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Smooth normal forms for integrable hamiltonian systems near a focus-focus singularity

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Abstract

We prove that completely integrable systems are normalisable in the C^∞ category near focus-focus singularities.

1 Introduction and exposition of the result

In his PhD Thesis [4], Eliasson proved some very important results about symplectic linearisation of completely integrable systems near non-degenerate singularities, in the C^∞ category. However, at that time the so-called elliptic singularities were considered the most important case, and the case of focus-focus singularities was never published. It turned out that focus-focus singularities became crucially important in the last 15 years in the topological, symplectic, and even quantum study of Liouville integrable systems. The aim of this article is to fill in some non-trivial gaps in the original treatment, in order to provide the reader with a complete and robust proof of the fact that a C^∞ completely integrable system is linearisable near a focus-focus singularity.

Note that in the holomorphic category, the result is already well established (see Vey [7]).

Let us first recall the result.

Thoughout the paper, we shall denote by (q_1, q_2) the quadratic focus-focus model system on $\mathbb{R}^4 = T^*\mathbb{R}^2$ equipped with the canonical symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$:

$$q_1 := x_1\xi_1 + x_2\xi_2 \quad \text{and} \quad q_2 := x_1\xi_2 - x_2\xi_1. \tag{1}$$

Let f_1, f_2 be C^∞ functions on a 4-dimensional symplectic manifold M, such that $\{f_1, f_2\} = 0$. Here the bracket $\{\cdot,\cdot\}$ is the Poisson bracket induced by the symplectic structure. We assume that the differentials df_1 and df_2 are independent

almost everywhere on M. Thus (f_1, f_2) is a completely (or "Liouville") integrable system.

If $m \in M$ is a critical point for a function $f \in C^\infty(M)$, we denote by $\mathcal{H}_m(f)$ the Hessian of f at m, which we view as a quadratic form on the tangent space T_mM.

Definition 1.1. For $F = (f_1, f_2)$ an integrable system on a symplectic 4-manifold M, m is a critical point of focus-focus type if

- $dF(m) = 0$;
- the Hessians $\mathcal{H}_m(f_1)$ and $\mathcal{H}_m(f_2)$ are linearly independent;
- there exist canonical symplectic coordinates on T_mM such that these hessians are linear combinations of the focus-focus quadratic forms q_1 and q_2.

Concretely, this definition amounts to requiring the existence of a linear symplectomorphism $U : \mathbb{R}^4 \to T_mM$ such that :

$$U^*(\mathcal{H}_m(F)) = \begin{pmatrix} aq_1 + bq_2 \\ cq_1 + dq_2 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}).$$

From a dynamical viewpoint, this implies that there exists $(\alpha, \beta) \in \mathbb{R}^2$, such that the linearization at m of the hamiltonian vector field associated to the hamiltonian $\alpha f_1 + \beta f_2$ has four distinct complex eigenvalues. Thus the Lie algebra spanned by the Hessians of f_1 and f_2 is generic (open and dense) within 2-dimensional abelian Lie algebras of quadratic forms on $(T_mM, \{\cdot, \cdot\})$. This is the non-degeneracy condition as defined by Williamson [9].

The purpose of this paper is to give a complete proof of the following theorem, which was stated in [4].

Theorem 1.2. Let (M, ω) be a symplectic 4-manifold, and $F = (f_1, f_2)$ an integrable system on M (ie. $\{f_1, f_2\} = 0$). Let m be a non-degenerate critical point of F of focus-focus type.

Then there exist a local symplectomorphism $\Psi : (\mathbb{R}^4, \omega_0) \to (M, \omega)$, defined near the origin, and sending the origin to the point m, and a local diffeomorphism $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}^2$, defined near 0, and sending 0 to $F(m)$, such that

$$F \circ \Psi = \tilde{G}(q_1, q_2)$$

The geometric content of this normal form theorem becomes clear if, given any completely integrable system F, one considers the singular foliation defined by the level sets of F. Thus, the theorem says that the foliation defined by F may, in suitable symplectic coordinates, be made equal to the foliation given by the quadratic part of F. With this in mind, the theorem can be viewed as a "symplectic Morse lemma for singular lagrangian foliations".

The normal form $\tilde{G}$ and the normalization Ψ are not unique. However, the degrees of liberty are well understood. We cite the following results for the reader's interest, but they are not used any further in this article.

Theorem 1.3 ([8]). If φ is a local symplectomorphism of $(\mathbb{R}^4, 0)$ preserving the focus-focus foliation $\{q := (q_1, q_2) = \text{const}\}$ near the origin, then there exists a unique germ of diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$q \circ \varphi = G \circ q, \tag{2}$$

and G is of the form $G = (G_1, G_2)$, where $G_2(c_1, c_2) = \varepsilon_2 c_2$ and $G_1(c_1, c_2) - \varepsilon_1 c_1$ is flat at the origin, with $\varepsilon_j = \pm 1$.

Theorem 1.4 ([6]). If φ is a local symplectomorphism of $(\mathbb{R}^4, 0)$ preserving the map $q = (q_1, q_2)$, then φ is the composition $A \circ \chi$ where A is a linear symplectomorphism preserving q and χ is the time-1 flow of a smooth function of (q_1, q_2).

1.1 The formulation in complex variables

Since Theorem 1.2 is a local theorem, we can always invoke the Darboux theorem and formulate it in local coordinates (x_1, ξ_1, x_2, ξ_2). Throughout the whole paper, we will switch whenever necessary to complex coordinates. But here, the complex coordinates are not defined in the usual way $z = x + i\xi$ but instead we set $z_1 := x_1 + ix_2$ and $z_2 := \xi_1 + i\xi_2$. We set then :

$$\frac{\partial}{\partial z_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \quad , \quad \frac{\partial}{\partial \bar{z}_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

$$\frac{\partial}{\partial z_2} := \frac{1}{2}\left(\frac{\partial}{\partial \xi_1} - i\frac{\partial}{\partial \xi_2}\right) \quad , \quad \frac{\partial}{\partial \bar{z}_2} := \frac{1}{2}\left(\frac{\partial}{\partial \xi_1} + i\frac{\partial}{\partial \xi_2}\right)$$

Introducing such notation is justified by the next properties :

Proposition 1.5.

- $q := q_1 + iq_2 = \overline{z}_1 z_2$.
- The hamiltonian flows of q_1 and q_2 in these variables are

$$\varphi^t_{q_1} : (z_1, z_2) \mapsto (e^t z_1, e^{-t} z_2) \text{ and } \varphi^s_{q_2} : (z_1, z_2) \mapsto (e^{is} z_1, e^{is} z_2) \tag{3}$$

- In complex coordinates, the Poisson bracket for real-valued functions is :

$$\{f, g\} = 2\left(-\frac{\partial f}{\partial \bar{z}_1}\frac{\partial g}{\partial z_2} + \frac{\partial f}{\partial z_2}\frac{\partial g}{\partial \bar{z}_1} - \frac{\partial f}{\partial z_1}\frac{\partial g}{\partial \bar{z}_2} + \frac{\partial f}{\partial \bar{z}_2}\frac{\partial g}{\partial z_1}\right)$$

2 Birkhoff normal form for focus-focus singularities

In this section we show 1.2 in a formal context (i.e. : with formal series instead of functions), and use the formal result to solve the problem modulo a flat function. For people familiar with Birkhoff normal forms, we compute here simultaneously the Birkhoff normal forms of 2 commuting hamiltonians.

2.1 Formal series

We consider the space $\mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$ of formal power expansions in x_1, x_2, ξ_1, ξ_2. We recall that this is a topological space for the U-adic topology, where U is the maximal ideal generated by the variables.

If $\mathring{f} \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$, we write

$$\mathring{f} = \sum_{N=0}^{+\infty} \mathring{f}^N \text{ , with } \mathring{f}^N = \sum_{i+j+k+l=N} \mathring{f}_{ijkl}\ x_1^i \xi_1^j x_2^k \xi_2^l \text{ and } \mathring{f}_{ijkl} \in \mathbb{R}$$

For $f \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$, $\mathcal{T}(f) \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$ designates the Taylor expansion of f at 0. We have the following definitions

Definition 2.1. $\mathring{\mathcal{O}}(N) := \{\mathring{f} \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!] \mid \quad \mathring{f}_{ijkl} = 0,\ \forall i + j + k + l < N\}$

Definition 2.2. $f \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$ is $\mathcal{O}(N)$ (note the difference with the previous definition) if one of the 3 equivalent conditions is fulfilled :

1. f and all its derivatives of order $< N$ at 0 are 0.
2. There exists a constant $C_N > 0$ such that, in a neighbourhood of the origin,
$$|f(x_1, x_2, \xi_1, \xi_2)| \leqslant C_N (x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)^{N/2}. \tag{4}$$
3. $\mathcal{T}(f) \in \mathring{\mathcal{O}}(N)$.

The equivalence of the above conditions is a consequence of the Taylor expansion of f. Recall, however, that if f were not supposed to be smooth at the origin, then the estimates (4) alone would not be sufficient for implying the smoothness of f.

Definition 2.3. $f \in \mathcal{C}^\infty(\mathbb{R}^4, 0)$ is flat at the origin or $\mathcal{O}(\infty)$ if for all $N \in \mathbb{N}$, it is $\mathcal{O}(N)$. Its Taylor expansion is equally zero as a formal series.

Smooth functions can be flat and yet non-zero in a neighbourhood of 0. The most classical example is the function $x \mapsto \exp(-1/x^2)$, which is in fact used in some proofs of the following Borel lemma.

Lemma 2.4 ([1]). Let $\mathring{f} \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R})[\![X_1,\dots,X_\ell]\!]$. Then there exists a function $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^{\ell+n})$ whose Taylor expansion in the $(x_1,\dots,x_\ell)$ variables is $\mathring{f}$. This function is unique modulo the addition of a function that is flat in the $(x_1,\dots,x_\ell)$ variables.

