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Stefano Morra

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On some representations of the Iwahori subgroup

Stefano Morra

ABSTRACT

Let $p \geq 5$ be a prime number. In [BL94] Barthel and Livné gave a classification for irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$, for F a p -adic field, discovering some objects, referred to as “supersingular”, which appear as subquotients of a universal representations $\pi(\underline{r}, 0, 1)$. In this paper we give a detailed description the Iwahori structure of such universal representations for F an unramified extension of \mathbf{Q}_p . We determine a fractal structure which shows how and why the techniques used for \mathbf{Q}_p fail and which let us determine “natural” subrepresentations of the universal object $\pi(\underline{r}, 0, 1)$. As a corollary, we get the Iwahori structure of tamely ramified principal series.

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1. Introduction

Let p be a prime number and F a p -adic field. In their works [BL94], [BL95] Barthel and Livné studied a classification (recently generalized for general $\mathrm{GL}_n(F)$ by Herzing in [Her]) for the representations of $\mathrm{GL}_2(F)$ with coefficients in an algebraic closure of \mathbf{F}_p . Besides characters, principal unramified series and special series, they found a new class of irreducible objects referred as “supersingular”, which are defined, up to twist, as subquotients of a universal representation, which we will note $\pi(\underline{r}, 0, 1)$ (and $\underline{r} = (r_0, \dots, r_{f-1})$ if f is the residual degree of F). The existence of supersingular representations is assured by a Zorn-type argument (see [BL95], Proposition 11) and

a complete exhaustive study for supersingular representations is a relevant open problem in the emerging p -adic Langlands program. Indeed, in a conjectural mod p -Langlands correspondence it is expected that the supersingular objects are those $\mathrm{GL}_2(F)$ representations which should naturally be attached to Galois representations arising from elliptic curves with supersingular reduction.

This is actually the case if $F = \mathbf{Q}_p$ (when the universal representations are indeed irreducible). Such result is due to Breuil [Bre] where he reaches a complete classification of supersingular representations thanks to direct computations on the ring of Witt vectors of \mathbf{F}_p . If $F \neq \mathbf{Q}_p$ the situation is not clear. For the time being, the problem of classifying supersingular representations looks to be infinitely more involved compared to its Galois analogue (known from the works of Serre [Ser72]). The methods of Paskunas [Pas] and Breuil-Paskunas [Br-Pa] let us associate an *infinite* family $\Pi(\rho)$ of supersingular representations to a single Galois object ρ , a major progress in this direction, but it is not clear, especially after the work of Hu [Hu1], how to distinguish in a canonical way a privileged supersingular representation inside $\Pi(\rho)$. We remark that the methods of [Pas] and [Br-Pa] have been improved by Hu's canonical diagrams in [Hu2]; unfortunately canonical diagrams are difficult to calculate explicitly.

Another approach to the problem has been treated by Schein in [Sch] where he studies the universal representations for a totally ramified extension F/\mathbf{Q}_p . He detects a natural quotient V_{e-1} of $\pi(\underline{r}, 0, 1)$ which enjoys an universal property with respect to supersingular representations whose $\mathrm{GL}_2(\mathcal{O}_F)$ -socle respects a certain combinatoric conjecturally associated to suitable Galois representations arising from elliptic curves with supersingular reduction (the modular weights introduced in [BDJ] and generalised in [Sch1])

In this paper we describe the Iwahori structure for the universal representation $\pi(\underline{r}, 0, 1)$ in the case where F/\mathbf{Q}_p is unramified generalizing Breuil's method (in particular, our result give the irreducibility for $F = \mathbf{Q}_p$ and shows how and why the universal representations fail to be irreducible otherwise). With "Iwahori structure" we mean that we are able to detect the Iwahori-socle filtration for $\pi(\underline{r}, 0, 1)$ as well as the extension between two consecutive graded pieces. As a byproduct we will deduce the Iwahori structure of principal and special series and the presence of a natural injection $c\text{-Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r}, 0, 1)$. The reader will find out that, as soon as $F \neq \mathbf{Q}_p$, the Iwahori-socle filtration for the universal representation relies on an extremely complicated combinatoric.

The main result of this paper is to show that such combinatoric can be handled with the help of some simple euclidean data; such a method -a far reaching generalisation of the techniques of [Bre]- can be briefly described as follow. We detect a natural \mathbf{F}_p -basis \mathcal{B} of $\pi(\underline{r}, 0, 1)$ as well as an injection:

$$\mathcal{B} \hookrightarrow \mathbf{Z}^{[F:\mathbf{Q}_p]};$$

as we will show, its image \mathfrak{R} is explicitly known. For $v \in \mathcal{B}$ we define the set of antecedents \mathfrak{S}_v of v as the set of $v' \in \mathcal{B}$ such that $v' = v - e_s$ where e_s is the s -th element of the canonical base of $\mathbf{Z}^{[F:\mathbf{Q}_p]}$. When we claim that the Iwahori structure for $\pi(\underline{r}, 0, 1)$ is described by \mathfrak{R} we mean the following facts:

- i) the Iwahori-socle filtration is obtained from \mathfrak{R} by successively removing the points with empty antecedents;
- ii) if $v_0, v_1 \in \mathcal{B}$ and $J \in \mathbf{N}$ is such that v_i is an eigenvector for the $J - i$ -th graded piece $(\pi(\underline{r}, 0, 1))_{J-i}$ of the socle filtration of the universal representation then we have a nontrivial extension *inside the quotient* $\pi(\underline{r}, 0, 1)/(\pi(\underline{r}, 0, 1))_{J-1}$ if and only if v_0 is an antecedent of v_1 .

According to this terminology the main result is the following (see Proposition 5.16):

THEOREM 1.1. *The Iwahori structure of the universal representations is described by \mathfrak{R} .*

We give in figure 1 the idea of such structure for the quadratic unramified extension of \mathbf{Q}_p .

As announced, we get some other byproducts as

THEOREM 1.2. *The Iwahori structure of tamely ramified principal series is described by two copies of $\mathbf{N}^{[F:\mathbf{Q}_p]}$.*

and

THEOREM 1.3. *Let $\underline{r} \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$ and let χ^s be the conjugate character of $(\sigma_{\underline{r}})^{U(\mathbf{F}_q)}$. There is a sub KZ -representation $V \leq \pi(\underline{r}, 0, 1)|_{KZ}$ isomorphic to the kernel of the natural map*

$$\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s / \mathrm{soc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s) \twoheadrightarrow \mathrm{cosoc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s)$$

and such that the map (induced by Frobenius reciprocity)

$$c\text{-Ind}_{KZ}^G V \rightarrow \pi(\underline{r}, 0, 1)$$

is injective.

We remark that a similar phenomenon has already been discovered by Paskunas in an unpublished draft.

Such results rely on an heavy formalism and they need preparation to be handled. In particular, from section §4 we start using the euclidean dictionary as a key tool to manage the combinatoric of the representation under study. In order to guide the reader the statements are preceded by a detailed translation in geometric terms (otherwise they would sound as empty exercices of combinatoric) and each section opens with an exhaustive description of the euclidean strategy adopted to reach our aims.

The reasons which make such strategy work are essentially three:

- i) we detect a suitable basis \mathcal{B} of the universal representation which is well behaved with respect to the action of the Iwahori subgroup and the canonical Hecke operator $T \in \mathrm{End}_G(c\text{-Ind}_{KZ}^G \sigma_{\underline{r}})$;
- ii) the action of the Iwahori subgroup on the elements of \mathcal{B} can be read through certain universal Witt polynomials whose homogeneous degree is known;
- iii) the correspondence between the elements of the basis \mathcal{B} and integers points in $\mathbf{R}^{[F:\mathbf{Q}_p]}$ is compatible with the homogeneous degree of the polynomials of ii).

The structure of the paper is then the following.

First two sections §2 and §3 are formal and do not need the hypothesis F/\mathbf{Q}_p unramified. Section §2 is essentially a dictionary which let us detect a natural KZ -filtration on the KZ -restriction of the universal representation. We first introduce a family of KZ -representations $\{R_n\}_{n \in \mathbf{N}}$. Through some convenient Hecke operators $T_n^\pm : R_n \rightarrow R_{n \pm 1}$ we define inductively a direct system of amalgamed sums (each of them endowed with a natural filtration) which leads to explicit isomorphism (Proposition 2.9):

$$\pi(\sigma_{\underline{r}}, 0, 1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n \text{ odd}} (R_0 \oplus R_1 \cdots \oplus R_n \oplus R_{n+1}) \oplus \varinjlim_{n \text{ even}} (R_1/R_0 \oplus R_2 \cdots \oplus R_n \oplus R_{n+1}).$$

We remark that such isomorphism was already draft by Breuil in [Bre].

In section 3 we start from an Iwahori-splitting $R_{n+1} = R_{n+1}^+ \oplus R_{n+1}^-$ to deduce, in the same flavour of the preceding section, an inductive system of amalgamed sums $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$. Such amalgamed sums are endowed with a natural Iwahori-filtration revealed by a short exact sequence

$$0 \rightarrow \cdots \oplus_{R_{n-2}^\pm} R_{n-1}^\pm \rightarrow \cdots \oplus_{R_n^\pm} R_{n+1}^\pm \rightarrow R_{n+1}^\pm / R_n^\pm \rightarrow 0. \quad (1)$$

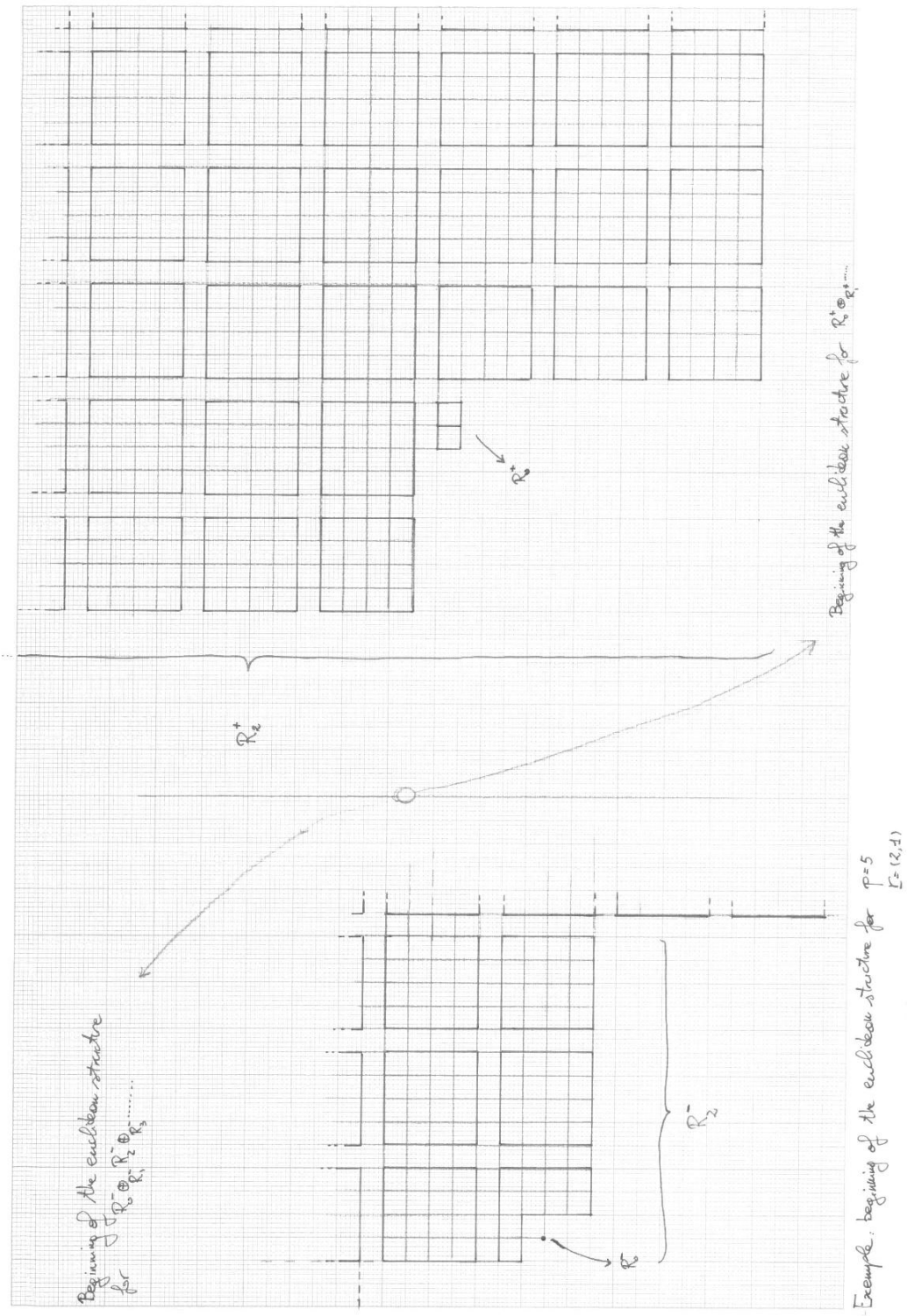


FIGURE 1. Part of the euclidean structure for $f = 2$, $\underline{r} = (2, 1)$.

The resulting inductive limits are related to the universal representation by the following

PROPOSITION 1.4. *We have an exact Iwahori-equivariant sequence*

$$0 \rightarrow \langle (v_+, v_-) \rangle_{\overline{\mathbf{F}}_p} \rightarrow \left(\lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+ \right) \oplus \left(\lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^- \right) \rightarrow \\ \rightarrow \left(\lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \right) |_{K_0(p)} \rightarrow 0$$

where $v_{\pm} \in \lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0^{\pm} \oplus_{R_1^{\pm}} \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$ (and are explicitly known).

We have an analogous result in the even case.

It will therefore be enough to focus our attention on the inductive limits of section §3. The euclidean dictionary is developed in section 4. Thanks to the natural filtration on the inductive limits, we are primarily concerned with the Iwahori structure of the representations R_{n+1}^{\pm} . We detect a convenient $\overline{\mathbf{F}}_p$ -basis \mathcal{B}_{n+1}^{\pm} (Lemma 2.6) and determine a natural way to identify the elements of \mathcal{B}_{n+1}^{\pm} to integer valued points of $\mathbf{R}^{[F:\mathbf{Q}_p]}$ (see section 4.1.1 for details). If we write \mathcal{R}_{n+1}^{\pm} to denote the image of \mathcal{B}_{n+1}^{\pm} in the $[F:\mathbf{Q}_p]$ -dimensional real euclidean space (such an image looks as a parallelepiped of side $p^{n+\epsilon}(\underline{r} + \underline{1})$ for $\epsilon \in \{0, 1\}$ according to the cases R_{n+1}^+ , R_{n+1}^-) then

PROPOSITION 1.5. *The Iwahori structure of R_{n+1}^{\pm} is described by \mathcal{R}_{n+1}^{\pm} .*

Because of the geometry of the polytope \mathcal{R}_{n+1}^{\pm} we indeed see that the socle filtration can be detected by successive cuttings by a suitable hyperplanes (parallel to the antidiagonal $X_0 + \cdots + X_{f-1} = 0$).

We similarly deduce the structure of tamely ramified principal series given in Proposition 1.2. Unfortunately, these results rely on a careful analysis of the behaviour of some universal Witt polynomials, contained in the two appendices A and B.

Section §5 deals finally with the universal representation $\pi(\underline{r}, 0, 1)$. We are first concerned with the graded pieces of the natural filtrations introduced in §3: it is the object of §5.1. Thanks to the behaviour of the canonical basis \mathcal{B}_n^{\pm} with respect to the Hecke operators of §3 we easily determine a natural basis $\mathcal{B}_{n+1/n}^{\pm}$ for each R_{n+1}^{\pm}/R_n^{\pm} and associate an euclidean structure \mathcal{R}_{n+1}^{\pm} to it. Such a structure is more complicated than the previous \mathcal{R}_{n+1}^{\pm} and can not be determined directly by Proposition 1.5 but a suitable decomposition of $\mathcal{B}_{n+1/n}^{\pm}$ as a union of increasing polytopes enable us to state the

PROPOSITION 1.6. *The Iwahori structure of R_{n+1}^{\pm}/R_n^{\pm} is described by $\mathcal{R}_{n+1/n}^{\pm}$.*

The euclidean image of $\mathcal{R}_{n+1/n}^{\pm}$ is more or less given in figure 2.

As a byproduct, the natural filtrations of section §3 and the previous description of the basis \mathcal{B}_{n+1}^{\pm} let us deduce Proposition 1.3.

The conclusion is in section §5.2 where we study the amalgamated sums $\cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$. Again, the behaviour of the canonical base \mathcal{B}_n^{\pm} with respect to the Hecke operators let us deduce, by induction on the exact sequence (1), an euclidean structure, say $\mathfrak{A}_{\text{even,odd}}^{\pm}$. Such a structure has a regular fractal nature, due to a convenient glueing of the blocks $\mathcal{R}_{n+1/n}^{\pm}$ and *simple* remarks on the geometry of $\mathfrak{A}_{\text{even,odd}}^{\pm}$, as well as the fact that $\cdots \oplus_{R_{n-2}^{\pm}} R_{n-1}^{\pm}$ is a Iwahori-subrepresentation of $\cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$, let us deduce the main result of Proposition 1.1.

We introduce now the basic conventions and notations of the paper (we essentially use the formalism and notations of [Bre]).

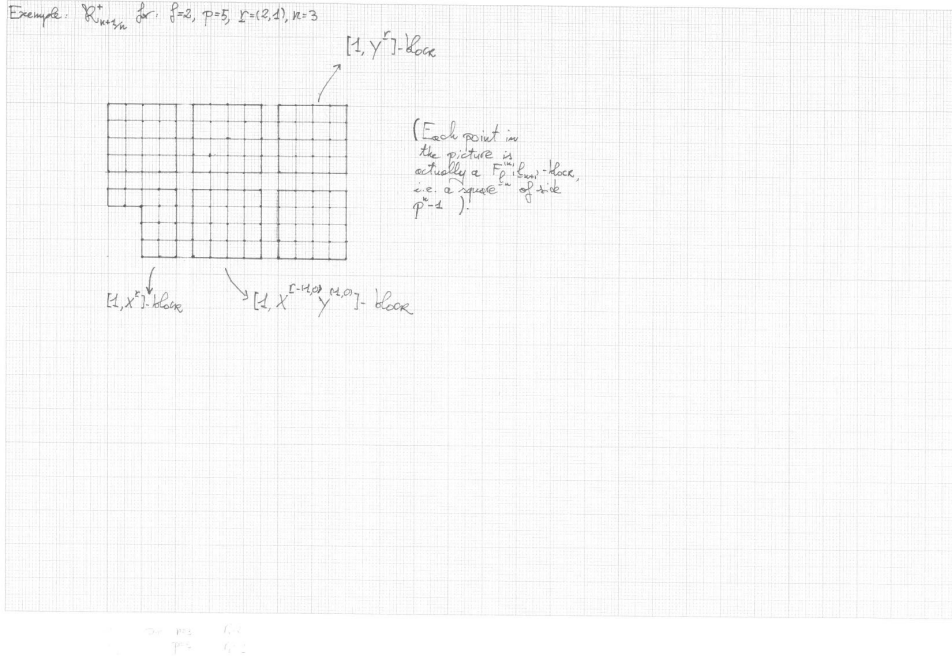


FIGURE 2. The structure of the quotients R_{n+1}^{\pm}/R_n^{\pm} .

Fix a prime $p \geq 5$ and let F be a finite unramified extension of \mathbf{Q}_p ; let $f \stackrel{\text{def}}{=} [F : \mathbf{Q}_p]$ be the residue degree. We write \mathcal{O}_F to denote the ring of integers of F and fix the uniformizer $p \in \mathcal{O}_F$; let k_F be the residue field; it is a finite field with $q \stackrel{\text{def}}{=} p^f$ elements. We fix an isomorphism $k_F \cong \mathbf{F}_q$; as F is unramified, we deduce an isomorphism $\mathcal{O}_F \cong W(\mathbf{F}_q)$ where $W(\mathbf{F}_q)$ denote the ring of Witt vectors of \mathbf{F}_q . We will write $[\cdot] : \mathbf{F}_q^{\times} \rightarrow W(\mathbf{F}_q)^{\times}$ to denote the Teichmüller character (putting $[0] \stackrel{\text{def}}{=} 0$). We finally fix an algebraic closure $\overline{\mathbf{F}}_p$ of \mathbf{F}_q .

For any $k \in \mathbf{N}$ the natural action of $\text{GL}_2(\mathbf{F}_q)$ on \mathbf{F}_q^2 let us determine, by functoriality of the k -th symmetric power, the $\text{GL}_2(\mathbf{F}_q)$ -representation $\text{Sym}^k \mathbf{F}_q^2$. It is isomorphic (up to a choice of an \mathbf{F}_q -basis for \mathbf{F}_q^2) to $\mathbf{F}_q[X, Y]_k^h$, the homogeneous component of degree k of the ring $\mathbf{F}_q[X, Y]$, endowed with the usual modular action:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} X^{k-i} Y^i = (aX + cY)^{k-i} (bX + dY)^i.$$

We recall that for $s \in \mathbf{N}$ $(\mathbf{F}_q[X, Y]_k^h)^{Frob^s}$ is the representation obtained by functoriality, in the evident way, from the field automorphism $x \mapsto x^{p^s}$ defined on \mathbf{F}_q .

For $\tau \in \text{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ and $r_{\tau}, t_{\tau} \in \{0, \dots, p-1\}$ we consider the $\text{GL}_2(\mathbf{F}_q)$ -representation

$$\sigma_{\{r_{\tau}, \{t_{\tau}\}} \stackrel{\text{def}}{=} \bigotimes_{\tau \in \text{Gal}(\mathbf{F}_q/\mathbf{F}_p)} (\det^{t_{\tau}} \otimes_{\mathbf{F}_q} \text{Sym}^{r_{\tau}} \mathbf{F}_q^2) \otimes_{\tau} \overline{\mathbf{F}}_p;$$

such representations exhaust all irreducible $\text{GL}_2(\mathbf{F}_q)$ -representations with coefficients in $\overline{\mathbf{F}}_p$ (and they are pairwise non isomorphic if we impose $t_{\tau} < p-1$ for at least one element $\tau \in \text{Gal}(\mathbf{F}_q/\mathbf{F}_p)$).

We fix once for all an immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$. Such a choice determines, up to twist, a manifest

isomorphism

$$\sigma_{\{r_\tau\}, \{t_\tau\}} \cong \sigma_{(r_0, \dots, r_{f-1})} \stackrel{\text{def}}{=} \bigotimes_{s=0}^{f-1} (\overline{\mathbf{F}}_p[X_s, Y_s]_{r_s}^h)^{Frob^s}$$

for a convenient $\underline{r} \stackrel{\text{def}}{=} (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$; such an isomorphism will be assumed to be fixed once for all throughout the paper. We notice that the choice of another immersion acts on the right hand side by a circular permutation on the indexes s in the obvious sense.

Write $G \stackrel{\text{def}}{=} \text{GL}_2(F)$, $K \stackrel{\text{def}}{=} \text{GL}_2(\mathcal{O}_F)$ and $Z \stackrel{\text{def}}{=} Z(G)$. We write $K_0(p)$ to denote the Iwahori subgroup of K . The $\text{GL}_2(\mathbf{F}_q)$ -representation $\sigma_{\underline{r}}$ will be seen, by the inflation map $K \rightarrow \text{GL}_2(\mathbf{F}_q)$, as a smooth representation of K . By imposing $p \in Z$ to act trivially, the smooth K -action on $\sigma_{\underline{r}}$ extends to a smooth action of KZ : by abuse of notation we will write $\sigma_{\underline{r}}$ to denote either the $\text{GL}_2(\mathbf{F}_q)$ or the K or the KZ -representation obtained by this procedure (or, as usual, the underlying vector space of $\sigma_{\underline{r}}$).

Similarly, the character

$$\begin{aligned} \chi_{\underline{r}} : B(\mathbf{F}_q) &\rightarrow \overline{\mathbf{F}}_p^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto a \sum_{s=0}^{f-1} p^s r_s \end{aligned}$$

will be considered, by inflation as a character of any open subgroup of $K_0(p)$. We write then $\chi_{\underline{r}}^s$ to denote the conjugate character of $\chi_{\underline{r}}$. We denote by \mathfrak{a} the character

$$\begin{aligned} B(\mathbf{F}_q) &\rightarrow \overline{\mathbf{F}}_p^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto ad^{-1}. \end{aligned}$$

Recall the compact induction:

$$c\text{-Ind}_{KZ}^G \sigma_{\underline{r}}$$

defined as the $\overline{\mathbf{F}}_p$ -linear space of functions $f : G \rightarrow \sigma_{\underline{r}}$, compactly supported modulo Z , verifying $f(kg) = k \cdot f(g)$ for any $k \in K$, $g \in G$; it is endowed with the smooth left action of G defined by right translations.

For $g \in G$, $v \in \sigma_{\underline{r}}$ we define $[g, v] \in c\text{-Ind}_{KZ}^G \sigma_{\underline{r}}$ as the unique function f supported in KZg^{-1} and such that $f(g) = v$. Then we have

$$\begin{aligned} g' \cdot [g, v] &= [g'g, v] && \text{for } g' \in G \\ [gk, v] &= [g, k \cdot v] && \text{for } k \in KZ. \end{aligned}$$

Each function $f \in c\text{-Ind}_{KZ}^G \sigma_{\underline{r}}$ can be written as a $\overline{\mathbf{F}}_p$ -linear combination of a finite family of functions $[g, v]$; if g varies in a fixed system of coset for G/KZ and v varies in a fixed $\overline{\mathbf{F}}_p$ -basis of $\sigma_{\underline{r}}$ the aforementioned writing is then unique.

We leave to the reader the task to adapt the previous definitions and remarks to such objects as

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \tau$$

where $K_0(p^{n+1}) \stackrel{\circ}{\leq} K_0(p^m) \stackrel{\circ}{\leq} K$ are open subgroups of K and τ is a smooth representation of $K_0(p^{n+1})$.

From [BL94], Proposition 8-(1) there exists a canonical Hecke operator (depending on \underline{r}) $T \in \text{End}_G(c\text{-Ind}_{KZ}^G \sigma_{\underline{r}})$. It realizes an isomorphism of the $\overline{\mathbf{F}}_p$ -algebra of endomorphisms $\text{End}_G(c\text{-Ind}_{KZ}^G \sigma_{\underline{r}})$ with the ring of polynomials in one variable over $\overline{\mathbf{F}}_p$. We then define the universal representation of

$\mathrm{GL}_2(F)$ as the cokernel of the canonical operator T :

$$\pi(\underline{r}, 0, 1) \stackrel{\mathrm{def}}{=} \mathrm{coker}(T).$$

We recall some conventions on the multiindex notations. For $\alpha_s \in \mathbf{N}$ we write $\underline{\alpha} \stackrel{\mathrm{def}}{=} (\alpha_0, \dots, \alpha_{f-1})$ to denote an f -tuple $\underline{\alpha} \in \mathbf{N}^f$. If $\underline{\alpha}, \underline{\beta}$ are f -tuples we define

- i) $\underline{\alpha} + \underline{\beta} \stackrel{\mathrm{def}}{=} (\alpha_s + \beta_s)_{s=0}^{f-1}$;
- ii) $\underline{\alpha} \geq \underline{\beta}$ if and only if $\alpha_s \geq \beta_s$ for all $s \in \{0, \dots, f-1\}$;
- iii) $\binom{\underline{\alpha}}{\underline{\beta}} \stackrel{\mathrm{def}}{=} \prod_{s=0}^{f-1} \binom{\alpha_s}{\beta_s}$.

For $n \in \mathbf{N}$ we will write $\underline{n} \stackrel{\mathrm{def}}{=} (n, \dots, n) \in \mathbf{N}^f$.

If $\underline{\alpha} + \underline{\beta} = \underline{r}$ we define the following element of $\sigma_{\underline{r}}$:

$$X^{\underline{\alpha}} Y^{\underline{\beta}} \stackrel{\mathrm{def}}{=} \otimes_{s=0}^{f-1} X_s^{\alpha_s} Y_s^{\beta_s};$$

for $\lambda \in \mathbf{F}_q$ and $\underline{\alpha} \in \{0, \dots, p-1\}^f$ we put

$$\lambda^{\underline{\alpha}} \stackrel{\mathrm{def}}{=} \lambda^{\sum_{s=0}^{f-1} p^s \alpha_s}.$$

For an integer $n \in \mathbf{N}$ we define $[n] \in \{0, \dots, f-1\}$ as the unique integer $m \in \{0, \dots, f-1\}$ congruent to n modulo f . Similarly, if $n \neq 0$ we define $[n] \in \{1, \dots, q-1\}$ as the unique integer $m \in \{1, \dots, q-1\}$ congruent to n modulo $q-1$; we set $[0] \stackrel{\mathrm{def}}{=} 0$.

Finally, for a smooth representation R of $K_0(p)$ over $\overline{\mathbf{F}}_p$ we write $\{\mathrm{soc}_N(R)\}_{N \in \mathbf{N}}$ to denote its socle filtration (with the convention $\mathrm{soc}(R)_0 \stackrel{\mathrm{def}}{=} \mathrm{soc}(R)$).

Let \mathcal{B} be an $\overline{\mathbf{F}}_p$ -basis of R and P a bijection of \mathcal{B} onto a subset \mathcal{R} in \mathbf{Z}^f . Let $\mathcal{B}' \subseteq \mathcal{B}$ be a subset and \mathcal{R}' denotes its image through the bijection P ; for $v \in \mathcal{B}'$ we define the set of antecedents of v in \mathcal{R}' as:

$$\mathfrak{S}_v(\mathcal{B}') \stackrel{\mathrm{def}}{=} \{w \in \mathcal{B}' \text{ s.t. } P(w) = P(v) - e_s \text{ for } s \in \{0, \dots, f-1\}\}$$

(where $(e_s)_{s=0}^{f-1}$ is the canonical basis of \mathbf{Z}^f).

