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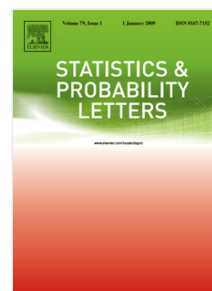
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Remarks and corrections on "An existence theorem for stochastic functional differential equations with delays under weak assumptions, Statistics and Probability Letters, 78, 2008." by N. Halidias-Y. Ren

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Abstract

In this note we would like to give some remarks and open problems concerning comparison results for stochastic functional differential equations. Our starting point was the paper of Halidias and Ren 2008 where one can find some wrong arguments in the comparison result. However, as we shall prove in this note, the results in that paper are true under some more restrictive conditions on the data.

AMS Subject Classification: 60H10, 60H20, 60H30.

Key words: stochastic functional differential equations, comparison theorem.

1 Introduction

Let t_0 be a positive constant and (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions. Assume that $B(t)$ is a Brownian motion defined on (Ω, \mathcal{F}, P) . Let $C([-\tau, 0]; \mathbb{R})$ denote the family of bounded continuous \mathbb{R} -valued functions φ defined on $[-\tau, 0]$ with norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. We denote by $\mathcal{M}^2([-\tau, 0]; \mathbb{R})$ the family of all \mathcal{F}_{t_0} -measurable, \mathbb{R} -valued process $\psi(t) = \psi(t, \omega)$, $t \in [-\tau, 0]$ such that $E \int_{-\tau}^0 |\psi(t)|^2 dt < \infty$.

In this note we will study stochastic functional differential equations like the following,

$$d[X(t) - D(X(t))] = f(t, X_t) dt + g(t, X(t)) dB(t), \quad t_0 \leq t \leq T, \quad (1)$$

where $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$ could be considered as a $C([-\tau, 0]; \mathbb{R})$ -value stochastic process. We define the initial value of (1) as follows:

$$X_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \text{ is } \mathcal{F}_{t_0} \text{-measurable,} \\ C([-\tau, 0]; \mathbb{R}) \text{-value random variable} \\ \text{such that } \xi \in \mathcal{M}^2([-\tau, 0]; \mathbb{R}). \quad (2)$$

Here, $f : [t_0, T] \times C([-\tau, 0], \mathbb{R}) \rightarrow \mathbb{R}$, $g : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $D : \mathbb{R} \rightarrow \mathbb{R}$ and assume that are Borel measurable.

We will study also the following stochastic functional differential equation of neutral type,

$$d[X(t) - D(X_t)] = f(t, X_t) dt + dB(t), \quad t_0 \leq t \leq T, \quad (3)$$

with the initial value defined as before. Here, $D : C([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$, $f(t, x) : [t_0, T] \times C([-\tau, 0]; \mathbb{R})$, are Borel measurable.

In the following two sections we prove comparison theorems for problems (1) and (3) with initial conditions like (2). In section 4 we give some remarks and conclusions.

2 Comparison theorem for sfdes

Our goal here is to give a comparison result for problem (1),(2). So, consider the following problems,

$$X(t) - D(X(t)) = \xi^1(0) - D(X(t_0)) + \int_{t_0}^t f_1(s, X_s) ds + \int_{t_0}^t g(s, X(s)) dB(s), \quad t_0 \leq t \leq T, \quad (4)$$

$$X(t) - D(X(t)) = \xi^2(0) - D(X(t_0)) + \int_{t_0}^t f_2(s, X_s) ds + \int_{t_0}^t g(s, X(s)) dB(s), \quad t_0 \leq t \leq T. \quad (5)$$

Theorem 1 *Suppose that problems (4) and (5) have unique strong solutions, $X(t), Y(t)$ respectively. Suppose that $f_1(t, x) \leq f_2(t, x)$ for all $x \in C([-\tau, 0]; \mathbb{R})$, $\xi^1(0) \leq \xi^2(0)$ and that $x \rightarrow f_1(t, x)$ or $x \rightarrow f_2(t, x)$ is increasing, in the sense that if $x \leq y$ then $f(t, x) \leq f(t, y)$ for $x, y \in C([-\tau, 0]; \mathbb{R})$. Then, we have that $X(t) \leq Y(t)$ a.s. for all $t > 0$.*

Proof.

