

in the right hand side of (4.1). The way to obtain this bound is different depending on the scheme $D_i^F(\mathbf{c}^n, \mathbf{c}^{n+1})$ which discretize the non linear terms of the Cahn-Hilliard system. This was largely discussed in reference [6].

REMARK 6. In the statement of theorem 6.1, the inequality (6.11) is not a stability condition. It only means that to obtain convergence towards weak solution of continuous problem, it is necessary that the time step goes to zero faster than the mesh size.

The proof of theorem 6.1 is inspired from the references [11] and [14] which deal with the homogeneous diphasic Cahn-Hilliard/Navier-Stokes system. Excluding the fact that we are interesting in a triphasic model, the major difference with these work is the taking into account of the transport term in the Cahn-Hilliard equation. We have to prove that the additional term do not disturb the consistency. This is true provided that the time step goes to zero faster than the mesh size. This condition is less restrictive than the stability condition introduced in [14] (cf remarque 6).

Basically, the proof of theorem 6.1 is split in three step:

- first, the energy equality (6.10) allows to prove that the sequences $(\mathbf{c}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$, $(\boldsymbol{\mu}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$ and $(\mathbf{u}_{h_K}^{N_K})_{K \in \mathbb{N}^*}$ are bounded in some suitable norms.
- it is then possible to apply compactness theorems to extract some convergent subsequences.
- the third step consists in proving that the obtained limit is a weak solution of problem (1.11).

We separately detailed each of these three steps below.

In the sequel (section 6.1, 6.2 and 6.3), we assume that assumptions of theorem 6.1 are satisfied and in particular the notation $\mathbf{c}_h^n, \boldsymbol{\mu}_h^n, \mathbf{u}_h^n, p_h^n \dots$ denote solutions of the discrete problem 2, and $(\mathbf{c}_{h_K}^{N_K})_{K \in \mathbb{N}^*}, (\boldsymbol{\mu}_{h_K}^{N_K})_{K \in \mathbb{N}^*}, (\mathbf{u}_{h_K}^{N_K})_{K \in \mathbb{N}^*} \dots$ the associated sequences.

6.1. Bounds on discrete solution. In this section, we assume that K is fixed and to simplify notation we omit the index K in the notation h_K and N_K . The first estimates stated in proposition 6.2 are directly derived from the energy estimate (6.10).

PROPOSITION 6.2. We have the following inequality:

$$\sup_{n \leq N} \left(|\mathbf{c}_h^n|_{(H^1(\Omega))^3} \right) + \sup_{n \leq N} \left(|\mathbf{u}_h^n|_{(L^2(\Omega))^d} \right) \leq K_1, \quad (6.15)$$

$$\left(\sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 |\mu_{ih}^{n+1}|_{H^1(\Omega)}^2 \right) + \left(\sum_{n=0}^{N-1} \Delta t |\mathbf{u}_h^{n+1}|_{(H^1(\Omega))^d}^2 \right) \leq K_2, \quad (6.16)$$

$$\Delta t \left(\sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 \left| \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \right|_{H^1(\Omega)}^2 \right) + \left(\sum_{n=0}^{N-1} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_{(L^2(\Omega))^d}^2 \right) \leq K_3, \quad (6.17)$$

where K_1, K_2 and K_3 are three constants independent of Δt and h .

Proof. This proof is very similar to the proof of proposition 4.2 in [6]. Nevertheless, it use additional ingredients (Korn lemma (cf [3, lemma VII.3.5]), the lower

bound for the viscosity $\eta(c)$ and the fact that the density is constant) to deal with the terms which involve the velocity.

Let $\Sigma_m = \min_{i=1,2,3} |\Sigma_i|$ and $\Sigma_M = \max_{i=1,2,3} |\Sigma_i|$.

(i) First, the inequality (6.10) implies that, for all $n \in \llbracket 0; N-1 \rrbracket$,

$$\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^{n+1}) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^{n+1}|^2 dx \leq \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^n) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^n|^2 dx.$$

Thus, we have, for all $n \in \llbracket 0; N \rrbracket$,

$$\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^n) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^n|^2 dx \leq \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^0) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^0|^2 dx. \quad (6.18)$$

Furthermore, thanks to the assumption of polynomial growth (1.10) on F and thanks to the definitions of \mathbf{c}_h^0 and \mathbf{u}_h^0 , the initial energy can be bounded independently of h :

$$\begin{aligned} \mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^0) + \frac{1}{2} \varrho_0 \int_{\Omega} |\mathbf{u}_h^0|^2 dx &\leq B_1 \left(|\Omega| + |\mathbf{c}^0|_{H^1}^p \right) + \Sigma_M |\mathbf{c}^0|_{H^1}^2 + \frac{1}{2} \varrho_0 |\mathbf{u}^0|_{H^1(\Omega)}^2 \\ &:= K_0. \end{aligned}$$

Since F is a positive function and using the proposition 1.1, the inequality (6.18) allows to deduce:

$$\forall n \in \llbracket 0; N \rrbracket, \quad \frac{3}{8} \varepsilon \underline{\Sigma} |\nabla \mathbf{c}_h^n|_{(L^2(\Omega))^3}^2 + |\mathbf{u}_h^n|_{(L^2(\Omega))^d}^2 \leq K_0.$$

The inequality (6.15) is obtained using the discrete volume conservation and the Poincaré inequality.

(ii) We obtain (6.16) and (6.17) by summing the equations (6.10) for n from 0 to $N-1$:

$$\begin{aligned} &\left[\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^{n=N}) + \frac{1}{2} \int_{\Omega} \varrho_0 |\mathbf{u}_h^{n=N}|^2 dx \right] - \left[\mathcal{F}_{\Sigma, \varepsilon}^{\text{triph}}(\mathbf{c}_h^0) + \frac{1}{2} \int_{\Omega} \varrho_0 |\mathbf{u}_h^0|^2 dx \right] \\ &+ C \sum_{n=0}^{N-1} \sum_{i=1}^3 \left[\Delta t \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} |\nabla \mu_{ih}^{n+1}|^2 dx + \frac{3}{8} (2\beta - 1) \varepsilon \int_{\Omega} \Sigma_i |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx \right] \\ &+ \sum_{n=0}^{N-1} \left[\Delta t \int_{\Omega} 2\eta_h^{n+1} |D\mathbf{u}_h^{n+1}|^2 dx + \frac{1}{4} \int_{\Omega} \varrho_0 |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 dx \right] \leq \sum_{n=0}^{N-1} \Delta t \int_{\Omega} \varrho_0 \mathbf{g} \cdot \mathbf{u}_h^{n+1} dx. \end{aligned}$$

Since the discrete energy is positive, using the proposition 1.1, the lower bounds for the mobility and the viscosity, we obtain:

$$\begin{aligned} &C \left[\frac{M_1 \underline{\Sigma}}{(\Sigma_M)^2} \sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 |\nabla \mu_{ih}^{n+1}|^2 + \frac{3}{8} (2\beta - 1) \varepsilon \underline{\Sigma} \sum_{n=0}^{N-1} \int_{\Omega} \sum_{i=1}^3 |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx \right] \\ &+ \sum_{n=0}^{N-1} \left[2\eta_{\min} \Delta t \int_{\Omega} |D\mathbf{u}_h^{n+1}|^2 dx + \frac{1}{4} \varrho_0 \int_{\Omega} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 dx \right] \\ &\leq K_0 + \varrho_0 |\mathbf{g}|_2 |\Omega|^{1/2} \sum_{n=0}^{N-1} \Delta t |\mathbf{u}_h^{n+1}|_{(L^2(\Omega))^d}. \end{aligned}$$