We say that the socle filtration $\{\mathrm{soc}_N(R)\}_{N \in \mathbf{N}}$ of R is described by \mathcal{R} if the following holds: it exists an increasing family $\{\mathcal{B}_N\}_{N \in \mathbf{N}}$ of subsets of \mathcal{B} such that

- i) for all $N \in \mathbf{N}$ the family \mathcal{B}_N is an $\overline{\mathbf{F}}_p$ -basis of $\mathrm{soc}_N(R)$;
- ii) for all $N \in \mathbf{N}$ an $\overline{\mathbf{F}}_p$ -basis for $\mathrm{soc}(R/\mathrm{soc}_{N-1}(R))$ is described as

$$\{v \in \mathcal{B} \setminus \mathcal{B}_{N-1}, \text{ s.t. } \mathfrak{S}_v(\mathcal{B} \setminus \mathcal{B}_{N-1}) = \emptyset\}.$$

If the socle filtration of R is described by \mathcal{R} we will say that the extensions between two graded pieces are described by \mathcal{R} if the following holds true:

for all $N \in \mathbf{N}$ and $v \in \mathcal{B}_{N+1}$ the $\overline{\mathbf{F}}_p$ -linear subspace $E_{v,N}$ of $R/\mathrm{soc}_{N-1}(R)$ generated by v , $\mathfrak{S}_v(\mathcal{B} \setminus \mathcal{B}_{N-1})$ is $K_0(p)$ -stable and for each $w \in \mathfrak{S}_v(\mathcal{B} \setminus \mathcal{B}_{N-1})$ the induced extension

$$0 \rightarrow \overline{w} \rightarrow E_{v,N} / \langle \mathfrak{S}_v(\mathcal{B} \setminus \mathcal{B}_{N-1}) \setminus \{w\} \rangle_{\overline{\mathbf{F}}_p} \rightarrow \overline{v} \rightarrow 0$$

is nonsplit (with the obvious meaning of $\overline{w}, \overline{v}$).

In euclidean terms the segments between v and the set of its antecedents let us determines all the nonsplit extensions between two graded pieces of the socle filtration.

2. Preliminaries

As we outlined in the introduction, the main aim of this section is to describe the Iwahori-structure of the universal representations $\pi(\underline{r}, 0, 1)$ of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$.

Such representations have a completely explicit description in terms of the Bruhat-Tits tree and of the Hecke operator T given in [Bre], §2 and their Iwahori structure can indeed be found by direct methods. Nevertheless, the extremely involved combinatoric of such results lead us to introduce an intermediary step -namely a suitable KZ -filtration- which let us handle, in a reasonable way, the high amount of technical computations. Precisely, we start (cf. definition 2.3) by introducing the KZ -representations

$$R_{n+1} \stackrel{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^K \sigma_{\underline{r}^{n+1}}$$

(where $\sigma_{\underline{r}^{n+1}}$ is a $K_0(p^{n+1})$ -representations obtained by twisting the action of $K_0(p^{n+1})$ on $\sigma_{\underline{r}}|_{K_0(p^{n+1})}$). Such objects are endowed with an action of suitable ‘‘Hecke’’ operators $T_n^\pm : R_n \rightarrow R_{n\pm 1}$ (cf. Lemma 2.7), with respect to which we are able to define (inductively) a direct system of amalgamed sums $\cdots \oplus_{R_n} R_{n+1}$ (cf. Proposition 2.8). Such amalgamed sums fit in a natural commutative diagram (see Proposition 2.8) which let us deduce a natural filtration on the resulting inductive limits. The final result is then the isomorphism of Proposition 2.9, which relies the KZ -restriction of the universal representation $\pi(\underline{r}, 0, 1)|_{KZ}$ to the inductive limits constructed above; in particular, we have a natural KZ -equivariant filtration on the universal representation $\pi(\underline{r}, 0, 1)$.

In Lemma 2.6 we introduce a ‘‘canonical’’ basis for the representations R_{n+1} . Such basis is well behaved with respect to both the action of the Hecke operators and the action of the Iwahori subgroup: this will be the key observation which lead us to the description of the Iwahori structure for $\pi(\underline{r}, 0, 1)$.

We remark that the isomorphism of Proposition 2.9 does not rely on the fact that F/\mathbf{Q}_p is unramified: the content of this section can be generalised in the evident manner for any finite extension F of \mathbf{Q}_p .

Reminders on the universal representations $\pi(\underline{r}, 0, 1)$. For $n \in \mathbf{N}_{\geq 1}$ we define

$$I_n \stackrel{\text{def}}{=} \left\{ \sum_{j=0}^{n-1} p^j [\lambda_j] \quad \text{for } \lambda_j \in \mathbf{F}_q \right\}$$

and we put $I_0 \stackrel{\text{def}}{=} \{0\}$. The sets I_n ’s let us describe the Bruhat-Tits tree in the following way: if $n, m \in \mathbf{N}$, $\lambda \in I_n$ and

$$g_{n,\lambda}^0 \stackrel{\text{def}}{=} \begin{bmatrix} p^n & \lambda \\ 0 & 1 \end{bmatrix}, \quad g_{n,\lambda}^1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p\lambda & p^{n+1} \end{bmatrix}$$

we get a decomposition

$$KZ\alpha^{-m}KZ = \prod_{\lambda \in I_m} g_{m,\lambda}^0 KZ \prod_{\lambda \in I_{m-1}} g_{m,\lambda}^1 KZ \quad (2)$$

thus describing the vertex of the tree having distance m from KZ (where we have written $\alpha \stackrel{\text{def}}{=}} g_{0,0}^1$). The canonical Hecke operator $T \in \mathrm{End}_G(\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}})$, defined in [Bre] §2.7, is then characterized as follow:

LEMMA 2.1. For $n \in \mathbf{N}_{>}$, $\lambda \in I_n$ and $0 \leq j \leq \underline{r}$ we have:

$$T([g_{n,\lambda}^0, X^{r-j}Y^j]) = \sum_{\lambda_n \in \mathbf{F}_q} [g_{n+1,\lambda+p^n[\lambda_n]}^0, (-\lambda_n)^j X^r] + [g_{n-1, [\lambda]_{n-1}}^0, \delta_{j,\underline{r}}(\lambda_{n-1}X + Y)^r]$$

$$T([g_{n,\lambda}^1, X^{r-j}Y^j]) = \sum_{\lambda_n \in \mathbf{F}_q} [g_{n+1,\lambda+p^n[\lambda_n]}^1, (-\lambda_n)^{r-j} Y^r] + [g_{n-1, [\lambda]_{n-1}}^0, \delta_{j,0}(X + \lambda_{n-1}Y)^r].$$

If $n = 0$ we have

$$T([1_G, X^{r-j}Y^j]) = \sum_{\lambda_0 \in \mathbf{F}_q} [g_{1, [\lambda_0]}^0, (-\lambda_0)^j X^r] + [\alpha, \delta_{j,\underline{r}} Y^r]$$

$$T([\alpha, X^{r-j}Y^j]) = \sum_{\lambda_1 \in \mathbf{F}_q} [g_{1, [\lambda_1]}^1, (-\lambda_1)^{r-j} Y^r] + [1_G, \delta_{j,0} X^r]$$

Proof. A computation shows that the statement of lemma 3.1.1 in [Bre] has an obvious generalisation for $f > 1$. The result follows then from *Ibid.*, §2.5. \square

For $n \in \mathbf{N}$ we define the $\overline{\mathbf{F}}_p$ -subspace of $\text{Ind}_{KZ}^G \sigma_r$:

$$W(n) \stackrel{\text{def}}{=} \{f \in \text{Ind}_{KZ}^G \sigma_r, \quad \text{s.t. the support of } f \text{ is contained in } KZ\alpha^{-n}KZ\}.$$

By Cartan decomposition the subspaces $W(n)$ are KZ -stable for all $n \in \mathbf{N}$ and therefore

LEMMA 2.2. There is a natural KZ -equivariant isomorphism

$$\text{Ind}_{KZ}^G \sigma_r \xrightarrow{\sim} \bigoplus_{n \in \mathbf{N}} W(n).$$

The representations R_n 's and the dictionary. Let $n \in \mathbf{Z}_{\geq -1}$; we define the open subgroups of K :

$$K_0(p^{n+1}) \stackrel{\text{def}}{=} \{g \in K, \text{ s.t. } g = \begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix} \text{ for } a, b, c, d \in \mathcal{O}_F\}.$$

As $\begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix}$ normalizes $K_0(p^{n+1})$, the representation $\sigma_r|_{K_0(p^n)}$ induces, by conjugation, a $K_0(p^{n+1})$ -representation which will be denoted as σ_r^{n+1} (or simply σ_r if there is no risk of confusion). Explicitly, we have

$$\sigma_r^{n+1}\left(\begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix}\right) = \sigma_r\left(\begin{bmatrix} d & c \\ p^{n+1}b & a \end{bmatrix}\right).$$

We can therefore introduce the representations R_{n+1} 's:

DEFINITION 2.3. Let $n \in \mathbf{Z}_{\geq -1}$. The K -representation R_{n+1} is defined as

$$R_{n+1} \stackrel{\text{def}}{=} \text{Ind}_{K_0(p^{n+1})}^K \sigma_r^{n+1}.$$

We can extend the action of K on R_{n+1} to an action of KZ by letting $p \in Z$ act trivially; the resulting representation will be denoted again by R_{n+1} and we will pass from the one to the other without commentary.

Thanks to the decomposition (2) we get the following, elementary, description of the R_n 's:

LEMMA 2.4. Let $n \in \mathbf{Z}_{\geq -1}$ Then:

i) right translation by $\alpha^{n+1}w$ induces a bijection

$$K/K_0(p^{n+1}) \xrightarrow{\sim} KZ\alpha^{-n-1}KZ/KZ;$$

ii) we have a decomposition

$$K = \coprod_{\lambda \in I_{n+1}} \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} K_0(p^{n+1}) \coprod_{\lambda' \in I_n} \begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix} K_0(p^{n+1});$$

Moreover, if $1 \leq m \leq n$ we have a decomposition

$$K_0(p^m) = \coprod_{\lambda' \in I_{n+1-m}} \begin{bmatrix} 1 & 0 \\ p^m\lambda' & 1 \end{bmatrix} K_0(p^{n+1});$$

iii) the family

$$\left\{ \left[\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j \right], \left[\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^{r-j}Y^j \right] \text{ for } \lambda \in I_{n+1}, \lambda' \in I_n, 0 \leq j \leq r \right\}$$

defines an $\overline{\mathbf{F}}_p$ -basis for the representation R_{n+1} . Moreover, if $1 \leq m \leq n$, the family

$$\left\{ \left[\begin{bmatrix} 1 & 0 \\ p^m\lambda' & 1 \end{bmatrix}, X^{r-j}Y^j \right] \text{ for } \lambda' \in I_{n+1-m}, 0 \leq j \leq r \right\}$$

defines an $\overline{\mathbf{F}}_p$ -basis for the representation $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \sigma_r$.

Proof. Omissis. □

The relation between the representations R_n 's and the compact induction $\text{Ind}_{KZ}^G \sigma_r|_{KZ}$ is then described by the following

PROPOSITION 2.5. *Let $n \in \mathbf{Z}_{\geq -1}$. We have a KZ -equivariant isomorphism*

$$\Phi_{n+1} : W(n+1) \xrightarrow{\sim} R_{n+1}$$

such that

$$\begin{aligned} \Phi_{n+1}([g_{n+1,\lambda}^0, X^{r-j}Y^j]) &= \left[\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j \right] \\ \Phi_{n+1}([g_{n,\lambda'}^1, X^{r-j}Y^j]) &= \left[\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^jY^{r-j} \right] \end{aligned}$$

for $n \geq 0$ and

$$\Phi_0([1_G, X^{r-j}Y^j]) = X^jY^{r-j}$$

for $n = 0$.

In particular, we have a KZ -equivariant isomorphism

$$\text{Ind}_{KZ}^G \sigma_r \xrightarrow{\sim} \bigoplus_{n \in \mathbf{N}} R_n$$

Proof. Elementary (see for instance [Mo], Proposition 3.4, whose proof generalizes line by line). □

We introduce now a convenient $\overline{\mathbf{F}}_p$ -basis for the representation R_{n+1} . Thanks to the transitivity

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \sigma_r \cong \text{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} \text{Ind}_{K_0(p^{n+1})}^{K_0(p^{m+1})} \sigma_r$$

(where $0 \leq m \leq n$) we see that a Vandermonde argument together with an immediate induction give us the following:

LEMMA 2.6 (Definition). *Let $n \in \mathbf{N}$. An $\overline{\mathbf{F}}_p$ basis for the K -representation R_{n+1} is described by the elements*

$$F_{l_1, \dots, l_n}^{(1, n)}(l_{n+1}) \stackrel{\text{def}}{=} \sum_{i=1}^n \sum_{\lambda_i \in \mathbf{F}_q} (\lambda_i^{\frac{1}{p^i}})^{l_i} \begin{bmatrix} 1 & 0 \\ p^i [\lambda_i^{\frac{1}{p^i}}] & 1 \end{bmatrix} [1, X^{r-l_{n+1}} Y^{l_{n+1}}]$$

$$F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1, \dots, l_n}^{(1, n)}(l_{n+1})]$$

for $l_i \in \{0, \dots, p-1\}^f$ (where $i \in \{0, \dots, n\}$) and $l_{n+1} \leq r$, with the obvious conventions that if $n=0$ we have

$$F_{\emptyset}^{(1, 0)}(l_1) \stackrel{\text{def}}{=} [1, X^{r-l_{n+1}} Y^{l_{n+1}}].$$

For notational convenience we define

$$F_{\emptyset}^{(0, -1)}(l_0) \stackrel{\text{def}}{=} (-1)^{l_0} X^{l_0} Y^{r-l_0}$$

$$F_{\emptyset}^{(1, -1)}(\emptyset) \stackrel{\text{def}}{=} Y^r.$$

Such basis will be denoted by \mathcal{B}_{n+1} .

The subset $\mathcal{B}_{n+1}^+ \subset \mathcal{B}_{n+1}$ described by the elements of the form $F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1})$ will be referred to as the set of positive elements of R_{n+1} ; the $\overline{\mathbf{F}}_p$ -linear subspace generated by the positive elements will be denoted as R_{n+1}^+ .

Similarly the subset $\mathcal{B}_{n+1}^- \subset \mathcal{B}_{n+1}$ described by elements of the form $F_{l_1, \dots, l_n}^{(1, n)}(l_{n+1})$ will be referred to as the set of negative elements of R_{n+1} ; the $\overline{\mathbf{F}}_p$ -linear subspace generated by the negative elements will be denoted as R_{n+1}^- .

Hecke operators on the R_{n+1} 's. Let $n \in \mathbf{N}$. Thanks to Lemma 2.1 the $W(n)$ -restriction of the operator T gives the $\overline{\mathbf{F}}_p$ -linear morphism

$$T|_{W(n)} : W(n) \rightarrow W(n-1) \oplus W(n+1).$$

Such restriction is KZ -equivariant (by Cartan decomposition) and composition by the natural projections gives us the KZ -equivariant operators

$$T_n^+ : W(n) \rightarrow W(n+1) \quad T_n^- : W(n) \rightarrow W(n-1).$$

By transport of structure (via the isomorphisms of Lemma 2.5) we get morphisms

$$T_n^+ : R_n \rightarrow R_{n+1} \quad T_n^- : R_n \rightarrow R_{n-1}$$

(where we used the same notations for the operators on $W(n)$ and R_n). Their description in terms of the canonical basis of R_{n+1} is immediate, following from Lemmas 2.1 and 2.5:

LEMMA 2.7. *Let $n > 0 \in \mathbf{N}$. The KZ -equivariant operators T_n^+, T_n^- are characterized by*

$$T_n^+ : R_n \rightarrow R_{n+1}$$

$$[1, X^{r-l_n} Y^{l_n}] \mapsto (-1)^{l_n} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{l_n} \begin{bmatrix} 1 & 0 \\ p^n [\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, X^r]$$

$$T_n^- : R_n \rightarrow R_{n-1}$$

$$[1, X^{r-l_n} Y^{l_n}] \mapsto \begin{cases} \delta_{r, l_n} [1, Y^r] & \text{if } n > 1 \\ \delta_{r, l_n} Y^r & \text{if } n = 1. \end{cases}$$

For $n = 0$ we have

$$R_0 \hookrightarrow R_1$$

$$X^{r-l_0} Y^{l_0} \mapsto \sum_{\lambda_0 \in \mathbf{F}_q} (-1)^{r-l_0} \lambda_0^{r-l_0} \begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix} [1, X^r] + \delta_{l_0, 0} [1, X^r].$$

Moreover, the operators T_n^+ are monomorphisms for all $n \in \mathbf{N}$ and the operators T_n^- are epimorphisms for all $n \in \mathbf{N}_{\geq 1}$.

Proof. The characterisation of the operators T_n^\pm follows by the explicit descriptions given in Lemmas 2.1 and 2.5.

As T_n^+ maps the basis \mathcal{B}_n into a subset of \mathcal{B}_{n+1} , the operator is injective for $n \geq 1$. As $[1, Y^r]$ (resp. Y^r) is a K -generator for R_{n-1} (resp. R_0) for $n \geq 2$ (resp. $n = 1$), the operator T_n^- is surjective. \square

We identify R_n as a K -subrepresentation of R_{n+1} via the monomorphism T_n^+ without any further commentary. For any odd integer $n \geq 1$ we use the Hecke operators T_n^\pm to define (inductively) the amalgamated sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ via the following co-cartesian diagram

$$\begin{array}{ccc} R_n & \xrightarrow{T_n^+} & R_{n+1} \\ \downarrow -pr_{n-1} \circ T_n^- & & \downarrow pr_{n+1} \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} & \dashrightarrow & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1} \end{array}$$

(where we define pr_0 to be the identity map). Similarly we define the amalgamated sums $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for any positive even integer $n \in \mathbf{N}_{>}$. The following result is then formal

PROPOSITION 2.8. *For any odd integer $n \in \mathbf{N}$, $n \geq 1$ we have a natural commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_n & \xrightarrow{T_n^+} & R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\ & & \downarrow -pr_{n-1} \circ T_n^- & & \downarrow pr_{n+1} & & \parallel \\ 0 & \longrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} & \longrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \end{array}$$

with exact lines.

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N} \setminus \{0\}}.$$

Proof. Formal. See for instance [Mo], Proposition 4.1. \square

The following result let us complete the dictionary

PROPOSITION 2.9. *We have a KZ equivariant isomorphism*

$$\pi(\sigma_{\underline{r}}, 0, 1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}).$$

Proof. The proof is formal and identical to [Mo], Proposition 3.9. \square

REMARK 2.10. *We can give analogous (in the evident way) definitions in the case where F is any finite extension of \mathbf{Q}_p : we would then get a statement completely analogous to Proposition 2.9.*

3. First description of the Iwahori structure

The goal of this section is to give a first, general description for the $K_0(p)$ -representation $\pi(\underline{r}, 0, 1)|_{K_0(p)}$. The endpoint is Proposition 3.6, which is the ‘‘Iwahori analogue’’ of Proposition 2.9 of the preceding section. More precisely, for each $n \in \mathbf{N}$ the block R_{n+1} has a natural $K_0(p)$ -equivariant splitting

$$R_{n+1} = R_{n+1}^+ \oplus R_{n+1}^-$$

which is compatible with the Hecke operators T_n^\pm in the obvious sense (cf. Lemma/definition 3.2). This will enable us to repeat the constructions of §2, i.e. the construction of the (inductive family of) amalgamed sums $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$, endowed with a natural filtration (cf. Lemma 3.5).

Thanks to Proposition 3.6 we see that we can content ourselves to the study of the amalgames sums $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$: actually we have a $K_0(p)$ -equivariant surjection

$$\begin{aligned} & \left(\lim_{\substack{\longrightarrow \\ n \text{ odd}}} \cdots \oplus_{R_n^+} R_{n+1}^+ \right) \oplus \left(\lim_{\substack{\longrightarrow \\ n \text{ odd}}} \cdots \oplus_{R_n^-} R_{n+1}^- \right) \oplus \left(\lim_{\substack{\longrightarrow \\ n \text{ even}}} \cdots \oplus_{R_n^+} R_{n+1}^+ \right) \oplus \left(\lim_{\substack{\longrightarrow \\ n \text{ even}}} \cdots \oplus_{R_n^-} R_{n+1}^- \right) \\ & \qquad \qquad \qquad \downarrow \\ & \pi(\underline{r}, 0, 1)|_{K_0(p)} \end{aligned}$$

whose kernel is ‘‘small’’ (and explicitly determined).

The following elementary result will be crucial.

LEMMA 3.1. *Let $a \in \{0, \dots, q-1\}$. Then*

$$\sum_{\lambda \in \mathbf{F}_q} \lambda^a = \begin{cases} 0 & \text{if } a \neq q-1 \\ -1 & \text{if } a = q-1. \end{cases}$$

Proof. Omissis. □

The representations R_{n+1}^\pm and the Hecke operators $(T_n^\pm)^{\text{pos, neg}}$. Fix $n \in \mathbf{N}$; the $\overline{\mathbf{F}}_p$ -linear decomposition

$$R_{n+1} \cong R_{n+1}^+ \oplus R_{n+1}^- \tag{3}$$

is easily checked to be $K_0(p)$ -equivariant (realising the Mackey decomposition for $R_{n+1}|_{K_0(p)}$) and we clearly have a $K_0(p)$ -equivariant isomorphism

$$R_{n+1}^- \xrightarrow{\sim} \text{Ind}_{K_0(p^{n+1})}^{K_0(p)} \sigma_{\underline{r}}^{n+1}.$$

We moreover define the following $K_0(p)$ -representations:

$$R_0^+ \stackrel{\text{def}}{=} R_0, \quad R_0^- \stackrel{\text{def}}{=} \langle Y^{\underline{r}} \rangle_{\overline{\mathbf{F}}_p}, \quad (R_1/R_0)^+ \stackrel{\text{def}}{=} \text{Im}(R_1^+ \hookrightarrow R_1 \twoheadrightarrow R_1/R_0).$$

The decomposition given in (3) and the description of Lemma 2.7 let us define the Hecke operators $(T_n^\pm)^{\text{pos, neg}}$ on the representations R_{n+1}^\pm :

LEMMA 3.2 (Definition). *Let $n \in \mathbf{N}_{\geq 1}$.*

- i) *The restriction of Hecke operator T_n^+ on the $K_0(p)$ -subrepresentations R_n^+, R_n^- of R_n induces two $K_0(p)$ -equivariant monomorphisms,*

$$\begin{aligned} (T_n^+)^{\text{pos}} &: R_n^+ \hookrightarrow R_{n+1}^+ \\ (T_n^+)^{\text{neg}} &: R_n^- \hookrightarrow R_{n+1}^- \end{aligned}$$

- ii) *The restriction of Hecke operator T_n^- on the $K_0(p)$ -subrepresentations R_n^+, R_n^- of R_n induces two $K_0(p)$ -equivariant epimorphisms,*

$$\begin{aligned} (T_n^-)^{\text{pos}} &: R_n^+ \twoheadrightarrow R_{n-1}^+ \\ (T_n^-)^{\text{neg}} &: R_n^- \twoheadrightarrow R_{n-1}^- \end{aligned}$$

Proof. Except for the operator $(T_1^-)^{\text{pos}}$, the result follows immediately from the decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ and the properties and characterisations of the Hecke operators T_n^\pm .

Concerning $(T_1^-)^{\text{pos}} : R_1^+ \rightarrow R_0$ we notice that

$$(T_1^-)^{\text{pos}}(F_{l_0(r)}^{(0)}) = \sum_{i \leq r} X^{r-i} Y^i \left(\sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{l_0+i} \right)$$

and the result follows from Lemma 3.1. \square

COROLLARY 3.3. *The natural $K_0(p)$ -equivariant map*

$$R_2^+ \rightarrow (R_1/R_0)^+$$

is an epimorphism.

Proof. Omissis. \square

REMARK 3.4. *The notation $(T_n^\pm)^{\text{pos,neg}}$ may look a bit awkward. We believe, though, that a notation of the kind $(T_n^\pm)^\pm$, even if it could be more convenient for statements (see Lemma 3.5), it can be disagreeable for the computations (and especially misprints!)*

Amalgamed sums and first description of the Iwahori structure. Using the Hecke operators defined in Lemma 3.2 we can introduce the following amalgamed sums, analogously to the constructions of §2.

Let $n \in \mathbf{N}$ be odd and $\bullet \in \{+, -\}$. We can define inductively a natural $K_0(p)$ -representation $R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ by the condition that the diagram

$$\begin{array}{ccc} R_n^\bullet & \xrightarrow{(T_n^+)^\bullet} & R_{n+1}^\bullet \\ \downarrow \scriptstyle -(pr_{n-1})^\bullet \circ (T_n^-)^\bullet & & \downarrow \scriptstyle \exists! \ (pr_{n+1})^\bullet \\ R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \xrightarrow[\exists!]{\iota_{n-1}^\bullet} & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet \end{array}$$

is co-cartesian (with the convention that $(T_j^\pm)^+ \stackrel{\text{def}}{=} (T_j^\pm)^{\text{pos}}$ and $(T_j^\pm)^- \stackrel{\text{def}}{=} (T_j^\pm)^{\text{neg}}$).

For $n \in \mathbf{N}$ even and $\bullet \in \{+, -\}$ we can define the amalgamed sums $(R_1/R_0)^\bullet \oplus_{R_2^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$, together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ in the evident analogous way (with the convention that $(R_1/R_0)^- = R_1^-$.)

The following result is similar to Proposition 2.8:

LEMMA 3.5. *Let $n \in \mathbf{N}$ be odd, $\bullet \in \{+, -\}$. Then ι_{n-1}^\bullet is a monomorphism, pr_{n+1}^\bullet is an epimorphism and we have a $(K_0(p)$ -equivariant) commutative diagram with exact lines:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_n^\bullet & \xrightarrow{(T_n^+)^\bullet} & R_{n+1}^\bullet & \xrightarrow{\pi_{n+1}} & R_{n+1}^\bullet/R_n^\bullet \longrightarrow 0 \\ & & \downarrow \scriptstyle -(T_n^-)^\bullet & & \downarrow \scriptstyle pr_{n+1}^\bullet & & \parallel \\ & & R_{n-1}^\bullet & & & & \\ & & \downarrow \scriptstyle pr_{n-1}^\bullet & & & & \\ 0 & \longrightarrow & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \xrightarrow[\exists!]{\iota_{n-1}^\bullet} & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet & \xrightarrow{\pi_{n+1}} & R_{n+1}^\bullet/R_n^\bullet \longrightarrow 0 \end{array}$$

We have an analogous (in the evident way) result in the case $n \in \mathbf{N}_>$ is even.

Proof. The proof is identical to Proposition 2.8, provided that the maps $R_1^\bullet \xrightarrow{(T_1^-)^\bullet} R_0^\bullet$ and $R_2^\bullet \xrightarrow{(T_2^-)^\bullet} (R_1/R_0)^\bullet$ are epimorphisms. \square

In order to give a first description of the $K_0(p)$ -representation $\pi(\underline{x}, 0, 1)|_{K_0(p)}$ we are now left to determine the relations between the amalgamated sums $\cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ (where $\bullet \in \{+, -\}$) and the restriction $(\cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$.

We will treat in detail the analysis of the limit $(\varinjlim_{n, \text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$. The case n even is proved in a similar way and is left to the reader.

PROPOSITION 3.6. *The decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ induces the following $K_0(p)$ -equivariant exact sequences:*

$$\begin{aligned} 0 \rightarrow \langle (F_\emptyset^{(0,-1)}(\underline{0}), F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbb{F}}_p} &\rightarrow (\varinjlim_{n \text{ odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n \text{ odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \rightarrow \\ &\rightarrow (\varinjlim_{n \text{ odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \langle (F_{\underline{r}}^{(0)}(\underline{0}), -F_\emptyset^{(1,0)}(\underline{0})) \rangle_{\overline{\mathbb{F}}_p} &\rightarrow (\varinjlim_{n \text{ even}} (R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n \text{ even}} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \rightarrow \\ &\rightarrow (\varinjlim_{n \text{ even}} (R_1/R_0) \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \rightarrow 0. \end{aligned}$$

Proof. Let us assume n odd, leaving the case n even to the reader (the proof being analogous). Since the functor \varinjlim is exact if the index category is filtrant and since the forgetful functor $F_{or} :$

$\mathcal{Rep}_{K_0(p)} \rightarrow \mathcal{Vect}_{\overline{\mathbb{F}}_p}$ commutes with \varinjlim it is enough to show that we have an inductive system of exact sequences

$$\begin{aligned} 0 \rightarrow \langle (F_\emptyset^{(0,-1)}(\underline{0}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbb{F}}_p} &\rightarrow (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \rightarrow \\ &\rightarrow (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \rightarrow 0. \end{aligned}$$

The proof is an induction on n .