We define the Poisson bracket for formal series the same way we do in the smooth context : for $A, B \in \mathbb{R}[\![x_1,x_2,\xi_1,\xi_2]\!]$,

$$\{A,B\} = \sum_{i=1}^{n} \frac{\partial A}{\partial \xi_i}\frac{\partial B}{\partial x_i} - \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial \xi_i}.$$

The same notation will designate the smooth and the formal bracket, depending on the context. We have that Poisson bracket commutes with taking Taylor expansions : for formal series A, B,

$$\{\mathcal{T}(A),\mathcal{T}(B)\} = \mathcal{T}(\{A,B\}). \tag{5}$$

From this we deduce, if we denote $\mathcal{D}_N$ the subspace of homogeneous polynomials of degree N in the variables (x_1,x_2,ξ_1,ξ_2):

$$\{\mathcal{O}(N),\mathcal{O}(M)\} \subset \mathcal{O}(N+M-2) \quad \text{and} \quad \{\mathcal{D}_N,\mathcal{D}_M\} \subset \mathcal{D}_{N+M-2}$$

We also define $\mathrm{ad}_A : f \mapsto \{A,f\}$. We still have two preliminary lemmas needed before starting the actual proof of the Birkhoff normal form.

Lemma 2.5. For $f \in C^\infty(\mathbb{R}^4;\mathbb{R})$ and $A \in \mathcal{O}(3)$ a smooth function, we have

$$\mathcal{T}((\varphi_A^t)^* f) = \exp(t\ \mathrm{ad}_{\mathcal{T}(A)})f,$$

for each $t \in \mathbb{R}$ for which the flow on the left-hand side is defined.

Notice that, since $\mathcal{T}(A) \in \mathring{\mathcal{O}}(3)$, the right-hand side

$$\exp(t\ \mathrm{ad}_{\mathcal{T}(A)})f = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left(\mathrm{ad}_{\mathcal{T}(A)}\right)^k f$$

is always convergent in $\mathbb{R}[\![x_1,x_2,\xi_1,\xi_2]\!]$. In order to prove this lemma, we shall also use the following result which we prove immediately :

Lemma 2.6. For $f \in C^\infty(\mathbb{R}^4;\mathbb{R}^4)$ and $g \in C^\infty(\mathbb{R}^4;\mathbb{R})$, if $f(0)=0$ and $g \in \mathcal{O}(N)$, then $g \circ f \in \mathcal{O}(N)$. Moreover if f and g depend on a parameter in such a way that their respective estimates (4) are uniform with respect to that parameter, then the corresponding $\mathcal{O}(N)$-estimates for $g \circ f$ are uniform as well.

Proof. Let $\|\cdot\|_2$ denote the Euclidean norm in $\mathbb{R}^4$. In view of the estimates (4), given any two neighbourhoods of the origin U and V, there exist, by assumption, some constants C_f and C_g such that

$$\|f(X)\|_2 \leqslant C_f \|X\|_2 \quad \text{and} \quad |g(Y)| \leqslant C_g \|Y\|_2^N ,$$

for $X \in U$ and $Y \in V$. Since $f(0) = 0$, we may choose V such that $f(U) \subset V$. So we may write

$$|g(f(X))| \leqslant C_g \|f(X)\|_2^N \leqslant C_g C_f^N \|X\|_2^N ,$$

which proves the result.

□

Proof of Lemma 2.5. We write the transport equation for f

$$\frac{d}{dt}\left((\varphi_A^t)^* f\right)|_{t=s} = (\varphi_A^s)^*\{A, f\}.$$

When integrated, it comes as

$$(\varphi_A^t)^* f = f + \int_0^t (\varphi_A^s)^*\{A, f\} ds \tag{6}$$

We now show by induction that for all $N \in \mathbb{N}$:

$$(\varphi_A^s)^* f = \sum_{k=0}^{N} \frac{s^k}{k!} (\mathrm{ad}_A)^k(f) + \mathcal{O}(N+1),$$

uniformly for $s \in [0, t]$.

Initial step $\mathbf{N = 0}$: Under the integral sign in (6), $\{A, f\} \in \mathcal{O}(1)$ (at least) and $\varphi_A^s(0) = 0$, so we have with lemma 2.6 that $\{A, f\} \circ \varphi_A^s \in \mathcal{O}(1)$, uniformly for $s \in [0, t]$. After integrating the estimate (4), the result follows from (6).

Induction step : We suppose, for a given N that

$$(\varphi_A^s)^* f = \sum_{k=0}^{N} \frac{s^k}{k!} ad_A^k(f) + \mathcal{O}(N+1),$$

uniformly for $s \in [0, t]$.

Composing by ad_A on the left and by φ_A^s on the right, and then integrating, we get for any $\sigma \in [0, t]$,

$$(\varphi_A^\sigma)^* f - f = \int_0^\sigma \frac{d}{ds}\left((\varphi_A^s)^* f\right) ds = \int_0^\sigma (\varphi_A^s)^*\{A, f\} ds = \int_0^\sigma \{A, (\varphi_A^s)^* f\} ds.$$

The last equality uses that φ_A^s is symplectic and $(\varphi_A^s)^* A = A$. The induction hypothesis gives

$$\begin{aligned}(\varphi_A^\sigma)^* f &= f + \int_0^\sigma \sum_{k=0}^{N} \frac{s^k}{k!} (\mathrm{ad}_A)^{k+1}(f) ds + \int_0^\sigma \mathrm{ad}_A(\mathcal{O}(N+1)) ds \\ &= \sum_{k=0}^{N+1} \frac{\sigma^k}{k!} \ (\mathrm{ad}_A)^k(f) + \mathcal{O}(N+2),\end{aligned}$$

uniformly for $\sigma \in [0, t]$, which concludes the induction. □

The next lemma will be needed to propagate through the induction the commutation relations.

Lemma 2.7. For f_1, f_2, A formal series and $A \in \mathcal{O}(3)$

$$\{\exp(\mathrm{ad}_A) f_1, \exp(\mathrm{ad}_A) f_2\} = \exp(\mathrm{ad}_A)\{f_1, f_2\}$$

Proof. The Borel lemma gives us $\tilde{A} \in \mathcal{C}^\infty$ whose Taylor expansion is A. Since a hamiltonian flow is symplectic, we have

$$(\varphi_{\tilde{A}}^t)^* \{f_1, f_2\} = \{(\varphi_{\tilde{A}}^t)^* f_1, (\varphi_{\tilde{A}}^t)^* f_2\}. \tag{7}$$

We know from Lemma 2.5 that the Taylor expansion of $(\varphi_{\tilde{A}}^t)^* f$ is $\exp(\mathrm{ad}_A) f$. Because the Poisson bracket commutes with the Taylor expansion (see equation (5)), we can simply conclude by taking the Taylor expansion in the above equality (7). □

2.2 Birkhoff normal form

We prove here a formal Birkhoff normal form for commuting Hamiltonians near a focus-focus singularity. Recall that the focus-focus quadratic forms q_1 and q_2 were defined in (1).

Theorem 2.8. Let $f_i \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$, $i = 1, 2$, such that

- $\{f_1, f_2\} = 0$
- $f_1 = a q_1 + b q_2 + \mathcal{O}(3)$
- $f_2 = c q_1 + d q_2 + \mathcal{O}(3)$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$

Then there exists $A \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$, with $A \in \mathcal{O}(3)$, and there exist $g_i \in \mathbb{R}[\![t_1, t_2]\!]$, $i = 1, 2$ such that :

$$\exp(\mathrm{ad}_A)(f_i) = g_i(q_1, q_2), \qquad i = 1, 2 \tag{8}$$

Proof. Let's first start by left-composing $F := (f_1, f_2)$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$, to reduce to the case where $f_1 = q_1 + \mathcal{O}(3)$ and $f_2 = q_2 + \mathcal{O}(3)$. Let $z_1 := x_1 + ix_2$ and $z_2 := \xi_1 + i\xi_2$.

Of course, an element in $\mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$ can be written as a formal series in the variables $z_1, \bar{z}_1, z_2, \bar{z}_2$. We consider a generic monomial $\mathbf{z}^{\alpha\beta} = z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$. Using (3), it is now easy to compute :

$$\begin{aligned} \{q_1, \mathbf{z}^{\alpha\beta}\} &= \frac{d}{dt}\left((\varphi_{q_1}^t)^* z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}\right)|_{t=0} \\ &= \frac{d}{dt}\left(e^{(\alpha_1-\alpha_2+\beta_1-\beta_2)t}\mathbf{z}^{\alpha\beta}\right)|_{t=0} = (\alpha_1 - \alpha_2 + \beta_1 - \beta_2)\mathbf{z}^{\alpha\beta}. \end{aligned} \tag{9}$$

For the same reasons, we have :

$$\{q_2, \mathbf{z}^{\alpha\beta}\} = i(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)\mathbf{z}^{\alpha\beta}. \tag{10}$$

Hence, the action of q_1 and q_2 is diagonal on this basis. A first consequence of this is the following remark: any formal series $f \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$ such that $\{f, q_1\} = \{f, q_2\} = 0$ has the form $f = g(q_1, q_2)$ with $g \in \mathbb{R}[\![t_1, t_2]\!]$ (and the reverse statement is obvious). Indeed, let

$$f = \sum f_{\alpha\beta}\mathbf{z}^{\alpha\beta};$$

If $\{q_1, f\} = \{q_2, f\} = 0$, then all $f_{\alpha\beta}$ must vanish, except when $\alpha_1 - \alpha_2 + \beta_1 - \beta_2 = 0$ and $\alpha_1 + \alpha_2 - \beta_1 - \beta_2 = 0$. For these monomials, $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$, that is, they are of the form $(\bar{z}_1 z_2)^\lambda (z_1 \bar{z}_2)^\mu = q^\lambda \bar{q}^\mu$ (remember that $q = q_1 + iq_2 = \bar{z}_1 z_2$). Thus we can write

$$f = \sum c_{k\ell} q_1^k q_2^\ell, \qquad c_{k\ell} \in \mathbb{C}.$$

Specializing to $x_2 = 0$ we have $q_1^k q_2^\ell = x_1^{k+\ell}\xi_1^k\xi_2^\ell$. Since $f \in \mathbb{R}[\![x_1, x_2, \xi_1, \xi_2]\!]$, we see that $c_{k\ell} \in \mathbb{R}$, which establishes our claim.

We are now going to prove the theorem by constructing A and g by induction.

Initial step : Taking $A = 0$, the equation (8) is already satisfied modulo $\mathcal{O}(3)$, by assumption.