Using Poincaré and Korn lemma, using the Young inequality, we find:

$$\begin{aligned}
& C \left[\frac{M_1 \underline{\Sigma}}{(\underline{\Sigma}_M)^2} \sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 |\nabla \mu_{ih}^{n+1}|^2 + \frac{3}{8} (2\beta - 1) \varepsilon \underline{\Sigma} \sum_{n=0}^{N-1} \int_{\Omega} \sum_{i=1}^3 |\nabla c_{ih}^{n+1} - \nabla c_{ih}^n|^2 dx \right] \\
& + C_K \eta_{\min} \sum_{n=0}^{N-1} \Delta t \int_{\Omega} |\nabla \mathbf{u}_h^{n+1}|^2 dx + \frac{1}{4} \varrho_0 \sum_{n=0}^{N-1} \int_{\Omega} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 dx \\
& \leq K_0 + t_f \frac{\varrho_0^2 |\mathbf{g}|_2^2 |\Omega| (C_p)^2 C_K}{\eta_{\min}}.
\end{aligned}$$

This inequality gives both (6.16) and (6.17).

□

To pass to the limit in non linear equations (*cf* section 6.3), we need strong convergence of the subsequences. For this reason, it is usefull to obtain more accurate estimates.

PROPOSITION 6.3. *There exist two constants K_4 and K_5 independent of h and Δt such that:*

$$\left(\sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 \left| \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \right|_{(\mathbf{H}^1(\Omega))'}^2 \right) + \left(\frac{1}{\sqrt{\Delta t}} \sum_{n=0}^{N-1} \sum_{i=1}^3 |c_{ih}^{n+1} - c_{ih}^n|_{(\mathbf{L}^2(\Omega))}^2 \right) \leq K_4, \quad (6.19)$$

$$\sum_{n=0}^{N-i-1} \Delta t |\mathbf{u}^{n+i} - \mathbf{u}^n|_{(\mathbf{L}^2(\Omega))^d}^2 \leq K_5 (t^i)^{\frac{1}{4}}, \quad \forall i \in \llbracket 0; N-1 \rrbracket. \quad (6.20)$$

Proof.

(i) The estimate (6.19) is obtained from the first equation of Cahn-Hilliard system.

(α) Consider $\nu_h^\mu \in \mathcal{V}_h^\mu$. The first equation of (2.4) reads:

$$\begin{aligned}
\int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx &= - \int_{\Omega} \frac{M_{0h}^{n+\alpha}}{\Sigma_i} \nabla \mu_{ih}^{n+1} \cdot \nabla \nu_h^\mu dx \\
&+ \int_{\Omega} [c_{ih}^n - \alpha_i] [\mathbf{u}_h^n - \frac{\Delta t}{\varrho_0} \sum_{j=1}^3 (c_{jh}^n - \alpha_j) \nabla \mu_{jh}^{n+1}] \cdot \nabla \nu_h^\mu dx.
\end{aligned}$$

Thus, the inverse inequality (2.15) yields:

$$\begin{aligned}
\left| \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx \right| &\leq \left[\frac{M_2}{\Sigma_m} |\mu_{ih}^{n+1}|_{\mathbf{H}^1} + (|\alpha_i| + |c_{ih}^n|_{\mathbf{H}^1}) |\mathbf{u}_h^n|_{\mathbf{H}^1} \right] |\nu_h^\mu|_{\mathbf{H}^1} \\
&+ \frac{\Delta t}{\varrho_0} (|\alpha_i| + C_{\text{inv}}(h)^{\frac{1}{2}} |c_{ih}^n|_{\mathbf{H}^1}) \sum_{j=1}^3 \left(1 + C_{\text{inv}}(h)^{\frac{1}{2}} |c_{jh}^n|_{\mathbf{H}^1} \right) |\mu_{jh}^{n+1}|_{\mathbf{H}^1} |\nu_h^\mu|_{\mathbf{H}^1}.
\end{aligned}$$

Finally, thanks to (6.11) and (6.15), we obtained that there exists a constant K (independent of h and Δt) such that:

$$\left| \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu dx \right| \leq K \left[|\mathbf{u}_h^n|_{\mathbf{H}^1(\Omega)} + \sum_{i=1}^3 |\mu_{ih}^{n+1}|_{\mathbf{H}^1(\Omega)} \right] |\nu_h^\mu|_{\mathbf{H}^1(\Omega)}. \quad (6.21)$$

We are now going to use this intermediate inequality to prove (6.19).

(β) Let $\nu \in H^1(\Omega)$. Let ν_h^μ be the L^2 projection of ν on \mathcal{V}_h^μ . Owing to (2.14), we have:

$$|\nu_h^\mu|_{H^1(\Omega)} \leq C|\nu|_{H^1(\Omega)}.$$

Thus, using (6.21), we obtain

$$\left| \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu \, dx \right| = \left| \int_{\Omega} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \nu_h^\mu \, dx \right| \leq KC \left[|\mathbf{u}_h^n|_{H^1} + \sum_{i=1}^3 |\mu_{ih}^{n+1}|_{H^1} \right] |\nu|_{H^1}.$$

Since this inequality is true for all $\nu \in H^1(\Omega)$, we have

$$\begin{aligned} \left| \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \right|_{(H^1(\Omega))'} &= \sup_{\nu \in H^1(\Omega)} \frac{\left| \left(\frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t}, \nu \right)_{L^2(\Omega)} \right|}{|\nu|_{H^1(\Omega)}} \\ &\leq KC \left[|\mathbf{u}_h^n|_{H^1(\Omega)} + \sum_{i=1}^3 |\mu_{ih}^{n+1}|_{H^1(\Omega)} \right]. \end{aligned}$$

Consequently, using (6.16) yields:

$$\sum_{n=0}^{N-1} \Delta t \sum_{i=1}^3 \left| \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} \right|_{(H^1(\Omega))'}^2 \leq 18K^2 C^2 K_2. \quad (6.22)$$

(γ) We now take $\nu_h^\mu = \Delta t(c_{ih}^{n+1} - c_{ih}^n)$ in (6.21). This yields:

$$\sum_{i=1}^3 \left| \int_{\Omega} |c_{ih}^{n+1} - c_{ih}^n|^2 \, dx \right| \leq K \Delta t \left[|\mathbf{u}_h^n|_{H^1} + \sum_{i=1}^3 |\mu_{ih}^{n+1}|_{H^1} \right] \sum_{i=1}^3 |c_{ih}^{n+1} - c_{ih}^n|_{H^1}.$$

and so, using (6.16) and (6.17), we have:

$$\sum_{n=0}^{N-1} \sum_{i=1}^3 |c_{ih}^{n+1} - c_{ih}^n|_{L^2(\Omega)}^2 \leq 2\sqrt{K_2} \sqrt{K_3} \sqrt{\Delta t}. \quad (6.23)$$

The inequality (6.19) is readily deduced from equations (6.22) and (6.23) by defining the constant $K_4 = \max(18K^2 C^2 K_2, 2\sqrt{3K_2} \sqrt{K_3})$.