For a discussion about the existence and uniqueness of strong solutions to problem like (3) one can see Mao (1997).

We define now the following operator, $p(\cdot, \cdot) : C([-\tau, 0]; \mathbb{R}) \times C([-\tau, 0]; \mathbb{R}) \rightarrow C([-\tau, 0]; \mathbb{R})$, with

$$p(x, y)(\theta) = \max_{\theta \in [-\tau, 0]} \{x(\theta), y(\theta)\}$$

It is clear that $y \rightarrow p(x, y)$ satisfies growth and Lipschitz conditions with constant one.

Suppose that f_2 is increasing. Consider the following problem,

$$Z(t) - D(p(X(t), Z(t))) = \xi(0) - D(p(X(t_0), Z(t_0))) + \int_{t_0}^t f_2(s, p(X_s, Z_s)) + \int_{t_0}^t g(s, p(X(s), Z(s))) dB(s), \quad t_0 \leq t \leq T.$$

It's clear that this problem has a unique solution noting that $y \rightarrow p(x, y)$ satisfies growth and Lipschitz conditions. We shall prove that $Z(t) \geq X(t)$ a.s.

Suppose that this is not true. Then, the following stopping time, τ_1

$$\tau_1 = \inf\{t \in [t_0, T] : X(t) - Z(t) > 0\},$$

is such that $\tau_1 < T$. In this case we define another stopping time τ_2 defined as

$$\tau_2 = \inf\{t \in [\tau_1, T] : Z(t) - X(t) > 0\},$$

Thus, we have produced the stochastic interval (τ_1, τ_2) in which $X(\tau_1) = Z(\tau_1)$ and $X(t) \geq Z(t)$ on (τ_1, τ_2) . For a discussion about stopping times and stochastic intervals one can see Mao (1997) and Karatzas and Shreve (1998).

Then we have

$$X(t) - Z(t) = X(\tau_1) - Z(\tau_1) - D(X(\tau_1)) + D(p(X(\tau_1), Z(\tau_1))) + D(X(t)) - D(p(X(t), Z(t))) + \int_{\tau_1}^t (f_1(s, X_s) - f_2(s, p(X_s, Z_s))) ds + \int_{\tau_1}^t (g(s, X(s)) - g(s, p(X(s), Z(s)))) dB(s), \quad t \in (\tau_1, \tau_2).$$

It is obvious that,

$$\begin{aligned} D(X(\tau_1)) - D(p(X(\tau_1), Z(\tau_1))) &= 0 \text{ because } X(\tau_1) = Z(\tau_1), \\ D(X(t)) - D(p(X(t), Z(t))) &= 0, \text{ because } X(t) \geq Z(t) \text{ on } t \in (\tau_1, \tau_2) \\ \int_{\tau_1}^t (f_1(s, X_s) - f_2(s, p(X_s, Z_s))) ds &\leq \int_{\tau_1}^t (f_1(s, X_s) - f_2(s, X_s)) ds \leq 0, t \in (\tau_1, \tau_2) \\ \int_{\tau_1}^t g(s, X(s)) - g(s, p(X(s), Z(s))) dB(s) &= 0, t \in (\tau_1, \tau_2), \end{aligned}$$

noting that always $p(X(t), Z(t)) \geq X(t)$.

Thus in fact we have equality, i.e. $Z(t) = X(t)$. So, always $Z(t) \geq X(t)$ and thus $Z(t) = Y(t)$ the unique solution of (4). \square

3 Comparison theorem for neutral sfdes with additive noise

In this section we consider the following stochastic functional differential equation of neutral type,

$$d[X(t) - D(X_t)] = f(t, X_t) dt + dB(t), \quad t_0 \leq t \leq T,$$

with the initial value defined as before. Here, $D : C([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$, $f(t, x) : [t_0, T] \times C([-\tau, 0]; \mathbb{R})$, are Borel measurable. Our goal is to obtain a comparison theorem for this kind of problem.