Induction step : Let $N \geqslant 2$ and suppose now that the equation is solved modulo $\mathcal{O}(N+1)$: we have then constructed polynomials $\mathcal{A}^{(N)}$, $g_i^{(N)}$ such that

$$\exp(\mathrm{ad}_{\mathcal{A}^{(N)}})(f_i) \equiv \underbrace{g_i^{(N)}(q_1, q_2)}_{d^\circ < N+1} + \underbrace{r_i^{N+1}(z_1, z_2, \bar{z}_1, \bar{z}_2)}_{\in \mathcal{D}_{N+1}} \mod \mathcal{O}(N+2).$$

With the lemma 2.7, we have that

$$\{exp(\mathrm{ad}_{\mathcal{A}^{(N)}})f_1, exp(\mathrm{ad}_{\mathcal{A}^{(N)}})f_2\} = exp(\mathrm{ad}_{\mathcal{A}^{(N)}})\{f_1, f_2\} = 0$$

Hence

$$\{g_1^{(N)}, g_2^{(N)}\} + \{g_1^{(N)}, r_2^{N+1}\} + \{r_2^{N+1}, g_1^{(N)}\} + \{r_1^{N+1}, r_2^{N+1}\} \equiv 0 \mod \mathcal{O}(N+2).$$

Since $g_1^{(N)}$ and $g_2^{(N)}$ are polynomials in (q_1, q_2), they must commute : $\{g_1^{(N)}, g_2^{(N)}\} = 0$. On the other hand, $\{r_1^{N+1}, r_2^{N+1}\} \in \mathcal{D}(2N)$ so

$$\{ \underbrace{g_1^{(N)}}_{=q_1+\mathcal{O}(3)}, r_2^{N+1}\} + \{r_1^{N+1}, \underbrace{g_2^{(N)}}_{=q_2+\mathcal{O}(3)} \} \equiv 0 \mod \mathcal{O}(N+2)$$

which implies $\{r_1^{N+1}, q_2\} = \{r_2^{N+1}, q_1\}$.

One then looks for $\mathcal{A}^{(N+1)} - \mathcal{A}^{(N)} = A_{N+1} \in \mathcal{D}_{N+1}$, and $g_i^{(N+1)} - g_i^{(N)} = g_i^{N+1} \in \mathcal{D}_{N+1}$, with $\{g_i^{N+1}, q_j\} = 0$ for $i, j = 1, 2$, such that

$$\exp(\mathrm{ad}_{\mathcal{A}^{(N+1)}})(f_i) \equiv g^{(N+1)} \mod \mathcal{O}(N+2). \tag{11}$$

We have

$$\exp(\mathrm{ad}_{A^{(N+1)}}) = \exp(\mathrm{ad}_{\mathcal{A}^{(N)}} + \mathrm{ad}_{A_{N+1}}) = \sum_{k=0}^{+\infty} \frac{(\mathrm{ad}_{\mathcal{A}^{(N)}} + \mathrm{ad}_{A_{N+1}})^k}{k!}$$

Here, one needs to expand a non-commutative binomial. We set

$$\langle (\mathrm{ad}_{\mathcal{A}^{(N)}})^{k-l} (\mathrm{ad}_{A_{N+1}})^l \rangle$$

for the summand of all possible words formed with $k-l$ occurrences of $\mathrm{ad}_{\mathcal{A}^{(N)}}$ and l occurences of $\mathrm{ad}_{A_{N+1}}$. We have then the formula

$$\begin{aligned}(\mathrm{ad}_{\mathcal{A}^{(N)}} + \mathrm{ad}_{A_{N+1}})^k = \mathrm{ad}^k_{\mathcal{A}^{(N)}} &+ \langle (\mathrm{ad}_{\mathcal{A}^{(N)}})^{k-1} \mathrm{ad}_{A_{N+1}} \rangle \\ &+ \ldots + \langle \mathrm{ad}_{\mathcal{A}^{(N)}} (\mathrm{ad}_{A_{N+1}})^{k-1} \rangle + (\mathrm{ad}_{A_{N+1}})^k\end{aligned}$$

What are the terms that we have to keep modulo $\mathcal{O}(N+2)$? As far as we only look here to modifications of the total valuation, the order of the composition between *ad*'s doesn't matter here. Let's examinate closely :
$\mathrm{ad}^k_{A_{N+1}}(\mathcal{O}(n)) \subset \mathcal{O}(k(N+1)+n-2)$, and so for any $h \in \mathcal{O}(3)$ and any $k \geqslant 1$,

$$\mathrm{ad}^k_{A_{N+1}}(h) \in \mathcal{O}(N+2).$$

Thus

$$\exp(\mathrm{ad}_{A^{(N+1)}})(f_i) \equiv \underbrace{\sum_{k=0}^{+\infty} \frac{\mathrm{ad}^k_{\mathcal{A}^{(N)}}(f_i)}{k!}}_{=exp(\mathrm{ad}_{\mathcal{A}^{(N)}})(f_i)} + \mathrm{ad}_{A_{N+1}}(q_i) \mod \mathcal{O}(N+2).$$

Therefore, equation (11) becomes

$$g_i^{(N)} + r_i^{N+1} + \mathrm{ad}_{A_{N+1}}(q_i) = g^{(N+1)} \mod \mathcal{O}(N+2),$$

and hence

$$(11) \iff \begin{cases} \{A_{N+1}, q_1\} + r_1^{N+1} = g_1^{N+1} \\ \{A_{N+1}, q_2\} + r_2^{N+1} = g_2^{N+1} \end{cases} \tag{12}$$

Using the notation $A_{N+1} = \sum A_{N+1,\alpha\beta}\mathbf{z}^{\alpha\beta}$, $r_i^{N+1} = \sum r_{i,\alpha\beta}^{N+1}\mathbf{z}^{\alpha\beta}$, and $g_i^{N+1} = \sum g_{i,\alpha\beta}^{N+1}\mathbf{z}^{\alpha\beta}$, we have

$$(12) \iff \begin{cases} A_{N+1,\alpha\beta}(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) + r_{1,\alpha\beta}^{N+1} = g_{1,\alpha\beta}^{N+1} \\ iA_{N+1,\alpha\beta}(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) + r_{2,\alpha\beta}^{N+1} = g_{2,\alpha\beta}^{N+1} \end{cases}$$

We can then solve the first equation by setting,

a) when $\alpha_1 - \alpha_2 + \beta_1 - \beta_2 \neq 0$:
$g_{1,\alpha\beta}^{N+1} := 0$ and $A_{\alpha\beta} := \frac{-r_{1,\alpha\beta}^{N+1}}{\alpha_1-\alpha_2+\beta_1-\beta_2}$ (these choices are necessary);

b) when $\alpha_1 - \alpha_2 + \beta_1 - \beta_2 = 0$:
$g_{1,\alpha\beta}^{N+1} := r_{1,\alpha\beta}^{N+1}$ (necessarily), and $A_{\alpha\beta} := 0$ (this one is arbitrary).

Similarly, we solve the second equation by setting,

c) when $\alpha_1 + \alpha_2 - \beta_1 - \beta_2 \neq 0$:
$g_{1,\alpha\beta}^{N+1} := 0$ and $A_{\alpha\beta} := \frac{-r_{2,\alpha\beta}^{N+1}}{i(\alpha_1+\alpha_2-\beta_1-\beta_2)}$ (necessarily);

d) when $\alpha_1 + \alpha_2 - \beta_1 - \beta_2 = 0$:
$g_{2,\alpha\beta}^{N+1} := r_{2,\alpha\beta}^{N+1}$ (necessarily), and $A_{\alpha\beta} := 0$ (arbitrarily).

Notice that (a) and (c) imply (in view of (9) and (10)) that g_i^{N+1} commutes with q_1 and q_2.

Of course we need to check that the choices for $A_{\alpha\beta}$ in (a) and (c) are compatible with each other. This is ensured by the "cross commuting relation" $\{r_1^{N+1}, q_2\} = \{r_2^{N+1}, q_1\}$, which reads

$$i(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)r_{1,\alpha\beta}^{N+1} = (\alpha_1 - \alpha_2 + \beta_1 - \beta_2)r_{2,\alpha\beta}^{N+1}.$$

□

Remark 2.9 The relation $\{r_1^{N+1}, q_2\} = \{r_2^{N+1}, q_1\}$ is a cocycle condition if we look at (12) as a cohomological equation. The relevant complex for this cohomological theory is a deformation complex "à la Chevalley-Eilenberg", described, for instance, in [5]. △

As a corollary of the Birkhoff normal form, we get a statement concerning C^∞ smooth functions, up to a flat term.

Lemma 2.10. Let $F = (f_1, f_2)$, where f_1 and f_2 satisfy the same hypothesis as in the Birkhoff theorem 2.8. Then there exist a symplectomorphism χ of $\mathbb{R}^4 = T^*\mathbb{R}^2$, tangent to the identity, and a smooth local diffeomorphism $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, tangent to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that :

$$\chi^* F = \tilde{G}(q_1, q_2) + \mathcal{O}(\infty).$$

Proof. We use the notation of (8). Let $\tilde{g}_j$, $j = 1, 2$ be Borel summations of the formal series g_j, and let $\tilde{A}$ be a Borel summation of A. Let $\tilde{G} := (\tilde{g}_1, \tilde{g}_2)$ and $\chi := \varphi^1_{\tilde{A}}$. Applying Lemma 2.5, we see that the Taylor series of

$$\chi^* F - \tilde{G}(q_1, q_2)$$

is flat at the origin.

□

We can see that Lemma 2.10 gives us the main theorem modulo a flat function. The rest of the paper is devoted to absorbing this flat function.

3 A Morse lemma in the focus-focus case

One of the key ingredients of the proof is a (smooth, but non symplectic) equivariant Morse lemma for commuting functions. In view of the Birkhoff normal form, it is enough to state it for flat perturbations of quadratic forms, as follows.

Theorem 3.1. Let h_1, h_2 be functions in $C^\infty(\mathbb{R}^4, 0)$ such that

$$\begin{cases} h_1 = q_1 + \mathcal{O}(\infty) \\ h_2 = q_2 + \mathcal{O}(\infty). \end{cases}$$

Assume $\{h_1, h_2\} = 0$ for the canonical symplectic form on $\mathbb{R}^4 = T^*\mathbb{R}^2$.

Then there exists a local diffeomorphism Υ of $(\mathbb{R}^2, 0)$ of the form $\Upsilon = id + \mathcal{O}(\infty)$ such that

$$\Upsilon^* h_i = q_i \ , \ i = 1, 2$$

Moreover, we can choose Υ such that the symplectic gradient of q_2 for ω_0 and for $\omega = \Upsilon^*\omega_0$ are equal, which we can write as

$$(\mathcal{P}) : \quad \imath_{X_2^0}(\Upsilon^*\omega_0) = -dq_2$$

That is to say, Υ preserves the $\mathbb{S}^1$-action generated by q_2.