(ii) To obtain estimate (6.20), we begin with bounding the term: $|\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(L^2)^d}^2$ for $n \in \llbracket 0; N-i-1 \rrbracket$. We choose $\boldsymbol{\nu}_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}$ such that

$$\int_{\Omega} \nu_h^p \operatorname{div} \boldsymbol{\nu}_h^{\mathbf{u}} \, dx = 0, \quad \forall \nu_h^p \in \mathcal{V}^p, \quad (6.24)$$

as function test in (6.2) and we sum up the equation to obtain:

$$\begin{aligned}
& \int_{\Omega} \varrho_0 (\mathbf{u}_h^{n+i} - \mathbf{u}_h^n) \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx + \underbrace{\frac{1}{2} \sum_{k=n}^{n+i-1} \Delta t \int_{\Omega} \varrho_0 (\mathbf{u}_h^k \cdot \nabla) \mathbf{u}_h^{k+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx}_{T_1} \\
& - \underbrace{\frac{1}{2} \sum_{k=n}^{n+i-1} \Delta t \int_{\Omega} \varrho_0 (\mathbf{u}_h^k \cdot \nabla) \boldsymbol{\nu}_h^{\mathbf{u}} \cdot \mathbf{u}_h^{k+1} dx}_{T_2} + \underbrace{\sum_{k=n}^{n+i-1} \Delta t \int_{\Omega} 2\eta_h^{k+1} D\mathbf{u}_h^{k+1} : D\boldsymbol{\nu}_h^{\mathbf{u}} dx}_{T_3} \\
& = \underbrace{\sum_{k=n}^{n+i-1} \Delta t \int_{\Omega} \varrho_0 \mathbf{g} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx}_{T_4} - \underbrace{\sum_{k=n}^{n+i-1} \Delta t \int_{\Omega} \sum_{j=1}^3 (c_{jh}^k - \alpha_j) \nabla \mu_{jh}^{k+1} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx}_{T_5}.
\end{aligned}$$

We then separately estimate each term of this inequality. For term T_1 , by using the Hölder inequality and an interpolation inequality, we obtain:

$$\begin{aligned}
T_1 & \leq \frac{1}{2} \varrho_0 \Delta t \sum_{k=n}^{n+i-1} |\mathbf{u}^k|_{(\mathbf{L}^3)^d} |\mathbf{u}^{k+1}|_{(\mathbf{H}^1)^d} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{L}^6)^d} \\
& \leq \frac{1}{2} \varrho_0 |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} \Delta t \sum_{k=n}^{n+i-1} |\mathbf{u}^k|_{(\mathbf{L}^2)^d}^{\frac{1}{2}} |\mathbf{u}^k|_{(\mathbf{L}^6)^d}^{\frac{1}{2}} |\mathbf{u}^{k+1}|_{(\mathbf{H}^1)^d}.
\end{aligned}$$

Using the bound (6.15) and the Young inequality yields:

$$T_1 \leq \frac{1}{2} \varrho_0 K_1^{\frac{1}{2}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} \Delta t \sum_{k=n}^{n+i-1} \frac{2}{3} \left[|\mathbf{u}^k|_{(\mathbf{H}^1)^d}^{\frac{3}{2}} + |\mathbf{u}^{k+1}|_{(\mathbf{H}^1)^d}^{\frac{3}{2}} \right]$$

We conclude by using the Hölder inequality and the bound (6.16):

$$T_1 \leq \frac{2}{3} \varrho_0 K_1^{\frac{1}{2}} K_2^{\frac{3}{4}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} (t_f)^{\frac{3}{4}} (t^i)^{\frac{1}{4}}.$$

The term T_2 is bounded in the same way:

$$\begin{aligned}
T_2 & \leq \frac{1}{2} \varrho_0 \Delta t \sum_{k=n}^{n+i-1} |\mathbf{u}^k|_{(\mathbf{L}^3)^d} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} |\mathbf{u}^{k+1}|_{(\mathbf{L}^6)^d} \\
& \leq \frac{1}{2} \varrho_0 |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} \Delta t \sum_{k=n}^{n+i-1} |\mathbf{u}^k|_{(\mathbf{L}^2)^d}^{\frac{1}{2}} |\mathbf{u}^k|_{(\mathbf{L}^6)^d}^{\frac{1}{2}} |\mathbf{u}^{k+1}|_{(\mathbf{H}^1)^d} \\
& \leq \frac{2}{3} \varrho_0 K_1^{\frac{1}{2}} K_2^{\frac{3}{4}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} (t_f)^{\frac{3}{4}} (t^i)^{\frac{1}{4}}.
\end{aligned}$$

For the viscous term T_3 , we derive the following estimate:

$$\begin{aligned}
T_3 & \leq 2\eta_{\max} \Delta t \sum_{k=n}^{n+i-1} |\mathbf{u}_h^{k+1}|_{(\mathbf{H}^1)^d} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} \\
& \leq 2\eta_{\max} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} \Delta t i^{\frac{1}{2}} \left(\sum_{k=n}^{n+i-1} |\mathbf{u}_h^{k+1}|_{(\mathbf{H}^1)^d}^2 \right)^{\frac{1}{2}} \\
& \leq 2\eta_{\max} K_2^{\frac{1}{2}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} (t_f)^{\frac{1}{2}} (t^i)^{\frac{1}{2}}.
\end{aligned}$$

It remains the terms T_4 and T_5 of right hand side:

$$T_4 \leq \varrho_0 |\mathbf{g}|_2 |\Omega|^{\frac{1}{2}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{\mathbf{L}^2} t^i,$$

and

$$\begin{aligned} T_5 &\leq \sum_{k=n}^{n+i-1} \Delta t \sum_{j=1}^3 |c_{jh}^k - \alpha_j|_{\mathbf{L}^4} |\mu_{jh}^{k+1}|_{\mathbf{H}^1} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{L}^4)^d} \\ &\leq |\Omega| \left[K_1 + \max_{i=1,2,3} |\alpha_i| \right] K_2^{\frac{1}{2}} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} (t^i)^{\frac{1}{2}}. \end{aligned}$$

Finally, we obtain the following result: there exists a positive constant K such that, for all $\boldsymbol{\nu}_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}$ satisfying (6.24), we have

$$\left| \int_{\Omega} (\mathbf{u}_h^{n+i} - \mathbf{u}_h^n) \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \right| \leq K |\boldsymbol{\nu}_h^{\mathbf{u}}|_{(\mathbf{H}^1)^d} (t^i)^{\frac{1}{4}}.$$

In particular, for $\boldsymbol{\nu}_h^{\mathbf{u}} = \mathbf{u}_h^{n+i} - \mathbf{u}_h^n$ (which satisfies (6.24) owing to (6.2)), we find

$$|\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(\mathbf{L}^2)^d}^2 \leq K |\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(\mathbf{H}^1)^d} (t^i)^{\frac{1}{4}}.$$

Thus, we obtain

$$\sum_{n=0}^{N-i-1} \Delta t |\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(\mathbf{L}^2(\Omega))^d}^2.$$

This leads to the conclusion with $K_5 = 2KK_2(t_f)^{\frac{1}{2}}$.

□

6.2. Compactness argument, convergence of subsequences. The estimates proved in section 6.1 (proposition 6.2 and 6.3), allow to obtained (up to subsequences) the convergence of sequences: $\mathbf{c}_{h_K}^{N_K}$, $\bar{\mathbf{c}}_{h_K}^{N_K}$, $\underline{\mathbf{c}}_{h_K}^{N_K}$, $\boldsymbol{\mu}_{h_K}^{N_K}$, $\mathbf{u}_{h_K}^{N_K}$, $\bar{\mathbf{u}}_{h_K}^{N_K}$ and $\underline{\mathbf{u}}_{h_K}^{N_K}$. The following propositions give the space in which these convergences hold.