Let $\bullet \in \{+, -\}$. By the universal property of the push out we deduce the following commutative diagramm

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_1^\bullet & \longrightarrow & R_2^\bullet & \longrightarrow & R_2^\bullet/R_1^\bullet \longrightarrow 0 \\ & & \swarrow & & \swarrow & & \parallel \\ 0 & \longrightarrow & R_1 & \longrightarrow & R_2 & \longrightarrow & R_2/R_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & R_0^\bullet & \longrightarrow & R_0^\bullet \oplus_{R_1^\bullet} R_2^\bullet & \longrightarrow & R_2^\bullet/R_1^\bullet \longrightarrow 0 \\ & & \downarrow & & \swarrow \scriptstyle f_\bullet & & \parallel \\ 0 & \longrightarrow & R_0 & \longrightarrow & (R_0 \oplus_{R_1} R_2)|_{K_0(p)} & \longrightarrow & R_2/R_1 \longrightarrow 0 \end{array}$$

and the morphism f_\bullet is injective by the four Lemma applied to the “bottom” diagram: recall that $(T_0^+)^\bullet$ is injective and we check easily the injectivity of the morphism $R_2^\bullet/R_1^\bullet \rightarrow R_2/R_1$. We deduce

the commutative diagram with exact lines

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R_0^+ \oplus R_0^- & \longrightarrow & (R_0^+ \oplus_{R_1^+} R_2^+) \oplus (R_0^- \oplus_{R_1^-} R_2^-) & \longrightarrow & (R_2^+/R_1^+) \oplus (R_2^-/R_1^-) \longrightarrow 0 & (4) \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & R_0 & \longrightarrow & (R_0 \oplus_{R_1} R_2)|_{K_0(p)} & \longrightarrow & R_2/R_1 \longrightarrow 0.
 \end{array}$$

The isomorphism $(R_2^+/R_1^+) \oplus (R_2^-/R_1^-) \xrightarrow{\sim} R_2/R_1$ and the exact sequence

$$0 \rightarrow \langle (F_\emptyset^{(0,-1)}(\underline{\emptyset}), -F_\emptyset^{(1,-1)}(\emptyset)) \rangle \rightarrow R_0^+ \oplus R_0^- \rightarrow R_0 \rightarrow 0$$

give the result, via the snake Lemma applied to the diagramm (4).

We treat now the inductive step. By the inductive hypothesis and the definition of the Hecke operators $(T_n^\pm)^{\text{pos,neg}}$, we dispose of the commutative diagrams

$$\begin{array}{ccc}
 R_n^\bullet & \hookrightarrow & R_n \\
 \downarrow & & \downarrow \\
 R_{n-1}^\bullet & \hookrightarrow & R_{n-1} \\
 \downarrow & & \downarrow \\
 R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \hookrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1}
 \end{array}$$

(the inductive hypothesis being used for the injectivity of the lower arrow) from which we deduce the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R_n^\bullet & \longrightarrow & R_{n+1}^\bullet & \longrightarrow & R_{n+1}^\bullet/R_n^\bullet \longrightarrow 0 \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R_n & \longrightarrow & R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet & \longrightarrow & \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet & \longrightarrow & R_{n+1}^\bullet/R_n^\bullet \longrightarrow 0 \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (\cdots \oplus_{R_{n-2}} R_{n-1})|_{K_0(p)} & \longrightarrow & (\cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0.
 \end{array}$$

f_\bullet
 $\exists!$

Again, the morphism f_\bullet is injective and we deduce as well the following commutative diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-1}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-1}^-} R_{n-1}^-) & \longrightarrow & (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-1}} R_{n-1})|_{K_0(p)} \\
 \downarrow & & \downarrow \\
 (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n+1}^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n+1}^-} R_{n+1}^-) & \longrightarrow & (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \\
 \downarrow & & \downarrow \\
 (R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) & \longrightarrow & R_{n+1}/R_n \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

As the natural morphism $(R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) \rightarrow R_{n+1}/R_n$ is an isomorphism, the conclusion follows by applying the snake lemma and using the exact sequence

$$\begin{aligned}
 0 \rightarrow \langle (F_\emptyset^{(0,-1)}(\underline{0}), F_\emptyset^{(1,-1)}(\emptyset)) \rangle_{\overline{\mathbf{F}}_p} &\rightarrow (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-2}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-) \rightarrow \\
 &\rightarrow (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1})|_{K_0(p)} \rightarrow 0.
 \end{aligned}$$

coming from the inductive hypothesis. \square

4. Representations of the Iwahori subgroups

We start here the technical computations which should lead us (in section §5) to the Iwahori-structure of the universal representations $\pi(\underline{r}, 0, 1)$. The aim is to describe the $K_0(p)$ -representations R_{n+1}^\pm which appeared in the preceding section §3.

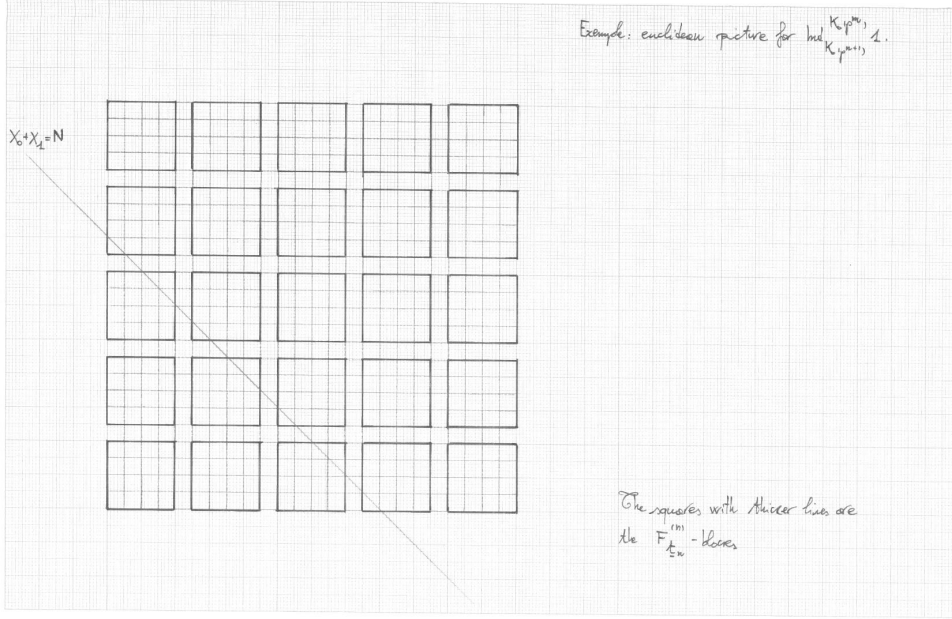
We focus our attention on the representations $\text{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$: the description of R_{n+1}^\pm can be obtained with identical techniques (cf. sections §4.1.3 or 4.2). The Iwahori structure of such objects -given by Proposition 4.2- may look complicated, but the keypoint is its combinatoric can be controlled by an easy euclidean method which can be outlined as follow.

First of all we detect a “canonical” $\overline{\mathbf{F}}_p$ -basis \mathcal{B} for the representation $\text{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ (definition 4.1). We see that each element $F_{\underline{l}_1, \dots, \underline{l}_n}^{(1,n)} \in \mathcal{B}$ is parametrized by a family of f -tuples $\underline{l}_i \in \{0, \dots, p-1\}^f$, family which can be used to define a point (in the naïve sense) $(x_0, \dots, x_{f-1}) \in \mathbf{R}^{f-1}$. In this way, we can associate, bijectively, the elements of the basis \mathcal{B} to the integer points of an f -hypercube of side $p^n - 1$ in \mathbf{R}^{f-1} : this is detailed in paragraph 4.1.1.

With this gloss, the $K_0(p)$ -socle filtration for $\text{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ can be simply described by the successive intersections of the f -hypercube with the antidiagonals $X_0 + \cdots + X_{f-1} = \text{constant}$, as illustrated in figure 3.

This is the content of Proposition 4.2 where we verify, by direct computation on Witt vectors, that the behaviour of the canonical elements $F_{\underline{l}_1, \dots, \underline{l}_{f-1}}^{(1,n)}$ fits the previous euclidean picture. It is the technical part of the paper and rely, as announced in the introduction, on the following three key facts (whose meaning will be clear to the reader of paragraph §4.1.2):

- i)* the elements of the canonical basis \mathcal{B} are “well behaved” with respect to the action of $g \in$


 FIGURE 3. The structure of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$.

- $K_0(p)$, i.e. one can naturally describe $gF_{l_1, \dots, l_{f-1}}^{(1, n)}$ as a linear combination of elements of \mathcal{B} ;
- ii) one can compute the homogeneous (pseudo-)degree of the universal Witt polynomials appearing in the development of $gF_{l_1, \dots, l_{f-1}}^{(1, n)}$;
 - iii) the correspondence between the elements of \mathcal{B} and the points in the associated hypercube is well behaved with respect to the homogeneous degree of the universal Witt polynomials.

As announced the same techniques let us detect the $K_0(p)$ -structure for the representations R_{n+1}^\pm : the involved combinatoric can be handled with the help of a simple euclidean picture (an f -parallelepiped). The precise statements are Propositions 4.9 and 4.10 which deal with R_{n+1}^- and R_{n+1}^+ respectively.

The constructions and computations of this section let us, as an application, determine the Iwahori structure for principal and special series: this is the object of §4.3. Again, in terms of euclidean space, we see that the successive layers for the $K_0(p)$ -socle filtration are detected by the intersections of \mathbf{N}^f (the “hypercube” associated to such series) with the hyperplans $X_0 + \dots + X_{f-1} = \text{constant}$.

4.1 The negative case.

Let $1 \leq m \leq n$ be integers. In this section we examine the $K_0(p)$ -socle filtration (and the extensions between two consecutive graded pieces) for the representations $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi$ where $\chi : K_0(p^{n+1}) \rightarrow \overline{\mathbf{F}}_p^\times$ is a smooth character of $K_0(p^{n+1})$ (i.e. the inflation of a character of the finite Borel $B(\mathbf{F}_q)$ by the morphism $K_0(p^{n+1}) \twoheadrightarrow B(\mathbf{F}_q)$). Thanks to the canonical isomorphism :

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi \cong (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1) \otimes \chi$$

we can assume that $\chi = 1$ is the trivial character. Finally, let $\{e\}$ be an $\overline{\mathbf{F}}_p$ -basis for the underlying vector space associated to the character χ .

We introduce now the canonical base of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ and its interpretation in terms of lattices of \mathbf{R}^f .

DEFINITION 4.1. For $j \in \{m, \dots, n\}$ let $l_j = (l_j^{(0)}, \dots, l_j^{(f-1)}) \in \{0, \dots, p-1\}^f$ be a f -tuple. We define the element $F_{l_m, \dots, l_n}^{(m,n)} \in \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ as

$$F_{l_m, \dots, l_n}^{(m,n)} \stackrel{\text{def}}{=} \sum_{j=m}^n \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, e].$$

For a notational convenience, we define $F_{l_{n+1}, \dots, l_n}^{(n+1,n)} \stackrel{\text{def}}{=} [1, e]$ and $l_{n+1} \stackrel{\text{def}}{=} \underline{0}$.

The set

$$\mathcal{B} \stackrel{\text{def}}{=} \left\{ F_{l_m, \dots, l_n}^{(m,n)} \in \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1, \text{ for } (l_m, \dots, l_n) \in \{\{0, \dots, p-1\}^f\}^{n+1-m} \right\}$$

is an $\overline{\mathbf{F}}_p$ -basis for $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$.

The fact that \mathcal{B} is an $\overline{\mathbf{F}}_p$ basis for $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ is again an induction together with a Vandermonde argument as for Lemma 2.6.

4.1.1 *Interpretation in terms of lattices.* As anticipated in the introduction, each element of \mathcal{B} can be seen as a ‘‘point’’ of a \mathbf{Z} -lattice in the standard euclidean f -dimensional space \mathbf{R}^f : such correspondence is given by the injective map

$$\begin{aligned} \mathcal{B} &\xrightarrow{P} \mathbf{R}^f & (5) \\ F_{l_m, \dots, l_n}^{m,n} &\mapsto \left(\sum_{j=m}^n p^{j-m} l_j^{(\lfloor j-m \rfloor)}, \dots, \sum_{j=m}^n p^{j-m} l_j^{(\lfloor f-1+j-m \rfloor)} \right) \end{aligned}$$

whose image will be denoted by \mathcal{R} . We notice that \mathcal{R} is a f -hypercube of side $p^{n-m+1} - 1$. It has a natural recurrent structure: for a fixed f -tuple $l_n \in \{0, \dots, p-1\}^f$ the subset

$$\{F_{l_m, \dots, l_{n-1}, l_n}^{(m,n)} \in \mathcal{B} \mid l_j \in \{0, \dots, p-1\}^f, \text{ for } m \leq j \leq n-1\}$$

is mapped onto an f -hypercube of side $p^{n-m} - 1$, which will be referred as the $F_{l_n}^{(n)}$ -block. The hypercube \mathcal{R} is then obtained as the juxtaposition of the $F_{l_n}^{(n)}$ -blocks for varying $l_n \in \{0, \dots, p-1\}^f$.

We are therefore allowed to apply the terminology of real euclidean spaces to the elements of \mathcal{B} , meaning their image through the map P . In particular if $e_i \stackrel{\text{def}}{=} (\delta_{0,i}, \dots, \delta_{f-1,i}) \in \{0, 1\}^f$ we define $F_{(l_m, \dots, l_n) - e_i}^{m,n}$ by

$$F_{(l_m, \dots, l_n) - e_i}^{m,n} = \begin{cases} 0 & \text{if } P^{\leftarrow}(P(F_{l_m, \dots, l_n}^{m,n}) - e_i) = \emptyset \\ \text{the only element of } P^{\leftarrow}(P(F_{l_m, \dots, l_n}^{m,n}) - e_i) & \text{otherwise.} \end{cases}$$

In order to give the statement concerning the $K_0(p^m)$ -structure of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi$ we still need

some notation. If $(\underline{l}_m, \dots, \underline{l}_n)$ is a $(n+1-m)f$ -tuple, we define

$$N_{m,n}(\underline{l}_m, \dots, \underline{l}_n) \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} l_m^{(s)} + p \left(\sum_{s=0}^{f-1} l_{m+1}^{(s)} \right) + \dots + p^{n-m} \left(\sum_{s=0}^{f-1} l_n^{(s)} \right)$$

$$e(\underline{l}_m, \dots, \underline{l}_n) \stackrel{\text{def}}{=} \left(\sum_{s=0}^{f-1} p^s l_m^{(s)} \right) + \dots + \left(\sum_{s=0}^{f-1} p^s l_n^{(s)} \right);$$

in particular any $F_{\underline{l}_m, \dots, \underline{l}_n}^{(m,n)}$ lies on the antidiagonal $X_0 + \dots + X_{f-1} = N_{m,n}(\underline{l}_m, \dots, \underline{l}_n)$.

Let $N \in \mathbf{N}$. We define the $\overline{\mathbf{F}}_p$ -linear subspace

$$(\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_N \stackrel{\text{def}}{=} \langle F_{\underline{l}_m, \dots, \underline{l}_n}^{m,n} \in \mathcal{B} \quad \text{s.t.} \quad N_{m,n}(\underline{l}_m, \dots, \underline{l}_n) < N \rangle_{\overline{\mathbf{F}}_p};$$

it is the subspace generated by the functions lying strictly below the antidiagonal $X_0 + \dots + X_{f-1} = N$.

We refer the reader to figure 3 to have the euclidean interpretation in the case $f = 2$.

Let $(\underline{l}_m, \dots, \underline{l}_n)$ a fixed f -tuple. For $s \in \{0, \dots, f-1\}$, we define

$$\Xi_s \stackrel{\text{def}}{=} \{a \in \{m, \dots, n\}, \quad \text{s.t.} \quad l_a^{[s+a-m]} \neq 0\}$$

and we set

$$a_0(s) \stackrel{\text{def}}{=} \begin{cases} \min(\Xi_s) & \text{if } \Xi_s \neq \emptyset \\ n+1 & \text{otherwise.} \end{cases}$$

The euclidean meaning of $a_0(s)$ is clear: if we consider the $F_{\underline{l}_{a_0(s)}, \dots, \underline{l}_n}^{(a_0(s), n)}$ -block then the function $F_{\underline{l}_m, \dots, \underline{l}_n}^{(m,n)}$ lies on its s -th face (which is a $(f-1)$ -hypercube of side $p^{a_0(s)-m} - 1$).

The $K_0(p^m)$ -structure of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi$ is then given by the following

PROPOSITION 4.2. *Let $\underline{r} \stackrel{\text{def}}{=} (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^{f-1}$ be a f -tuple, m, n be integers such that $1 \leq m \leq n$ and let $F_{\underline{l}_m, \dots, \underline{l}_n}^{(m,n)} \in \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi_{\underline{r}}^s$ be as in definition 4.1. If $a, b, c, d \in \mathcal{O}_F$ are integers such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ p^m c & d \end{bmatrix} \in K_0(p^m)$ we have*

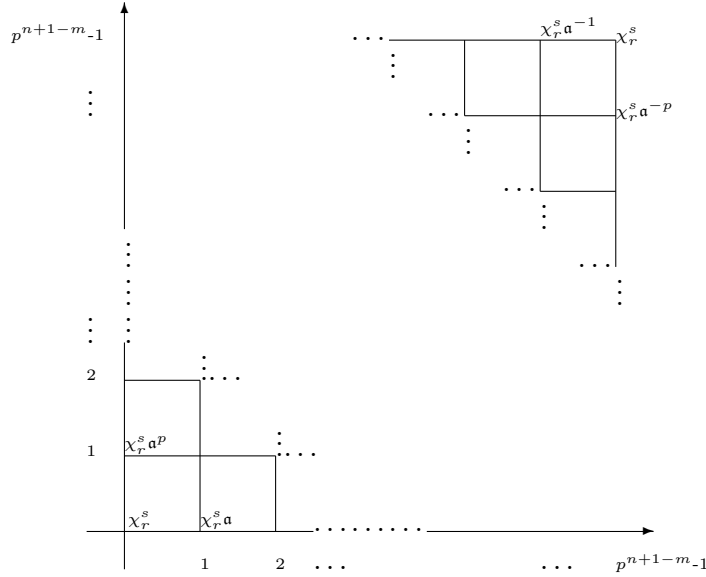
$$g F_{\underline{l}_m, \dots, \underline{l}_n}^{(m,n)} = \mathbf{a}^{e(\underline{l}_m, \dots, \underline{l}_n)} \chi_{\underline{r}}^s(g) (F_{\underline{l}_m, \dots, \underline{l}_n}^{(m,n)} - \sum_{s=0}^{f-1} (\overline{ca}^{-1}) p^s l_{a_0(s)}^{[s+a_0(s)-m]} F_{\underline{l}_m, \dots, \underline{l}_n - e_s}^{(m,n)} + y)$$

where, putting $N \stackrel{\text{def}}{=} N_{m,n}(\underline{l}_m, \dots, \underline{l}_n)$, we have $y \in (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi_{\underline{r}}^s)_{N-1}$.

In particular, the $K_0(p)$ -socle filtration, as well as the extensions between two consecutive graded pieces, of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi_{\underline{r}}^s$ is described by the associated lattice \mathcal{R} .

We emphasise again the meaning of Proposition 4.2 in terms of lattices in \mathbf{R}^f : the socle filtration of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \chi$ is given by cutting up the hypercube \mathcal{R} by the antidiagonals $X_0 + \dots + X_{f-1} = N$ (precisely, soc_N is obtained by cutting the antidiagonal $X_0 + \dots + X_{f-1} = N$); the extensions between two consecutive graded pieces are visualized by the segments of length 1 obtained by cutting \mathcal{R} by two consecutive antidiagonals $X_0 + \dots + X_{f-1} = N$, $X_0 + \dots + X_{f-1} = N - 1$.

Here below an exemple for $f = 2$.



Here, each “point” in the lattice corresponds to a function $F_{l_m, \dots, l_n}^{m,n} \in \mathcal{B}$ according to the map P described in (5). The N -th composition factor $\text{soc}_N(\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)$ of the socle filtration can be read as the intersection of \mathbf{R} with the semispace $X_0 + \dots + X_{f-1} \leq N$, and the N -th graded piece $\text{soc}_N(\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1) / \text{soc}_{N-1}(\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)$ as the intersection with the antidiagonal $X_0 + \dots + X_{f-1} = N$. Finally, a “point” of coordinates $(\sum_{j=m}^n p^{j-m} l_j^{\lfloor j-m \rfloor}, \sum_{j=m}^n p^{j-m} l_j^{\lfloor 1+j-m \rfloor})$ should be understood as the character $\chi_r^s \mathbf{a}^{e(l_m, \dots, l_n)}$.

4.1.2 Proof of Proposition 4.2. The section is devoted to the proof of Proposition 4.2. Thanks to the decomposition

$$K_0(p^m) = H \cdot \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + p\mathcal{O}_F \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^m \mathcal{O}_F & 1 \end{bmatrix} \quad (6)$$

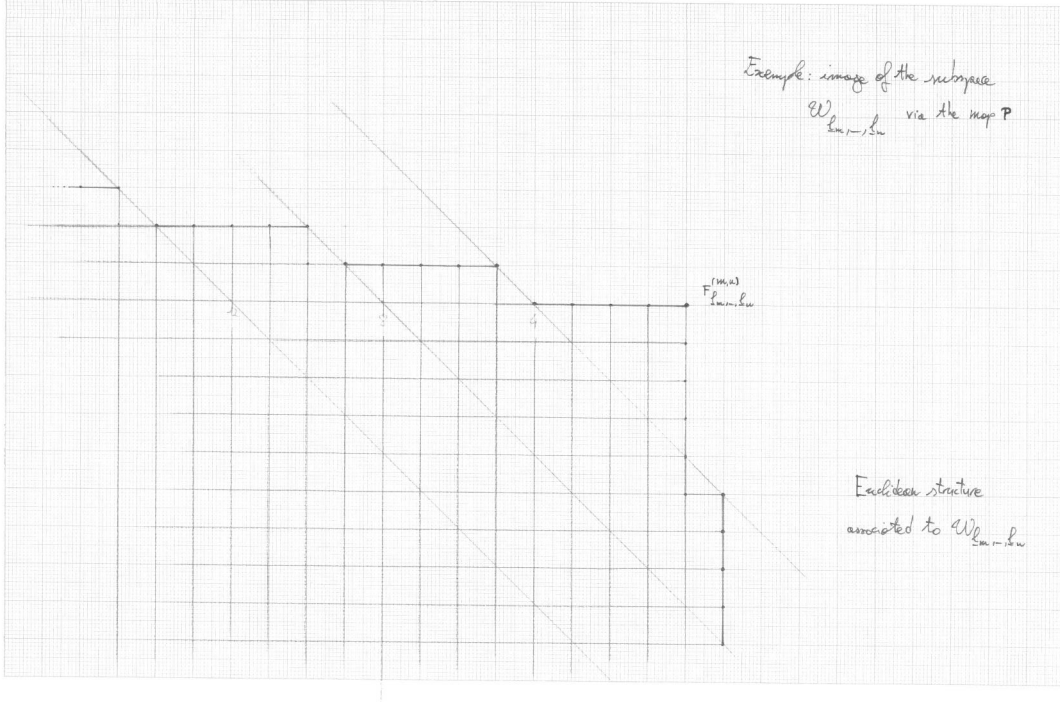
for $m \geq 1$ we are led to study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of the canonical basis \mathcal{B} : this will be the object of the next three paragraphs.

The action of lower unipotents matrices. We study here the action of the closed subgroup $\begin{bmatrix} 1 & 0 \\ p^m \mathcal{O}_F & 1 \end{bmatrix}$ of $K_0(p^m)$ on $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$; we first need to introduce a family of $\overline{\mathbf{F}}_p$ -subspaces of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$.

Let $F_{l_m, \dots, l_n}^{(m,n)} \in \mathcal{B}$ and set $(x_0, \dots, x_{f-1}) \stackrel{\text{def}}{=} P(F_{l_m, \dots, l_n}^{(m,n)}) \in \mathcal{R}$. We define the $\overline{\mathbf{F}}_p$ -subspace $\mathfrak{W}_{(l_m, \dots, l_n)}$ of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ via

$$P(\mathfrak{W}_{(l_m, \dots, l_n)}) \stackrel{\text{def}}{=} \{(x'_0, \dots, x'_{f-1}) \in \mathcal{R} \quad \text{s.t. it exists } n \geq 0 \text{ for which} \\ n(p-1) \leq \sum_{s=0}^{f-1} (x_s - x'_s) < (n+1)(p-1) \text{ and } x'_j \leq x_j + n \text{ for all } j = 0, \dots, f-1\}.$$

The image $P(\mathfrak{W}_{(l_m, \dots, l_n)}) \subseteq \mathbf{R}^f$ looks as a snowflake: in figure 4 an exemple for $f = 2$ (and $p = 5$).


 FIGURE 4. Euclidean interpretation of $\mathfrak{W}_{(l_m, \dots, l_n)}$.

It is immediate to check that if $F_{l'_m, \dots, l'_n}^{(m,n)} \in \mathfrak{W}_{(l_m, \dots, l_n)}$ then $\mathfrak{W}_{(l'_m, \dots, l'_n)} \subseteq \mathfrak{W}_{(l_m, \dots, l_n)}$. The action of $\begin{bmatrix} 1 & 0 \\ p^m \mathcal{O}_F & 1 \end{bmatrix}$ is then described in the following

PROPOSITION 4.3. Let $F_{l_m, \dots, l_n}^{(m,n)} \in \mathcal{B}$, and write $N \stackrel{\text{def}}{=} N_{m,n}(l_m, \dots, l_n)$. Let $g = \begin{bmatrix} 1 & 0 \\ p^m c & 1 \end{bmatrix} \in \begin{bmatrix} 1 & 0 \\ p^m \mathcal{O}_F & 1 \end{bmatrix}$ for $c \in \mathcal{O}_F$. Then we have

$$g \cdot F_{l_m, \dots, l_n}^{(m,n)} = F_{l_m, \dots, l_n}^{(m,n)} - \sum_{s=0}^{f-1} c^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-m \rfloor} F_{(l_m, \dots, l_n)-e_s}^{(m,n)} + y$$

for a suitable $y \in (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-1}$. More precisely, via the projection

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \xrightarrow{pr} \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-(p^f+2)},$$

the image of the element y is contained in the image of the subspace $\mathfrak{W}_{(l_m, \dots, l_n)}$.

Proof. As the action of $\begin{bmatrix} 1 & 0 \\ p \mathcal{O}_F & 1 \end{bmatrix}$ is continuous, we can assume that c belongs to a set of topological generators (for the additive structure) of \mathcal{O}_F ; in particular, we can assume $c = [\mu^{\frac{1}{p^m}}]$ for $\mu \in \mathbf{F}_q$.

Using the notations of §6.2, we can write the following equality in $p^m \mathcal{O}_F / p^{n+1} \mathcal{O}_F$:

$$p^m[\mu] + \sum_{j=m}^n p^j[\lambda_j^{\frac{1}{p^j}}] = \sum_{j=m}^n p^j[\lambda_j^{\frac{1}{p^j}} + (\tilde{S}_{j-m}^{\frac{1}{p^j}})] \quad (7)$$

A direct computation describes the action of g on the function $F_{l_m, \dots, l_n}^{(m,n)}$:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ p^m[\mu] & 1 \end{bmatrix} F_{l_m, \dots, l_n}^{(m,n)} = \\ & = \sum_{j=m}^{n-1} \sum_{i_j \leq l_j} \binom{l_j}{i_j} (-s_0(\tilde{S}_0)^{i_m}) \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j - i_j} (-s_{j-m}(\tilde{S}_{j-m+1})^{\frac{1}{p^{j+1}}})^{i_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, F_{l_n - i_n}^n]. \end{aligned}$$

As $\deg(s_{j-1}(\tilde{S}_j)) \leq p^j$ for each $j \in \{1, \dots, n-m\}$ we can apply Proposition 7.3 (with $T_{m+j} = s_{j-1}(\tilde{S}_j)$) to conclude that

$$g \cdot F_{l_m, \dots, l_n}^{(m,n)} = F_{l_m, \dots, l_n}^{(m,n)} + \sum_{s=0}^{f-1} \beta_s F_{(l_m, \dots, l_n) - e_s}^{(m,n)} + y$$

where $y \in \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ is the element described in the statement, for suitable elements $\beta_s \in \mathbf{F}_q$.

We are now left to prove that $\beta_s = -(\mu^{\frac{1}{p^m}})^{p^s} l_{a_0(s)}^{[s+a_0(s)-m]}$.