3.1 Proof of the classical flat Morse lemma

In a first step, we will establish the flat Morse lemma without the $(\mathcal{P})$ equivariance property. This result, that we call the "classical" flat Morse lemma, will be used in the next section to show the equivariant result.

Proof. Using Moser's path method, we shall look for Υ as the time-1 flow of a time-dependent vector field X_t, which should be uniformly flat for $t \in [0,1]$.

We define : $H_t := (1-t)Q + tH = (1-t)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + t\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. We want X_t to satisfy

$$(\varphi^t_{X_t})^* H_t = Q \ , \ \forall t \in [0,1].$$

Differentiating this equation with respect to t, we get

$$(\varphi^t_{X_t})^* \left[\frac{\partial H_t}{\partial t} + \mathcal{L}_{X_t} H_t\right] = (\varphi^t_{X_t})^* \left[-Q + H + (\imath_{X_t} d + d\imath_{X_t}) H_t\right] = 0.$$

So it is enough to find a neighbourood of the origin where one can solve, for $t \in [0,1]$, the equation

$$dH_t(X_t) = Q - H =: R. \tag{13}$$

Notice that R is flat and $dH_t = dQ - tdR = dQ + \mathcal{O}(\infty)$.

Let Ω be an open neighborhood of the origin in $\mathbb{R}^4$. Let's consider dQ as a linear operator from $\mathcal{X}(\Omega)$, the space of smooth vector fields, to $C^\infty(\Omega)^2$, the space of pairs of smooth functions. This operator sends flat vector fields to flat functions.

Before going on, we wish to add here a few words concerning flat functions. Assume Ω is contained in the euclidean unit ball. Let $C^\infty(\Omega)_{\text{flat}}$ denote the vector space of flat functions defined on Ω. For each integer $N \geqslant 0$, and each $f \in C^\infty(\Omega)_{\text{flat}}$, the quantity

$$p_N(f) = \sup_{z \in \Omega} \frac{|f(z)|}{\|z\|_2^N}$$

is finite due to (4), and thus the family (p_N) is an increasing[(1)] family of norms on $C^\infty(\Omega)_{\text{flat}}$. We call the corresponding topology the "local topology at the origin", as opposed to the usual topology defined by suprema on compact subsets of Ω. Thus, a linear operator A from $C^\infty(\Omega)_{\text{flat}}$ to itself is continuous in the local topology if and only if

$$\forall N \geqslant 0, \quad \exists N' \geqslant 0, \exists C > 0, \forall f \quad p_N(Af) \leqslant C p_{N'}(f). \tag{14}$$

(1) $p_{N+1} \geqslant p_N$

For such an operator, if f depends on an additional parameter and is uniformly flat, in the sense that the estimates (4) are uniform with respect to that parameter, then Af is again uniformly flat.

Lemma 3.2. Restricted to flat vector fields and flat functions, dQ admits a linear right inverse $\Psi : C^\infty(\Omega)^2_{\text{flat}} \to \mathcal{X}(\Omega)_{\text{flat}}$: for every $U = (u_1, u_2) \in C^\infty(\Omega)^2$ with $u_j \in \mathcal{O}(\infty)$, one has

$$\begin{cases} dq_1(\Psi(U)) = u_1 \\ dq_2(\Psi(U)) = u_2. \end{cases}$$

Moreover Ψ is a multiplication operator, explicitly :

$$\Psi(U) = \frac{1}{x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \\ x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{15}$$

In (15) the right-hand side is a matrix product; we have identified $\Psi(U)$ with a vector of 4 functions corresponding to the coordinates of $\Psi(U)$ in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$.

An immediate corollary of this lemma is that Ψ is continuous in the local topology. Indeed, if $|u_j| \leqslant C(x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)^{N/2}$ for $j = 1, 2$, then

$$\|\Psi(U)(x_1, x_2, \xi_1, \xi_2)\| \leqslant d(\Omega) C(x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2)^{N/2-1}. \tag{16}$$

Here $\|\Psi(U)(x_1, x_2, \xi_1, \xi_2)\|$ is the supremum norm in $\mathbb{R}^4$ and $d(\Omega)$ is the diameter of Ω.

Now assume the lemma holds, and let $A := \Psi \circ dR$, where $R = (r_1, r_2)$ was defined in (13). A is a linear operator from $\mathcal{X}(\Omega)_{\text{flat}}$ to itself, sending a vector field v to the vector field $\Psi(dr_1(v), dr_2(v))$. From the lemma, we get :

$$dQ(A(v)) = dR(v).$$

We claim that for Ω small enough, the operator $(\text{Id} - tA)$ is invertible, and its inverse is continuous in the local topology, uniformly for $t \in [0, 1]$. Now let

$$X_t := (\text{Id} - tA)^{-1} \circ \Psi(R).$$

We compute :

$$dH_t(X_t) = dQ(X_t) - tdR(X_t) = dQ(X_t) - tdQ(A(X_t)) = dQ(\text{Id} - tA)(X_t).$$

Hence

$$dH_t(X_t) = dQ(\Psi(R)) = R.$$

Thus equation (13) is solved on Ω. Since $X_t(0) = 0$ for all t, the standard Moser's path argument shows that, up to another shrinking of Ω, the flow of X_t is defined up to time 1. Because of the continuity in the local topology, X_t is uniformly flat, which implies that the flow at time 1, Υ, is the identity modulo a flat term. $\square$

To make the above proof complete, we still need to prove Lemma 3.2 and the claim concerning the invertibility of $(\mathrm{Id} - tA)$.

Proof of Lemma 3.2. Let $u := u_1 + iu_2$ and $q := q_1 + iq_2$. Thus we want to find a real vector field Y such that

$$dq(Y) = u. \tag{17}$$

We invoke again the complex structure used in the Birkhoff theorem, and look for Y in the form $Y = a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial \bar{z}_1} + c\frac{\partial}{\partial z_2} + d\frac{\partial}{\partial \bar{z}_2}$. The vector field Y is real if and only if $a = \bar{b}$ and $c = \bar{d}$. Writing

$$dq = d(\bar{z}_1 z_2) = z_2 d\bar{z}_1 + \bar{z}_1 dz_2$$

we see that (17) is equivalent to

$$z_2 b + \bar{z}_1 c = u.$$

Since u is flat, there exists a smooth flat function $\tilde{u}$ such that $u = (|z_1|^2 + |z_2|^2)\tilde{u} = z_1\bar{z}_1\tilde{u} + z_2\bar{z}_2\tilde{u}$.

Thus we find a solution by letting $\begin{cases} b = \bar{z}_2\tilde{u} \\ c = z_1\tilde{u} \end{cases}$. Back in the original basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$, we have then

$$\Psi(U) := Y = \frac{1}{|z_1|^2 + |z_2|^2}\left(\Re e(\bar{z}_2 u), \Im m(\bar{z}_2 u), \Re e(z_1 u), \Im m(z_1 u)\right)$$

Thus, Ψ is indeed a linear operator and its matrix on this basis is

$$\frac{1}{x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2}\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \\ x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}$$

□

Now consider the operator $A = \Psi \circ dR$. We see from equation (15) and the fact that the partial derivatives of R are flat that, when expressed in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$ of $\mathcal{X}(\Omega)_{\text{flat}}$, A is a 4×4 matrix with coefficients in $C^\infty(\Omega)_{\text{flat}}$. In particular one may choose Ω small enough such that $\sup_{z\in\Omega} \|A(z)\| < 1/2$. Thus, for all $t \in [0, 1]$, the matrix $(\mathrm{Id} - tA)$ is invertible and its inverse is of the form $\mathrm{Id} + t\tilde{A}_t$, where $\tilde{A}_z$ has smooth coefficients and $\sup_{z\in\Omega} \left\|\tilde{A}_t(z)\right\| < 1$. Now $p_N(f + t\tilde{A}_t f) \leqslant p_N(f) + p_N(\tilde{A}_t f) \leqslant 2p_N(f)$: $(\mathrm{Id} + t\tilde{A}_t)$ is uniformly continuous in the local topology.

With this the proof of Theorem 3.1, without the $(\mathcal{P})$ property, is now complete.

3.2 Proof of the equivariant flat Morse lemma

We will now use the flat Morse lemma we've just shown to obtain the equivariant version, ie. Theorem 3.1.

The main ingredient will be the construction of a smooth hamiltonian S^1 action on the symplectic manifold $(\mathbb{R}^4, \omega_0)$ that leaves the original moment map (h_1, h_2) invariant. The natural idea is to define the moment map H through an action integral on the lagrangian leaves, but because of the singularity, it is not obvious that we get a smooth function.

Let γ_z be the loop in $\mathbb{R}^4$ equal to the S^1-orbit of z for the action generated by q_2 with canonical symplectic form ω_0. Explicitly (see (3)), we can write $z = (z_1, z_2)$ and

$$\gamma_z(t) = (e^{2\pi i t} z_1, e^{2\pi i t} z_2), \qquad t \in [0, 1].$$

Notice that if $z = 0$ then the "loop" γ_z is in fact just a point. A key formula is

$$q_2 = \frac{1}{2\pi} \int_{\gamma_z} \alpha_0.$$

This can be verified by direct computation, or as a consequence of the following lemma.

Lemma 3.3. Let α_0 be the Liouville 1-form on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ (thus $\omega_0 = d\alpha_0$). If H is a hamiltonian which is homogeneous of degree n in the variables $(\xi_1, \ldots, \xi_n)$, defining $\mathcal{X}_H^0$ as the symplectic gradient of H induced by ω_0, we have :

$$\alpha_0(\mathcal{X}_H^0) = nH$$

Proof. Consider the $\mathbb{R}_+^*$-action on $T^*\mathbb{R}^n$ given by multiplication on the cotangent fibers: $\varphi^t(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = (x_1, \ldots, x_n, t\xi_1, \ldots, t\xi_n)$. Since $\alpha_0 = \sum_{i=1}^n \xi_i dx_i$, we have : $\varphi^{t*}\alpha_0 = t\alpha_0$. Taking the derivative with respect to t gives $\mathcal{L}_\Xi \alpha_0 = \alpha_0$ where Ξ is the infinitesimal action of φ^t : $\Xi = (0, \ldots, 0, \xi_1, \ldots, \xi_n)$. By Cartan's formula, $\imath_\Xi d\alpha_0 + d(\imath_\Xi \alpha_0) = \alpha_0$. But $\imath_\Xi \alpha_0 = \sum_{i=1}^n \xi_i dx_i(\Xi) = 0$ so $\alpha_0 = \imath_\Xi \omega_0$.