PROPOSITION 6.4. *Up to subsequences, we have the following convergences when $K \rightarrow +\infty$:*

$$\mathbf{c}_{h_K}^{N_K} \rightharpoonup \mathbf{c} \quad \text{in } \mathbf{L}^\infty(0, t_f, (\mathbf{H}^1(\Omega))^3) \text{ weak-}, \quad (6.25)$$

$$\boldsymbol{\mu}_{h_K}^{N_K} \rightharpoonup \boldsymbol{\mu} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))^3) \text{ weak}, \quad (6.26)$$

$$\frac{\partial \mathbf{c}_{h_K}^{N_K}}{\partial t} \rightharpoonup \frac{\partial \mathbf{c}}{\partial t} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))') \text{ weak}, \quad (6.27)$$

$$\mathbf{u}_{h_K}^{N_K} \rightharpoonup \mathbf{u} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))^d) \text{ weak}. \quad (6.28)$$

Proof. The convergences (6.25), (6.26), (6.27) and (6.28) are direct consequences of proposition 6.2. Indeed, it is easy to verify that the estimates stated in this proposition prove that the sequences $\mathbf{c}_{h_K}^{N_K}$, $\boldsymbol{\mu}_{h_K}^{N_K}$, $\partial_t \mathbf{c}_{h_K}^{N_K}$ and $\mathbf{u}_{h_K}^{N_K}$ are respectively bounded in the following norm: $\mathbf{L}^\infty(0, t_f, (\mathbf{H}^1(\Omega))^3)$, $\mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))^3)$, $\mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))')$, $\mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))^d)$. □

The weak convergences we write above are not sufficient to pass to the limit in the non linear terms of Cahn-Hilliard and Navier-Stokes systems. We prove in the two next proposition that it is possible to obtain strong convergence for order parameters and velocity in some suitable functional spaces.

PROPOSITION 6.5. *Up to subsequences, we have the following convergences when $K \rightarrow +\infty$:*

$$\mathbf{c}_{h_K}^{N_K} \rightarrow \mathbf{c} \quad \text{in } \mathcal{C}^0(0, t_f, (\mathbf{L}^q(\Omega))^3) \text{ strong, for all } 1 \leq q < +\infty \text{ if } d = 2, \quad (6.29)$$

$$\text{or } 1 \leq q < 6 \text{ if } d = 3,$$

$$\mathbf{c}_{h_K}^{N_K} \rightarrow \mathbf{c} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{L}^2(\Omega))^3) \text{ strong,} \quad (6.30)$$

$$\underline{\mathbf{c}}_{h_K}^{N_K} \rightarrow \mathbf{c} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{L}^2(\Omega))^3) \text{ strong,} \quad (6.31)$$

$$\bar{\mathbf{c}}_{h_K}^{N_K} \rightarrow \mathbf{c} \quad \text{in } \mathbf{L}^2(0, t_f, (\mathbf{L}^2(\Omega))^3) \text{ strong.} \quad (6.32)$$

Proof. The sequence $\mathbf{c}_{h_K}^{N_K}$ is bounded in $\mathbf{L}^\infty(0, t_f, (\mathbf{H}^1(\Omega))^3)$ and its time derivative $\partial_t \mathbf{c}_{h_K}^{N_K}$ is bounded in $\mathbf{L}^2(0, t_f, (\mathbf{H}^1(\Omega))')$. We obtain the strong convergence (6.29) of order parameters by applying the Aubin–Lions–Simon compactness theorem [19]. From this convergence, we deduce the strong convergence (6.30), and then using the inequality (6.17), the strong convergences (6.31) and (6.32) of functions $\underline{\mathbf{c}}_{h_K}^{N_K}$ and $\bar{\mathbf{c}}_{h_K}^{N_K}$. \square

To prove the result of strong convergence on the velocity, we need to apply a more precise compactness result since we do not have any estimate on its time derivative. We apply a compactness theorem due to Simon [19, Théorème 5, p.84] in which the condition on the time derivative is replaced by an estimation on time translates.

First, we write the term to estimate. This term is defined from the discrete function $\underline{\mathbf{u}}_h^N$ which is piecewise constant (in time) and its time translate. We link this term to the values \mathbf{u}_h^n of the function on each time intervals in order to exploit estimates proved in section 6.1. To simplify the notation, we omit in this lemma, the index K in the notation h_K and N_K .

LEMMA 6.6. *Let $\tau \in]0, t_f[$. We denote by $i \in \llbracket 0; N-1 \rrbracket$ the unique index such that $t^i \leq \tau < t^{i+1}$. Then, we have:*

(i) *if $\tau < \Delta t$ then*

$$\int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(\mathbf{L}^2(\Omega))^d}^2 ds = \tau \sum_{n=0}^{N-2} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_{(\mathbf{L}^2(\Omega))^d}^2, \quad (6.33)$$

(ii) *in all cases, we have:*

$$\begin{aligned} & \int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(\mathbf{L}^2(\Omega))^d}^2 ds \\ & \leq \sum_{n=0}^{N-i-1} \Delta t |\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(\mathbf{L}^2(\Omega))^d}^2 + \sum_{n=0}^{N-i-2} \Delta t |\mathbf{u}_h^{n+i+1} - \mathbf{u}_h^n|_{(\mathbf{L}^2(\Omega))^d}^2. \end{aligned} \quad (6.34)$$

Proof. Begin with writing the left hand side in the form:

$$\begin{aligned} \int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(\mathbf{L}^2(\Omega))^d}^2 ds &= \sum_{n=0}^{N-i-2} \int_{t^n}^{t^{n+1}} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(\mathbf{L}^2(\Omega))^d}^2 ds \\ &+ \int_{t^{N-i-1}}^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(\mathbf{L}^2(\Omega))^d}^2 ds. \end{aligned}$$

It only remains to identify the values of the translates of the function in the intervals $]t^n, t^{n+1}[$ for $n \in \llbracket 0; N-i-2 \rrbracket$ and $]t^{N-i-1}, t_f - \tau[$.

We introduce the real $\bar{\tau}$ defined by $\bar{\tau} = \tau - t^i$, we choose $n \in \llbracket 0; N-i-2 \rrbracket$ and consider the two following cases:

- either $s \in [t^n, t^{n+1} - \bar{\tau}]$, then we have $t^{n+i} \leq t^n + \tau \leq s + \tau \leq t^{n+1} - \bar{\tau} + \tau \leq t^{n+i+1}$ and so

$$\underline{\mathbf{u}}_h^N(s + \tau, \cdot) = \underline{\mathbf{u}}_h^{n+i}(\cdot).$$

- either $s \in [t^{n+1} - \bar{\tau}, t^{n+1}]$, then we have $t^{n+i+1} \leq t^{n+1} + \tau - \bar{\tau} \leq s + \tau \leq t^{n+1} + \tau \leq t^{n+i+2}$ and so

$$\underline{\mathbf{u}}_h^N(s + \tau, \cdot) = \underline{\mathbf{u}}_h^{n+i+1}(\cdot).$$