We use the notations of Proposition 7.3 and we recall that, for $b = m+1, \dots, n$, a polynomial $-s_{b-m-1}(\tilde{S}_{b-m}(\underline{X}, Y))$ is homogeneous of degree p^{b-m} if X_a has degree p^a , Y degree p^0 (and $\tilde{S}_0 = Y$). In particular if we pick an element

$$x \stackrel{\text{def}}{=} \sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

appearing in the development of $gF_{l_m, \dots, l_n}^{(m,n)}$ we have, for $b \in \{m+1, \dots, n\}$,

$$\sum_{a=m}^{b-1} p^{a-m} \kappa_a^{(b),s} = i_b^{(s)} p^{b-m} - \alpha_b^{(s)}$$

where $i_b^{(s)}(p^{b-m} - 1) \geq \alpha_b^{(s)} \geq i_b^{(s)}$ is the exponent of Y in the fixed monomial of $-s_{b-1-m}(\tilde{S}_{b-m})^{i_b^{(s)}}$ (recall that any monomial $Y^c \prod_{i=0}^{b-1-m} X_i^{a_i}$ with $c = 0$ appears in the development of $-s_{b-1-m}(\tilde{S}_{b-m})$)

with coefficient zero). Considering that $p \geq 3$ the inequalities

$$\begin{aligned}
 & \mathfrak{s}(\kappa_m) + p\mathfrak{s}(\kappa_{m+1}) + \cdots + p^{n-m}\mathfrak{s}(\kappa_n) \leq \\
 & \leq (\mathfrak{s}(l_m - \underline{i}_m) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_m^{(m+1)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m) \rfloor} \kappa_m^{(n)})) + \\
 & \quad + p(\mathfrak{s}(l_{m+1} - \underline{i}_{m+1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_{m+1}^{(m+2)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m-1) \rfloor} \kappa_{m+1}^{(n)})) + \cdots \\
 & \quad \cdots + p^{n-m-1}(\mathfrak{s}(l_{n-1} - \underline{i}_{n-1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_{n-1}^{(n)})) + p^{n-m}(\mathfrak{s}(l_n - \underline{i}_n)) \leq \\
 & \leq \mathfrak{s}(l_m - \underline{i}_m) + \sum_{s=0}^{f-1} \mathfrak{s}(\kappa_m^{(m+1),s}) + \\
 & p(\mathfrak{s}(l_{m+1} - \underline{i}_{m+1})) + \left(\sum_{s=0}^{f-1} (\mathfrak{s}(\kappa_m^{(m+2),s}) + p\mathfrak{s}(\kappa_{m+1}^{(m+2),s})) \right) + \cdots \\
 & \cdots + \left(\sum_{s=0}^{f-1} (\mathfrak{s}(\kappa_m^{(n),s}) + p\mathfrak{s}(\kappa_{m+1}^{(n),s}) + \cdots + p^{n-m-1}\mathfrak{s}(\kappa_{n-1}^{(n),s})) \right) + p^{n-m}\mathfrak{s}(l_n - \underline{i}_n) \leq \\
 & \leq \sum_{a=m}^n p^{a-m}(\mathfrak{s}(l_a - \underline{i}_a)) + \sum_{b=m+1}^n (p^{b-m}(\mathfrak{s}(\underline{i}_b)) - \sum_{s=0}^{f-1} \alpha_b^{(s)})
 \end{aligned}$$

have to be equalities if we furthermore require our element to lie on the hyperplane $X_0 + \cdots + X_{f-1} = N - 1$; in particular we must have $i_b^{(s)} = 0$ for all couples $(b, s) \in \{m, \dots, n\} \times \{0, \dots, f-1\}$ except one and only one, say (b_0, s_0) , for which we must have $i_{b_0}^{(s_0)} = 1$.

We notice that for $b_0 \neq m$ we require furthermore that $\alpha_{b_0} = 1$ i.e. the exponent of Y appearing in the fixed monomial of $-s_{b_0-m-1}(\tilde{S}_{b_0-m})$ is 1. Thanks to Lemmas 6.3 and 6.4 we check that

$$x = -(\mu^{\frac{1}{p^m}})^{p^{s_0}} (l_{a_0(s)}^{\lfloor s+a_0(s)-m \rfloor}) F_{l_m, \dots, l_n - \epsilon_{s_0}}^{(m,n)}$$

as required. \square

The action of diagonal matrices. We are going to study the action of the subgroup

$$\begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + p\mathcal{O}_F \end{bmatrix}$$

on the elements of \mathcal{B} . If $z \in p^m\mathcal{O}_F/p^{n+1}\mathcal{O}_F$, an elementary computation shows that

$$\begin{bmatrix} 1 + pa & 0 \\ 0 & 1 + pd \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+1})$ is upper unipotent modulo p and $z' \in p^m\mathcal{O}_F/p^{n+1}\mathcal{O}_F$ is determined by the condition

$$z' \equiv (1 + pa)^{-1}(1 + pd)z \pmod{p^{n+1}}. \quad (8)$$

We can therefore content ourself studying the action of an element of the form $x \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 + p\alpha \end{bmatrix}$ for $\alpha \in \mathcal{O}_F$.

PROPOSITION 4.4. *Let $g \in \begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + p\mathcal{O}_F \end{bmatrix}$ and fix $F_{l_m, \dots, l_n}^{(m,n)} \in \mathcal{B}$; write $N \stackrel{\text{def}}{=} N_{m,n}(l_m, \dots, l_n)$.*

We then have the equality

$$g \cdot F_{l_m, \dots, l_n}^{(m,n)} = F_{l_m, \dots, l_n}^{m,n} + y$$

where $y \in \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1_{N-1}$.

More precisely, via the projection

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \xrightarrow{pr} \text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-(p^f+2)},$$

the image of y is contained in the image of the subspace $\mathfrak{W}_{(l_m, \dots, l_n)}$ and writing

$$y = \sum_{i \in I} \beta_i F_{l_m(i), \dots, l_n(i)}^{(m,n)}$$

(for a suitable set of indexes I and scalars $\beta_i \in \overline{\mathbf{F}}_p^\times$) we have that each function $F_{l_m(i), \dots, l_n(i)}^{(m,n)}$ which is not in the kernel $\ker(pr)$ lies on an hyperplane

$$X_0 + \dots + X_{f-1} = N - t(p-1)$$

for some $t \in \mathbf{N}_{>}$.

Proof. The proof is completely analogous to the proof of Proposition 4.3. As remarked above, it is enough to consider the case $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 + p\alpha \end{bmatrix}$ where $\alpha = \sum_{j=0}^{\infty} p^j [\alpha_j^{\frac{1}{p^j}}]$. Using the notations of §6.3 we see that

$$(1 + p\alpha) \left(\sum_{j=m}^n p^j [\lambda_m^{\frac{1}{p^j}}] \right) \equiv \sum_{j=m}^n p^j [\lambda_j^{\frac{1}{p^j}} + \tilde{Q}_j^{\frac{1}{p^j}}] \pmod{p^{n+1}}$$

and we deduce

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ 0 & 1 + p\alpha \end{bmatrix} F_{l_m, \dots, l_n}^{m,n} = \\ & = \sum_{j=m}^{n-1} \sum_{\underline{i}_j \leq \underline{l}_j} \binom{\underline{l}_j}{\underline{i}_j} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (-q_{j-m}(\tilde{Q}_{j+1-m}))^{\underline{i}_j+1} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, F_{\underline{l}_n - \underline{i}_n}^{(n)}] \end{aligned} \quad (9)$$

(where we convene that $\underline{i}_m = \underline{0}$ and with the obvious conventions if $n \in \{m, m+1\}$). As each polynomial $(-q_{j-1}(\tilde{Q}_j)) \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{j-1-m}]$, for $1 \leq j \leq n-m$ is homogeneous of degree p^j (for the shifted grading for which λ_{m+h} is homogeneous of degree p^h for $h \geq 0$) we can apply Proposition 7.3 with $T_{m+j} = (-q_{j-1}(\tilde{Q}_j))$ to get the first part of the statement.

We are left to prove 2). Consider an integer $t \in \mathbf{N}$ and an hyperplane $\mathfrak{H} : X_0 + \dots + X_{f-1} = N - t$. Following the proof of Proposition 7.3, a necessary condition for an element

$$\sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m} \begin{bmatrix} 1 & 0 \\ p^m [\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n} \begin{bmatrix} 1 & 0 \\ p^n [\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

appearing in the development of (9) to lie in \mathfrak{H} is then

$$\sum_{j=m}^n p^{j-m} \mathfrak{s}(\kappa_j) \equiv N - t \pmod{p-1}.$$

Again, as each polynomial $(-q_{j-1}(\tilde{Q}_j))$, for $1 \leq j \leq n-m$ is homogeneous of degree p^j , and $\mathfrak{s}(h) \equiv h \pmod{p-1}$ we deduce that inequalities 20, 21, 22 and 23 appearing in the proof of Proposition

7.3 are actually *equalities* in $\mathbf{Z}/(p-1)$ so that we get

$$\sum_{j=m}^n p^{j-m} \mathfrak{s}(\kappa_j) \equiv N - \mathfrak{s}(i_m) \pmod{p-1} = N.$$

The conclusion follows. \square

The action of upper unipotent matrices. We are left to study the action of the closed subgroup $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ on the elements of \mathcal{B} . We recall that the action of $K_0(p^m)$ is continuous on $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ and the natural topology on $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ coincides with the topology induced (via the natural immersion) by $K_0(p^m)$. Thanks to the isomorphisms of abelian topological groups

$$\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix} \cong \mathcal{O}_F \cong (\mathbf{Z}_p)^f$$

where the latter isomorphism is determined by the choice of a primitive element $\alpha \in \mathbf{F}_q$ of \mathbf{F}_q over \mathbf{F}_p (cf. Serre [Ser], Proposition 16 Ch.I) it is enough to study the action of elements $g \in \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ of the form $g = \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}$ for $\mu \in \mathbf{F}_q$.

We start with an elementary computation:

LEMMA 4.5. *Let $z \in p^m \mathcal{O}_F / p^{n+1} \mathcal{O}_F$ and $\mu \in \mathbf{F}_q$. We have the following equality:*

$$\begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+1})$ is upper unipotent modulo p and $z' \in p^m \mathcal{O}_F / p^{n+1} \mathcal{O}_F$ is uniquely determined by the condition

$$z' \equiv z(1 + z[\mu])^{-1} \pmod{p^{n+1}} \equiv \sum_{j=0}^N (z^{j+1}[\mu^j]) \pmod{p^{n+1}}$$

for $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$.

Proof. Omissis. \square

We are now left to use Lemma 4.5 and the results of §6.4 in order to describe the required action of $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$:

PROPOSITION 4.6. *Let $g \in \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ and fix $F_{l_m, \dots, l_n}^{(m,n)} \in \mathcal{B}$. Write ${}^1 N \stackrel{\text{def}}{=} N_{m,n}(l_m, \dots, l_n)$. In the quotient space*

$$\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / (\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-(p^m-2)+1}$$

we have the equality

$$g \cdot F_{l_m, \dots, l_n}^{(m,n)} = F_{l_m, \dots, l_n}^{(m,n)}.$$

¹of course, this N does not have anything to do with $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. We believe this conflict of notations will not give rise to any confusion, as the meaning of N will be clear from the context.

Proof. As remarked at the beginning of this paragraph, we can assume $g = \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}$ where $\mu \in \mathbf{F}_q$.

Using Lemma 4.5 and the results (and notations) of §6.4.1 we get the following equality in $\mathcal{O}_F/(p^{n+1})$:

$$\sum_{j=0}^N z^{j+1}[\mu^j] \equiv \sum_{j=m}^n p^j [\lambda_j^{\frac{1}{p^j}} + \tilde{U}_j^{\frac{1}{p^j}}] \pmod{p^{n+1}}$$

so that, inside $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$, we have:

$$gF_{l_m, \dots, l_n}^{m,n} = \sum_{j=m}^{n-1} \sum_{\substack{i_j \leq l_j \\ i_j \leq l_j}} \binom{l_j}{i_j} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j - i_j} (-u_j(\tilde{U}_{j+1}^{\frac{1}{p^{j+1}}}))^{i_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, F_{l_n - i_n}^{(n)}]$$

where we convene that $i_m = 0$ and we recall that $\tilde{U}_j = 0$ for $m \leq j \leq 2m-1$. As for each $2m \leq j \leq n$ the polynomial $-u_{j-1}(\tilde{U}_j)$ is pseudo-homogeneous of degree $p^j - p^m(p^m - 2)$ the conclusion follows from Proposition 7.4, with $V_j = -u_{j-1}(\tilde{U}_j)$. \square

Proof of Proposition 4.2. The last step in order to complete the proof of Proposition 4.2 is immediate:

PROPOSITION 4.7. Let $F_{l_m, \dots, l_n}^{(m,n)} \in \mathcal{B}$ and let $a, d \in \mathbf{F}_q$. We then have the following equality in $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$:

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} F_{l_m, \dots, l_n}^{(m,n)} = \mathbf{a}^{e(l_m, \dots, l_n)} \left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \right) F_{l_m, \dots, l_n}^{(m,n)}.$$

In particular

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} F_{l_m, \dots, l_n - e_s}^{(m,n)} = \mathbf{a}^{e(l_m, \dots, l_n) - p^s} \left(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \right) F_{l_m, \dots, l_n - e_s}^{(m,n)}.$$

Proof. We just remark that for $z = \sum_{j=m}^n p^j [\lambda_j] \in p^m \mathcal{O}_F / p^{n+1} \mathcal{O}_F$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z[a^{-1}d] & 1 \end{bmatrix} \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}$$

and that

$$z[a^{-1}d] = \sum_{j=m}^n p^j [\lambda_j(a^{-1}d)].$$

\square

Finally, for $a, b, c, d \in \mathcal{O}_F$ as in the statement of Proposition 4.2, we recall the matrix equality

$$\begin{bmatrix} a & b \\ p^m c & d \end{bmatrix} = \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^m z & 1 \end{bmatrix} \begin{bmatrix} 1 + px & 0 \\ 0 & 1 + pw \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

where $x, y, z, w \in \mathcal{O}_F$ are suitable integers verifying $\bar{z} = \overline{cd^{-1}}$. The result follows now from Propositions 4.3, 4.4, 4.6 and Lemma 4.7. \square

REMARK 4.8. We note that the bijection (5) depends on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ fixed in the introduction and should be noted as P_τ . As another immersion $\tau' : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ is obtained by composing τ with a power ϕ^a of the Frobenius on \mathbf{F}_q we see that the map $P_{\tau'}$ is obtained by composing P_τ with a power Φ^a , where $\Phi \in \text{End}(\mathbf{R}^f)$ is defined by $\Phi(e_s) = e_{[s+1]}$. Hence, as the antidiagonal is fixed under Φ , Proposition 4.2 does not depend on τ .

4.1.3 *The structure of the representations R_n^- .* Fix an integer $n \in \mathbf{N}$. We describe here the socle filtration (and the extensions between two consecutive graded pieces) for the $K_0(p)$ -representations R_{n+1}^- . Again, we can identify the negative elements of R_{n+1}^- with the points of a lattice of \mathbf{R}^f according to the following injective map

$$\begin{aligned} \mathcal{B}_{n+1}^- &\hookrightarrow \mathbf{R}^f \\ F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) &\mapsto \left(\sum_{a=1}^{n+1} p^{a-1} l_a^{\lfloor s+a-1 \rfloor} \right)_{s \in \{0, \dots, f-1\}} \end{aligned}$$

whose image will be denoted by \mathcal{R}_{n+1}^- ; we define in the evident way the subspaces $(R_{n+1}^-)_N$ for $N \in \mathbf{N}$.

The structure of R_{n+1}^- is then summarized in the following

PROPOSITION 4.9. *Let $n \in \mathbf{N}$, $F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) \in \mathcal{B}_{n+1}^-$ and let $a, b, c, d \in \mathcal{O}_F$ be such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \stackrel{\text{def}}{=} N_{1, n+1}(l_1, \dots, l_{n+1})$.*

We have the equality

$$gF_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) = \mathbf{a}^{e(l_1, \dots, l_{n+1})} \chi_r^s(g) (F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) - \sum_{s=0}^{f-1} (\overline{ca}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-1 \rfloor} (-1)^{\delta_{a_0(s), n+1}} F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) + y)$$

where $y \in (R_{n+1}^-)_{N-1}$.

In particular, the $K_0(p)$ -socle filtration of R_{n+1}^- , as well as the extensions between two consecutive graded pieces, are described by the associated lattice \mathcal{R}_{n+1}^- .

Proof. We notice that we have a $K_0(p^{n+1})$ -equivariant monomorphism

$$\begin{aligned} \sigma_r^{(n+1)} &\hookrightarrow \text{Ind}_{K_0(p^{n+2})}^{K_0(p^{n+1})} \chi_r^s \\ X^{r-l_{n+1}} Y^{l_{n+1}} &\mapsto (-1)^{l_{n+1}} \sum_{\lambda_{n+1} \in \mathbf{F}_q} (\lambda_{n+1}^{\frac{1}{p^{n+1}}})^{l_{n+1}} \begin{bmatrix} 1 & 0 \\ p^{n+1} [\lambda_{n+1}^{\frac{1}{p^{n+1}}}] & 1 \end{bmatrix} [1, e]. \end{aligned}$$

By transitivity and exactness of the induction functor $\text{Ind}_{K_0(p^{n+1})}^{K_0(p)}(\bullet)$ we get a $K_0(p)$ -equivariant monomorphism

$$\begin{aligned} R_{n+1}^- &\hookrightarrow \text{Ind}_{K_0(p^{n+2})}^{K_0(p)} \chi_r^s \\ F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) &\mapsto (-1)^{l_{n+1}} F_{l_1, \dots, l_n, l_{n+1}}^{(1, n+1)}. \end{aligned}$$

The conclusion is now immediate from Proposition 4.2. \square

4.2 The positive case

This section is again divided into two parts. We begin with the study of the $K_0(p)$ -representations R_{n+1}^+ , for $n \in \mathbf{N}$: they are described in Proposition 4.10. We subsequently switch our attention introducing other $K_0(p)$ representations (the $(\text{Ind}_{K_0(p^{n+1})}^K \chi^s)^+$, defined in §4.3) which will let us describe the $K_0(p)$ -restriction of principal and special series (see §4.3).

The philosophy is completely analogous to the one of the previous paragraph: we verify by a direct computation on the ring of Witt vectors that the $K_0(p)$ -structure of such objects can be described in terms of f -parallelepipoids in the euclidean space \mathbf{R}^f .

Fix $n \in \mathbf{N}$. We introduce the injective map

$$\mathcal{B}_{n+1}^+ \hookrightarrow \mathbf{R}^f$$

$$F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) \mapsto \left(\sum_{i=0}^{n+1} p^i l_i^{\lfloor l^{s+i} \rfloor} \right)_{s \in \{0, \dots, f-1\}}$$

which let us interpret the positive elements of R_{n+1}^+ as points in a convenient lattice of \mathbf{R}^f . The image of such map (which is a parallelepiped of side $p^{n+1}(r_s + 1) - 1$) will be denoted as \mathcal{R}_{n+1}^+ . We still need the following notations (see also §4.1.1):

i) for a $(n+2)f$ -tuple $(l_0, \dots, l_{n+1}) \in \{0, \dots, p-1\}^{n+2}$ define the integers

$$N_{0, n+1}(l_0, \dots, l_{n+1}) \stackrel{\text{def}}{=} \sum_{a=0}^{n+1} p^a \mathfrak{s}(l_a)$$

$$e(l_0, \dots, l_{n+1}) \stackrel{\text{def}}{=} \left(\sum_{s=0}^{f-1} p^s l_0^{(s)} \right) + \dots + \left(\sum_{s=0}^{f-1} p^s l_{n+1}^{(s)} \right);$$

ii) for $N \in \mathbf{N}$ we define the $\overline{\mathbf{F}}_p$ -linear subspace

$$(R_{n+1}^+)_N \stackrel{\text{def}}{=} \left\langle F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) \in \mathcal{B}_{n+1}^+ \quad \text{s.t.} \quad N_{0, n+1}(l_0, \dots, l_{n+1}) < N \right\rangle_{\overline{\mathbf{F}}_p};$$

iii) for $s \in \{0, \dots, f-1\}$, we define

$$\Xi_s \stackrel{\text{def}}{=} \{a \in \{0, \dots, n+1\}, \quad \text{s.t.} \quad l_a^{\lfloor l^{s+a} \rfloor} \neq 0\}$$

and we set

$$a_0(s) \stackrel{\text{def}}{=} \begin{cases} \min(\Xi_s) & \text{if } \Xi_s \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For a given positive element $F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1})$ we define the subspace $\mathfrak{W}_{(l_0, \dots, l_{n+1})}$ in the evident, similar way.

The structure of R_{n+1}^+ is then given by

PROPOSITION 4.10. *Let $n \in \mathbf{N}$, $F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) \in \mathcal{B}_{n+1}^+$ and let $a, b, c, d \in \mathcal{O}_F$ be such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \stackrel{\text{def}}{=} N_{0, n+1}(l_0, \dots, l_{n+1})$. We then have*

$$gF_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) = (\mathfrak{a}^{-1})^{e(l_0, \dots, l_{n+1})} \chi_{\mathcal{R}}(g) (F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) - \sum_{s=0}^{f-1} (\overline{bd}^{-1}) p^s l_{a_0(s)}^{\lfloor l^{s+a_0(s)} \rfloor} (-1)^{\delta_{a_0(s), n+1}} F_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) + y)$$

where $y \in (R_{n+1}^+)_{N-1}$.

In particular, the $K_0(p)$ -filtration, as well as the extensions between two consecutive pieces, is described by the associated lattice \mathcal{R}_{n+1}^+ .

Proof. The proof is analogous to the proof of Proposition 4.2, using this time Lemma 6.17 and Proposition 7.5. The details are left as an exercise to the reader. \square

4.2.1 On some other $K_0(p)$ -representations. As announced in the introduction, we define and study some $K_0(p)$ -representations (denoted as $\text{Ind}_{K_0(p^{n+1})}^K \chi^+$) which naturally appear dealing with the Iwahori structure of principal and special series. The reader will realize soon that the

behaviour of the representations $(\text{Ind}_{K_0(p^{n+1})}^K \chi)^+$ can be treated with the same methods of §4.2 and 4.1; the proofs will be therefore omitted.

Fix an integer $n \in \mathbf{N}$, a smooth character $\chi : K_0(p^{n+1}) \rightarrow \overline{\mathbf{F}}_p^\times$ and an $\overline{\mathbf{F}}_p$ -basis $\{e\}$ for the underlying vector space of χ . The $K_0(p)$ -representation $(\text{Ind}_{K_0(p^{n+1})}^K \chi)^+$ is defined as the $K_0(p)$ -subrepresentation induced by $\text{Ind}_{K_0(p^{n+1})}^K \chi$ on the $\overline{\mathbf{F}}_p$ -subspace

$$\langle \left[\begin{array}{cc} [z] & 1 \\ 1 & 0 \end{array} \right], e \rangle \in \text{Ind}_{K_0(p^{n+1})}^K \chi, \quad z \in I_{n+1}_{\overline{\mathbf{F}}_p}$$

(the $K_0(p)$ -stability of such $\overline{\mathbf{F}}_p$ -linear space is immediately verified). Again, we have the

DEFINITION 4.11. *Let $j \in \{0, \dots, n\}$ and let $\underline{l}_j \in \{0, \dots, p-1\}^f$ be a f -tuple. We define the following element of $(\text{Ind}_{K_0(p^{n+1})}^K \chi)^+$:*

$$F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)} \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \sum_{j=1}^n \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j} \left[\begin{array}{cc} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{array} \right] [1, e].$$

The family

$$\mathcal{B}^+ \stackrel{\text{def}}{=} \{F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)} \in (\text{Ind}_{K_0(p^{n+1})}^K \chi)^+, \quad \underline{l}_j \in \{0, \dots, p-1\}^f \text{ for all } j \in \{0, \dots, n\}\}$$

is an $\overline{\mathbf{F}}_p$ -basis for $(\text{Ind}_{K_0(p^{n+1})}^K \chi)^+$.

Exactly as we did for R_{n+1}^+ , each given element $F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}$ of \mathcal{B}^+ will be read as a point in a convenient lattice \mathcal{R} of \mathbf{R}^f and the integers $a_0(s)$ (for $s \in \{0, \dots, f-1\}$) can be assigned. Moreover, if $N \in \mathbf{N}$, the subspaces $((\text{Ind}_{K_0(p^{n+1})}^K \chi)^+)_N$ are defined in the similar, evident way (see the introduction of §4.2 for details).

The structure of the representations $(\text{Ind}_{K_0(p^{n+1})}^K \chi)^+$ is then described in the next

PROPOSITION 4.12. *Let $\underline{r} \in \{0, \dots, p-1\}^f$ be an f -tuple, $n \in \mathbf{N}$ an integer and let $a, b, c, d \in \mathcal{O}_F$ be such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Fix an element $F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)} \in \mathcal{B}^+$ and set $N \stackrel{\text{def}}{=} N_{0,n}(\underline{l}_0, \dots, \underline{l}_n)$.*

Then

$$g \cdot F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)} = (\mathfrak{a}^{-1})^{e(\underline{l}_0, \dots, \underline{l}_n)} \chi_{\underline{r}}(g) F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)} - \sum_{s=0}^{f-1} (\overline{bd}^{-1})^{p^s} l_{\underline{a}_0(s)}^{\lfloor s + a_0(s) \rfloor} F_{(\underline{l}_0, \dots, \underline{l}_n) - e_s}^{0,n} + y$$

where $y \in (\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^{s+})_{N-1}$.

In particular the $K_0(p)$ -socle filtration of $(\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+$, as well as the extensions of two consecutive graded pieces, are described by the associated lattice \mathcal{R} .

Proof. Omissis. □

4.3 The Iwahori structure of Principal and Special Series

We are now able to describe easily the Iwahori-structure of principal and special series for $\text{GL}_2(F)$. Such result is essentially a formal consequence of the previous sections §4.1 and §4.2.1.

For $\lambda \in \overline{\mathbf{F}}_p^\times$ and $\underline{r} \in \{0, \dots, p-1\}^f$ we consider the smooth parabolic induction

$$\text{Ind}_{B(F)}^{\text{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

where ω denotes the mod p cyclotomic character and μ_λ the unramified character verifying $\mu_\lambda(p) = \lambda$. It is well known that for $(\underline{r}, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ such inductions are irreducible, while, if $(\underline{r}, \lambda) \in \{(0, \pm 1), (p-1, \pm 1)\}$ they have length 2 and a unique infinite dimensional factor, the Steinberg representation (see also [BL94]). Thanks to the Iwahori decomposition and Mackey theorem we have

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_K \xrightarrow{\sim} \mathrm{Ind}_{B(\mathcal{O}_F)}^{\mathrm{GL}_2(\mathcal{O}_F)} \chi_{\underline{r}}^s$$

and, since the elements $f \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$ are locally constant functions and $B(\mathcal{O}_F) \backslash \mathrm{GL}_2(\mathcal{O}_F)$ is compact we have a natural isomorphism

$$\mathrm{Ind}_{B(\mathcal{O}_F)}^{\mathrm{GL}_2(\mathcal{O}_F)} \chi_{\underline{r}}^s \xrightarrow{\sim} \varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s.$$

Again, we can use Mackey decomposition to deduce

$$\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s|_{K_0(p)} \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \oplus (\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+$$

so that, by exactness property of filtrant inductive limit, we get

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} \left(\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \right) \oplus \left(\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+ \right). \quad (10)$$

The isomorphism (10) let us reduce to the case of the finite inductions $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$, $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s$, whose structure is completely described in Propositions 4.2 and 4.12. Therefore

THEOREM 4.13. *Let $\lambda \in \overline{\mathbf{F}}_p^\times$ and $\underline{r} \in \{0, \dots, p-1\}^f$ an f -tuple. For any $m \in \mathbf{N}_{>}$ we write*

$$F_{\underline{0}, \dots, \underline{0}, \dots}^{(m, \infty)} \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

to denote the characteristic function of $K_0(p^m)$.

The $K_0(p)$ -restriction of the parabolic induction admits a natural splitting

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} \left(\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \right) \oplus \left(\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+ \right).$$

Moreover an $\overline{\mathbf{F}}_p$ -basis \mathcal{B}^- for $\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$ (resp. \mathcal{B}^+ for $\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+$) is described by

the elements

$$F_{\underline{l}_1, \dots, \underline{l}_n, \dots}^{(1, \infty)} \stackrel{\text{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \cdots$$

(resp. the elements

$$F_{\underline{l}_0, \dots, \underline{l}_n, \dots}^{(0, \infty)} \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{l_0} \begin{bmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{l_1} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \cdots)$$

for a varying sequence $(\underline{l}_n)_{n \in \mathbf{N}_{>}} \in \{0, \dots, p-1\}^{(\mathbf{N}_{>})}$ (resp. $(\underline{l}_n)_{n \in \mathbf{N}} \in \{0, \dots, p-1\}^{(\mathbf{N})}$).

If we associate the elements of such basis to points in \mathbf{R}^f according to the law

$$F_{\underline{l}_1, \dots, \underline{l}_n, \dots}^{(1, \infty)} \mapsto \left(\sum_{i=1}^{\infty} p^{i-1} l_i^{\lfloor s+i-1 \rfloor} \right)_{s \in \{0, \dots, f-1\}}$$

$$F_{\underline{l}_1, \dots, \underline{l}_n, \dots}^{(0, \infty)} \mapsto \left(\sum_{i=0}^{\infty} p^i l_i^{\lfloor s+i \rfloor} \right)_{s \in \{0, \dots, f-1\}}$$

and write \mathcal{R}^- (resp \mathcal{R}^+) to denote the image of \mathcal{B}^- (resp. \mathcal{B}^+) by this map, then the $K_0(p)$ -socle filtration for $\varinjlim_{n \in \mathbf{N}} \text{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$ (resp. for $\varinjlim_{n \in \mathbf{N}} (\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s)^+$), as well as the extensions between two graded pieces, is described by the associated lattice \mathcal{R}^- (resp. \mathcal{R}^+).

The Iwahori structure of irreducible principal series follows.

As far as the Steinberg representation is concerned, we just need to notice the following fact:

LEMMA 4.14. *Assume $\underline{r} \in \{0, \underline{p} - 1\}$ and let $n \in \mathbf{N}$. We have a $K_0(p)$ -equivariant exact sequence*

$$0 \rightarrow \langle (F_{\underline{0}}^{(0)}, F_{\emptyset}^{(1,0)}) \rangle \rightarrow \text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^{s+} \oplus \text{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \rightarrow (\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s / \langle 1 \rangle) |_{K_0(p)} \rightarrow 0.$$

Proof. The proof is an induction on n , the case $n = 0$ being well known (cf. [Br-Pa], Lemma 2.6).