Thus, $\alpha_0(\mathcal{X}_H^0) = \omega_0(\Xi, \mathcal{X}_H^0) = dH(\Xi)$, and since H is a homogeneous function of degree n with respect to Ξ, Euler's formula gives $\mathcal{L}_\Xi H = dH(\Xi) = nH$. Therefore we get, as required :

$$dH(\Xi) = nH(\Xi)$$

□

From this lemma, we deduce, since q_2 is invariant under $\mathcal{X}_{q_2}^0$:

$$\frac{1}{2\pi} \int_{\gamma_z} \alpha_0 = \int_0^1 \alpha_{0\gamma_2(t)}(\mathcal{X}_{q_2}^0) dt = \int_0^1 q_2(\gamma_z(t)) = q_2$$

By the classic flat Morse lemma we have a local diffeomorphism $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\Phi^* h_j = q_j$, $j = 1, 2$. Let $\alpha := (\Phi^{-1})^* \alpha_0$, and let

$$K(z) := \frac{1}{2\pi} \int_{\gamma_z} \alpha,$$

and let $H := K \circ \Phi$. Note that

$$H(m) = \int_{\gamma_2[m]} \alpha_0$$

where $\gamma_2[m] := \Phi^{-1} \circ \gamma_2[z = \Phi(m)]$, and $H(0) = 0$.

We prove now that H is a hamiltonian moment map for an S^1-action on M leaving (h_1, h_2) invariant.

Lemma 3.4. $H \in C^\infty(\mathbb{R}^4, 0)$.

Proof. Equivalently, we prove that $K \in C^\infty(\mathbb{R}^4, 0)$. The difficulty lies in the fact that the family of "loops" γ_z is not locally trivial: it degenerates into a point when $z = 0$. However it is easy to desingularize K, as follows. Again we identify $\mathbb{R}^4$ with $\mathbb{C}^2$; we introduce the maps :

$$\begin{array}{llll} j : \mathbb{C} \times \mathbb{C}^2 & \longrightarrow \mathbb{C}^2 & j_z : & \mathbb{C} \to M \\ (\zeta, (z_1, z_2)) & \mapsto (\zeta z_1, \zeta z_2) & & \zeta \mapsto (\zeta z_1, \zeta z_2) \end{array}$$

so that $\gamma_z = (j_z)_{|U(1)}$. Let $D \subset \mathbb{C}$ be the closed unit disc $\{\zeta \leqslant 1\}$. Thus

$$\int_{\gamma_z} \alpha = \int_{j_z(U(1))} \alpha = \int_{U(1)} j_z^* \alpha = \int_{\partial D} j_z^* \alpha.$$

Let $\omega := d\alpha$. By Stokes' formula,

$$\int_{\partial D} j_z^* \alpha = \int_D j_z^* \omega = \int_D \omega_{j(z,\zeta)}(d_\zeta j(z, \zeta)(\cdot), d_\zeta j(z, \zeta)(\cdot)).$$

Since D is a fixed compact set and ω, j are smooth, we get $K \in C^\infty(\mathbb{R}^4, 0)$. $\square$

Consider now the integrable system (h_1, h_2). Since H is an action integral for the Liouville 1-form α_0, it follows from the action-angle theorem by Liouville-Arnold-Mineur that the hamiltonian flow of H preserves the regular Liouville tori of (h_1, h_2). Thus, $\{H, h_j\} = 0$ for $j = 1, 2$ on every regular torus. The function $\{H, h_j\}$ being smooth hence continuous, $\{H, h_j\} = 0$ everywhere it is defined : H is locally constant on every level set of the joint moment map (h_1, h_2). Equivalently, K is locally constant on the level sets of $q = (q_1, q_2)$. It is easy to check that these level sets are locally connected near the origin. Thus there exists a map $g : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that

$$K = g \circ q.$$

It is easy to see that g must be smooth : indeed, K itself is smooth and, in view of (1), one can write

$$g(c_1, c_2) = K(x_1 = c_1, x_2 = -c_2, \xi_1 = 1, \xi_2 = 0). \tag{18}$$

We claim that the function $(c_1, c_2) \mapsto g(c_1, c_2) - c_2$ is flat at the origin : since $\Phi = id + \mathcal{O}(\infty)$, we have : $\alpha = \Phi^* \alpha_0 = \alpha_0 + \mathcal{O}(\infty)$ so $K(z) = \int_{\gamma_2[z]} \alpha_0 + \mathcal{O}(\infty) = q_2 + \eta(z)$ with η a flat function of the 4 variables. We show now the lemma :

Lemma 3.5. Let $\eta \in C^\infty(\mathbb{R}^4; \mathbb{R})$ be a flat function at the origin in $\mathbb{R}^4$ such that $\eta(z) = \mu(q_1, q_2)$ for some map $\mu : \mathbb{R}^2 \to \mathbb{R}$. Then μ is flat at the origin in $\mathbb{R}^2$.

Proof. We already know from (18) that μ has to be smooth. Thus, it is enough to show the estimates : $\forall N \in \mathbb{N}, \exists C_N \in \mathbb{R}$ such that

$$\forall (c_1, c_2) \in \mathbb{R}^2, \qquad |\eta(c_1, c_2)| \leqslant C_N \left\|(c_1, c_2)\right\|^N = C_N(|c_1|^2 + |c_2|^2)^{N/2}.$$

Since η is flat, we have, for some constant C_N,

$$|\eta(x_1, x_2, \xi_1, \xi_2)| \leqslant C_N \left\|(x_1, x_2, \xi_1, \xi_2)\right\|^N.$$

But for any $c = (c_1 + ic_2) \in \mathbb{C}$, there exists $(z_1, z_2) \in q^{-1}(c)$ such that $2|c|^2 = |z_1|^2 + |z_2|^2$: if $c = 0$ we take $z = 0$, otherwise take $z_1 := |c|^{1/2}$ and $z_2 := c/z_1$, so that $q(z_1, z_2) = \bar{z}_1 z_2 = c$.

Therefore, for all $(c_1, c_2) \in \mathbb{R}^2$ we can write

$$|\mu(c_1, c_2)| = |\eta(z_1, z_2)| \leqslant C_N \|(z_1, z_2)\|^N \leqslant 2C_N |c|^N,$$

which finishes the proof. □

We have now :

$$\begin{pmatrix} h_1 \\ H \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 + \mu(h_1, h_2) \end{pmatrix}$$

By the implicit function theorem, the function $V : (x, y) \mapsto (x, y + \mu(x, y)$ is locally invertible around the origin, since μ is flat; moreover, V^{-1} is infinitely tangent to the identity. Therefore, in view of the statement of Theorem 3.1, we can replace our initial integrable system (h_1, h_2) by the system $V \circ (h_1, h_2) = (h_1, H)$.

Thus, we have reduced our problem to the case where $h_2 = H$ is a hamiltonian for a smooth S^1 action on $\mathbb{R}^4$. We denote by S^1_H this action. The origin is a fixed point, and we denote by $\mathrm{lin}(S^1_H)$ the action linearized at the origin. We now invoke an equivariant form of the Darboux theorem.

Theorem 3.6 (Darboux-Weinstein [2]). There exists φ a diffeomorphism of $(\mathbb{R}^4, 0)$ such that :

$$\left(\mathbb{R}^4, \omega_0, S^1_H\right) \xrightarrow{\varphi} \left(T_0\mathbb{R}^4, T_0\omega_0, \mathrm{lin}(S^1_H)\right)$$

The linearization $T_0\omega_0$ of ω_0 is ω_0 : φ is a symplectomorphism, and the linearization of the S^1-action of H is the S^1-action of the quadratic part of H, which is q_2. Hence $H \circ \varphi^{-1}$ and q_2 have the same symplectic gradient, and both vanish at the origin : so $H \circ \varphi^{-1} = q_2$. So we have got rid of the flat part of h_2 without modifying the symplectic form. The last step is to give a precised version of the equivariant flat Morse lemma :

Lemma 3.7. Let h_1, h_2 be functions in $C^\infty(\mathbb{R}^4, 0)$ such that

$$\begin{cases} h_1 = q_1 + \mathcal{O}(\infty) \\ h_2 = q_2 \end{cases}$$

Then there exists a local diffeomorphism Υ of $(\mathbb{R}^2, 0)$ of the form $\Upsilon = id + \mathcal{O}(\infty)$ such that

$$\Upsilon^* h_i = q_i \ , \ i = 1, 2$$

Moreover, we can choose Υ such that the symplectic gradient of q_2 for ω_0 and for $\Upsilon^*\omega_0$ are equal, which we can write as

$$(\mathcal{P}) : \quad \imath_{\mathcal{X}^0_{q_2}}(\Upsilon^*\omega_0) = -dq_2$$

That is, Υ preserves the S^1-action generated by q_2.

Proof. Following the same Moser's path method we used in the proof of the classical flat Morse lemma, we come up with the following cohomological equation (13)

$$(Z) \begin{cases} (dq_1 + tdr_1)(X_t) = r_1 \\ dq_2(X_t) = 0 \end{cases}$$

The classical flat Morse lemma ensures the existence of a solution X_t to this system. We have then that $\{r_1, q_2\} = \{r_2, q_1\} = 0$, because here $r_2 = 0$. We have also that : $\{q_1, q_2\} = 0$, $\{q_2, q_2\} = 0$, so r_1,q_1 and q_2 are invariant by the flow of q_2. So we can average (Z) by the action of q_2 : let $\varphi^s_2 := \varphi^s_{\mathcal{X}^0_{q_2}}$ be the time s-flow of the vector field $\mathcal{X}^0_{q_2}$ and let

$$\langle X_t \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\varphi^s_2)^* X_t ds.$$

If a function f is invariant under φ^s_2, i.e. $(\varphi^s_2)^* f = f$, then

$$((\varphi^s_2)^* X_t) f = ((\varphi^s_2)^* X_t)((\varphi^s_2)^* f) = (\varphi^s_2)^*(X_t f).$$

Integrating over $s \in [0, 2\pi]$, we get $\langle X_t \rangle f = \langle X_t f \rangle$, where the latter is the standard average of functions. Therefore $\langle X_t \rangle$ satisfies the system (Z) as well.