Finally, consider now the case where $s \in [t^{N-i-1}, t_f - \tau]$. We have $t^{N-i-1} \leq s \leq t^{N-i}$ and $t^{N-1} \leq t^{N-1} + \bar{\tau} \leq t^{N-i-1} + \tau \leq s + \tau \leq t^N$. Thus, for all $s \in [t^{N-i-1}, t_f - \tau]$, we have

$$\underline{\mathbf{u}}_h^N(s + \tau, \cdot) = \underline{\mathbf{u}}_h^{N-1}(\cdot).$$

We deduce the following equality:

$$\begin{aligned} \int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(L^2)^d}^2 ds \\ = \sum_{n=0}^{N-i-1} (\Delta t - \bar{\tau}) |\underline{\mathbf{u}}_h^{n+i} - \underline{\mathbf{u}}_h^n|_{(L^2)^d}^2 + \sum_{n=0}^{N-i-2} \bar{\tau} |\underline{\mathbf{u}}_h^{n+i+1} - \underline{\mathbf{u}}_h^n|_{(L^2)^d}^2. \end{aligned}$$

We now examine the cases (i) and (ii) :

- (i) if $\tau < \Delta t$ then we have $i = 0$ and $\bar{\tau} = \tau$. the above equality exactly gives the conclusion.
- (ii) this second conclusion is also deduced from the above equality since $0 \leq \bar{\tau} \leq \Delta t$.

□

We can now state the proposition giving the strong convergence for the velocity.

PROPOSITION 6.7. *Up to subsequences, we have the following convergences when $K \rightarrow +\infty$:*

$$\underline{\mathbf{u}}_{h_K}^{N_K} \rightarrow \underline{\mathbf{u}} \quad \text{in } L^2(0, t_f, (L^2(\Omega))^d) \text{ strong,} \quad (6.35)$$

$$\underline{\mathbf{u}}_{h_K}^{N_K} \rightarrow \underline{\mathbf{u}} \quad \text{in } L^2(0, t_f, (L^2(\Omega))^d) \text{ strong,} \quad (6.36)$$

$$\bar{\mathbf{u}}_{h_K}^{N_K} \rightarrow \underline{\mathbf{u}} \quad \text{in } L^2(0, t_f, (L^2(\Omega))^d) \text{ strong.} \quad (6.37)$$

Proof. The proof relies on a compactness theorem due to Simon [19, Theorem 5, p.84] which states that the embedding

$$L^2\left(]0, t_f[, (H^1(\Omega))^d\right) \cap N_2^{\frac{1}{8}}\left(]0, t_f[, (L^2(\Omega))^d\right) \hookrightarrow L^2\left(]0, t_f[, (L^2(\Omega))^d\right)$$

is compact. The Nikolskii space $N_2^{\frac{1}{8}}\left(]0, t_f[, (L^2(\Omega))^d\right)$ is defined by:

$$\begin{aligned} N_2^{\frac{1}{8}}\left(]0, t_f[, (L^2(\Omega))^d\right) &= \left\{ \mathbf{v} \in L^2\left(]0, t_f[, (L^2(\Omega))^d\right); \right. \\ &\quad \left. \exists C > 0, \forall \tau \in]0, t_f[, \|\mathbf{v}(\cdot + \tau, \cdot) - \mathbf{v}\|_{L^2(]0, t_f - \tau[, (L^2(\Omega))^d)} \leq C\tau^{\frac{1}{8}} \right\}, \end{aligned}$$

with the norm

$$\begin{aligned} |\mathbf{v}|_{N_2^{\frac{1}{8}}([0, t_f], (L^2(\Omega))^d)} &= \left(|\mathbf{v}|_{L^2([0, t_f], (L^2(\Omega))^d)}^2 \right. \\ &\quad \left. + \sup_{0 < \tau < t_f} \left(\frac{1}{\tau^{\frac{1}{8}}} |\mathbf{v}(\cdot + \tau, \cdot) - \mathbf{v}|_{L^2([0, t_f - \tau], (L^2(\Omega))^d)} \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, since the sequence $\mathbf{u}_{h_K}^{N_K}$ is bounded in the spaces $L^2([0, t_f], (H^1(\Omega))^d)$ and $L^2([0, t_f], (L^2(\Omega))^d)$ (cf equations (6.15) and (6.16)), it is sufficient to prove that it is bounded in the space $N_2^{\frac{1}{8}}([0, t_f], (L^2(\Omega))^d)$, to obtain the conclusion. Let $\tau \in]0, t_f[$. We still omit the index K in the notation h_K and N_K .

(i) If $\tau < \Delta t$ then owing to lemma 6.6, we have:

$$\int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(L^2(\Omega))^d}^2 ds = \tau \sum_{n=0}^{N-2} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_{(L^2(\Omega))^d}^2 \leq K_3 \tau.$$

(ii) If $\tau \geq \Delta t$ then owing to lemma 6.6, and then using the inequality (6.20), we have:

$$\begin{aligned} &\int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(L^2(\Omega))^d}^2 ds \\ &\leq \sum_{n=0}^{N-i-1} \Delta t |\mathbf{u}_h^{n+i} - \mathbf{u}_h^n|_{(L^2(\Omega))^d}^2 + \sum_{n=0}^{N-i-2} \Delta t |\mathbf{u}_h^{n+i+1} - \mathbf{u}_h^n|_{(L^2(\Omega))^d}^2 \\ &\leq K_5 \left[(t^i)^{\frac{1}{4}} + (t^{i+1})^{\frac{1}{4}} \right] \leq K_5 \left[1 + 2^{\frac{1}{4}} \right] \tau^{\frac{1}{4}}, \end{aligned}$$

since we have $t^i \leq \tau$ and $t^{i+1} = t^i + \Delta t \leq 2\tau$.

In all cases, we have obtained the existence of a positive constant K_6 (independent of h and Δt) such that:

$$\int_0^{t_f - \tau} |\underline{\mathbf{u}}_h^N(s + \tau, \cdot) - \underline{\mathbf{u}}_h^N(s, \cdot)|_{(L^2(\Omega))^d}^2 ds \leq K_6 \tau^{\frac{1}{4}}, \quad \forall \tau \in]0, t_f[.$$

This concludes the proof of convergence (6.36). The convergences (6.35) and (6.37) are then obtained thanks to the inequality (6.17). \square

6.3. Passing to the limit in the scheme. The convergences obtained in section 6.2 allows to pass to the limit in the discrete system.

For Cahn-Hilliard system (without the transport term), this work was already done in details in reference [6]. We focus here on the transport term and on the Navier-Stokes equation.

To simplify the notation, we still omit the index K in the notation N_K and h_K but when we say “convergence” it means $K \rightarrow +\infty$ (and consequently $N_K \rightarrow +\infty$ and $h_K \rightarrow 0$).

6.3.1. Transport term in Cahn-Hilliard equation. Let $\nu^\mu \in \mathcal{C}^\infty(\overline{\Omega})$ a given function and $\tau \in \mathcal{C}_c^\infty([0, t_f])$. We define ν_h^μ as the H^1 projection of the function ν^μ

on \mathcal{V}_h^μ . We have to prove the following convergence:

$$\begin{aligned} & \int_0^{t_f} \int_{\Omega} \left[\underline{c}_{ih}^N - \alpha_i \right] \left[\underline{\mathbf{u}}_h^N - \frac{\Delta t}{\varrho_0} \sum_{j=1}^3 (\underline{c}_{jh}^N - \alpha_j) \nabla \mu_{jh}^N \right] \cdot \nabla \nu_h^\mu dx \tau(t) dt \\ & \longrightarrow \int_0^{t_f} \int_{\Omega} (c_i - \alpha_i) \mathbf{u} \cdot \nabla \nu^\mu dx \tau(t) dt. \end{aligned} \quad (6.38)$$

We proceed in two steps, separately considering two terms of the left hand side: the standard transport term and the additional term which ensures the inconditionnal stability.