For the general case, we leave to the reader the easy task to check that we have a natural commutative diagram with exact lines

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \text{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^s \oplus \text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^{s+} & \longrightarrow & (\text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^s / \langle 1 \rangle) \\ \downarrow & & \downarrow \\ \text{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \oplus \text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^{s+} & \longrightarrow & (\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s / \langle 1 \rangle) \\ \downarrow & & \downarrow \\ (\text{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s / \text{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^s) \oplus (\text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^{s+} / \text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^{s+}) & \xrightarrow{\cong} & \text{Ind}_{K_0(p^{n+1})}^K \chi_{\underline{r}}^s / \text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^s \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

so that the snake lemma and the inductive hypothesis, giving an exact sequence

$$0 \rightarrow \langle (F_{\underline{0}}^{(0)}, F_{\emptyset}^{(1,0)}) \rangle \rightarrow \text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^{s+} \oplus \text{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^s \rightarrow (\text{Ind}_{K_0(p^n)}^K \chi_{\underline{r}}^s / \langle 1 \rangle) |_{K_0(p)} \rightarrow 0,$$

let us conclude. \square

5. The structure of the universal representation

In this section we show how the technical results of §4 concerning the representations R_{n+1}^{\pm} and the formalism of §3 let us describe the Iwahori structure for the universal representation $\pi(\underline{r}, 0, 1)$. Again we develop an euclidean dictionary which enable us to handle the involved combinatoric of $\pi(\underline{r}, 0, 1) |_{K_0(p)}$: the conclusion is then Proposition 5.16, which loosely speaking shows that the required structure is obtained by a juxtaposition of the blocks R_{n+1}^{\pm} in a fractal way. As a byproduct, we will exhibit a natural injective map

$$c\text{-Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r}, 0, 1)$$

where $V \leq \pi(\underline{r}, 0, 1) |_{KZ}$ is a convenient KZ -subrepresentation of $\pi(\underline{r}, 0, 1) |_{KZ}$. We remark that a similar injective map has been detected independently by Paskunas in an unpublished draft.

We give here a more precise description of this section. Thanks to Proposition 3.6 we can content ourselves to the study of the representations $\varinjlim_{n \text{ odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+$ and $\varinjlim_{n \text{ odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-$.

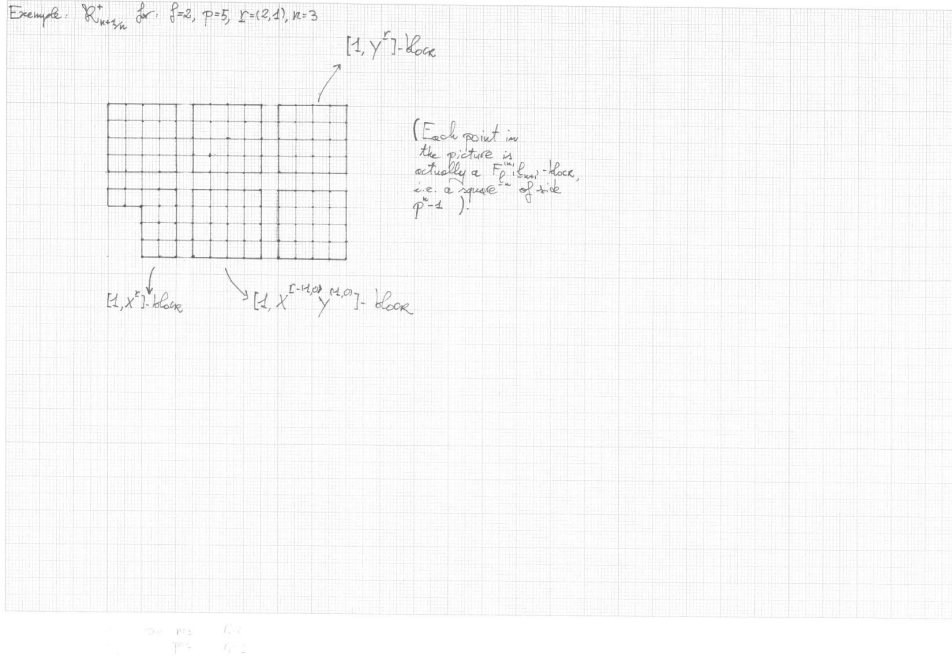


FIGURE 5. Euclidean structure for R_{n+1}^\pm / R_n^\pm .

As seen in Proposition 3.5, such $K_0(p)$ -representations have a natural filtration whose graded pieces are isomorphic to the quotients R_{n+1}^+ / R_n^+ , R_{n+1}^- / R_n^- respectively.

Such quotients are studied in §5.1. As we did in sections §4.1.3 and §4.2 -concerning the $K_0(p)$ -structure of R_{n+1}^+ and R_{n+1}^- - we introduce a natural correspondence between a “canonical” $\overline{\mathbf{F}}_p$ -base $\mathcal{B}_{n+1/n}^\pm$ for R_{n+1}^\pm / R_n^\pm and a convenient lattice (denoted as $\mathcal{R}_{n+1/n}^\pm$) in \mathbf{R}^f . Thanks to the behaviour of the canonical Hecke operator $(T_n^+)^{\text{pos,neg}}$ with respect to the elements of $\mathcal{B}_{n+1/n}^\pm$ we see that such a lattice is in fact the set-theoretic difference of the lattices \mathcal{R}_{n+1}^\pm and \mathcal{R}_n^\pm (cf. Lemma 5.1): figure 5 shows this phenomenon for $f = 2$.

Unfortunately, we can not use directly the results of sections 4 to conclude that the $K_0(p)$ -structure of R_{n+1}^\pm / R_n^\pm is predicted by the lattice $\mathcal{R}_{n+1/n}^\pm$: in fact Propositions 4.9 and 4.10 describe the extensions detected by functions $f_1, f_2 \in \mathcal{B}_{n+1}^\pm$ lying on adjacent antidiagonals.

It is therefore necessary to perfect the estimates made in the proofs of Propositions 4.9, 4.10: this is the object of §5.1.1. We remark that the behaviour of $(R_1/R_0)^+$ (resp. $R_0^- \oplus_{R_1^-} R_2^-$) is slightly different from that of R_{n+1}^+ / R_n^+ for $n \geq 1$ (resp. R_{n+1}^- / R_n^- for $n \geq 2$) (and treated in §5.1.2).

In section §5.2 we determine the structure of the amalgamated sums $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$: their structure can be *easily* determined from the results concerning of R_{n+1}^\pm / R_n^\pm . Indeed, thanks to the behaviour of the canonical basis of R_n^\pm with respect to the Hecke operators $(T_n^\pm)^{\text{pos,neg}}$ we see that the convenient euclidean pictured is obtained by gluing the lattice $\mathcal{R}_{n+1/n}^\pm$ with (a suitable translation of) the lattice associated to $\cdots R_{n-2}^\pm R_{n-1}^\pm$ (which we assume inductively to have been described). Again, the $K_0(p)$ -socle filtration is expected to be obtained by successive intersections of such lattice with parallels antidiagonals, as it was for R_{n+1}^\pm / R_n^\pm , but a simple computation shows that the hyperplanes giving the J -th layer of the socle filtration of $\mathcal{R}_{n+1/n}^\pm$ lie always below the hyperplanes giving the J -th layer of the socle filtration for $\cdots R_{n-2}^\pm R_{n-1}^\pm$. As $\cdots R_{n-2}^\pm R_{n-1}^\pm$ is a $K_0(p)$ -subrepresentation of

$\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$ we are able to deduce the desired result of Proposition 5.16.

In figure 6 an exemple of the glueing of blocks ² and their fractal stucture.

As announced, we can combine Lemma 5.1 and Proposition 3.5 to exhibit a natural *injective* morphism -whose existence was known informally by an unpublished work of Paskunas-

$$c\text{-Ind}_{KZ}^G V \hookrightarrow \pi(\mathcal{r}, 0, 1)|_{KZ}$$

where $V \leq \pi(\mathcal{r}, 0, 1)|_{KZ}$ is a convenient KZ -subrepresentation of $\pi(\mathcal{r}, 0, 1)|_{KZ}$: this is the object of Proposition 5.10.

As the cutting hyperplanes are fixed by the linear transformation $e_s \mapsto e_{[s+1]}$ of \mathbf{R}^f the results of §5.1 and §5.2 do not depend on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$, see remark 4.8.

5.1 The structure of the quotients $R_{n+1}^\bullet/R_n^\bullet$

In the flavour of §4.1.3 and §4.2 we start by describing a suitable $\overline{\mathbf{F}}_p$ -basis for the quotients $R_{n+1}^\bullet/R_n^\bullet$.

LEMMA 5.1. *Let $n \in \mathbf{N}_{\geq 1}$.*

- 1) An $\overline{\mathbf{F}}_p$ -basis $\mathcal{B}_{n+1/n}^+$ for R_{n+1}^+/R_n^+ is described as the homomorphic image (via the natural projection $R_{n+1}^+ \twoheadrightarrow R_{n+1}^+/R_n^+$) of the elements

$$F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) \in \mathcal{B}_{n+1}^+$$

such that $l_n \not\leq \mathcal{r}$ if $l_{n+1} = \mathcal{0}$.

- 2) An $\overline{\mathbf{F}}_p$ -basis $\mathcal{B}_{n+1/n}^-$ for R_{n+1}^-/R_n^- is described as the homomorphic image (via the natural projection $R_{n+1}^- \twoheadrightarrow R_{n+1}^-/R_n^-$) of the elements

$$F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) \in \mathcal{B}_{n+1}^-$$

such that $l_n \not\leq \mathcal{r}$ if $l_{n+1} = \mathcal{0}$.

If $n = 0$ then an $\overline{\mathbf{F}}_p$ -basis for $(R_1/R_0)^+$ is described as the homomorphic image (via the natural projection $R_1^+ \twoheadrightarrow (R_1/R_0)^+$) of the elements

$$F_{l_0}^{(0)}(l_1)$$

such that $l_1 \not\leq \mathcal{r}$ if $l_1 = \mathcal{0}$ and of the element $F_{\mathcal{r}}^{(0)}(\mathcal{0})$.

Proof. The result follows immediately from the definition of the operators $(T_n^+)^{\text{pos,neg}}$. Indeed, for $n \geq 1$ we have (with the obvious conventions if $n = 1$):

$$\begin{aligned} (T_n^+)^{\text{pos}}(F_{l_0, \dots, l_{n-1}}^{(0,n-1)}(l_n)) &= (-1)^{l_n} F_{l_0, \dots, l_n}^{(0,n)}(\mathcal{0}); \\ (T_n^+)^{\text{neg}}(F_{l_1, \dots, l_{n-1}}^{(1,n-1)}(l_n)) &= (-1)^{l_n} F_{l_1, \dots, l_n}^{(1,n)}(\mathcal{0}) \end{aligned}$$

while, for $n = 0$ we have

$$T_0(F_{\emptyset}^{(0,-1)}(l_0)) = F_{l_0}^{(0)}(\mathcal{0}) + (-1)^{\mathcal{r}} \delta_{l_0, \mathcal{0}} F_{\emptyset}^{(1,0)}(\mathcal{0}).$$

□

As usual the elements of the basis $\mathcal{B}_{n+1/n}^\pm$ will be read as the elements of a convenient lattice $\mathcal{R}_{n+1/n}^\pm$ of \mathbf{R}^f .

²strictly speaking, the figure gives the glueing of blocks R_{n-1}^+/R_{n-2}^+ and R_{n+1}^+/R_n^+ , i.e. the structure of $R_{n-1}^+/R_{n-2}^+ \oplus_{R_n^+} R_{n+1}^+$. If we want to get the picture of the *whole* amalgamated sum $\cdots \oplus_{R_n^+} R_{n+1}^+$ we should insert a “even smaller” structure inside the point (1, 2) of the rectangle drawn on the left in figure 6.

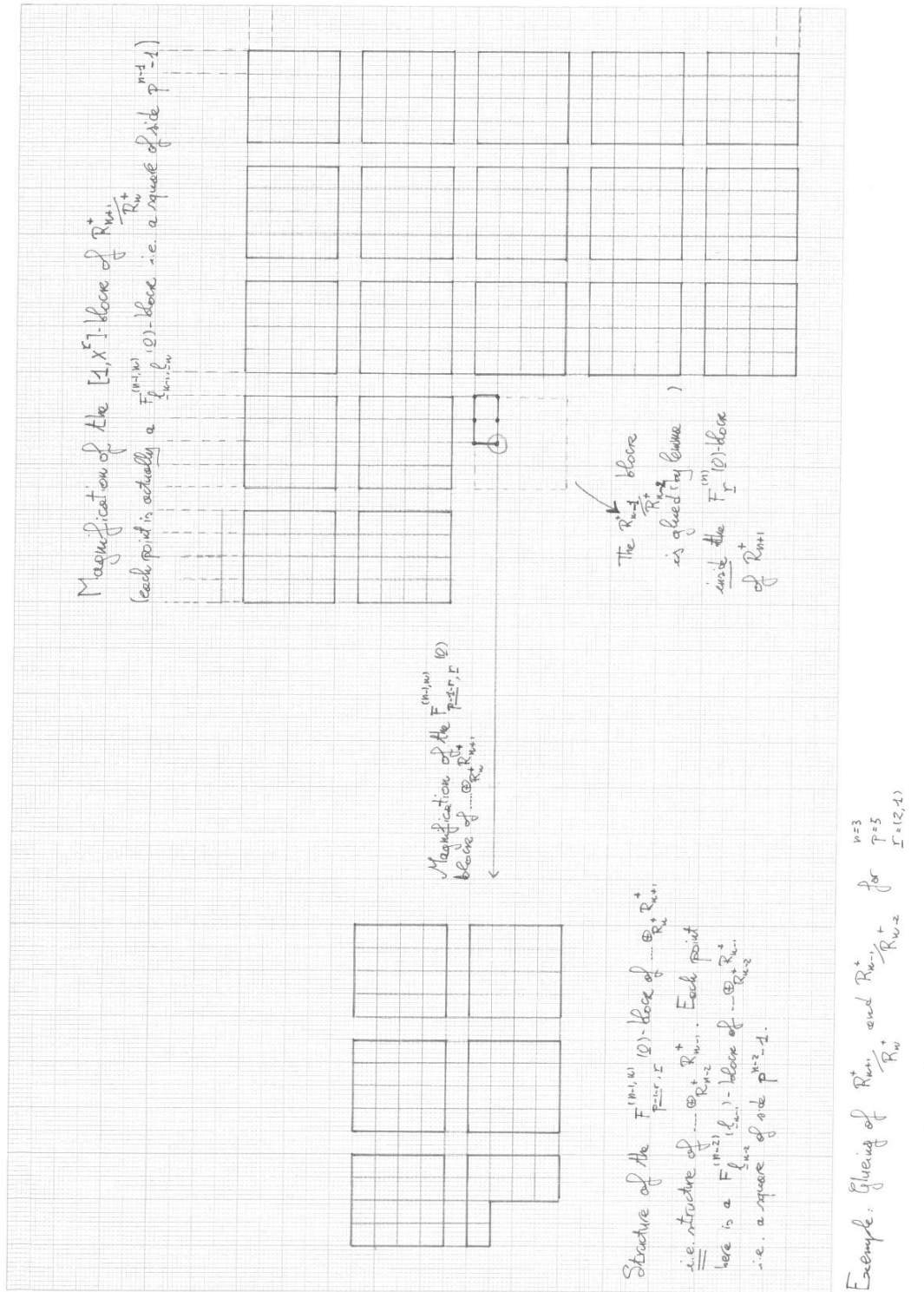


FIGURE 6. The gluing and the fractal structure.

Interpretation in terms of euclidean data. Exactly as we did in sections §4.1.3 and §4.2 we have natural injections $\mathcal{B}_{n+1/n}^\pm \hookrightarrow \mathbf{R}^f$ which let us interpret the elements of $\mathcal{B}_{n+1/n}^\pm$ as points in a convenient lattice $\mathcal{R}_{n+1/n}^\pm$ of \mathbf{R}^f : the details can safely be left to the reader.

The euclidean interpretation of Lemma 5.1 is therefore clear: for $n \geq 1$ the lattice $\mathcal{R}_{n+1/n}^+$ (resp. $\mathcal{R}_{n+1/n}^-$) of \mathbf{R}^f , which is expected to describe the $K_0(p)$ -structure of R_{n+1}^+/R_n^+ (resp. R_{n+1}^-/R_n^-), is obtained from the lattice of R_{n+1}^+ (resp. R_{n+1}^-) by removing the simplex

$$\{(x_0, \dots, x_{f-1}) \in \mathcal{R}_{n+1}^+ \quad \text{s.t. } x_s < p^n(r_{\lfloor n+s \rfloor} + 1) \text{ for all } s = 0, \dots, f-1\}$$

(resp.

$$\{(x_0, \dots, x_{f-1}) \in \mathcal{R}_{n+1}^- \quad \text{s.t. } x_s < p^{n-1}(r_{\lfloor n+s-1 \rfloor} + 1) \text{ for all } s = 0, \dots, f-1\})$$

(equivalently, $\mathcal{R}_{n+1/n}^\pm$ is obtained as the set-theoretical difference of $\mathcal{R}_{n+1}^\pm \setminus \mathcal{R}_n^\pm$).

We refer the reader to figure 5 for an exemple in residual degree $f = 2$.

The lattice $\mathcal{R}_{1/0}^+$ associated to $(R_1/R_0)^+$ similarly obtained from the lattice associated to R_1^+ , by removing the subset

$$\{(x_0, \dots, x_{f-1}) \in \mathcal{R}_{n+1}^+ \quad \text{s.t. } x_s < (r_{\lfloor n+s \rfloor} + 1) \text{ for all } s = 0, \dots, f-1\} \setminus \{(r_0, \dots, r_{f-1})\}.$$

To be precise, the lattice $\mathcal{R}_{1/0}^+$ (resp. the lattice naturally associated to $R_0^- \oplus_{R_1^-} R_2^-$) does *not* describe the $K_0(p)$ -structure of $(R_1/R_0)^+$ (resp. $R_0^- \oplus_{R_1^-} R_2^-$) *sic et simpliciter*. But a harmless modification of the formalism used for $\mathcal{R}_{n+1/n}^+$ if $n \geq 1$ (resp. $\mathcal{R}_{n+1/n}^-$ if $n \geq 2$) let us detect their $K_0(p)$ -socle filtration: see section §5.1.2 and Propositions 5.6, 5.7 and 5.8 for details.

We will describe in detail the $K_0(p)$ -structure of R_{n+1}^+/R_n^+ for $n \geq 1$; as announced, the negative case (for $n \geq 2$) will be left to the reader.

Preliminaries: partitioning the lattice. As announced in the introduction to §5, the mere knowledge of the $K_0(p)$ -socle filtration for R_{n+1}^+ does *not* allow us determine the structure of the quotient R_{n+1}^+/R_n^+ . Indeed Proposition 4.9 let us determine the extensions detected by functions $F_{l_0, \dots, l_n}^{(0,n)}, F_{l'_0, \dots, l'_n}^{(0,n)} \in \mathcal{B}_{n+1}^+$ lying on adjacent antidiagonals. We could therefore get, *a priori*, a nontrivial extension between them if $l_j = l'_j = \underline{0}$ for all $j \neq n$ and $l_n = (0, \dots, 0, r_s, 0, \dots, 0)$ $l'_n = (0, \dots, 0, r_{s'}, 0, \dots, 0)$ for $s \neq s'$ as illustrated in the figure 7.

Notice that this phenomena happens only if $F \neq \mathbf{Q}_p$: if $F = \mathbf{Q}_p$ the structure of the quotients is immediate from the structure of R_{n+1}^+ .

We modify the strategy of section 4.2. We show that the $K_0(p)$ -structure of R_{n+1}^+ is again predicted by \mathcal{R}_{n+1}^+ but each cutting antidiagonal $X_0 + \dots + X_{f-1} = \text{constant}$ of section §4.2 is now replaced by f -antidiagonals of the form $X_0 + \dots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor} + 1) + \text{constant}$: we will say that $X_0 + \dots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor} + 1) + \text{constant}$ is the s -th cutting hyperplane of R_{n+1}^+/R_n^+ . This means that we naturally divide the lattice $\mathcal{R}_{n+1/n}^+$ into sub-blocks $\mathfrak{V}_{s_{m+k}}$ of increasing size for $k \in \{0, \dots, f-1\}$ (cf. definition 5.2); the J -th composition factor for the $K_0(p)$ -socle filtration of R_{n+1}^+/R_n^+ is then obtained as the sum of the subspaces determined by the intersection of the block $\mathfrak{V}_{s_{m+k}}$ with the antidiagonal $X_0 + \dots + X_{f-1} = p^n(r_{s_{m+k}} + 1) + \text{constant}$, for varying $k \in \{0, \dots, f-1\}$. This is the content of Proposition 5.3. In figure 8, an exemple of how the increasing block (and successive cuttings) look like.

We determine the decomposition of $\mathcal{R}_{n+1/n}^+$ into increasing blocks. Fix $n \geq 0$ and define $s_m \in$

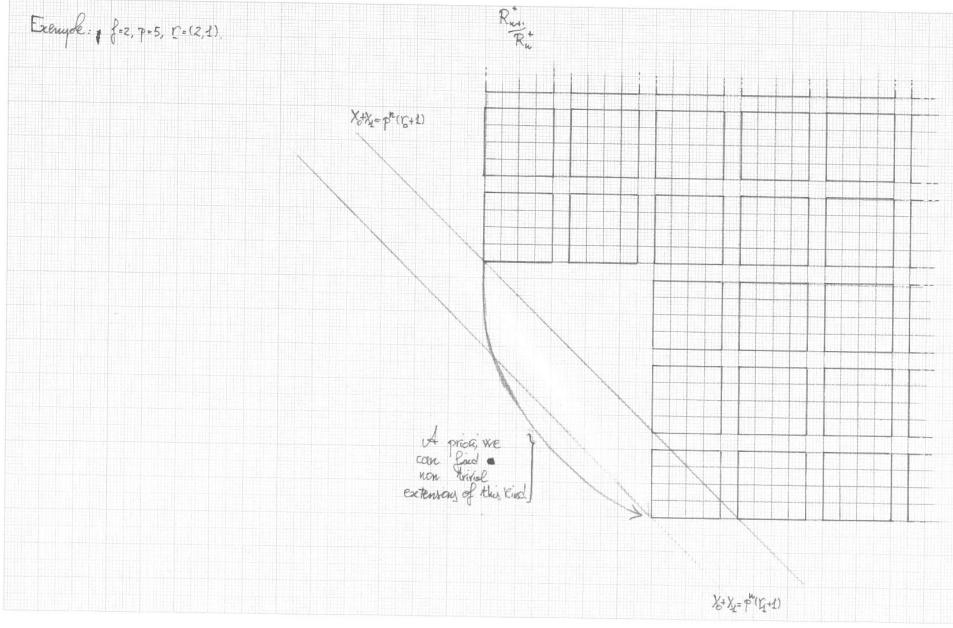


FIGURE 7. A priori, we can have disagreeable glueing phenomena.

$\{0, \dots, f-1\}$ by the condition

$$r_{\lfloor s_m+n \rfloor} = \max\{r_{\lfloor s+n \rfloor}\}.$$

We fix an ordering

$$p-1 \geq r_{\lfloor s_m+n \rfloor} \geq r_{\lfloor s_{m+1}+n \rfloor} \geq \dots \geq r_{\lfloor s_{m+f-1}+n \rfloor} \geq 0$$

and define the following $\overline{\mathbf{F}}_p$ -subspaces of R_{n+1}^+/R_n^+ :

DEFINITION 5.2. For $k \in \{0, \dots, f-1\}$ define $\mathfrak{V}_{s_{m+k}}$ as the $\overline{\mathbf{F}}_p$ -subspace of R_{n+1}^+/R_n^+ generated by the elements $F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) \in \mathcal{B}_{n+1/n}^+$ verifying the properties:

i) for $s \notin \{s_m, \dots, s_{m+k}\}$ we have

$$l_n^{\lfloor s+n \rfloor} \leq r_{\lfloor s+n \rfloor};$$

ii) for $s \notin \{s_m, \dots, s_{m+k}\}$ we have

$$l_{n+1}^{\lfloor s+n+1 \rfloor} = 0.$$

By abuse of notation, we will also write $\mathfrak{V}_{s_{m+k}}$ to denote the image of the canonical basis (in the obvious sense) of $\mathfrak{V}_{s_{m+k}}$ in the lattice $\mathcal{R}_{n+1/n}^+$. The geometric meaning of the previous definition is the following: the block $\mathfrak{V}_{s_{m+k}}$ is described as the intersection of the subset

$$\{X_{s_{m+k+1}} < p^n(r_{\lfloor s_{m+k+1}+n \rfloor} + 1)\} \cap \dots \cap \{X_{s_{m+f-1}} < p^n(r_{\lfloor s_{m+f-1}+n \rfloor} + 1)\}$$

with the lattice $\mathcal{R}_{n+1/n}^+$: in other words, we give restrictions on the coordinates $x_{s_{m+k+1}}, \dots, x_{s_{m+f-1}}$ of a point $(x_0, \dots, x_{f-1}) \in \mathcal{R}_{n+1/n}^+$ to lie in the block $\mathfrak{V}_{s_{m+k}}$.

Notice that in order to detect if a function $F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) \in \mathcal{B}_{n+1/n}^+$ belongs to the subspace $\mathfrak{V}_{s_{m+k}}$ we only need to study the last two f -tuples l_n, l_{n+1} .

Obviously, the subspaces $\mathfrak{V}_{s_{m+k}}$ describe (for $n \geq 1$) an exhaustive increasing filtration on R_{n+1}^+/R_n^+ as a $\overline{\mathbf{F}}_p$ -vector space.

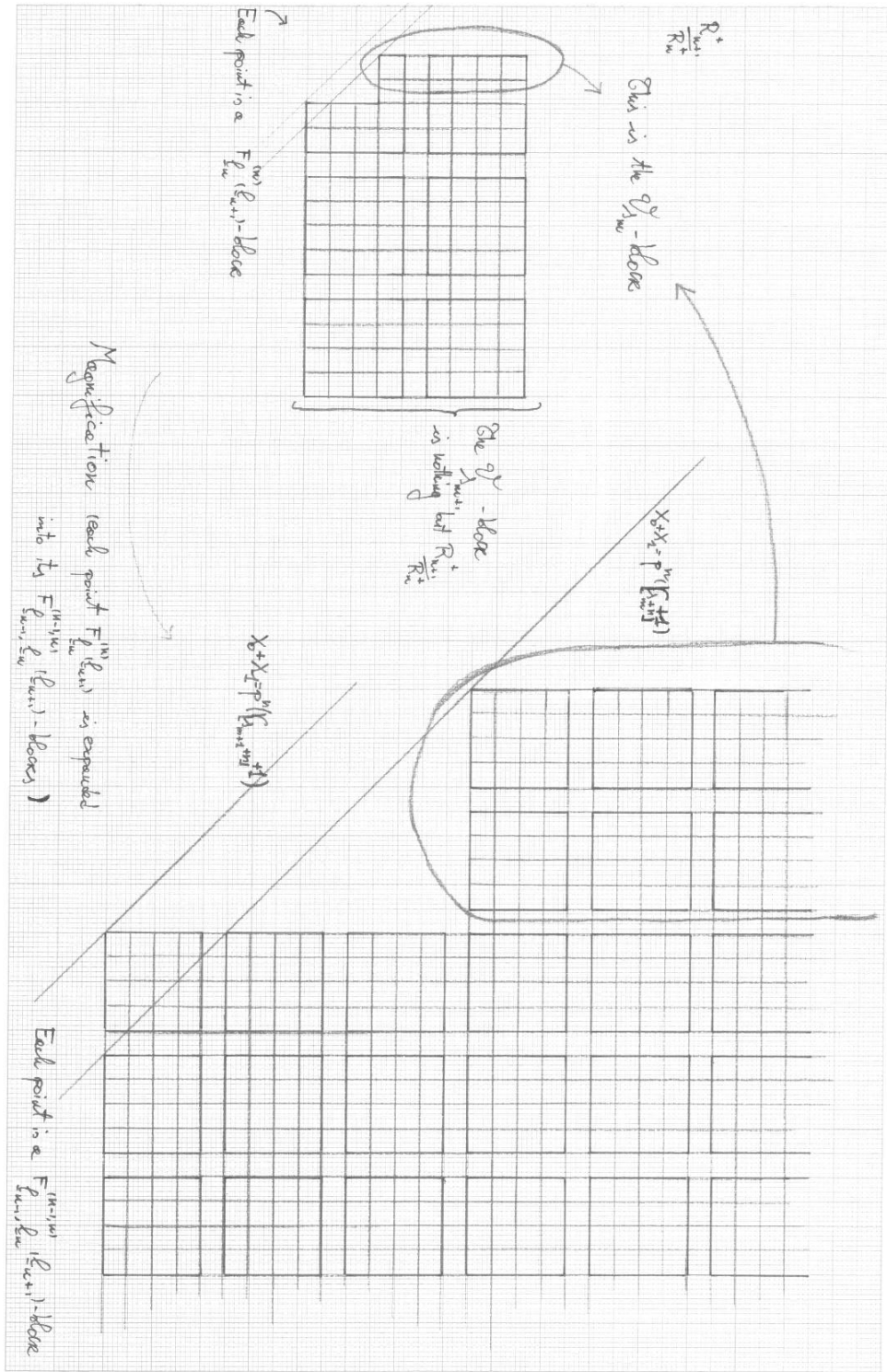


FIGURE 8. Exemple of blocs subdivision and cutting.

The following crucial result shows that the lattice $\mathcal{R}_{n+1/n}^+$ let us detect the required $K_0(p)$ -structure for $n \geq 1$.