Finally, we have, for any s, $(\varphi_2^t)^* \langle X_t \rangle = \langle X_t \rangle$ which implies

$$[\mathcal{X}_{q_2}^0, \langle X_t \rangle] = 0; \tag{19}$$

in turn, if we let $\varphi_{\langle X_t \rangle}^t$ be the flow of the non-autonomous vector field $\langle X_t \rangle$, integrating (19) with respect to t gives $(\varphi_{\langle X_t \rangle}^t)^* \mathcal{X}_{q_2}^0 = \mathcal{X}_{q_2}^0$. For $t = 1$ we get $\Upsilon^* \mathcal{X}_{q_2}^0 = \mathcal{X}_{q_2}^0$. But, by naturality $\Upsilon^* \mathcal{X}_{q_2}^0$ is the symplectic gradient of $\Upsilon^* q_2 = q_2$ with respect to the symplectic form $\Upsilon^* \omega_0 = \omega$, so property $(\mathcal{P})$ is satisfied. □

4 Principal lemma

4.1 Division lemma

The following cohomological equation, formally similar to to (12), is the core of Theorem 1.2.

Theorem 4.1. Let $r_1, r_2 \in \mathcal{C}^\infty((\mathbb{R}^4, 0); \mathbb{R})$, flat at the origin such that $\{r_1, q_2\} = \{r_2, q_1\}$. Then there exists $f \in \mathcal{C}^\infty((\mathbb{R}^4, 0); \mathbb{R})$ and $\phi_2 \in C^\infty((\mathbb{R}^2, 0); \mathbb{R})$ such that

$$\begin{cases} \{f, q_1\}(x, \xi) = r_1 \\ \{f, q_2\}(x, \xi) = r_2 - \phi_2(q_1, q_2), \end{cases} \tag{20}$$

and f and ϕ_2 are flat at the origin. Moreover ϕ_2 is unique and given by

$$\forall z \in \mathbb{R}^4, \quad \phi_2(q_1(z), q_2(z)) = \frac{1}{2\pi} \int_0^{2\pi} (\varphi_{q_2}^s)^* r_2(z) ds, \tag{21}$$

where $s \mapsto \varphi_{q_2}^s$ is the hamiltonian flow of q_2.

One can compare the difficulty to solve this equation in the 1D hyperbolic case with elliptic cases. If the flow is periodic, that is, if $SO(q)$ is compact, then one can solve the cohomological equation by averaging over the action of $SO(q)$. This is what happens in the elliptic case. But for a hyperbolic singularity, $SO(q)$ is not compact anymore, and the solution is more technical (see [3]). In our focus-focus case, we have to solve simultaneously two cohomological equations, one of which yields a compact group action while the other doesn't. This time again, it is the "cross commuting relation" $\{r_i, q_j\} = \{r_j, q_i\}$ that we already encountered in the formal context that will allow us to solve simultaneously both equations.

Proof. Let $\varphi_{q_1}^s, \varphi_{q_2}^s$ be respectively the flows of q_1 and q_2. From (3), we see φ_{q_2} is 2π-periodic : q_2 is a momentum map of a hamiltonian $\mathbb{S}^1$-action. Since

$(\varphi^s_{q_2})^*\{f,q_2\} = -\frac{d}{dt}\left[(\varphi^s_{q_2})^* f\right]$, the integral of $\{f,q_2\}$ along a periodic orbit vanishes, and (21) is a necessary consequence of (20). We set, for all $z \in \mathbb{R}^4$:

$$h_2(z) = \frac{1}{2\pi}\int_0^{2\pi} (\varphi^s_{q_2})^* r_2(z) ds.$$

Notice that h_2 is smooth and flat at the origin. One has $\{h_2, q_2\} = 0$ and

$$\begin{aligned}\{h_2, q_1\} &= \frac{1}{2\pi}\int_0^{2\pi} \{(\varphi^s_{q_2})^* r_2, q_1\} ds = \frac{1}{2\pi}\int_0^{2\pi} \{(\varphi^s_{q_2})^* r_2, (\varphi^s_{q_2})^* q_1\} ds \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\varphi^s_{q_2})^* \underbrace{\{r_2, q_1\}}_{=\{r_1,q_2\}} ds = \left\{\frac{1}{2\pi}\int_0^{2\pi} (\varphi^s_{q_2})^* r_1 ds, q_2\right\} = 0\end{aligned}$$

Thus dh_2 vanishes on the vector fields $\mathcal{X}_{q_1}$ and $\mathcal{X}_{q_2}$, which implies that h_2 is locally constant on the smooth parts of the level sets $(q_1, q_2) = \text{const}$. As before (Lemma 3.5) this entails that there is a unique germ of function ϕ_2 on $(\mathbb{R}^2, 0)$ such that $h_2 = \phi_2(q_1, q_2)$; what's more ϕ_2 is smooth in a neighbourhood of the origin, and flat at the origin.

Next, we define $\check{r}_2 = r_2 - h_2$ and

$$f_2(z) = -\frac{1}{2\pi}\int_0^{2\pi} s(\varphi^s_{q_2})^*(\check{r}_2(z)) ds.$$

Then $f_2 \in \mathcal{C}^\infty((\mathbb{R}^4, 0); \mathbb{R})_{\text{flat}}$, and we compute :

$$\begin{aligned}\{q_2, f_2\} &= -\frac{1}{2\pi}\int_0^{2\pi} s\{q_2, (\varphi^s_{q_2})^*\check{r}_2\} ds \\ &= -\frac{1}{2\pi}\int_0^{2\pi} s(\varphi^s_{q_2})^*\{q_2, \check{r}_2\} ds = -\frac{1}{2\pi}\int_0^{2\pi} s\frac{d}{ds}((\varphi^s_{q_2})^*\check{r}_2) ds \\ &= -\frac{1}{2\pi}\left[s(\varphi^s_{q_2})^*\check{r}_2\right]_{s=0}^{s=2\pi} + \frac{1}{2\pi}\underbrace{\int_0^{2\pi} (\varphi^s_{q_2})^*\check{r}_2 ds}_{=0} = -\check{r}_2 = \phi_2(q_1, q_2) - r_2.\end{aligned}$$

In the last line we have integrated by parts and used that $\check{r}_2$ has vanishing $\mathbb{S}^1$-average. Hence f_2 is solution of $\{f_2, q_2\} = r_2 - \phi_2(q_1, q_2)$ and $\{f_2, q_1\} = 0$. So

$$(20) \Leftrightarrow \begin{cases}\{f - f_2, q_1\} = r_1 \\ \{f - f_2, q_2\} = 0\end{cases}$$

So we managed to reduce the initial cohomological equations (20) to the case $r_2 = \phi_2 = 0$; In other words, upon replacing $f - f_2$ by f, we need to solve

$$\begin{cases}\{f, q_1\} = r_1 \\ \{f, q_2\} = 0\end{cases} \tag{22}$$

and the cocycle condition simply becomes $\{r_1, q_2\} = 0$.

Now, in order to solve the first cohomological equation of (22), we shall reduce our problem to the case where r_1 is a flat function on the planes $z_1 = 0$ and $z_2 = 0$. For this we shall adapt the technique used by Colin de Verdière and Vey in [3], but in a 4-dimensional setting.

Let $z_1 = re^{it}$ and $z_2 = \rho e^{i\theta}$. We introduce the usual vector fields :

$$r\frac{\partial}{\partial r} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\ , \qquad \frac{\partial}{\partial t} = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$$

$$\rho\frac{\partial}{\partial \rho} = \xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2}\ , \qquad \frac{\partial}{\partial \theta} = -\xi_2\frac{\partial}{\partial \xi_1} + \xi_1\frac{\partial}{\partial \xi_2}.$$

Remember that in complex notation $q = q_1 + iq_2 = \bar{z}_1 z_2$. In polar coordinates, our system becomes

$$(22) \Leftrightarrow \begin{cases} r\frac{\partial f}{\partial r} - \rho\frac{\partial f}{\partial \rho} = r_1 \\ \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta}\right) = 0, \end{cases} \tag{23}$$

and the cross-commuting relation becomes: $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right) r_1 = 0$.

We first determine the Taylor series of f along the $z_2 = 0$ plane; we denote for any function $h \in \mathcal{C}^\infty(\mathbb{R}^4; \mathbb{R})$

$$[h]_{\ell_1,\ell_2}(x_1, x_2) := \frac{\partial^{\ell_1+\ell_2} h}{\partial \xi_1^{\ell_1} \partial \xi_2^{\ell_2}}(x_1, x_2, 0, 0).$$

Now, when applying ℓ_1 times $\frac{\partial}{\partial \xi_1}$ and ℓ_2 times $\frac{\partial}{\partial \xi_2}$ to the equations (23), and then setting $\xi_1 = \xi_2 = 0$, one gets

$$\begin{cases} r\frac{\partial [f]_{\ell_1,\ell_2}}{\partial r}(x_1, x_2) - (\ell_1 + \ell_2)[f]_{\ell_1,\ell_2}(x_1, x_2) = [r_1]_{\ell_1,\ell_2}(x_1, x_2) \\ \frac{\partial [f]_{\ell_1,\ell_2}}{\partial t} = 0 \\ \frac{\partial [r_1]_{\ell_1,\ell_2}}{\partial t} = 0 \end{cases}$$

So the second and third equation tells us that $[r_1]_{\ell_1,\ell_2}$ and $[f]_{\ell_1,\ell_2}$ can be written as continuous functions of r. We set $[r_1]_{\ell_1,\ell_2}(x_1, x_2) = [R_1]_{\ell_1,\ell_2}(\sqrt{x_1^2 + x_2^2})$ and $[f]_{\ell_1,\ell_2}(x_1, x_2) = [F]_{\ell_1,\ell_2}(\sqrt{x_1^2 + x_2^2})$. The equation satisfied by $[F]_{\ell_1,\ell_2}$ and $[R_1]_{\ell_1,1_2}$ is actually an ordinary differential equation of the real variable r, which admits as a solution

$$[F]_{\ell_1,\ell_2}(r) = \int_0^1 t^{-(\ell_1+\ell_2+1)}[R_1]_{\ell_1,\ell_2}(tr)dt$$