The following inequalities allows to identify the limit of the first term:

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} (\underline{c}_{ih}^N - \alpha_i) \underline{\mathbf{u}}_h^N \cdot \nabla \nu_h^\mu dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} (c_i - \alpha_i) \mathbf{u} \cdot \nabla \nu^\mu dx \tau(t) dt \right| \\ & \leq \left| \int_0^{t_f} \int_{\Omega} (\underline{c}_{ih}^N - \alpha_i) \underline{\mathbf{u}}_h^N \cdot \nabla (\nu_h^\mu - \nu^\mu) dx \tau(t) dt \right| \\ & + \left| \int_0^{t_f} \int_{\Omega} (\underline{c}_{ih}^N - \alpha_i) (\underline{\mathbf{u}}_h^N - \mathbf{u}) \cdot \nabla \nu^\mu dx \tau(t) dt \right| + \left| \int_0^{t_f} \int_{\Omega} (\underline{c}_{ih}^N - c_i) \mathbf{u} \cdot \nabla \nu^\mu dx \tau(t) dt \right| \\ & \leq |\tau|_{L^\infty(0, t_f)} |\nu_h^\mu - \nu^\mu|_{H^1(\Omega)} |\underline{c}_{ih}^N - \alpha_i|_{L^2(0, t_f, H^1(\Omega))} |\underline{\mathbf{u}}_h^N|_{L^2(0, t_f, (H^1(\Omega))^d)} \\ & + |\tau|_{L^\infty(0, t_f)} |\nabla \nu^\mu|_{L^3(\Omega)} |\underline{c}_{ih}^N - \alpha_i|_{L^2(0, t_f, H^1(\Omega))} |\underline{\mathbf{u}}_h^N - \mathbf{u}|_{L^2(0, t_f, (L^2(\Omega))^d)} \\ & + |\tau|_{L^\infty(0, t_f)} |\nabla \nu^\mu|_{L^3(\Omega)} |\underline{c}_{ih}^N - c_i|_{L^2(0, t_f, L^2(\Omega))} |\mathbf{u}|_{L^2(0, t_f, (H^1(\Omega))^d)} \\ & \longrightarrow 0, \end{aligned}$$

since \underline{c}_{ih}^N is bounded in $L^2(0, t_f, H^1(\Omega))$, $\underline{\mathbf{u}}_h^N$ is bounded in $L^2(0, t_f, (H^1(\Omega))^d)$, \underline{c}_{ih}^N (strongly) converges in $L^2(0, t_f, L^2(\Omega))$ towards c_i (cf equation (6.31)), $\underline{\mathbf{u}}_h^N$ (strongly) converges in $L^2(0, t_f, (L^2(\Omega))^d)$ towards \mathbf{u} (cf equation (6.36)) and, owing to assumption (2.9), $|\nu^\mu - \nu_h^\mu|_{H^1(\Omega)} = \inf_{\nu_h \in \mathcal{V}_h^\mu} |\nu^\mu - \nu_h|_{H^1(\Omega)} \xrightarrow{h \rightarrow 0} 0$.

We now use the fact than the sequences \underline{c}_{ih}^N are μ_{jh}^N are respectively bounded in $L^\infty(0, t_f, H^1(\Omega))$ and $L^2(0, t_f, H^1(\Omega))$ norm, the inverse inequality (2.15) and the condition (6.11) on the sequences h_K and N_K to show that the second term convergences towards 0:

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} \left[\underline{c}_{ih}^N - \alpha_i \right] \left[\frac{\Delta t}{\varrho_0} \sum_{j=1}^3 (\underline{c}_{jh}^N - \alpha_j) \nabla \mu_{jh}^N \right] \cdot \nabla \nu_h^\mu dx \tau(t) dt \right| \\ & \leq \frac{\Delta t}{\varrho_0} |\nabla (\nu_h^\mu - \nu^\mu)|_{L^2} \int_0^{t_f} |\underline{c}_{ih}^N - \alpha_i|_{L^\infty} \sum_{j=1}^3 |\underline{c}_{jh}^N - \alpha_j|_{L^\infty} |\nabla \mu_{jh}^N|_{L^2} \tau(t) dt \\ & + \frac{\Delta t}{\varrho_0} |\nabla \nu^\mu|_{L^\infty} \int_0^{t_f} |\underline{c}_{ih}^N - \alpha_i|_{L^4} \sum_{j=1}^3 |\underline{c}_{jh}^N - \alpha_j|_{L^4} |\nabla \mu_{jh}^N|_{L^2} \tau(t) dt \\ & \leq \frac{\Delta t C_{\text{inv}}(h)}{\varrho_0} |\tau|_{L^2} |\nu_h^\mu - \nu^\mu|_{H^1} |\underline{c}_{ih}^N - \alpha_i|_{L^\infty(H^1)} \sum_{j=1}^3 |\underline{c}_{jh}^N - \alpha_j|_{L^\infty(H^1)} |\mu_{jh}^N|_{L^2(H^1)} \\ & + \frac{\Delta t}{\varrho_0} |\tau|_{L^2} |\nabla \nu^\mu|_{L^\infty} |\underline{c}_{ih}^N - \alpha_i|_{L^\infty(H^1)} \sum_{j=1}^3 |\underline{c}_{jh}^N - \alpha_j|_{L^\infty(H^1)} |\mu_{jh}^N|_{L^2(H^1)} \\ & \longrightarrow 0. \end{aligned}$$

Thus, we proved that the convergence (6.38) holds. Re-using (exactly as it is) the reasoning presented in [6] allows to pass to the limit in the other terms of the Cahn-Hilliard system.

6.3.2. Navier-Stokes system. Let $\boldsymbol{\nu}^{\mathbf{u}} \in \mathcal{C}_c^\infty(\Omega)$ satisfying $\operatorname{div}(\boldsymbol{\nu}^{\mathbf{u}}) = 0$ and $\tau \in \mathcal{C}^1([0, t_f])$ such that $\tau(t_f) = 0$.

We introduce the space

$$Z_h = \left\{ \mathbf{z}_h \in \mathcal{V}_{h,0}^{\mathbf{u}}; \quad \forall \nu_h^p \in \mathcal{V}_h^p, \quad \int_{\Omega} \operatorname{div}(\mathbf{z}_h) \nu_h^p dx = 0 \right\}.$$

The inf-sup condition (2.12) implies that the function $\boldsymbol{\nu}^{\mathbf{u}} \in H_0^1(\Omega)$ which is divergence free can be “well approximated” with functions in Z_h . This is detailed in the proposition 6.8.