PROPOSITION 5.3. *Assume $n \in \mathbf{N}_{\geq 1}$. Let $a, b, c, d \in \mathcal{O}_F$, $g \stackrel{\text{def}}{=} \begin{bmatrix} 1+pa & b \\ pc & 1+pd \end{bmatrix} \in K_0(p)$, fix an element $F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) \in \mathfrak{V}_{s_{m+k}}$ for some $k \in \{0, \dots, f-1\}$ and write $N_{0,n+1}(l_0, \dots, l_{n+1}) = p^n(r_{[s_{m+k+n}]} + 1) + J$ for some $J \in \mathbf{N}$. Finally, consider the linear development*

$$gF_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) = \sum_{i \in I} \beta(i) F_{l_0(i), \dots, l_n(i)}^{(0,n)}(l_{n+1}(i))$$

(where I is a suitable set of indices $\beta(i) \in \overline{\mathbf{F}}_p^\times$ are scalars).

Fix an index $i_0 \in I$ and assume there exists $k' \in \{k+1, \dots, f-1\}$, minimal with respect to the property $F_{l_0(i_0), \dots, l_n(i_0)}^{(0,n)}(l_{n+1}(i_0)) \in \mathfrak{V}_{s_{m+k'}} \setminus \mathfrak{V}_{s_{m+k}}$.

Then we have

$$N_{0,n+1}(l_0(i_0), \dots, l_{n+1}(i_0)) \leq p^n(r_{[s_{m+k'+n}]} + 1) + J - 2. \quad (11)$$

In particular, the lattice $\mathcal{R}_{n+1/n}^+$ describes the $K_0(p)$ -socle filtration, as well as the extensions between two consecutive graded pieces, of R_{n+1}^+/R_n^+ .

We insist on the geometric meaning of Proposition 5.3: we pick a function in the k -th block $F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) \in \mathfrak{V}_{s_{m+k}}$, lying on the antidiagonal $X_0 + \dots + X_{f-1} = p^n(r_{s_{m+k}} + 1) + J$ and $F_{l_0(i_0), \dots, l_n(i_0)}^{(0,n)}(l_{n+1}(i_0))$ a function appearing (with nonzero linear coefficient) in the linear development of $gF_{l_0, \dots, l_n}^{(0,n)}(l_{n+1})$. If $F_{l_0(i_0), \dots, l_n(i_0)}^{(0,n)}(l_{n+1}(i_0))$ happens to belong to a strictly bigger block, say $\mathfrak{V}_{s_{m+k'}}$ with $k' > k$ and minimal with respect to this property, then it lies *strictly below* the antidiagonal $X_0 + \dots + X_{f-1} = p^n(r_{s_{m+k'}} + 1) + J - 1$.

Thanks to this phenomenon, we can invoke Proposition 4.10 to deduce the $K_0(p)$ -structure for R_{n+1}^+/R_n^+ from the associated lattice $\mathcal{R}_{n+1/n}^+$: the J -composition factor for the socle filtration of R_{n+1}^+/R_n^+ is determined as the sum of the f subspaces obtained by intersecting each block $\mathfrak{V}_{s_{m+k}}$ with the corresponding antidiagonal $X_0 + \dots + X_{f-1} = p^n(r_{s_{m+k}} + 1) + J$ (as in figure 8).

Notice moreover that the statement of Proposition 5.3 is empty if $f = 1$: in the rest of §5.1 we will assume $f \geq 2$.

5.1.1 Proof of Proposition 5.3. The rest of this section is devoted to the proof of Proposition 5.3. Thanks to decomposition (6) we can study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of R_{n+1}^+ : this will be the object of the next three paragraphs. The proofs are similar to the proofs of Propositions 4.3, 4.4 and 4.6, but need a delicate extra argument due to the irregular structure of the lattice $\mathcal{R}_{n+1/n}^+$.

The action of upper unipotent matrices. We study here the case where $g \in \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$, and

again we assume $g = \begin{bmatrix} 1 & [\mu] \\ 0 & 1 \end{bmatrix}$ for $\mu \in \mathbf{F}_q$. As in Proposition 4.3 we write

$$\begin{aligned} gF_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) &= \\ &= \sum_{j=1}^{n+1} \sum_{i_j \leq l_j} \binom{l_j}{i_j} \sum_{i_0 \leq l_0} \binom{l_0}{i_0} (T_0)^{i_0} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j - i_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{i_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, f_{l_{n+1} - i_{n+1}}] \end{aligned}$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set

$$f_{l_{n+1}-i_{n+1}} \stackrel{\text{def}}{=} (-1)^{i_{n+1}} X^{r-(l_{n+1}-i_{n+1})} Y^{l_{n+1}-i_{n+1}},$$

$T_0 \stackrel{\text{def}}{=} -s_0(\tilde{S}_0)$, $T_{j+1} \stackrel{\text{def}}{=} -s_j(\tilde{S}_{j+1})$ for $j \in \{0, \dots, n\}$.

Developing the polynomials T_{j+1} 's we write

$$gF_{l_0, \dots, l_n}^{(0,n)}(l_{n+1}) = \sum_{i \in I} \beta(i) F_{l_0(i), \dots, l_n(i)}^{(0,n)}(l_{n+1}(i))$$

(for a suitable set of indices I) and we pick a vector v appearing in the linear development of $gF_{l_0, \dots, l_n}^{(0,n)}(l_{n+1})$:

$$v \stackrel{\text{def}}{=} F_{[\underline{\kappa}_0], \dots, [\underline{\kappa}_n]}^{(0,n)}([\underline{\kappa}_{n+1}]);$$

where, as in Proposition 7.3, we write for $0 \leq a \leq n+1$

$$\underline{\kappa}_a = \underline{l}_a - \underline{i}_a + \sum_{b=a+1}^{n+1} p^{[a-b]} \kappa_a^{(b)}$$

and, for $a+1 \leq b \leq n+1$ we have

$$\kappa_a^{(b)} = \sum_{s=0}^{f-1} p^s \kappa_a^{(b),s}$$

where $\kappa_a^{(b),s}$ is the exponent of λ_a in $T_b^{i_b^{(s)}}$. By the definition of the subspace $\mathfrak{W}_{s_{m+k}}$ we see that

$$\begin{aligned} \underline{\kappa}_n &= \underline{l}_n - \underline{i}_n + p^{[-1]} \kappa_n^{(n+1)} = \\ &= \sum_{h=0}^k p^{[s_{m+h+n}]} (l_n^{[s_{m+h+n}]} - i_n^{[s_{m+h+n}]} + \kappa_n^{(n+1), [s_{m+h+n+1}]}) + \sum_{h=k+1}^{f-1} p^{[s_{m+h+n}]} (l_n^{[s_{m+h+n}]} - i_n^{[s_{m+h+n}]}) \end{aligned}$$

If $v \notin \mathfrak{W}_{s_{m+k}}$ then

$$k' \stackrel{\text{def}}{=} \min \{c \in \{k+1, \dots, f-1\}, \text{ s.t. } [\kappa_n^{[s_{m+c+n}]}] > r_{[s_{m+c+n}]} \} (> k)$$

and we necessarily have $\underline{\kappa}_n \neq 0$ and the equality

$$\mathfrak{s}(\underline{l}_n - \underline{i}_n + p^{[-1]} \kappa_n^{(n+1)}) = \sum_{s=0}^{f-1} l_n^{(s)} - i_n^{(s)} + \kappa_n^{(n+1), [s+1]} - \tilde{j}(p-1)$$

for a suitable $\tilde{j} \geq 1$. Following the inequalities (21), (22), (23) of Proposition 7.3 (i.e. using the subadditivity of the function \mathfrak{s} and the fact that the polynomials T_j are homogeneous of degree p^j if λ_i is defined to have degree p^j) we get

$$\mathfrak{s}(\underline{\kappa}_0) + \dots + p^{n+1} \mathfrak{s}(\underline{\kappa}_{n+1}) \leq p^n (r_{[s_{m+k+n}]} + 1) + J - \mathfrak{s}(\underline{i}_0) + p^n (p-1) \tilde{j}.$$

As $n \geq 1$ the inequality

$$p^n (r_{[s_{m+k+n}]} - r_{[s_{m+k'}+n]}) \leq \tilde{j} p^n (p-1) + \mathfrak{s}(\underline{i}_0) - 2$$

is then obvious if either $\tilde{j} \geq 2$ or $r_{[s_{m+k'}+n]} > 0$.

Assume finally $\tilde{j} = 1$ and $r_{[s_{m+k'}+n]} = 0$. Therefore the p -adic development of $[\kappa_n]$ has the form

$$(l_n^{(0)} - i_n^{(0)} + \kappa_n^{(n+1),1}, \dots, l_n^{(s)} - i_n^{(s)} + \kappa_n^{(n+1),s+1} - p, l_n^{(s+1)} - i_n^{(s+1)} + \kappa_n^{(n+1),s+2} + 1, \dots)$$

for a unique $s \in \{s_m, \dots, s_{m+k}\}$. The condition $x \notin \mathfrak{A}_{s_{m+k}}$ imposes $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$ and the minimality condition on k' imposes $\lfloor s_{m+k'}+n \rfloor = \lfloor s+1 \rfloor$, in particular $r_{\lfloor s+1 \rfloor} = 0$. As $\kappa_n^{(n+1), s+1}$ is the coefficient of $\lambda_n^{\frac{1}{p^n}}$ in the fixed monomial of $s(\tilde{S}_{n+1})^{i_{n+1}^{\lfloor s+1 \rfloor}}$ and $i_n^{\lfloor s+1 \rfloor} \leq r_{\lfloor s+1 \rfloor}$ we get an absurde.

The action of diagonal matrices. The next step is to study the action of an element $g \in \begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + p\mathcal{O}_F \end{bmatrix}$; again we can assume $g = \begin{bmatrix} 1 + p\alpha & 0 \\ 0 & 1 \end{bmatrix}$. The arguments are completely analogous to those of the previous paragraph, in this case using the fact that the polynomials $g_{j-1}(Q_j)$ of §6.3 are homogeneous of degree p^j . The details are left to the reader.

The action of lower unipotent matrices. In this section we deal with the action of an element $g \in \begin{bmatrix} 1 & 0 \\ p\mathcal{O}_F & 1 \end{bmatrix}$; again, we assume $g = \begin{bmatrix} 1 & 0 \\ p[\mu] & 1 \end{bmatrix}$. This case is more delicate than the previous and we need to recall and carry on the accurate estimates seen in the appendix A §6.4.2.

As for Proposition 4.6, we write

$$gF_{l_0, \dots, l_n}^{(0, n)}(l_{n+1}) = \sum_{j=0}^n \sum_{i_{j+1} \leq l_{j+1}} \binom{l_{j+1}}{i_{j+1}} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j - i_j} (V_{j+1}^{\frac{1}{p^{j+1}}})^{i_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j] & 1 \end{bmatrix} [1, f_{l_{n+1} - i_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set $i_0 \stackrel{\text{def}}{=} 0$,

$$f_{l_{n+1} - i_{n+1}} \stackrel{\text{def}}{=} (-1)^{i_{n+1}} X^{r - (l_{n+1} + i_{n+1})} Y^{l_{n+1} - i_{n+1}}$$

and $V_{j+1} \stackrel{\text{def}}{=} -u_j(\tilde{U}_{j+1})$ for $j \in \{0, \dots, n\}$. We develop the polynomials $V_{j+1}^{i_{j+1}}$, recognizing again a sum of elements of the basis $\mathcal{B}_{n+1/n}^+$: we pick a vector

$$v \stackrel{\text{def}}{=} F_{[\kappa_0], \dots, [\kappa_n]}^{(0, n)}([\kappa_{n+1}]);$$

as in the previous paragraph we write for $0 \leq a \leq n+1$

$$\underline{\kappa}_a = \underline{l}_a - \underline{i}_a + \sum_{b=a+1}^{n+1} p^{[a-b]} \kappa_a^{(b)}$$

and, for $a+1 \leq b \leq n+1$ we have

$$\kappa_a^{(b)} = \sum_{s=0}^{f-1} p^s \kappa_a^{(b), s}$$

where $\kappa_a^{(b), s}$ is the exponent of λ_a in $V_b^{i_b^{(s)}}$. Again, using the notations of Lemmas 6.19 and 6.20, we focus our attention on

$$\begin{aligned} \underline{\kappa}_n &= l_n - i_n + p^{\lfloor -1 \rfloor} \kappa_n^{(n+1)} = \\ &= \sum_{h=0}^k p^{\lfloor s_{m+h}+n \rfloor} (l_n^{\lfloor s_{m+h}+n \rfloor} - i_n^{\lfloor s_{m+h}+n \rfloor}) + B_n^{\lfloor s_{m+h}+1+n \rfloor}(0) + p B_{n+1}^{\lfloor s_{m+h}+1+n \rfloor}(1) + \\ &\quad + \sum_{h=k+1}^{f-1} p^{\lfloor s_{m+h}+n \rfloor} (l_n^{\lfloor s_{m+h}+n \rfloor} - i_n^{\lfloor s_{m+h}+n \rfloor}) \end{aligned}$$

(where we can again assume $\underline{\kappa}_n \neq 0$) and we distinguish the following four possibilities.

I). Assume $\sum_{h=0}^k B_{n+1}^{\lfloor s_{m+h}+1+n \rfloor}(1) = 0$. The condition $v \notin \mathfrak{V}_{s_{m+k}}$ imposes that

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1} l_n^{(s)} - i_n^{(s)} + B_n^{\lfloor s+1 \rfloor} - \tilde{j}(p-1)$$

for $\tilde{j} \in \mathbf{N}$, $\tilde{j} \geq 1$. We recall that for each $j \in \{0, \dots, n-1\}$ the polynomial V_j is pseudohomogeneous of degree $p^j - (p-2)$ so that the subadditivity of \mathfrak{s} and Lemma 6.20 give

$$\sum_{j=0}^{n+1} p^j \mathfrak{s}(\kappa_j) \leq \sum_{j=0}^{n+1} p^j \mathfrak{s}(l_j) - (p-2) \left(\sum_{j=0}^{n+1} \mathfrak{s}(i_j) \right) - p^n \tilde{j}(p-1)$$

and the conclusion follows.

II). Assume $\sum_{h=0}^k B_{n+1}^{\lfloor s_{m+h}+1+n \rfloor}(1) \geq 2$. Then we have

$$\sum_{s=0}^{f-1} \sum_{j=0}^n p^j \mathfrak{s}(\kappa_j^{(n+1),s}) \leq p^{n+1} \mathfrak{s}(i_{n+1}) - 2p^n(p-2).$$

The conclusion is now easy and left to the reader.

III). Assume $1 = \sum_{h=0}^k A_{n+1}^{\lfloor s_{m+h}+1+n \rfloor}(1) = \sum_{h=0}^k B_{n+1}^{\lfloor s_{m+h}+1+n \rfloor}(1) = 1$. Let $h_1 \in \{0, \dots, k\}$ the unique integer such that $B_{n+1}^{\lfloor s_{m+h_1}+1+n \rfloor}(1) = 1$. We can again distinguish the following two subcases:

III)_A Assume

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1} (l_n^{(s)} - i_n^{(s)} + B_n^{(s+1)}(0) + B_{n+1}^{(s)}(1)) - \tilde{j}(p-1)$$

for $\tilde{j} \in \mathbf{N}$, $\tilde{j} \geq 1$. In this case the reader can check that

$$\sum_{j=0}^{n+1} p^j \mathfrak{s}(\kappa_j) \leq \sum_{j=0}^{n+1} p^j \mathfrak{s}(l_j) - (p-2) \left(\sum_{j=0}^n \mathfrak{s}(i_j) \right) - p^n \tilde{j}(p-1) - (p-2)p^n$$

and the conclusion follows.

III)_B Assume finally

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1} (l_n^{(s)} - i_n^{(s)} + B_n^{(s+1)}(0) + B_{n+1}^{(s)}(1)).$$

Such condition, together with $v \notin \mathfrak{V}_{s_{m+k}}$ imposes that $\lfloor s_{m+h_1} + 1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$; by minimality of k' we conclude that $\lfloor s_{m+h_1} + 1 \rfloor = s_{m+k'}$; in particular $r_{s_{m+k'}} > 0$. We deduce that the chosen monomial of $u_n(\tilde{U}_{n+1}^{\frac{1}{p^{n+1}}})^{i_{n+1}}$ is of the form

$$\lambda_0^{\alpha'_0} \dots \lambda_n^{\alpha'_n} (\lambda_0 \lambda_n^{\frac{1}{p^n}})^{p^{\lfloor s_{m+h_1}+1+n \rfloor}}$$

where the integers α'_j verify

$$\sum_{j=0}^n p^j \mathfrak{s}(\alpha'_j) \leq (p^{n+1} - (p-2))(\mathfrak{s}(i_{n+1}) - 1).$$

By subadditivity of the function \mathfrak{s} we find finally

$$\begin{aligned} \sum_{j=0}^{n+1} p^j \mathfrak{s}(\kappa_j) &\leq \sum_{j=0}^{n+1} p^j \mathfrak{s}(l_j) - (p-2) \left(\sum_{j=0}^n \mathfrak{s}(i_j) \right) + (p^{n+1} - (p-2))(\mathfrak{s}(i_{n+1}) - 1) + \\ &\quad + (1 + p^n) - p^{n+1} \mathfrak{s}(i_{n+1}) \end{aligned}$$

(where the integer $1 + p^n$ is deduced from the monomial $\lambda_0 \lambda_n^{\frac{1}{p^n}}$) and the conclusion follows easily (notice that $\sum_{j=0}^{n+1} \mathfrak{s}(i_j) \geq 1$).

The proof of Proposition 5.3 is therefore complete.

REMARK 5.4. *The reader has noticed that if we assume $r_s \leq p - 2$ for all $s \in \{0, \dots, f - 1\}$ then the inequality (11) in the statement can be replaced by the following, stronger, inequality*

$$N_{0,n+1}(l_0(i_0), \dots, l_{n+1}(i_0)) \leq p^n + J - 2.$$

5.1.2 **The case $n = 0$.** In this section we show that the $K_0(p)$ -structure of $(R_1/R_0)^+$ is actually slightly more complicated than expected, at least under some particular conditions on the f -tuple r . The negative counterpart will be the $K_0(p)$ -structure of $R_0^- \oplus_{R_1^-} R_2^-$ which is left to the reader. The aim is to give an analogue of Proposition 5.3 in the case $n = 0$: in the next three paragraphs we will analyse where and how a statement of such a kind fails to hold true, detecting some condition on the f -tuple r . The main statements are Propositions 5.6, 5.7 and 5.8, where we see that the $K_0(p)$ -socle filtration for $(R_1/R_0)^+$ can be obtained from the associated lattice $\mathcal{R}_{1/0}^+$, with some harmless adjustment in few special cases (according to the combinatoric of r).

In what follows, we fix $k \in \{0, \dots, f - 1\}$ and an element $F_{l_0}^{(0)}(l_1) \in \mathfrak{V}_{s_{m+k}} \setminus \langle F_r^{(0)}(\underline{0}) \rangle_{\overline{\mathbf{F}}_p}$. Let $g \in K_0(p)$. We fix an element $v = F_{[\kappa_0]}^{(0)}([\kappa_1])$ appearing (with a nonzero linear coefficient) in the $\overline{\mathbf{F}}_p$ -linear development of $gF_{l_0}^{(0)}(l_1)$, for suitable integers $\kappa_0, \kappa_1 \in \mathbf{N}$.

We assume there exists an integer $k' \in \{k + 1, \dots, f - 1\}$ such that $v \notin \mathfrak{V}_{s_{m+k'}} \setminus \mathfrak{V}_{s_{m+k}}$ and k' is minimal with respect to this property.

The next lemma can be verified by an easy computation on the ring $\mathbf{W}_1(\mathbf{F}_q)$:

LEMMA 5.5. *In the previous hypothesis we have*

$$N_{0,1}(\kappa_0, \kappa_1) = N_{0,1}(l_0, l_1) - \epsilon$$

where

- 1) if $g \in \begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ then $\epsilon = \mathfrak{s}(i_0) + \mathfrak{s}(i_1) + \tilde{j}(p - 1)$ where $\tilde{j} \geq 1$ and $\mathfrak{s}(i_0) + \mathfrak{s}(i_1) \geq 1$;
- 2) if $g \in \begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + p\mathcal{O}_F \end{bmatrix}$ then $\epsilon = \mathfrak{s}(i_1)(p - 1) + \tilde{j}(p - 1)$ where $\mathfrak{s}(i_1) \geq 1$ and $\tilde{j} \in \mathbf{N}$;
- 3) if $g \in \begin{bmatrix} 1 & 0 \\ p\mathcal{O}_F & 1 \end{bmatrix}$ then $\epsilon = \mathfrak{s}(i_1)(p - 2) + \tilde{j}(p - 1)$ where $\mathfrak{s}(i_1) \geq 1$ and $\tilde{j} \in \mathbf{N}$.

Moreover:

- 1A) if in case 1) we have $\tilde{j} = 1$ then we necessarily have $s_{m+k'} = \lfloor s + 1 \rfloor$ for an index s verifying $s \in \{s_m, \dots, s_{m+k}\}$ and $\lfloor s + 1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$; moreover $r_{s_{m+k'}} > 0$;
- 2B) if in case 2) we have $\tilde{j} = 0$ and $\mathfrak{s}(i_1) = 1$ then we have

$$[\kappa_0] = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 1, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s verify $s \in \{s_m, \dots, s_{m+k}\}$ and $\lfloor s + 1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$. Furthermore $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$.

- 3B) if in case 3) we have $\tilde{j} = 0$ and $\mathfrak{s}(i_1) = 1$ then we have

$$[\kappa_0] = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 2, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s verify $s \in \{s_m, \dots, s_{m+k}\}$ and $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$. Furthermore $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$.

Proof. The proof, a direct computation, is left to the reader. \square

Thanks to its explicit nature, the description of the socle filtration for $(R_1/R_0)^+$ can be easily deduced from Lemma 5.5. We have to distinguish three cases, according to the combinatoric of the f -tuple \underline{r} ; the proofs are left as an exercise to the reader (see [Mo1] for details).

PROPOSITION 5.6. Assume that the f -tuple verifies one of the following hypothesis:

I_A). For each $s \in \{0, \dots, f-1\}$ the condition

$$\begin{cases} r_s \geq r_{\lfloor s+1 \rfloor} \geq 1 \\ r_s - r_{\lfloor s+1 \rfloor} \in \{p-2, p-3\} \end{cases}$$

is false.

I_B). The f -tuple is of the form $(0, \dots, 0, r_{s_m}, 0, \dots, 0)$.

Then the socle filtration, together with the extensions between two consecutive graded pieces, of $(R_1/R_0)^+$ is described by the associated lattice $\mathcal{R}_{1/0}^+$.

Proof. Omissis. \square

PROPOSITION 5.7. Assume that for all $s \in \{0, \dots, f-1\}$ we have $\sum_{s=0}^{f-1} (r_s) \geq r_s + 1$ and that the condition

$$\begin{cases} r_s \geq r_{\lfloor s+1 \rfloor} \geq 1 \\ r_s - r_{\lfloor s+1 \rfloor} = p-2 \end{cases}$$

is false.

Then the socle filtration for $(R_1/R_0)^+$ is described by the lattice $\mathcal{R}_{1/0}^+$.

Proof. Omissis. \square

We finally deal with the remaining case -the socle filtration is here slightly more complicated: in euclidean terms, the blocks $\mathfrak{V}_{s_{m+k}}$ for $r_{s_{m+k}} = p-1$ should be cutted by the hyperplanes $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J$ or $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J - 1$ according to a condition on $r_{s_{m+k}+1}$.

PROPOSITION 5.8. Assume there exists an index $s \in \{0, \dots, f-1\}$ such that $r_s = p-1$ and $r_{\lfloor s+1 \rfloor} = 1$. Up to reordering, we assume there exists integers $0 \leq k_1 \leq k_0$ such that $r_{s_{m+j}} = p-1$ for all $j \in \{0, \dots, k_0\}$ and

$$\begin{cases} r_{\lfloor s_{m+j}+1 \rfloor} \neq 1 & \text{if } 0 \leq j \leq k_1 - 1, \\ r_{\lfloor s_{m+j}+1 \rfloor} = 1 & \text{if } k_1 \leq j \leq k_0. \end{cases}$$

Then the J -th factor for the socle filtration of $(R_1/R_0)^+$ is described by the subspace

$$\mathcal{V}_J \stackrel{\text{def}}{=} \langle F_{\underline{r}}^{(0)}(\underline{0}) \rangle_{\overline{\mathbb{F}}_p} + \sum_{k=0}^{f-1} \langle F_{l_0}(l_1) \in \mathfrak{V}_{s_{m+k}}, \quad N_{(0,1)}(l_0, l_1) \leq (r_{s_{m+k}} + 1) + J - \delta_{k_1 \leq k \leq k_0} \rangle_{\overline{\mathbb{F}}_p}.$$

In particular, the socle filtration is deduced from the lattice $\mathcal{R}_{1/0}^+$ by cutting the k -th block by the hyperplane $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J - \delta_{k_1 \leq k \leq k_0}$.

Proof. Omissis. \square

5.1.3 Application: the universal representation contains infinitely many compact inductions. As announced in the introduction of §5 we are able to describe a G -equivariant natural injection

$$c\text{-Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r}, 0, 1)$$

for $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$ where V is a convenient KZ -subrepresentation of $\pi(\underline{r}, 0, 1)|_{KZ}$. An analogous result has been discovered by Paskunas in an unpublished draft.

The proof can be outlined as follow. Via the isomorphism of Proposition 2.9 we define the representation V as a suitable subrepresentation of R_1/R_0 : by Frobenius reciprocity we get a morphism $\phi : c\text{-Ind}_{KZ}^G V \rightarrow \pi(\underline{r}, 0, 1)$. From a basis of V we construct a convenient $\overline{\mathbf{F}}_p$ -basis for the compact induction $c\text{-Ind}_{KZ}^G V$ and therefore we only have to check that ϕ maps such basis into a linearly independent family of $\pi(\underline{r}, 0, 1)$.

This can be easily verified combining Proposition 3.5, Lemma 5.1 and Proposition 3.6.

We start from the following elementary fact:

LEMMA 5.9. *The K subrepresentation $\text{Fil}^0(R_1)$ of R_1 generated by $[1, X^{\underline{r}}]$ is naturally isomorphic to the finite principal series $\text{Ind}_{K_0(p)}^K \chi_{\underline{r}}^s$ and $\text{soc}(\text{Fil}^0(R_1)) \cong R_0$ via the monomorphism $R_0 \hookrightarrow R_1$.*

Proof. Obvious. \square

Let \tilde{V} denote the kernel of the natural map

$$\text{Fil}^0(R_1)/R_0 \twoheadrightarrow \text{cosoc}(\text{Fil}^0(R_1));$$

we define $V \leq \pi(\underline{r}, 0, 1)|_{KZ}$ as the homomorphic image of \tilde{V} via the isomorphism given in 2.9. Therefore, by Frobenius reciprocity, we get a morphism

$$\phi : c\text{-Ind}_{KZ}^G V \rightarrow \pi(\underline{r}, 0, 1).$$

We claim that

THEOREM 5.10. *Assume $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$. Then ϕ is a monomorphism.*

Proof. We show that the composite morphism of ϕ with the isomorphism (3)

$$c\text{-Ind}_{KZ}^G V \xrightarrow{\phi} \pi(\underline{r}, 0, 1) \xrightarrow{\sim} \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$$

maps an $\overline{\mathbf{F}}_p$ -basis of $c\text{-Ind}_{KZ}^G V$ onto a linearly independent family of the amalgamed sums on the right hand side.

By the well known results concerning the structure of finite principal series for $\text{GL}_2(\mathbf{F}_q)$ we have

LEMMA 5.11. *Assume $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$. For an f -tuple $\underline{t} \in \{0, \dots, p-1\}^f$ such that $\underline{t} \not\leq \underline{r}$ and $\underline{r} \not\leq \underline{t}$ the element $v_{\underline{t}} \in V$ is defined as*

$$v_{\underline{t}} \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_q} \mu_0^{\underline{t}} \begin{bmatrix} p & [\mu_0] \\ 0 & 1 \end{bmatrix} [1, X^{\underline{r}}].$$

An $\overline{\mathbf{F}}_p$ -basis \mathcal{V} for the compact induction is described by the elements

$$\begin{aligned} G_{\emptyset}^{(0,-1)}(\underline{t}) &\stackrel{\text{def}}{=} [1, v_{\underline{t}}] \\ G_{l_0, \dots, l_n}^{(1,n)}(\underline{t}) &\stackrel{\text{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{l_n} \begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix} [1, v_{\underline{t}}] \\ G_{l_0, \dots, l_n}^{(0,n)}(\underline{t}) &\stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, G_{l_0, \dots, l_n}^{(1,n)}(\underline{t})] \end{aligned}$$

where $n \in \mathbf{N}$, $l_j \in \{0, \dots, p-1\}^f$ for all $j \in \{0, \dots, n\}$, and $\underline{t} \in \{0, \dots, p-1\}^f$ verify the conditions $\underline{t} \not\leq \underline{r}$ and $\underline{r} \not\leq \underline{t}$.

Proof. It is elementary and left to the reader. See [Mo1], Lemma 5.13 for details. \square

We recall that the morphism ϕ is G -equivariant and the isomorphism (3) is KZ -equivariant. We deduce the equalities

$$\begin{aligned}\phi(G_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{t})) &= pr(F_{\underline{l}_0, \dots, \underline{l}_n, \underline{t}}^{(0,n+1)}(\underline{0})) \\ \phi(G_{\underline{l}_1, \dots, \underline{l}_n}^{(1,n)}(\underline{t})) &= pr(F_{\underline{l}_1, \dots, \underline{l}_n, \underline{t}}^{(1,n+1)}(\underline{0})) \\ \phi(G_{\emptyset}^{(0,-1)}(\underline{t})) &= pr(F_{\underline{t}}^{(0)}(\underline{0}))\end{aligned}$$

where we wrote pr to denote the natural epimorphisms of Proposition 3.6.