For any $t \geqslant 0$, $[R_1]_{\ell_1,1_2}(t) = [r_1]_{\ell_1,\ell_2}(t, 0)$; hence $[R_1]_{\ell_1,1_2}$ is smooth on $\mathbb{R}^+$, and flat at the origin. This implies that the above integral is convergent for any (ℓ_1, ℓ_2), and defines a smooth function of $r \geqslant 0$, which is also flat when $r \to 0^+$. We shall now check that $[F]_{\ell_1,\ell_2}(\sqrt{x_1^2 + x_2^2})$ is a smooth (and flat) as a function of x_1 and

x_2. Obviously, the only problem that can occur is at $(0,0)$, since the square root is not a smooth function at the origin. Yet, we can see with the help of the Faa Di Bruno formula that, for any smooth function F on $(0,\infty)$,

$$\frac{d^n}{dx^n}\left[F(\sqrt{x})\right] = \sum_{k=0}^{n} F^{(n)}(\sqrt{x}) B_{n,k}(\frac{d}{dx}(\sqrt{x}), \frac{d^2}{dx^2}(\sqrt{x}), \ldots, \frac{d^n}{dx^n}(\sqrt{x})).$$

(Here $B_{n,k}$ designate the (n,k)-th Bell polynomial; of course we don't need its exact value). Since $\frac{d^n}{dx^n}(\sqrt{x}) = \frac{(-1)^n(2n-1)!}{2^{2n-1}(n-1)!}x^{\frac{1}{2}-n}$, we have in fact that $\frac{d^n}{dx^n}\left[F\circ(\sqrt{x})\right]$ is a finite sum of terms of the form

$$C_j \frac{F^{(j)}(\sqrt{x})}{x^{\frac{N_j}{2}}} \quad \text{with } C_j \in \mathbb{R} \text{ and } N_j \in \mathbb{N}.$$

Since, in our case, F is in fact flat when $r \to 0^+$, all of these terms tends to 0 when x does, so $F \circ \sqrt{.}$ is a smooth, and actually flat function. Hence $[f]_{\ell_1,\ell_2}$ is a flat function of the variables (x_1, x_2). Now, invoking Borel's lemma, let $u_1 \in \mathcal{C}^\infty(\mathbb{R}^4;\mathbb{R})$, whose Taylor expansion in the ξ variables is

$$\mathcal{T}_{(\xi_1,\xi_2)} u_1 = \sum_{\ell_1,\ell_2} \frac{1}{\ell_1!\ell_2!}[f]_{\ell_1,\ell_2}(x_1,x_2)\xi_1^{\ell_1}\xi_2^{\ell_2}.$$

We have that u_1 is flat all along the axis $z_1 = 0$. One can always symmetrize this last function by the action of q_2. We still get a flat function on $z_1 = 0$. Let's review the properties of u_1

- $r_1 - \{u_1, q_1\}$ is flat on $z_2 = 0$ by construction of u_1
- u_1 and therefore $\{u_1, q_1\}$ are flat on $z_1 = 0$
- $\{u_1, q_2\} = 0$ on the whole space.

On can construct by the same process u_2, flat all along $z_2 = 0$ and such that $\{u_2, q_1\} - r_1$ is flat on $z_1 = 0$, and $\{u_2, q_2\} = 0$ the whole space. The cohomological equations (22) are now equivalent to

$$\begin{cases} \{f - u_1 - u_2, q_1\} = r_1 - \{u_1 + u_2, q_1\} \\ \{f - u_1 - u_2, q_2\} = -\{u_1 + u_2, q_2\} = 0. \end{cases}$$

Notice that $r_1 - \{u_1 + u_2, q_1\} = r_1 - \{u_1, q_1\} - \{u_2, q_1\}$ is flat on both planes $z_1 = 0$ and $z_2 = 0$. Thus, replacing $f - u_1 - u_2$ by f again, we are now reduced to the case where r_1 is a flat function on both planes $z_1 = 0$ and $z_2 = 0$. We'll

now solve the cohomological equation away from $z_1 = 0$ or $z_2 = 0$ and extend it to these planes. If f is a solution, it satisfies the transport equation

$$-\frac{d}{dt}\left[(\varphi_{q_1}^t)^* f\right]\Big|_{t=s} = (\varphi_{q_1}^s)^*\{f, q_1\} = (\varphi_{q_1}^s)^* r_1.$$

Integrating between 0 and a time T depending on z_1 and z_2, we get

$$f - (\varphi_{q_1}^T)^* f = \int_0^T (\varphi_{q_1}^s)^* r_1 ds.$$

Looking for a solution f of the form $f = \int_0^T (\varphi_{q_1}^s)^* r_1 ds$, we get

$$\{f, q_1\} = r_1 - (1 + \{q_1, T\})(\varphi_{q_1}^T)^* r_1$$

and, using $\{r_1, q_2\} = 0$,

$$\{f, q_2\} = \{T, q_2\}(\varphi_{q_1}^T)^* r_1.$$

Thus, such an f will give us a solution to both cohomological equations if T satisfies

$$\begin{cases} \{T, q_1\} = 1 \\ \{T, q_2\} = 0 \end{cases} \tag{24}$$

Notice that this can be easily understood in a geometrical manner near a regular point of the map (q_1, q_2), in the following way. We fix a hypersurface $\mathcal{T}_0$ transversal to the flow of q_1 and invariant under the flow of q_2; then $T(z)$ is the (locally unique) time such that $\varphi_{q_1}^{-T(z)}(z) \in \mathcal{T}_0$.

Here, we may take $T = \frac{1}{4}\left[\ln|z_2|^2 - \ln|z_1|^2\right]$, which corresponds to the hypersurface $|z_1| = |z_2|$, and gives a smooth solution on each connected component of $\mathbb{C}^2 \setminus \{q = 0\}$ (singularities of T are exactly the zero locus of q).

The last thing one has to check is that this solution can be extended to the whole space as a smooth solution; actually, f will be flat on the two complex axis $z_1 = 0$ and $z_2 = 0$. Let's denote a derivation of arbitrary degree in the four variables $z_1, \bar{z_1}, z_2, \bar{z_2}$ by

$$L_{\alpha\beta} := \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1} \partial \bar{z_1}^{\beta_1} \partial z_2^{\alpha_2} \partial \bar{z_2}^{\beta_2}}.$$

We can then write

$$\begin{aligned} L_{\alpha\beta} f = &\int_0^T (L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2) e^{ks} ds \\ &+ \left\langle (L_{\gamma\delta} T)^\ell \left(L_{\gamma'\delta'} r_1\right)(e^T z_1, e^{-T} z_2) e^{mT} \right\rangle \end{aligned} \tag{25}$$

with $k \in \mathbb{Z}$ depending on (α, β). The term between brackets designates a finite sum of terms of the generical form inside the bracket, where the number of terms and the exact values of $\gamma, \delta, \gamma', \delta', \ell$ and m depend on (α, β).

We will make use of the following fact : if a smooth function $r(z_1, z_2)$ is flat on $z_1 = 0$, then for any $N_1 \geqslant 0$, the function $r(z_1, z_2)/\left|z_1\right|^{N_1}$ is still smooth and flat on $z_1 = 0$. Of course the corresponding statement holds if r is flat on the plane $z_2 = 0$. Thus, when r is flat on both planes $z_1 = 0$, $z_2 = 0$, the function $r(z_1, z_2)/\left|z_1\right|^{N}\left|z_2\right|^{N_2}$ is again smooth and flat on both planes. Therefore, given any N_1, N_2 and any bounded region for (z_1, z_2), there exists a constant $C > 0$ such that

$$\left|r(z_1, z_2)\right| \leqslant C\left|z_1\right|^{N_1}\left|z_2\right|^{N_2}. \tag{26}$$

We return now to the terms in (25). Notice that, by construction,

$$\max(\left|e^s z_1\right|, \left|e^{-s} z_2\right|) \leqslant \max(\left|z_1\right|, \left|z_2\right|),$$

when s varies between 0 and T. This is clear also from the geometric picture. Thus, using that $L_{\alpha\beta} r_1$ is flat along $z_1 = 0$ and $z_2 = 0$ we have from (26), for bounded (z_1, z_2) and for any N_1, N_2, a constant $C > 0$ such that

$$\begin{aligned}\left|(L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2)\right| &\leqslant C(e^s\left|z_1\right|)^{N_1}(e^{-s}\left|z_2\right|)^{N_2} \\ &= C\left|q\right|(e^s\left|z_1\right|)^{N_1-1}(e^{-s}\left|z_2\right|)^{N_2-1}\end{aligned} \tag{27}$$

Choosing N_1 and N_2 large enough, and taking $(\alpha, \beta) = (\gamma', \delta')$, $s = T$, this proves that each of the terms between brackets (25) in tend to 0 as $|q| \to 0$.

We consider now the integral term. If $|z_2| \leqslant |z_1|$, then $T \leqslant 0$; letting $N_2 = 1$ in (27) we get

$$\forall s \in [T, 0], \qquad \left|(L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2)\right| \leqslant C\left|q\right|(e^{-|s|}\left|z_1\right|)^{N_1-1}.$$

If we suppose instead $|z_1| \leqslant |z_2|$, we get similarly

$$\forall s \in [0, T], \qquad \left|(L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2)\right| \leqslant C\left|q\right|(e^{-|s|}\left|z_2\right|)^{N_2-1}.$$

Therefore we can write, for any bounded (z_1, z_2) and any N,

$$\left|(L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2)\right| \leqslant C\left|q\right|(e^{-|s|}\left|z\right|)^{N}, \quad \text{with } \left|z\right| := \max(\left|z_1\right|, \left|z_2\right|). \tag{28}$$

Thus, with $N \geqslant |k| + 1$,

$$\left|\int_0^T (L_{\alpha\beta} r_1)(e^s z_1, e^{-s} z_2) e^{ks} ds\right| \leqslant C'\left|q\right| \int_0^{|T|} e^{-s} ds \leqslant C'\left|q\right|.$$

This shows that this term tends to 0 as $|q| \to 0$ as well.

These estimates conclude the proof : our solution on $\mathbb{C}^2 \setminus \{q = 0\}$ extends to a smooth function on $\mathbb{C}^2$. By continuity of Poisson brackets, this extension is a solution to our cohomological equation (22) on the whole space. $\square$

4.2 A Darboux lemma for focus-focus foliations

Here again $\mathbb{R}^4$ is endowed with the canonical symplectic form ω_0. Recall that the regular level sets of the map $q = (q_1, q_2) : \mathbb{R}^4 \to \mathbb{R}^2$ are lagrangian for ω_0.