PROPOSITION 6.8 (Approximation of divergence free functions, [7, eq. 12.5.17]). *We have the following inequality:*

$$\inf_{\mathbf{z}_h \in Z_h} |\boldsymbol{\nu}^{\mathbf{u}} - \mathbf{z}_h|_{H^1(\Omega)} \leq \frac{1}{\beta} \inf_{\nu_h^{\mathbf{u}} \in \mathcal{V}_{h,0}^{\mathbf{u}}} |\boldsymbol{\nu}^{\mathbf{u}} - \boldsymbol{\nu}_h^{\mathbf{u}}|_{H^1(\Omega)}. \quad (6.39)$$

Let $\boldsymbol{\nu}_h^{\mathbf{u}}$ be the H^1 projection of $\boldsymbol{\nu}^{\mathbf{u}}$ on the space Z_h . The proposition 6.8 and the assumption (2.10) show that

$$\boldsymbol{\nu}_h^{\mathbf{u}} \rightarrow \boldsymbol{\nu}^{\mathbf{u}}, \quad \text{in } (H^1(\Omega))^d \text{ strong.} \quad (6.40)$$

We use $\boldsymbol{\nu}_h^{\mathbf{u}}$ as a test function in the first equation of (6.2). We then multiply by $\tau(t)$, $t \in]t^n, t^{n+1}[$, integrate between t^n and t^{n+1} , and sum up for n from 0 to $N-1$ so that we rebuilt a variational formulation on $]0, t_f[\times \Omega$. The unsteady term is modified by a discrete integration by part:

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_{\Omega} \varrho_0 \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt &= - \varrho_0 \int_0^{t_f} \int_{\Omega} \underline{\mathbf{u}}_h^N(t, x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \frac{\tau(t) - \tau(t - \Delta t)}{\Delta t} dt \\ &\quad + \varrho_0 \int_{\Omega} \mathbf{u}_h^{n=N}(x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}}(x) dx \int_0^1 \tau(t_f - t \Delta t) dt \\ &\quad - \varrho_0 \int_{\Omega} \mathbf{u}_h^0(x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}}(x) dx \int_0^1 \tau(\Delta t(t - 1)) dt. \end{aligned}$$

Thus, we obtain the following formulation of the scheme in which we can pass to the limit:

$$\begin{aligned} &\underbrace{- \varrho_0 \int_0^{t_f} \int_{\Omega} \underline{\mathbf{u}}_h^N(t, x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \frac{\tau(t) - \tau(t - \Delta t)}{\Delta t} dt}_{T_1} - \underbrace{\varrho_0 \int_{\Omega} \mathbf{u}_h^0(x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}}(x) dx \int_0^1 \tau(\Delta t(t - 1)) dt}_{T_2} \\ &+ \underbrace{\frac{1}{2} \int_0^{t_f} \int_{\Omega} \varrho_0 (\underline{\mathbf{u}}_h^N \cdot \nabla) \bar{\mathbf{u}}_h^N \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt}_{T_3} - \underbrace{\frac{1}{2} \int_0^{t_f} \int_{\Omega} \varrho_0 (\underline{\mathbf{u}}_h^N \cdot \nabla) \boldsymbol{\nu}_h^{\mathbf{u}} \cdot \bar{\mathbf{u}}_h^N dx \tau(t) dt}_{T_4} \\ &+ \underbrace{\int_0^{t_f} \int_{\Omega} 2\eta(\bar{\mathbf{c}}_h^N) D\bar{\mathbf{u}}_h^N : D\boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt}_{T_5} = \underbrace{\int_0^{t_f} \int_{\Omega} \varrho_0 \mathbf{g} \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt}_{T_6} \\ &- \underbrace{\int_0^{t_f} \int_{\Omega} \sum_{j=1}^3 (\underline{\mathbf{c}}_{jh}^N - \alpha_j) \nabla \mu_{jh}^N \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt}_{T_7} - \underbrace{\varrho_0 \int_{\Omega} \mathbf{u}_h^{n=N}(x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}}(x) dx \int_0^1 \tau(t_f - t \Delta t) dt}_{T_8}. \end{aligned}$$

The limit of the term T_1 is readily obtained from strong convergences (6.36), (6.40) and those of functions $t \mapsto \frac{\tau(t) - \tau(t - \Delta t)}{\Delta t}$ towards τ' in $L^2(0, t_f)$ (obtained for instance with dominated convergence theorem since the function τ is in $\mathcal{C}^1([0, t_f])$):

$$T_1 \longrightarrow \varrho_0 \int_0^{t_f} \int_{\Omega} \mathbf{u} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau'(t) dt.$$

The term T_2 allows to show that \mathbf{u} satisfies the initial condition (1.16) in a weak sense. The convergences (2.18), (6.40) and the uniform convergence on $[0, t_f]$ of the function $t \mapsto \tau(\Delta t(t - 1))$ towards the constant function equal to $\tau(0)$ yields:

$$T_2 \rightarrow \varrho_0 \int_{\Omega} \mathbf{u}^0(x) \cdot \boldsymbol{\nu}^{\mathbf{u}}(x) dx \tau(0) dt.$$

Concerning the term T_3 , the following inequality allows to conclude:

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} (\underline{\mathbf{u}}_h^N \cdot \nabla) \bar{\mathbf{u}}_h^N \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \leq |\tau|_{L^\infty} \left[\left| \bar{\mathbf{u}}_h^N \right|_{L^2((H^1)^d)} |\boldsymbol{\nu}_h^{\mathbf{u}} - \boldsymbol{\nu}^{\mathbf{u}}|_{L^4} \left| \underline{\mathbf{u}}_h^N \right|_{L^2((L^4)^d)} + \left| \underline{\mathbf{u}}_h^N - \mathbf{u} \right|_{L^2((L^2)^d)} \left| \bar{\mathbf{u}}_h^N \right|_{L^2((H^1)^d)} |\boldsymbol{\nu}^{\mathbf{u}}|_{(L^\infty)^d} \right] \\ & \quad + \left| \int_0^{t_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) (\bar{\mathbf{u}}_h^N - \mathbf{u}) \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \longrightarrow 0. \end{aligned}$$

Indeed, since the sequences $(\underline{\mathbf{u}}_h^N)$ and $(\bar{\mathbf{u}}_h^N)$ are bounded in $L^2(0, t_f, (H^1(\Omega))^d)$, the convergences (6.36) and (6.40) shows that the two first terms of the above right hand side tends to 0. And the last one (the term involving the integral), it also tends to 0 by weak convergence of $\nabla \bar{\mathbf{u}}_h^N$ towards $\nabla \mathbf{u}$ in $L^2(0, t_f, (L^2(\Omega))^d)$ (a reasoning component by component gives the result since for all $1 \leq i, j \leq d$, the function $(t, x) \mapsto \mathbf{u}_i(x) \boldsymbol{\nu}_j^{\mathbf{u}}(x) \tau(t)$ is in $L^2(0, t_f, L^2(\Omega))$).

The term T_4 is treated in the same way:

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} (\underline{\mathbf{u}}_h^N \cdot \nabla) \boldsymbol{\nu}_h^{\mathbf{u}} \cdot \bar{\mathbf{u}}_h^N dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \boldsymbol{\nu}^{\mathbf{u}} \cdot \mathbf{u} dx \tau(t) dt \right| \\ & \leq |\tau|_{L^\infty} \left[\left| \underline{\mathbf{u}}_h^N \right|_{L^2((L^4)^d)} |\boldsymbol{\nu}_h^{\mathbf{u}} - \boldsymbol{\nu}^{\mathbf{u}}|_{H^1} \left| \bar{\mathbf{u}}_h^N \right|_{L^2((L^4)^d)} + \left| \underline{\mathbf{u}}_h^N \right|_{L^2((L^6)^d)} |\nabla \boldsymbol{\nu}^{\mathbf{u}}|_{(L^3)^d} \left| \bar{\mathbf{u}}_h^N - \mathbf{u} \right|_{L^2((L^2)^d)} \right] \\ & \quad + \left| \underline{\mathbf{u}}_h^N - \mathbf{u} \right|_{L^2(0, t_f, (L^2)^d)} |\nabla \boldsymbol{\nu}^{\mathbf{u}}|_{(L^3)^d} |\mathbf{u}|_{L^2((L^6)^d)} |\tau(t)|_{L^\infty}, \end{aligned}$$

the conclusion being now obtained using the convergences (6.36), (6.37), (6.40) and the fact that the two sequences $\underline{\mathbf{u}}_h^N$ and $\bar{\mathbf{u}}_h^N$ are bounded in $L^2(0, t_f, (H^1(\Omega))^d)$.