As the kernel of the epimorphism pr is known and we dispose of a suitable $\overline{\mathbf{F}}_p$ -basis of the inductive limits $\lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0^\pm \oplus_{R_1^\pm} \cdots \oplus_{R_n^\pm} R_{n+1}^\pm$, $\lim_{\substack{\longrightarrow \\ n \text{ even}}} (R_1/R_0)^\pm \oplus_{R_2^\pm} \cdots \oplus_{R_n^\pm} R_{n+1}^\pm$ we conclude that the elements $pr(F_{\underline{l}_0, \dots, \underline{l}_n, \underline{t}}^{(0,n+1)}(\underline{0}))$, $pr(F_{\underline{l}_1, \dots, \underline{l}_n, \underline{t}}^{(1,n+1)}(\underline{0}))$ and $pr(F_{\underline{t}}^{(0)}(\underline{0}))$ of the inductive limits $\lim_{\substack{\longrightarrow \\ n \text{ odd}}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$, $\lim_{\substack{\longrightarrow \\ n \text{ even}}} (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ are linearly independent, as required. \square

REMARK 5.12. Let \mathfrak{A} the image of the composite map obtained by ϕ and the isomorphism (3). By the proof of Proposition 5.10 the reader can easily describe, in terms of the lattices $\cdots \oplus_{\mathcal{R}_n^\pm} \mathcal{R}_{n+1}^\pm$, the inverse image of \mathfrak{A} by the natural epimorphism pr of Proposition 3.6.

5.2 The structure of the amalgamed sums

We are now ready to describe two blocks $R_{n+1}^\bullet/R_n^\bullet$ and $R_{n-1}^\bullet/R_{n-2}^\bullet$ should be glued together. We will see that such glueing is more or less a formal consequence of the geometric interpretation of the amalgamed sums, as announced in the introduction of §5.

Like in section 5.1 we will give the detailed proofs for the positive case: the negative part is deduced analogously.

First, we want to understand the image of an element $F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1}) \in R_{n+1}^+$ (resp. $F_{\underline{l}_1, \dots, \underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) \in R_{n+1}^-$) via the projection $(pr_{n+1})^{\text{pos}}$ (resp. $(pr_{n+1})^{\text{neg}}$) of Lemma 3.5.

LEMMA 5.13. Let $n \in \mathbf{N}_{\geq 1}$. The image of the element $F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1}) \in R_{n+1}^+$ via the projection pr_{n+1}^{pos} is described as follow:

- 1) If either $\underline{l}_{n+1} \neq \underline{0}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \not\leq \underline{r}$ then

$$\pi_{n+1}(pr_{n+1})^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1})) = \pi_{n+1}(F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1}));$$

- 2) If $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \geq \underline{p-1-r}$ then

$$(-1)^{\underline{r}}(pr_{n+1})^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1})) = \iota_{n-1}^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_{n-2}}^{(0,n-2)}(\underline{l}_{n-1} - \underline{p-1-r})) + \delta_{\underline{r}, \underline{p-1}} \delta_{\underline{l}_{n-1}, \underline{p-1}} \iota_{n-1}^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_{n-2}}^{(0,n-2)}(\underline{0}));$$

- 3) If either $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \not\geq \underline{p-1-r}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \leq \underline{r}$ then

$$(pr_{n+1})^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{l}_{n+1})) = 0.$$

Proof. Assertion 1) is clear by Lemma 5.1. We assume now that $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \leq \underline{r}$. Thus,

$$F_{\underline{l}_0, \dots, \underline{l}_n}^{(0,n)}(\underline{0}) = (-1)^{\underline{l}_n} (T_n^+)^{\text{pos}}(F_{\underline{l}_0, \dots, \underline{l}_{n-1}}^{(0,n-1)}(\underline{l}_n))$$

so that we get the following equality in the amalgamated sum $\cdots \oplus_{R_n^+} R_{n+1}^+$:

$$(pr_{n+1})^{\text{pos}}(F_{l_0, \dots, l_n}^{(0,n)}(\underline{0})) = \iota_{n-1}^+ \circ pr_{n-1}^+ \circ (-T_n^-)^{\text{pos}}((-1)^{l_n} (F_{l_0, \dots, l_{n-1}}^{(0,n-1)}(l_n))).$$

In order to get the statement, we are now left to describe

$$(T_n^-)^{\text{pos}}((F_{l_0, \dots, l_{n-1}}^{(0,n-1)}(l_n))).$$

Let assume $n \geq 2$ (the case $n = 1$ is treated in an analogous way and is left to the reader). By the characterisation of the operator T_n^- we have

$$(T_n^-)^{\text{pos}}((F_{l_0, \dots, l_{n-1}}^{(0,n-1)}(l_n))) = 0$$

if $l_n \neq \underline{r}$, while, for $l_n = \underline{r}$, we have

$$\begin{aligned} & (T_n^-)^{\text{pos}}((F_{l_0, \dots, l_{n-1}}^{(0,n-1)}(l_n))) = \\ &= \sum_{j=0}^{n-2} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \left[1, \sum_{\lambda_{n-1} \in \mathbf{F}_q} (\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{l_{n-1}} (\lambda_{n-1}^{\frac{1}{p^{n-1}}} X + Y) \underline{r} \right] = \\ &= \sum_{\underline{i} \leq \underline{r}} \binom{\underline{r}}{\underline{i}} \sum_{j=0}^{n-2} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \left[1, X^{\underline{r}-\underline{i}} Y^{\underline{i}} \sum_{\lambda_{n-1} \in \mathbf{F}_q} (\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{l_{n-1} + \underline{r} - \underline{i}} \right]. \end{aligned}$$

By Lemma 3.1, the quantity

$$\sum_{\lambda_{n-1} \in \mathbf{F}_q} (\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{l_{n-1} + \underline{r} - \underline{i}}$$

is non zero (indeed assuming the value -1) if and only if $l_{n+1} + \underline{r} - \underline{i} \equiv 0 \pmod{q-1}$ and $l_{n+1} + \underline{r} - \underline{i} \neq 0$. The result follows. \square

The result concerning the negative part is similar

LEMMA 5.14. *Let $n \in \mathbf{N}_{\geq 1}$. The image of the element $F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}) \in R_{n+1}^-$ via the projection pr_{n+1}^{neg} is described as follow:*

1) *If either $l_{n+1} \neq \underline{0}$ or $l_{n+1} = \underline{0}$ and $l_n \not\leq \underline{r}$ then*

$$\pi_{n+1}(pr_{n+1})^{\text{neg}}(F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1})) = \pi_{n+1}(F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1}));$$

2) *If $l_{n+1} = \underline{0}$, $l_n = \underline{r}$ and $l_{n-1} \geq \underline{p-1-r}$ (the latter condition being empty if $n = 1$) then*

$$(-1)^{\underline{r}} (pr_{n+1})^{\text{neg}}(F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1})) = \iota_{n-1}^{\text{neg}}(F_{l_1, \dots, l_{n-2}}^{(1,n-2)}(l_{n-1} - \underline{p-1-r})) + \delta_{\underline{r}, \underline{p-1}} \delta_{l_{n-1}, \underline{p-1}} \iota_{n-1}^{\text{pos}}(F_{l_1, \dots, l_{n-2}}^{(1,n-2)}(\underline{0}));$$

3) *If either $l_{n+1} = \underline{0}$, $l_n = \underline{r}$ and $l_{n-1} \not\geq \underline{p-1-r}$ (the latter condition being empty if $n = 1$) or $l_{n+1} = \underline{0}$ and $l_n \leq \underline{r}$ then*

$$(pr_{n+1})^{\text{pos}}(F_{l_1, \dots, l_n}^{(1,n)}(l_{n+1})) = 0.$$

Proof. It is analogous to the proof of Proposition 5.13 and it is left to the reader. \square

Interpretation in terms of euclidean data. We dispose of a canonical $\overline{\mathbf{F}}_p$ -basis for the representation $\cdots \oplus_{R_n^\pm} R_{n+1}^\pm$, which is obtained in the obvious way by an induction from Proposition 3.5 and Lemma 5.1.

Exactly as we did in §5.1 we have a natural way to associate an element of such canonical basis to a point in \mathbf{R}^f : again, we obtain a lattice, which we will denote by $\cdots \oplus_{\mathcal{R}_n^\pm} \mathcal{R}_{n+1}^\pm$.

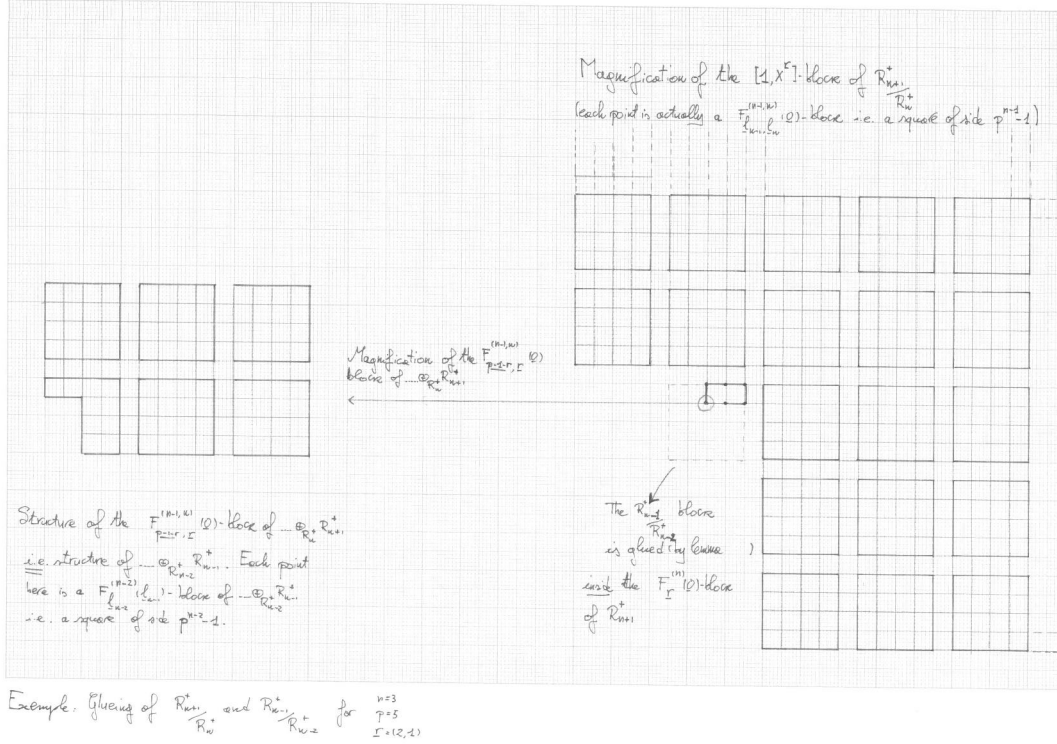


FIGURE 9. Again, the gluing and the fractal structure.

In such euclidean setting Proposition 5.13 is clear: it tells that lattice $\cdots \oplus_{\mathcal{R}_n^+} \mathcal{R}_{n+1}^+$ is obtained as the union of the lattice $\mathcal{R}_{n+1/n}^+$ associated to R_{n+1}^+/R_n^+ and the image of the lattice $\cdots \oplus_{\mathcal{R}_{n-2}^+} \mathcal{R}_{n-1}^+$ associated to the amalgamated sum $\cdots \oplus_{R_{n-2}^+} R_{n-1}^+$ (which, inductively, can be assumed to be known) by the translation

$$\begin{aligned} \mathbf{R}^f &\rightarrow \mathbf{R}^f \\ (x_i)_i &\mapsto (x_i + p^{n-1}(p-1 - r_{\lfloor i+n-1 \rfloor}) + p^n r_{\lfloor i+n \rfloor}). \end{aligned} \quad (12)$$

Notice that in particular the lattice $\cdots \oplus_{\mathcal{R}_{n-2}^+} \mathcal{R}_{n-1}^+$ is glued inside the $F_{\mathbb{Z}}^n(\underline{0})$ -block of \mathcal{R}_{n+1}^+ .

We stress again in figure 9 the gluing and the fractal structure for $f=2$ (noticing the gluing of $\cdots \oplus_{\mathcal{R}_{n-2}^+} \mathcal{R}_{n-1}^+$ inside the $F_{\mathbb{Z}}^{(n)}(\underline{0})$ -block of $\mathcal{R}_{n+1/n}^+$).

The evident analogous considerations for the negative part $\cdots \oplus_{\mathcal{R}_n^-} \mathcal{R}_{n+1}^-$ are left to the reader.

REMARK 5.15. Notice that if $f=1$ then it follows directly from Propositions 5.13 and 5.14 that the $K_0(p)$ -structure (and the extensions between two consecutive graded pieces) of the representations $\cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ are given by the associated lattices $\cdots \oplus_{\mathcal{R}_n^\bullet} \mathcal{R}_{n+1}^\bullet$. In particular, each of these representations has a space of I_1 invariants of dimension 1.

By remark 5.15 we can assume $f \geq 2$. In the next proposition we describe the socle filtration (and the extension between two consecutive graded pieces) of the $K_0(p)$ -representations $\cdots \oplus_{R_n^+} R_{n+1}^+$ for $n \geq 1$; the corresponding result for $\cdots \oplus_{R_n^-} R_{n+1}^-$ is similar and left to the reader.

The euclidean leitfaden which we are going to follow in order to prove the main result given in Proposition 5.16 is the following. As $\cdots \oplus_{R_{n-2}^+} R_{n-1}^+$ is a $K_0(p)$ -subrepresentation of $\cdots \oplus_{R_n^+} R_{n+1}^+$ the *only* thing we have to check is the following:

each of the J -th cutting hyperplanes $X_0 + \dots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor} + 1) + J$ of the lattice $\mathcal{R}_{n+1/n}^+$ lies strictly below ³ any of the $J - 1$ -cutting hyperplanes of the lattice $\dots \oplus_{\mathcal{R}_{n-2}^+} \mathcal{R}_{n-1}^+$.

Note that, as the cutting hyperplanes are parallel, we can assume $J = 0$.

Fix $n \geq 1$ and define

$$M_n \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} p^{n-1}(p-1-r_{\lfloor s+n-1 \rfloor}) + p^n r_{\lfloor s+n \rfloor}$$

(so that the hyperplane $X_0 + \dots + X_{f-1} = M_n$ contains the image of the point $\underline{0}$ via the translation (12)).

THEOREM 5.16. *Let $n \geq 1$ and consider the $K_0(p)$ -representation $\dots \oplus_{R_n^+} R_{n+1}^+$.*

The socle filtration and the extensions between two consecutive graded pieces are described by the associated lattice $\dots \oplus_{\mathcal{R}_n^+} \mathcal{R}_{n+1}^+$, with the conventions of section §5.1.2 and Propositions 5.6, 5.7 and 5.8 concerning the lattice associated to the $K_0(p)$ -structure of $(R_1/R_0)^+$.

Proof. By the euclidean interpretation of the $K_0(p)$ -structure of $\dots \oplus_{R_n^+} R_{n+1}^+$ and an immediate induction we see that it is enough to prove the inequalities

1) for $n \geq 3$

$$p^n(r_{s_0} + 1) < M_n + p^{n-2}(r_{s_1} + 1)$$

for any all indexes $s_0, s_1 \in \{0, \dots, f-1\}$;

2 for $n = 2$ and $s_0, s_1 \in \{0, \dots, f-1\}$

$$p^2(r_{s_0} + 1) < M_2 + (r_{s_1} + 1) - \delta$$

where $\delta \in \{0, 1\}$ is nonzero if and only if either the f -tuple \underline{r} verifies the hypothesis I_B) of Proposition 5.6 and $s_1 = s_m$ or the the f -tuple \underline{r} verifies the hypothesis of Proposition 5.8 $s_1 \in \{s_{m+k_1}, \dots, s_{m+k_0}\}$.

3) if $n = 1$

$$p(r_{s_0} + 1) \leq M_1.$$

Inequality 1) is immediately verified, and 2), 3) are trivial if $f \geq 3$ or $f = 2$ and $(r_0, r_1) \notin \{(p-1, 0), (0, p-1), (p-2, 0), (0, p-2)\}$. Notice that if $f = 2$ and $(r_0, r_1) \in \{(p-1, 0), (0, p-1)\}$ then $\mathfrak{V}_{s_m} = \{0\}$ so that it suffices to prove inequalities 2) and 3) only for $s_0 = s_{m+1}$, i.e. $r_{s_0} = 0$, which is true. The remaining case $f = 2$ and $(r_0, r_1) \in \{(p-2, 0), (0, p-2)\}$ is trivially checked and the proof is complete. \square

6. Appendix A: Some remarks on Witt polynomials

The aim of this appendix is to collect some technical results concerning Witt polynomials. After a section of general reminders (§6.1), we will treat in detail the case of the universal polynomials for the sum and the product (§6.2 and §6.3). In section §6.4 we study the Witt polynomials of a certain power series in the ring $W(\mathbf{F}_q)$: in this situation it is more complicate to keep track of the exponents of such polynomials. We are therefore led to introduce the notion of ‘‘pseudo homogeneity’’ (definition 6.11), a weak condition which nevetheless gives us a small control, sufficient for our aim (see also Proposition 7.4 and 7.5).

³if $f = 2$ and $n = 1$ we will see that, in few cases depending on the f -tuple \underline{r} , the J -th cutting hyperplane $X_0 + \dots + X_{f-1} = p(r_{\lfloor n+s \rfloor} + 1) + J$ of R_2^+/R_1^+ coincide with a J -th cutting hyperplane for R_0^+ . A direct check shows that the $K_0(p)$ -structure is the desired one.

6.1 Reminder on Witt polynomials

The description of the socle filtration for the aforementioned representations of $\mathrm{GL}_2(F)$ relies crucially on the behaviour of the universal Witt polynomials. After some generalities, we focus on specific situations related to the study of the action of lower unipotent, diagonal and upper unipotent matrices in $\mathrm{GL}_2(\mathcal{O}_F)$.

For $n \in \mathbf{N}$ the n -th Witt polynomial $W_n(\underline{X}) \in \mathbf{Z}[X_0, \dots, X_n]$ is defined by

$$W_n(\underline{X}) \stackrel{\text{def}}{=} \sum_{i=0}^n X_i^{p^{n-i}} p^i.$$

As the ring endomorphism

$$\begin{aligned} \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n] &\xrightarrow{\omega_n} \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n] \\ X_j &\longmapsto W_j(X_0, \dots, X_j) \end{aligned}$$

is bijective, we get a family of polynomials $M_0(X_0), \dots, M_n(X_0, \dots, X_n) \in \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n]$ which are uniquely determined by the condition:

$$M_j(W_0(\underline{X}), \dots, W_n(\underline{X})) = X_j.$$

They are of course described inductively by

$$M_n = \frac{1}{p^n} (X_n - p^{n-1} M_{n-1}(\underline{X})^p - \dots - p M_1(X_0, X_1)^{p^{n-1}} - M_0(X_0)^{p^n}).$$

The following lemma let us deduce the universal Witt polynomials describing the ring structure of $W(\mathbf{F}_q)$:

PROPOSITION 6.1. *Let $\Phi \in \mathbf{Z}[\zeta, \xi]$ be a polynomial in the variables ζ, ξ . For all $n \in \mathbf{N}$ there exist polynomials $\phi_n \in \mathbf{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$, uniquely determined by the conditions*

$$W_n(\phi_0, \dots, \phi_n) = \Phi(W_n(X_0, \dots, X_n), W_n(Y_0, \dots, Y_n)).$$

Sketch of the proof. The proof is constructive: we considering the commutative diagramm

$$\begin{array}{ccc} \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n] & \xrightarrow[\sim]{\omega_n} & \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n] \\ \downarrow f & & \downarrow \text{dotted} \\ \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n, Y_0, \dots, Y_n] & \xrightarrow[\sim]{\omega_n \otimes \omega_n} & \mathbf{Z}\left[\frac{1}{p}\right][X_0, \dots, X_n, Y_0, \dots, Y_n] \end{array}$$

where $f : \mathbf{Z}\left[\frac{1}{p}\right][\underline{X}] \rightarrow \mathbf{Z}\left[\frac{1}{p}\right][\underline{X}, \underline{Y}]$ is defined by $f(X_j) \stackrel{\text{def}}{=} \Phi(X_j, Y_j)$ for any $j \in \{0, \dots, n\}$; the polynomial ϕ_n is then given by

$$\phi_n(\underline{X}, \underline{Y}) \stackrel{\text{def}}{=} (\omega_n \otimes \omega_n) \circ f \circ \omega_n^{-1}(\underline{X}_n).$$

The fact that such ϕ_n 's have integer coefficients is then an induction on n . □

We apply then Proposition 6.1 to the polynomials

$$\Phi(\zeta, \xi) = \zeta + \xi, \quad \Phi(\zeta, \xi) = \zeta \xi$$

to get the universal polynomials for the sum and the product respectively. They will be denoted as

$S_n, Prod_n \in \mathbf{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ and are described inductively by

$$S_n(\underline{X}, \underline{Y}) = \frac{1}{p^n} (W_n(\underline{X}) + W_n(\underline{Y}) - p^{n-1} S_{n-1}(\underline{X}, \underline{Y})^p - \dots - p S_1(\underline{X}, \underline{Y})^{p^{n-1}} - S_0(\underline{X}, \underline{Y})^{p^n})$$

$$Prod_n(\underline{X}, \underline{Y}) = \frac{1}{p^n} (W_n(\underline{X}) W_n(\underline{Y}) - p^{n-1} Prod_{n-1}(\underline{X}, \underline{Y})^p - \dots - p Prod_1(\underline{X}, \underline{Y})^{p^{n-1}} - Prod_0(\underline{X}, \underline{Y})^{p^n}).$$

In section 4 we are interested in such operations as either rise to the N -power or the sum of N elements. We can of course adapt the arguments of Proposition 6.1 (or, use an induction on N) to determine the universal Witt polynomials associated to such operations. We will write $Pot_n^N(\underline{X}) \in \mathbf{Z}[X_0, \dots, X_n]$, $S_n^N(X(1), \dots, X(N)) \in \mathbf{Z}[X(1)_0, \dots, X(1)_n, \dots, X(N)_0, \dots, X(N)_n]$ for the n -th Witt polinomial associated to the rise to the N -power and the sum of N elements respectively. We have then the recursive relations:

$$Pot_n^N(\underline{X}) = \frac{1}{p^n} (W_n(\underline{X})^N - p^{n-1} Pot_{n-1}^N(\underline{X})^p - \dots - p Pot_1^N(\underline{X})^{p^{n-1}} - Pot_0^N(\underline{X})^{p^n})$$

$$S_n^N(\underline{X}(1), \dots, \underline{X}(N)) = \frac{1}{p^n} \left(\sum_{j=1}^N W_n(\underline{X}(j)) - p^{n-1} S_{n-1}^N(\underline{X}(1), \dots, \underline{X}(N))^p - \dots - p S_1^N(\underline{X}(1), \dots, \underline{X}(N))^{p^{n-1}} - S_0^N(\underline{X}(1), \dots, \underline{X}(N))^{p^n} \right).$$

6.2 Some special polynomials-I

In this paragraph we collect some technical results concerning some Witt polynomials which appear naturally in the study of the action of $\begin{bmatrix} 1 & 0 \\ \mathcal{O}_F & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & p\mathcal{O}_F \\ 0 & 1 \end{bmatrix}$) for the representations of §4.2 (resp. of §4.1).

For $n \in \mathbf{N}$ we define $S_n(\underline{X}, Y_0) \in \mathbf{Z}[X_0, \dots, X_n, Y_0]$ as the specialisation of $S_n(\underline{X}, \underline{Y})$ at $\underline{Y} = (Y_0, 0, \dots, 0, \dots)$. We recall

LEMMA 6.2. *For $n \in \mathbf{N}$ the polynomial $S_n(\underline{X}, \underline{Y})$ is an homogeneous polinomial in $\underline{X}, \underline{Y}$, of degree p^n if we define the elemets X_j, Y_j to be homogeneous of degree p^j .*

Proof. Elementary. □

Thus, if we set

$$\tilde{S}_n(\underline{X}, Y_0) \stackrel{\text{def}}{=} S_n(\underline{X}, Y_0) - X_n$$

we see that $\tilde{S}_j(\underline{X}, Y_0)$ is a polynomial in $\mathbf{Z}[X_0, \dots, X_{n-1}, Y_0]$, homogeneous of degree p^n . Moreover, as $\tilde{S}_n(\underline{X}, 0) = 0$ we see that $\tilde{S}_n(\underline{X}, Y_0)$ belongs to the ideal generated by Y_0 .

We define inductively the following family of automorphisms: we put

$$s_0 : \mathbf{Z}[X_0, Y_0] \rightarrow \mathbf{Z}[X_0, Y_0]$$

$$X_0 \mapsto X_0 - Y_0$$

$$Y_0 \mapsto Y_0$$

and, assuming $s_{j-1} : \mathbf{Z}[X_0, \dots, X_{j-1}, Y_0] \rightarrow \mathbf{Z}[X_0, \dots, X_{j-1}, Y_0]$ being constructed, we define

$$s_j : \mathbf{Z}[X_0, \dots, X_j, Y_0] \rightarrow \mathbf{Z}[X_0, \dots, X_j, Y_0]$$

$$X_j \mapsto X_j - s_{j-1}(\tilde{S}_j)$$

By their very construction, the s_j 's are graded homomorphisms; in particular $s_j(\tilde{S}_j)$ is homogeneous of degree p^j , and belongs to the ideal (Y_0) inside $\mathbf{Z}[X_0, \dots, X_j, Y_0]$. We can actually prove

the following result

LEMMA 6.3. For any $n \geq 1$ we have

$$s_{n-1}(S_n(\underline{X}, Y_0) - X_n) = -(S_n(\underline{X}, -Y_0) - X_n).$$

Proof. The case $n = 1$ is elementary:

$$s_0(S_1(X_0, X_1, Y_0) - X_1) = s_0\left(\frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p)\right) = \frac{1}{p}((X_0 - Y_0)^p + Y_0^p - X_0^p) = -(S_1(X_0, X_1, Y_0) - X_1).$$

Concerning the general case, we write

$$S_n(X_0, \dots, X_n, Y_0) - X_n = \frac{1}{p^n} [X_0^{p^n} + Y_0^{p^n} - p^{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \dots \quad (13)$$

$$\dots - p(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0 + Y_0)^{p^n}]. \quad (14)$$

For $j \in \{1, \dots, n-1\}$ we have

$$\begin{aligned} s_j(S_j(X_0, \dots, X_j, Y_0)^{p^{n-j}} - X_j^{p^{n-j}}) &= (s_{j-1}(S_j(X_0, \dots, X_j, Y_0) - X_j) + s_j(X_j))^{p^{n-j}} - (s_j(X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j - s_{j-1}(S_j(X_0, \dots, X_j, Y_0) - X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j + S_j(X_0, \dots, X_j, -Y_0) - X_j)^{p^{n-j}} \\ &= -(S_j(X_0, \dots, X_j, -Y_0)^{p^{n-j}} - X_j^{p^{n-j}}). \end{aligned}$$

As $s_{n-1}(S_n(X_0, \dots, X_n, Y_0) - X_n) = s_n(S_n(X_0, \dots, X_n, Y_0) - X_n)$ we are left compute

$$\begin{aligned} s_n\left(\frac{1}{p^n} [X_0^{p^n} + Y_0^{p^n} - p^{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \dots \right. \\ \left. \dots - p(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0 + Y_0)^{p^n}\right] = \\ \frac{1}{p^n} [(X_0 - Y_0)^{p^n} + Y_0^{p^n} - p^{n-1}s_{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \dots \\ \dots - ps_1(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0)^{p^n}] \end{aligned}$$

and the result follows as $s_j(S_j(X_0, \dots, X_j, Y_0)^{p^{n-j}} - X_j^{p^{n-j}}) = -(S_j(X_0, \dots, X_j, -Y_0)^{p^{n-j}} - X_j^{p^{n-j}})$ for all $j \in \{1, \dots, n-1\}$. \square

We will also need a cleaner statement concerning the monomials of $S_n(X_0, \dots, X_n, Y_0)$:

LEMMA 6.4. For all $n \geq 1$ the coefficient of the monomial $X_0^{p-1} \dots X_{n-1}^{p-1} Y_0$ appearing in the development of the universal Witt polynomial $S_n(X_0, \dots, X_n, Y_0)$ is 1.

Proof. The proof is again an induction on n : the case $n = 1$ is evident.