Proposition 4.2. Let ω be a symplectic form on $\mathbb{R}^4$ such that

(a) $\omega = \omega_0 + \mathcal{O}(\infty)$;

(b) the regular level sets of q are lagrangian for ω;

(c) for all $z \in \mathbb{R}^4$,

$$\int_{D_z} \omega - \omega_0 = 0,$$

where D_z is the disk given by

$$D_z := \{(\zeta z_1, \zeta z_2) \in \mathbb{C}^2; \quad \zeta \in \mathbb{C}, |\zeta| \leqslant 1\}$$

(here we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and denote $z = (z_1, z_2) \in \mathbb{C}^2$).

Then there exists a local diffeomorphism Φ of $(\mathbb{R}^4, 0)$ and $U = (U_1, U_2)$ a local diffeomorphism of $(\mathbb{R}^2, 0)$ such that

1. $\Phi^*\omega = \omega_0$
2. $\Phi^* q_1 = U_1(q_1, q_2)$
3. $\Phi^* q_2 = U_2(q_1, q_2)$
4. Both Φ and U are infinitely tangent to the identity.

Notice that conditions 2. and 3. together mean that Φ preserves the (singular) foliation defined by the level sets of q. Notice also that the hypothesis (a),(b),(c) are in fact necessary : for (a) and (b) this is obvious; for (c), remark that $\gamma_z := \partial D_z$ is an orbit of the S^1-action generated by q_2 for the canonical symplectic form ω_0, and thus is a homology cycle on the Liouville torus $q = \text{const}$. Since Φ is tangent to the identity, $\Phi_*\gamma_z$ is homologous to γ_z; thus, if α_0 is the Liouville 1-form on $\mathbb{R}^4$, which is closed on the Liouville tori, we have

$$\int_{\gamma_z} \alpha_0 = \int_{\Phi_*(\gamma_z)} \alpha_0 = \int_{\gamma_z} \Phi^*\alpha_0,$$

which by Stokes gives (c).

Proof of the proposition. We use again the standard deformation method by Moser. Let

$$\omega_s = (1-s)\omega_0 + s\omega.$$

We look for Y_s a time-dependant vector field defined for $s \in [0,1]$ whose flow $s \mapsto \varphi^s_{Y_s}$ satisfies $(\varphi^s_{Y_s})^*\omega_s = \omega_0$. Taking the derivative with respect to s gives

$$(\varphi^s_{Y_s})^* \left[\frac{\partial \omega_s}{\partial s} + \mathcal{L}_{Y_s}\omega_s\right] = (\varphi^s_{Y_s})^* \left[\omega - \omega_0 + d(\imath_{Y_s}\omega_s)\right] = 0.$$

ω and ω_0 being closed, we can find, in a neighbourhood of the origin, smooth 1-forms α and α_0 such that $\omega = d\alpha$ and $\omega_0 = d\alpha_0$. Using the standard constructive proof of the Poincaré lemma, we can choose α and α_0 such that $\alpha = \alpha_0 + \mathcal{O}(\infty)$. Let $\varphi^t_{q_2}$ be the hamiltonian flow of X^0_2 on $(\mathbb{R}^4, \omega_0)$.

Since $\omega_s(0) = \omega_0(0) = \omega_0$, one can find a neighbourhood of the origin on which ω_s is non-degenerate for all $s \in [0,1]$. This enables us to find a suitable Y_s by solving

$$\imath_{Y_s}\omega_s = -(\alpha - \alpha_0) + df, \tag{29}$$

for a suitable function f. Here, any function f such that $df(0) = 0$ will yield a vector field Y_s whose time-1 flow Φ solves the point 1. of the lemma. It turns out that properly choosing f will be essential in ensuring that Φ preserves the foliation (point 2. and 3.).

Let X^0_1, X^0_2 the hamiltonian vector fields associated to q_1, q_2 respectively, for ω_0. Since the level sets of q are lagrangian for ω_0, X^0_1, X^0_2 are commuting vector fields spanning the tangent space to regular leaves. Thus $\omega_0(X^0_1, X^0_2) = 0$. But, by assumption, the level sets of q are lagrangian for ω as well. This implies that $\omega(X^0_1, X^0_2) = 0$ as well. Thus $\omega_s(X^0_1, X^0_2) = 0$ for all s : the level sets of q are lagrangian for ω_s. This entails that the condition that Y_s be tangent to the leaves can be written

$$\begin{cases} \omega_s(Y_s, X^0_1) = 0 \\ \omega_s(Y_s, X^0_2) = 0. \end{cases} \tag{30}$$

We can expand this :

$$(30) \iff_{(29)} \begin{cases} -(\alpha - \alpha_0)(X^0_1) + df(X^0_1) = 0 \\ -(\alpha - \alpha_0)(X^0_2) + df(X^0_2) = 0. \end{cases}$$

Now we may let

$$\begin{cases} r_1 := (\alpha - \alpha_0)(X^0_1) \\ r_2 := (\alpha - \alpha_0)(X^0_2) \end{cases}$$

and the condition becomes

$$(30) \iff \begin{cases} \{f, q_1\} = r_1 \\ \{f, q_2\} = r_2. \end{cases}$$

(Here the Poisson brackets come from the canonical symplectic form ω_0). Notice that r_1 and r_2 are flat at the origin.

Next, recall the following formula for 1-forms : $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$. Thus

$$0 = \omega_0(X_1^0, X_2^0) = d\alpha_0(X_1^0, X_2^0) = X_1^0\alpha_0(X_2^0) - X_2^0\alpha_0(X_1^0) - \alpha_0([X_1^0, X_2^0]),$$

which implies

$$\imath_{X_1^0} d(\alpha_0(X_2^0)) = \imath_{X_2^0} d(\alpha_0(X_1^0)).$$

and similarly

$$\imath_{X_1^0} d(\alpha(X_2^0)) = \imath_{X_2^0} d(\alpha(X_1^0)).$$

Hence we may write the same equation again for $\alpha - \alpha_0$ which, in terms of ω_0-Poisson brackets, becomes

$$\{r_1, q_2\} = \{r_2, q_1\}.$$

Therefore, a solution f to this system (30) is precisely given by the division lemma (Theorem 4.1), provided we show that r_2 has vanishing $\varphi^t_{q_2}$-average. But, since $\frac{d}{dt}\varphi^t_{q_2} = X_2^0(\varphi^t_{q_2})$, we have,

$$\forall z \in \mathbb{R}^4, \quad \frac{1}{2\pi}\int_0^{2\pi} r_2(\varphi^t_{q_2}(z))dt = \int_{\gamma_z} \alpha - \alpha_0 = \int_{D_z} \omega - \omega_0 = 0.$$

Finally, we check that Y_s as defined with 29 vanishes at the origin and hence yields a flow up to time 1 on a open neighbourhood of the origin.

To conclude, the time-1 flow of Y_s is a local diffeomorphism Φ that preserves the q-foliation and such that $\Phi^*\omega = \omega_0$, which finishes the proof.

Notice that Y_s is uniformly flat, whence Φ is a flat perturbation of the identity. □

5 Proof of the main theorem

We summarize here all the steps that bring us to prove Theorem 1.2.

$$\begin{cases} F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ \omega_0 \end{cases} \xrightarrow{\text{Lemma 2.10}} \begin{cases} \chi^* F = G(q_1, q_2) + \mathcal{O}(\infty) \\ \chi^*\omega_0 = \omega_0 \end{cases}$$

$$\xrightarrow{\text{Theorem 3.1}} \begin{cases} \Upsilon^* G^{-1}(\chi^* F) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ \Upsilon^*(\chi^*\omega_0) = \omega = \omega_0 + \mathcal{O}(\infty) \end{cases}$$

$$\xrightarrow{\text{Proposition 4.2}} \begin{cases} \Phi^*\Upsilon^* G^{-1}(\chi^* F) = \begin{pmatrix} U(q_1, q_2) \\ U(q_1, q_2) \end{pmatrix} \\ \Phi^*\Upsilon^*\chi^*\omega_0 = \omega_0 \end{cases}$$

Only the last implication needs an explanation : indeed, even if the Morse lemma is not symplectic, the initial foliation by F is lagrangian for ω_0, and this implies that, under Υ, the target foliation by q becomes lagrangian for the target symplectic form ω. Thus the hypothesis (a) and (b) of the Darboux lemma (Proposition 4.2) are satisfied. That (c) is also satisfied follows from the equivariance property $(\mathcal{P})$ of Theorem 3.1. Indeed, let α_0 be the Liouville 1-form in $\mathbb{R}^4$, and $\alpha := \Upsilon^*\alpha_0$. Since Υ commutes with $\varphi^t_{q_2}$, we have

$$\mathcal{L}_{X_2^0}\alpha = \mathcal{L}_{X_2^0}\Upsilon^*\alpha_0 = \Upsilon^*\mathcal{L}_{X_2^0}\alpha_0.$$

On the other hand, since $\imath_{X_2^0}d\alpha_0 = -dq_2$ and (Lemma 3.3) $d\imath_{X_2^0}\alpha_0 = dq_2$, we get $\mathcal{L}_{X_2^0}\alpha_0 = 0$. Thus $\mathcal{L}_{X_2^0}\alpha = 0$ which, in turn, says that $d\imath_{\mathcal{X}^\omega_{q_2}}\alpha = -\imath_{\mathcal{X}^\omega_{q_2}}d\alpha = dq_2$, where we denote by $\mathcal{X}^\omega_{q_2}$ the ω-gradient of q_2. By property $(\mathcal{P})$, $\mathcal{X}^\omega_{q_2} = X_2^0$, so $d\imath_{X_2^0}\alpha = dq_2$. Hence $\imath_{X_2^0}\alpha = q_2 + \beta$, where β is a constant, which is actually equal to 0 since $\imath_{X_2^0}\alpha = q_2 + \mathcal{O}(\infty)$. We thus get $\imath_{X_2^0}(\alpha - \alpha_0) = 0$, which of course implies

$$\int_{\gamma_z} \alpha - \alpha_0 = 0.$$

Thus one may apply Proposition 4.2, and the main theorem 1.2 is shown for $\Psi := \Phi \circ \Upsilon \circ \chi$ and $\tilde{G} := G \circ U$.

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