The limit of the term (5) is obtained using the following convergence (up to a subsequence):

$$\eta(\bar{\mathbf{c}}_{h_K}^N) \rightarrow \eta(\mathbf{c}) \quad \text{in } L^2(0, t_f, (L^2(\Omega))^d) \text{ strong.} \quad (6.41)$$

This convergence is proved by using the dominated convergence theorem (the viscosity η is a bounded continuous function and $\bar{\mathbf{c}}_{h_K}^N$ strongly converge in $L^2(0, t_f, (L^2(\Omega))^3)$, so almost everywhere up to a subsequence).

Thus, using the convergences (6.40), (6.41), the fact that the sequence $\bar{\mathbf{u}}_h^N$ is bounded in $L^2(0, t_f, (H^1(\Omega))^d)$, and the weak convergence of $D\bar{\mathbf{u}}_h^N$ towards $D\mathbf{u}$ in $L^2(0, t_f, (L^2(\Omega))^d)$, we obtain

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} 2\eta(\bar{\mathbf{c}}_{h_K}^N) D\bar{\mathbf{u}}_h^N : D\boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} 2\eta(\mathbf{c}) D\mathbf{u} : D\boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \leq \left| \bar{\mathbf{u}}_h^N \right|_{L^2((H^1)^d)} \left[2\eta_{\max} |\boldsymbol{\nu}_h^{\mathbf{u}} - \boldsymbol{\nu}^{\mathbf{u}}|_{(H^1)^d} |\tau|_{L^2} + 2 \left| \eta(\bar{\mathbf{c}}_{h_K}^N) - \eta(\mathbf{c}) \right|_{L^2(0, t_f, L^2)} |\nabla \boldsymbol{\nu}^{\mathbf{u}}|_{L^\infty(\Omega)} |\tau|_{L^\infty} \right] \\ & \quad + \left| \int_0^{t_f} \int_{\Omega} 2\eta(\mathbf{c}) D(\bar{\mathbf{u}}_h^N - \mathbf{u}) : D\boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \longrightarrow 0. \end{aligned}$$

By (6.40), the convergence of term T_6 is immediate:

$$T_6 \rightarrow \int_{\Omega} \varrho_0 \mathbf{g} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt.$$

The convergence of the capillary force term T_7 is obtained as follows:

$$\begin{aligned} & \left| \int_0^{t_f} \int_{\Omega} \sum_{j=1}^3 (\mathcal{C}_{jh}^N - \alpha_j) \nabla \mu_{jh}^N \cdot \boldsymbol{\nu}_h^{\mathbf{u}} dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} \sum_{j=1}^3 (c_{jh} - \alpha_j) \nabla \mu_{jh} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \leq \sum_{j=1}^3 \left| \nabla \mu_{jh}^N \right|_{L^2((L^2)^d)} \left[\left| \mathcal{C}_{jh}^N - \alpha_j \right|_{L^\infty(L^4)} |\boldsymbol{\nu}_h^{\mathbf{u}} - \boldsymbol{\nu}^{\mathbf{u}}|_{L^4} |\tau|_{L^2} + \left| \mathcal{C}_{jh}^N - c_{jh} \right|_{L^2((L^2)^d)} |\boldsymbol{\nu}^{\mathbf{u}}|_{L^\infty} |\tau|_{L^\infty} \right] \\ & \quad + \left| \int_{\Omega} \sum_{j=1}^3 (c_{jh} - \alpha_j) \nabla (\mu_{jh}^N - \mu_j) \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \right| \\ & \longrightarrow 0. \end{aligned}$$

The two first terms of the right hand side tend to 0 thanks to convergences (6.31) and (6.40) since the sequences (\mathcal{C}_{jh}^N) and (μ_{jh}^N) are bounded in $L^2(0, t_f, H^1(\Omega))$ and $L^\infty(0, t_f, H^1(\Omega))$ respectively. The last term tends to 0 by weak convergence of $\nabla \mu_{jh}^N$ towards $\nabla \mu_j$ in $L^2(0, t_f, (L^2(\Omega))^d)$.

Finally, it only remains to prove that the residual term T_8 tends to 0. This simply comes from the fact that:

$$\left| \int_{\Omega} \mathbf{u}_h^N(x) \cdot \boldsymbol{\nu}_h^{\mathbf{u}}(x) dx \right| \leq \left| \mathbf{u}_h^N(\cdot) \right|_{L^2(\Omega)} |\boldsymbol{\nu}_h^{\mathbf{u}}|_{L^2(\Omega)} \leq K_1 |\boldsymbol{\nu}^{\mathbf{u}}|_{L^2(\Omega)},$$

and

$$\int_0^1 \tau(t_f - t\Delta t) dt \longrightarrow \tau(t_f) = 0.$$

In conclusion, we proved that:

$$\begin{aligned} & -\varrho_0 \int_0^{t_f} \int_{\Omega} \mathbf{u} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau' dt - \varrho_0 \int_{\Omega} \mathbf{u}^0 \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(0) \\ & \quad + \int_0^{t_f} \int_{\Omega} \frac{1}{2} \varrho_0 \left[(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\nu}^{\mathbf{u}} - (\mathbf{u} \cdot \nabla) \boldsymbol{\nu}^{\mathbf{u}} \cdot \mathbf{u} \right] + 2\eta(\mathbf{c}) D\mathbf{u} : D\boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt \\ & \quad = \int_0^{t_f} \int_{\Omega} \varrho_0 \mathbf{g} \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt - \int_0^{t_f} \int_{\Omega} \sum_{j=1}^3 (\mathcal{C}_j - \alpha_j) \nabla \mu_j \cdot \boldsymbol{\nu}^{\mathbf{u}} dx \tau(t) dt. \end{aligned}$$

To finish, passing to the limit in the constraint equation yields:

$$\operatorname{div}(\mathbf{u}) = 0.$$

7. Conclusion. We proposed in this article an original scheme for the discretization of the triphasic Cahn-Hilliard/Navier-Stokes model.

This scheme is unconditionally stable and preserves, at the discrete level, the main properties of the model, that is the volume conservation and the fact that the sum of the three order parameters remains equal to 1 during the time evolution.

We proved the existence of at least one solution of the discrete problem and, in the homogeneous case (*i.e.* three phases with the same densities), we proved the convergence of discrete solutions towards a weak solution of the model (whose existence is proven in the same time).

The main perspective is the study of the convergence in the case where the three fluids in presence have different densities. Even if the energy estimate (and the existence of discrete solutions) are still true in this case, it is delicate to obtain sufficient estimates which would lead, by compactness, to strong convergence on the velocity which is necessary to pass to the limit in non linear terms. Indeed, the Navier-Stokes equations involves a term of the form:

$$\mathbf{u} \partial_t \rho.$$

The time derivative of the density is not very smooth since it is a function of order parameters whose time derivative is in $L^2(0, t_f, (H^1(\Omega))')$.

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