Let the following problems,

$$X(t) - D(X_t) = \xi(0) - D(X_{t_0}) + \int_{t_0}^t f_1(s, X_s) ds + B_t - B_{t_0}, \quad t_0 \leq t \leq T, \quad (6)$$

$$X(t) - D(X_t) = \xi(0) - D(X_{t_0}) + \int_{t_0}^t f_2(s, X_s) ds + B_t - B_{t_0}, \quad t_0 \leq t \leq T. \quad (7)$$

Theorem 2 *Suppose that problems (6) and (7) have unique strong solutions. Suppose further that $x \rightarrow f_1(t, x), x \rightarrow D(x)$ are increasing operators in sense that we have described before. If, $f_1(t, x) \leq f_2(t, x)$ for all $x \in C([-\tau, 0]; \mathbb{R})$ then $X(t) \leq Z(t)$ a.s. for all t .*

Proof.

As before we consider the operator $p(\cdot, \cdot) : C([-\tau, 0]; \mathbb{R}) \times C([-\tau, 0]; \mathbb{R}) \rightarrow C([-\tau, 0]; \mathbb{R})$, with

$$p(x, y)(\theta) = \max_{\theta \in [-\tau, 0]} \{x(\theta), y(\theta)\}$$

We consider the following problem,

$$\begin{aligned} Z(t) - D(p(X_t, Z_t)) &= \xi(0) - D(p(X_{t_0}, Z_{t_0})) + \\ &\int_{t_0}^t f_2(s, p(X_s, Z_s)) ds + B_t - B_{t_0}, \quad t_0 \leq t \leq T. \end{aligned}$$

It's clear that this problem has a unique strong solution. We are going to show that $Z(t) \geq X(t)$ a.s. for all $t > 0$.

Suppose that this is not true. Then as before we can construct a stochastic interval (τ_1, τ_2) in which $X(\tau_1) = Z(\tau_1)$ and $X(t) \geq Z(t)$ on (τ_1, τ_2) .

Thus we have,

$$X(t) - Z(t) = X(t_0) - Z(t_0) - D(X_{t_0}) + D(p(X_{t_0}, Z_{t_0})) + D(X_t) - D(p(X_t, Z_t)) + \int_{t_0}^t (f_1(s, X_s) - f_2(s, p(X_s, Z_s))) ds, \quad t \in (\tau_1, \tau_2).$$

Arguing as before and noting that f_2, D are increasing operators we have that $X(t) \leq Z(t)$ a.s. for all t . But $Z(t) = Y(t)$ the solution of problem (7), so we have done. \square

The proof of Theorem 2 is essential the same with that of Theorem 1. But the key difference is that we have integrate over (t_0, t) with $t \in (\tau_1, \tau_2)$ in Theorem 2 while in Theorem 1 we have integrate over (τ_1, t) with $t \in (\tau_1, \tau_2)$.

We can have another comparison theorem with different D . Assume that $\xi = x_0 \in \mathbb{R}$ and suppose that $D^1(x_0) = D^2(x_0)$.

Consider the following problems,

$$X(t) - D^1(X_t) = x_0 - D^1(x_0) + \int_{t_0}^t f_1(s, X_s) ds + B_t - B_{t_0}, \quad t_0 \leq t \leq T, \quad (8)$$

$$X(t) - D^2(X_t) = x_0 - D^2(x_0) + \int_{t_0}^t f_2(s, X_s) ds + B_t - B_{t_0}, \quad t_0 \leq t \leq T. \quad (9)$$

Theorem 3 *Suppose that problems (8) and (9) have unique strong solutions. Suppose further that $x \rightarrow f_2(t, x), x \rightarrow D^2(x)$ are increasing operators in sense that we have described before. If, $f_1(t, x) \leq f_2(t, x)$ and $D^1(x) \leq D^2(x)$ for all $x \in C([-\tau, 0]; \mathbb{R})$ then $X(t) \leq Z(t)$ a.s. for all t .*

We omit the proof because is the same as Theorem 2.

4 Remarks

There exists some open problems, for example what can we say if both D, g are functionals depending on $X_t = \{X(t + \theta) : \theta \in [-\tau, 0]\}$. We would like also to point out that the results in Halidias and Ren 2008 does not hold as they are but are true if we assume further that $d = 1$, $x \rightarrow f_2(t, x)$ is increasing and that g is not a functional depending on X_t .

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