For the general case, consider

$$S_n(\underline{X}, Y_0) = \frac{1}{p^n} (W_n(\underline{X}) + Y_0^{p^n} - p^{n-1}S_{n-1}(\underline{X}, Y_0)^p - \dots - pS_1(\underline{X}, Y_0)^{p^{n-1}} - S_0(\underline{X}, Y_0)^{p^n}).$$

A monomial of the form $X_0^{p-1} \dots X_{n-1}^{p-1} Y_0$ lies therefore inside

$$-\frac{1}{p}(S_{n-1}(X_0, \dots, X_{n-1}, Y_0)^p - X_{n-1}^{p-1})$$

and the inductive hypothesis yields

$$S_{n-1}(X_0, \dots, X_{n-1}, Y_0) = X_{n-1} + X_0^{p-1} \dots X_{n-2}^{p-1} Y_0 + x(X_0, \dots, X_{n-2}, Y_0)$$

where $x(X_0, \dots, X_{n-2}, Y_0) \in \mathbf{Z}[X_0, \dots, X_{n-2}, Y_0]$ doesn't contains the monomial $X_0^{p-1} \dots X_{n-2}^{p-1} Y_0$. Finally, we have

$$(S_{n-1}(X_0, \dots, X_{n-1}, Y_0))^p = \sum_{\substack{i+j+k=p \\ 0 \leq i, j, k}} \frac{p!}{i!j!k!} X_{n-1}^i (X_0^{p-1} \dots X_{n-2}^{p-1} Y_0)^j (x(X_0, \dots, X_{n-2}, Y_0))^k$$

and the conclusion follows. \square

6.3 Some special polynomials -II

In this section we deal with some Witt polynomials which appear naturally when we study the action of the diagonal matrices $\begin{bmatrix} 1 + p\mathcal{O}_F & 0 \\ 0 & 1 + \mathcal{O}_F \end{bmatrix}$. Recall that

LEMMA 6.5. *Let $n \in \mathbf{N}$. The n -th universal Witt polynomial of the product $Prod_n(\underline{X}, \underline{Y})$ is an homogeneous element of $(\mathbf{Z}[\underline{Y}])[\underline{X}]$ (resp. $(\mathbf{Z}[\underline{X}])[\underline{Y}]$) provided that X_j (resp. Y_j) is homogeneous of degree p^j for any $0 \leq j \leq n$.*

Proof. Elementary. \square

REMARK 6.6. *In the present paragraph, we will be concerned with the image in $\mathbf{F}_p[\underline{X}, \underline{Y}]$ of the universal Witt polynomials $S_n(\underline{X}, \underline{Y}), Prod_n(\underline{X}, \underline{Y})$. Such images will be denoted again by $S_n(\underline{X}, \underline{Y}), Prod_n(\underline{X}, \underline{Y})$, in order not to overload notations. As $p \cdot 1 = 0$ multiplication by p is the composite of Frobenius and Verschiebung.*

For $N \in \mathbf{N}$, let $z' = (\lambda'_0, \dots, \lambda'_N, 0, \dots, 0, \dots) \in W(\mathbf{F}_q)$ and let $\alpha = (\alpha_0, \alpha_1, \dots) \in W(\mathbf{F}_q)$; we need to describe

$$z' + p\alpha \cdot z' \pmod{p^{N+1}} \tag{15}$$

in terms of the universal Witt polynomials.

LEMMA 6.7. *For $0 \leq j \leq N$ the j -th Witt polynomial of the development of (15) is an homogeneous element $Q_j(\underline{\lambda}', \underline{\alpha})$ of degree p^j in $(\mathbf{F}_p[\alpha_0, \dots, \alpha_{j-1}])[\lambda'_0, \dots, \lambda'_j]$ if we define, for $0 \leq s \leq j$, λ'_s to be homogeneous of degree p^s .*

Proof. It is a straightforward consequence of Lemmas 6.2 and 6.5. More precisely, from 6.5 we see that

$$p \cdot z' \cdot \alpha = (0, Prod_0(\lambda'_0, \alpha_0), \dots, \underbrace{Prod_{j-1}(\lambda'_0, \dots, \lambda'_{j-1}, \alpha_0, \dots, \alpha_{j-1})}_{j \text{ th entry}} \dots)$$

where each $Prod_{j-1}(\underline{\lambda}', \underline{\alpha})^p$ is homogeneous of degree p^j (provided that λ'_s is homogeneous of degree p^s for $0 \leq s \leq j-1$). Furthermore, $Q_j(\underline{\lambda}', \underline{\alpha})$ is the specialisation of $S_j(\underline{X}, \underline{Y})$ at $\underline{X} = z', \underline{Y} = p \cdot z' \cdot \alpha$ and we use Lemma 6.2 to get the desired result. \square

As we did in §6.2 we define (for $0 \leq j \leq N$)

$$\tilde{Q}_j \stackrel{\text{def}}{=} Q_j(\underline{\lambda}', \underline{\alpha}) - \lambda'_j.$$

For $j \neq 0$ it is a polynomial in $(\mathbf{F}_p[\alpha_0, \dots, \alpha_{j-1}])[\lambda'_0, \dots, \lambda'_{j-1}]$, homogeneous of degree p^j

We can finally define, inductively, a family of ring homomorphisms: we let

$$q_0 : \mathbf{F}_p[\lambda'_0] \rightarrow \mathbf{F}_p[\lambda'_0]$$

be the identity map, and, assuming q_{j-1} being constructed for $j \geq 1$, we define

$$q_j : \mathbf{F}_p[\lambda'_0, \dots, \lambda_j, \alpha_0, \dots, \alpha_{j-1}] \rightarrow \mathbf{F}_p[\lambda'_0, \dots, \lambda'_j, \alpha_0, \dots, \alpha_{j-1}]$$

by the condition

$$\begin{aligned} \lambda'_j &\mapsto \lambda'_j - q_{j-1}(\tilde{Q}_j) \\ \alpha_{j-1} &\mapsto \alpha_{j-1} \\ q_j|_{\mathbf{F}_p[\lambda'_0, \dots, \lambda_{j-1}, \alpha_0, \dots, \alpha_{j-2}]} &= q_{j-1} \end{aligned}$$

(and the obvious formalism: if $j = 1$ we just forget α_{j-2} from the formulas).

We deduce:

LEMMA 6.8. *For $0 \leq j \leq N$, the polynomial $q_{j-1}(\tilde{Q}_j)$ is homogeneous of degree p^j in $\lambda'_0, \dots, \lambda'_{j-1}$.*

Proof. The morphism q_{j-1} is a graded ring homomorphism. \square

6.4 Some special Witt polynomials -III

In this paragraph we study some Witt polynomials giving the action of $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & 0 \\ p\mathcal{O}_F & 1 \end{bmatrix}$) for the representations of §4.1 (resp. of §4.2). Such study is more delicate than the previous sections (§6.2 and §6.3) and relies crucially on the fact that we deal with Witt vectors $x \in W(\mathbf{F}_q)$ which are NOT invertible.

We start with a general remark

LEMMA 6.9. *Let $N, n \in \mathbf{N}$.*

- i) *The n -th universal Witt polynomial of the rise to the N -th power $Pot_n^N(\underline{X})$ is an homogeneous element of degree Np^n in $\mathbf{Z}[X_0, \dots, X_n]$ provided that X_j is homogeneous of degree p^j for any $0 \leq j \leq n$.*
- ii) *The n -th universal Witt polynomial associated to the sum of N elements $S_n^N(\underline{X}(1), \dots, \underline{X}(N))$ is an homogeneous element of degree p^n in $\mathbf{Z}[X(1)_0, \dots, X(1)_n, \dots, X(N)_0, \dots, X(N)_n]$ if we define $X(l)_j$ to be homogeneous of degree p^j , for any $l \in \{1, \dots, N\}$.*

Proof. Elementary. \square

As in §6.3 we have the following

REMARK 6.10. *In the present paragraph, we will be concerned with polynomials with coefficients in \mathbf{F}_p obtained by reducing modulo p the coefficients of the universal Witt polynomials $S_n^N(\underline{X}, \underline{Y})$, $Pot_n^N(\underline{X})$, $S_n(\underline{X}, \underline{Y})$, $Prod_n(\underline{X}, \underline{Y})$. In order not to overload notations, such images will be denoted again by $S_n^N(\underline{X}, \underline{Y}), \dots$. As $p \cdot 1 = 0$, recall that multiplication by p is the composite of Frobenius and Verschiebung.*

Fix $0 \leq m \leq n$ and consider the ring $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$.

DEFINITION 6.11. *Let $M \in \mathbf{N}$. A monomial $\lambda_m^{\alpha_m} \dots \lambda_n^{\alpha_n} \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be pseudo-homogeneous of degree M if the following holds:*

there exist an integer $L \in \mathbf{N}$ and integers $\beta_l(j)$ for $j \in \{1, \dots, L\}$, $l \in \{m, \dots, n\}$ such that

- i) *for all $l \in \{m, \dots, n\}$ we have*

$$\alpha_l = \sum_{j=1}^L p^{j-1} \beta_l(j)$$

ii) we have

$$p^m \left(\sum_{j=1}^L \beta_m(j) \right) + \cdots + p^n \left(\sum_{j=1}^L \beta_n(j) \right) \leq M.$$

A polynomial in $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be pseudo-homogeneous of degree M if it is a sum of monomials each of which is pseudo homogeneous of degree M .

The following result is imediate

LEMMA 6.12. Fix m, n as above. Then:

- i) If $P_1, P_2 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ are pseudo-homogeneous of degree M_1, M_2 respectively, then $P_1 P_2$ is pseudo-homogeneous of degree $M_1 + M_2$.
- ii) if $P_1 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous of degree M_1 then P_1^p is again pseudohomogeneous of degree M_1 .

Proof. Omissis. □

REMARK 6.13. If $P \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous and we specialise P on an element of \mathbf{F}_q^{n-m+1} , we see that the integer L in definition 6.11 can be assumed to verify $L \leq f$.

We are now ready to focus our attention some Witt vectors in $W(\mathbf{F}_q)$.

6.4.1 *The negative case.* For $1 \leq m \leq n$, let $z \stackrel{\text{def}}{=} (0, \dots, 0, \lambda_m, \dots, \lambda_n, 0, \dots)$ and $[\mu] \stackrel{\text{def}}{=} (\mu, 0, \dots)$ be elements of $W(\mathbf{F}_q)$. We are interested in the Witt development of

$$\sum_{j=0}^N z^{j+1} [\mu^j] \pmod{p^{n+1}} \tag{16}$$

where $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. For $j \in \{m, \dots, n\}$ write finally $U_j(\lambda, \mu) \in \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$ for the j -th polynomial of the Witt development of (16) and put

$$\tilde{U}_j(\lambda, \mu) \stackrel{\text{def}}{=} U_j - \lambda_j.$$

We notice that $\tilde{U}_j = 0$ if $m \leq j \leq 2m - 1$ and $\tilde{U}_{2m} = \lambda_m^{2p^m}$.

We have a rough estimate of the degree of the \tilde{U}_h

LEMMA 6.14. Let $h \in \{2m, \dots, n\}$. Then $\tilde{U}_h \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, \mu]$ and is pseudo homogeneous of degree $p^h - p^m(p^m - 2)$.

Proof. If $\tilde{z} \stackrel{\text{def}}{=} (\lambda_m^{\frac{1}{p^m}}, \dots, \lambda_n^{\frac{1}{p^m}}, 0, \dots)$ then we recall that $Pot_l^{j+1}(\tilde{z})$ is homogeneous of degree $(j+1)p^l$ (if λ_s is homogeneous of degree p^s). Thus the Witt development of $z^{j+1}[\mu^j]$ has the form

$$z^{j+1}[\mu^j] = (0, \dots, 0, \underbrace{Pot_0^{j+1}(\lambda_m^{p^{mj}})(\mu^j)^{p^{m(j+1)}}}_{\text{position } m(j+1)}, \dots, \underbrace{Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}}}_{\text{position } m(j+1)+l}, \dots)$$

and $Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}}$ is homogeneous of degree $(j+1)p^{l+m(j+1)}$ and actually is pseudo-homogeneous of degree $(j+1)p^{l+m}$.

Thus, if $a_{(j+1)m}(j), \dots, a_h(j)$ is an $h - (j+1)m + 1$ -tuple of integers, the polynomial

$$\prod_{l=0}^{h-(j+1)m} (\text{Pot}_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(\mu^j)^{p^{m(j+1)+l}})^{a_{(j+1)m+l}(j)}$$

is pseudo-homogeneous of degree

$$(j+1)(p^m a_{(j+1)m}(j) + \dots + p^{h-mj} a_h(j)).$$

By Lemma 6.9 we see that a monomial of $S_h^{N+1}(\underline{X}(1), \dots, \underline{X}(N+1))$ has the following form:

$$\mathfrak{X} \stackrel{\text{def}}{=} \prod_{l_0=0}^h X_{l_0}(1)^{a_{l_0}(0)} \dots \prod_{l_N=0}^h X_{l_N}(N+1)^{a_{l_N}(N)}$$

where

$$\sum_{l_0=0}^h p^{l_0} a_{l_0}(0) + \dots + \sum_{l_N=0}^h p^{l_N} a_{l_N}(N) = p^h.$$

As U_h is the specialisation of $S_h^{(N+1)}$ at

$$(\underline{X}(j+1))_{j \in \{0, \dots, N\}} = (z^{j+1}[\mu^j])_{j \in \{0, \dots, N\}}$$

we see in particular that $\widetilde{U}_h \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, \mu]$.

Assume now that

- 1) for $h \geq (j+1)m$ we have $a_{l_j}(j) = 0$ for all $l_j < (j+1)m$;
- 2) for $h < (j+1)m$ we have $a_{l_j}(j) = 0$.

Then Lemma 6.12 shows that the specialisation of \mathfrak{X} is pseudo-homogeneous of degree

$$d \stackrel{\text{def}}{=} \sum_{j=0}^N (j+1) \left(\sum_{i=(j+1)m}^h p^{i-jm} a_i(j) \right).$$

Letting

$$x_{j+1} \stackrel{\text{def}}{=} \sum_{i=(j+1)m}^h p^{i-mj} a_i(j)$$

for $j \in \{0, \dots, h\}$ we get

$$d = p^h - \sum_{j=0}^N (p^{jm} - (j+1)) x_j$$

and the conclusion follows from Lemma 6.15 below. □

LEMMA 6.15. *Let $j \in \{0, \dots, N\}$ and let*

$$\mathfrak{X} \stackrel{\text{def}}{=} \prod_{l_0=0}^h X_{l_0}(1)^{a_{l_0}(0)} \dots \prod_{l_N=0}^h X_{l_N}(N+1)^{a_{l_N}(N)}$$

be a monomial of $S_h^{(N+1)}(\underline{X}(1), \dots, \underline{X}(N+1))$.

If $a_{l_i}(i) = 0$ for all $i \neq j$ and $l_i \in \{0, \dots, h\}$ then

$$\mathfrak{X} = X_h(j).$$

Proof. An immediate induction on h shows that if we specialise $S_h^{(N+1)}$ at

$$(X_0(i), \dots, X_h(i)) = (0, \dots, 0)$$

for $i \neq j$ we get

$$S_h^{(N+1)}(\underline{0}, \dots, \underline{0}, \underline{X}(j), \underline{0}, \dots, \underline{0}) = X_h(j)$$

and the claim follows. \square

We finally introduce a family of ring homomorphisms, for $m \leq j \leq n$,

$$u_j : \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu] \rightarrow \mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$$

defined inductively as follow: u_m is the identity map and, assuming u_{j-1} being constructed, we define u_j as the unique extension of u_{j-1} to $\mathbf{F}_p[\lambda_m, \dots, \lambda_j, \mu]$ such that

$$\lambda_j \mapsto \lambda_j - u_{j-1}(\tilde{U}_j).$$

We have the

LEMMA 6.16. *Let $h \in \{2m, \dots, n\}$. Then $u_h(\tilde{U}_h)$ is pseudo homogeneous of degree $p^h - p^m(p^m - 2)$.*

Proof. Arguing by induction, we can assume that $u_l(\lambda_l)$ is pseudohomogeneous of degree p^l for all $l \in \{m, \dots, h-1\}$. As \tilde{U}_h is pseudohomogeneous of degree $p^h - p^m(p^m - 2)$ by Lemma 6.14, the claim follows from Lemma 6.12. \square

6.4.2 *The positive case* This section is essentially a re-edition of §6.4.1, where we take $m = 0$. The interest of this case will appear in §4.2, where we give a description of the $K_0(p)$ -representations R_{n+1}^+ .

Let $(\lambda_0, \dots, \lambda_n, 0, \dots) \in \mathbf{W}(\mathbf{F}_q)$.

We are interested in the Witt development $(U_0(\lambda_0, \mu), U_1(\lambda_0, \lambda_1, \mu), \dots, U_{n+1}(\lambda_0, \dots, \lambda_{n+1}, \mu), 0, \dots)$ of

$$z(1 + p[\mu]z)^{-1} \equiv \sum_{j=0}^{n+1} p^j[\mu]z^{j+1} \pmod{p^{n+2}}.$$

We check immediately that $U_0 = \lambda_0$ and $U_1 = \lambda_1 + \lambda_0^{2p}\mu$.

We define for $h = 0, \dots, n+1$ $\tilde{U}_h \stackrel{\text{def}}{=} U_h - \lambda_h$. The following result is the analogous of Lemma 6.14

LEMMA 6.17. *Let $h \in \{1, \dots, n+1\}$. Then $\tilde{U}_h \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{h-1}, \mu]$ is pseudohomogeneous of degree $p^h - (p-2)$.*

Proof. The proof is completely analogous to the proof of Lemma 6.14 and left to the reader (see [Mo1] for details). \square

As in section §6.4.1 we define inductively, for $h = 0, \dots, n+1$, the ring morphisms

$$u_h : \mathbf{F}_p[\lambda_0, \dots, \lambda_h, \mu] \rightarrow \mathbf{F}_p[\lambda_0, \dots, \lambda_h, \mu]$$

by the condition $u_h(\lambda_h) \stackrel{\text{def}}{=} \lambda_h - u_{h-1}(\tilde{U}_h)$ for $h \geq 1$ and $u_0 \stackrel{\text{def}}{=} id$. Then

LEMMA 6.18. *Let $1 \leq h \leq n+1$. Then $u_h(\tilde{U}_h)$ is pseudo homogeneous of degree $p^h - (p-2)$.*

Proof. As for Lemma 6.16 it is a consequence of Lemma 6.12 and Lemma 6.17. \square

Still others remarks on some universal Witt polynomials. In this paragraph we are going to pursue the technical computations of §6.4.2. Indeed, the structure of the quotients $R_{n+1}^\bullet/R_n^\bullet$ (cf. §5.1) can not be deduced simply from Lemma 6.18, as for R_{n+1}^\bullet . We should instead look more closely the structure of the polynomial \tilde{U}_{n+1} and $u_n(\tilde{U}_n + 1)$ (the notations being the same as for §6.4.2).

The following description is deduced as in the proof of Lemma 6.14. Let $z = (\lambda_0, \dots, \lambda_n, 0) \in \mathbf{W}_{n+1}(\mathbf{F}_q)$ and write

$$\sum_{j=0}^{n+1} p^j [\mu] z^{j+1} = (U_0, \dots, U_{n+1}).$$

for $U_j \in \mathbf{F}_p[\lambda_0, \dots, \lambda_j, \mu]$. We recall that U_h is obtained by specializing the universal polynomial $S_h^{n+2}(\underline{X}(1), \dots, \underline{X}(n+2))$ at

$$\underline{X}(j+1) = (0, \dots, 0, \underbrace{(Pot_0^{j+1}(\lambda))^{p^j} (\mu^j)^{p^j}}_{\text{position } j}, \dots, \underbrace{(Pot_l^{j+1}(\lambda))^{p^j} (\mu^j)^{p^{j+l}}}_{\text{position } j+l}, \dots).$$

We recall moreover that a monomial \mathfrak{X} of $S_h^{n+2}(\underline{X}(1), \dots, \underline{X}(n+2))$ has the form

$$\mathfrak{X} = \prod_{l_0=0}^h X_{l_0}(1)^{a_{l_0}(0)} \dots \prod_{l_{n+1}=0}^h X_{l_{n+1}}(n+2)^{a_{l_{n+1}}(n+1)}$$

where the integers $a_{l_i}(i)$ verify

$$\sum_{l_0=0}^h p^{l_0} a_{l_0}(0) + \dots + \sum_{l_{n+1}=0}^h p^{l_{n+1}} a_{l_{n+1}}(n+1) = p^h;$$

Therefore a monomial $\lambda_0^{\alpha_0} \dots \lambda_h^{\alpha_h}$ issued from U_h verifies

$$\sum_{j=0}^h p^j \mathfrak{s}(\alpha_j) \leq \sum_{j=0}^{n+1} (j+1) \left(\sum_{i=j}^h p^{i-j} a_i(j) \right) = p^h - \sum_{j=1}^h (p^j - (j+1)) x_j$$

where we have set

$$x_j \stackrel{\text{def}}{=} \sum_{i=j}^h p^{i-j} a_i(j).$$

We focus our attention for the case $h = n+1$, obtaining thus the following

LEMMA 6.19. *A monomial of \tilde{U}_{n+1} has the following form*

$$\lambda_n^{a_n(0)+pa_{n+1}(1)} \cdot \lambda_{n-1}^{\alpha_{n-1}} \dots \lambda_0^{\alpha_0}$$

whose the exponents verify the following properties:

- 1) we have $a_n(0) \in \{0, \dots, p-1\}$ and $a_{n+1}(1) \in \{0, 1\}$,
- 2) letting $x_j \stackrel{\text{def}}{=} \sum_{i=j}^{n+1} p^{i-j} a_i(j)$ we have

$$\sum_{j=0}^n p^j \mathfrak{s}(\alpha_j) + (a_n(0) + a_{n+1}(1)) \leq p^h - \sum_{j=1}^h (p^j - (j+1)) x_j$$

- 3) if $a_{n+1}(1) = 1$ then the monomial has the form

$$\lambda_0^{p^{n+1}} \lambda_n^p.$$

Proof. The fact that $\alpha_n(0) \neq p$ follows from the fact that in the polynomial S_{n+1}^{n+2} the coefficient of $X_n(1)^p$ is zero (the proof is the usual one: see Lemma 6.15). Assertion 2) is deduced from 1) (and the fact that $f \geq 2$). Assertion 3) follows noticing that $(Pot_n^2(z))^p = 2\lambda_0^{p^{n+1}} \lambda_n^p + x$ where $x \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{n-1}]$. \square

We recall the ring morphism $u_n : \mathbf{F}_p[\lambda_0, \dots, \lambda_n \mu] \rightarrow \mathbf{F}_p[\lambda_0, \dots, \lambda_n \mu]$ (cf. 6.4.2). If $i_{n+1}^{(s)} \in \mathbf{N}$ deduce the following

LEMMA 6.20. *In the preceding notations, a monomial issued from $u_n(\tilde{U}_{n+1})^{i_{n+1}^{(s)}}$ has the following form*

$$(\lambda_0^{p^{n+1}} \lambda_n^p)^{B_{n+1}^{(s)}(1)} \lambda_n^{B_n^{(s)}(0)} \lambda_{n-1}^{\beta_{n-1}} \dots \lambda_0^{\beta_0}$$

where the exponents verify the following properties:

1) we have

$$\mathfrak{s}(\beta_0 + p^{n+1} B_{n+1}^{(s)}(1)) + \sum_{j=1}^{n-1} p^j \mathfrak{s}(\beta_j) + p^n (B_n^{(s)}(0) + B_{n+1}^{(s)}(1)) \leq p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) A_i(j)$$

for suitable integers $A_i(j) \in \mathbf{N}$;

2) we have $A_i(j) = 0$ for all couples (i, j) if and only if $i_{n+1}^{(s)} = 0$;

3) we have $0 \leq B_{n+1}^{(s)}(1) \leq A_{n+1}^{(s)}(1) \leq i_{n+1}^{(s)}$.

Proof. By the argument appearing in the proof of Lemma 6.14 (see also [Mo1], Lemma 6.17) we see that a fixed monomial $\lambda_0^{\alpha_0} \dots \lambda_n^{\alpha_n}$ issued from $\tilde{U}_{n+1}^{i_{n+1}^{(s)}}$ is pseudohomogeneous of degree $d \stackrel{\text{def}}{=} p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) A_i(j)$, where the integers $A_i(j) \in \mathbf{N}$ are not all equal to zero, except if $i_{n+1}^{(s)} = 0$. By Lemma 6.19 we see moreover that a monomial issued from $\tilde{U}_{n+1}^{i_{n+1}^{(s)}}$ has the form

$$(\lambda_0 \lambda_n^p)^{A_{n+1}^{(s)}(1)} \lambda_n^{A_n^{(s)}(0)} \lambda_{n-1}^{\alpha_{n-1}} \dots \lambda_0^{\alpha_0}$$

and $A_{n+1}(1) \leq i_{n+1}^{(s)}$. Recall now that $u_j(\lambda_j)$ is pseudohomogeneous of degree p^j (Lemma 6.18) and $u_n(\lambda_n) = \lambda_n - u_{n-1}(\tilde{U}_n)$, where $\tilde{U}_n \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{n-1}, \mu]$. It follows then from Lemma 6.12 that a monomial issued from $u_n(\lambda_0^{\alpha_0} \dots \lambda_n^{\alpha_n})$ has the form

$$(\lambda_0^{p^{n+1}} \lambda_n^p)^{B_{n+1}^{(s)}(1)} \lambda_n^{B_n^{(s)}(0)} \lambda_{n-1}^{\beta_{n-1}} \dots \lambda_0^{\beta_0}$$

for convenient integers $B_{n+1}^{(s)}, B_n^{(s)}(0), \beta_j$, satisfying $B_{n+1}^{(s)}(1) \leq A_{n+1}^{(s)}(1)$ and, being pseudohomogeneous of degree d , it verifies

$$\sum_{j=0}^n p^j \mathfrak{s}(\beta_j) \leq p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) A_i(j).$$

□

7. Appendix B: Two rough estimates

In this appendix use the material of appendix A to estimate the behaviour of (the reduction modulo $p^f - 1$ of) some elements which appear naturally in the study of the socle filtration for the representations $R_{n+1}^{\pm}, \text{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$, etc...

The first tool is discussed in §7.1: it is an elementary description of the function \mathfrak{s} giving the cipher sum of the reduction modulo $p^f - 1$ of a natural number. In §7.2 the properties of the function \mathfrak{s} and the results on Witt polynomials stated in §6 will be used to describe in detail some explicit vectors of the aforementioned representations (Propositions 7.3, 7.4 and 7.5).

7.1 Remark on the proof of Stickelberger's theorem

In this section we recall the construction and the properties of a certain function $s : \mathbf{Z} \rightarrow \mathbf{N}$ which appears in the proof of Stickelberger's theorem.

If \mathfrak{p} is a prime of $\mathbf{Q}(\zeta_{q-1})$ lying above p , the reduction modulo \mathfrak{p} , $\mathbf{Z}[\zeta_{q-1}] \rightarrow \mathbf{F}_q$ admits a multiplicative section

$$\omega_{\mathfrak{p}} : \mathbf{F}_p^\times \rightarrow \mathbf{Z}[\zeta_{q-1}]$$

which induces an isomorphisms on the group μ_{q-1} of $q-1$ -th roots of unity. If \mathfrak{P} is the prime of $\mathbf{Q}(\zeta_{q-1}, \zeta_p)$ lying above \mathfrak{p} , we define a function $s : \mathbf{Z} \rightarrow \mathbf{N}$ by

$$s(n) \stackrel{\text{def}}{=} \text{val}_{\mathfrak{P}}(g(\omega_{\mathfrak{p}}^{-n}))$$

where $\text{val}_{\mathfrak{P}}$ denotes the \mathfrak{P} -adic valuation and $g(\omega_{\mathfrak{p}}^{-n})$ denotes the Gauss sum of the character $\omega_{\mathfrak{p}}^{-n} : \mathbf{F}_q^\times \rightarrow \mu_{q-1}$.

We need to modify slightly this function as follow:

$$\mathfrak{s} : \mathbf{N} \rightarrow \mathbf{N}$$

$$n \mapsto \begin{cases} s(n) & \text{if either } n \not\equiv 0 \pmod{q-1} \text{ or } n = 0 \\ f(p-1) & \text{otherwise} \end{cases}$$

The following lemma is then easily deduced from the well known properties of the function s (cf. [Was], §6.2):

LEMMA 7.1. *Let $n, m \in \mathbf{N}$. Then:*

- a) $\mathfrak{s}(0) = 0$ and $\mathfrak{s}(1) = 1$;
- b) $0 \leq \mathfrak{s}(m+n) \leq \mathfrak{s}(n) + \mathfrak{s}(m)$;
- c) $\mathfrak{s}(pn) = \mathfrak{s}(n)$;
- d) if $0 \leq n \leq q-1$ and (a_0, \dots, a_{f-1}) are the cyphers of the p -adic development of n , we have

$$\mathfrak{s}(n) = a_0 + a_1 + \dots + a_{f-1}.$$

In particular, $\mathfrak{s}(n) \leq n$ for any $n \in \mathbf{N}$, with equality if and only if $n \in \{0, \dots, p-1\}$.

We can improve the statement of b):

LEMMA 7.2. *Let $b_0, \dots, b_{f-1} \in \mathbf{N}$ be integers.*

Then there exists integers m_s, n_s , where $s \in \{0, \dots, f-1\}$ such that:

- 1) for all $s \in \{0, \dots, f-1\}$

$$c_s \stackrel{\text{def}}{=} b_s - pm_s + n_{[s-1]} \in \{0, \dots, p-1\};$$

- 2) we have

$$\tilde{j} \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} m_s = \sum_{s=0}^{f-1} n_s;$$

- 3) we have

$$\sum_{s=0}^{f-1} p^s b_s \equiv \sum_{s=0}^{f-1} p^s c_s \pmod{p^f - 1};$$

- 4) we have the equality

$$\mathfrak{s}\left(\sum_{s=0}^{f-1} p^s b_s\right) = \sum_{s=0}^{f-1} b_s - \tilde{j}(p-1).$$

Proof. Assume first that $b_s \in \{0, \dots, p-1\}$ for all $s \geq 1$ and $b_0 \geq p$. There exist (unique) integers m_s , for $s = 0, \dots, f-1$ such that

- i) $b_s + m_{s-1} - pm_s \in \{0, \dots, p-1\}$ for all $s \geq 1$ and $b_0 - pm_0 \in \{0, \dots, p-1\}$;

ii) we have the equality

$$\sum_{s=0}^{f-1} b_s p^s = (b_0 - pm_0) + \sum_{s=0}^{f-1} p^s (b_s + m_{s-1} - pms) + p^{f-1} m_{f-1}. \quad (17)$$

As we work modulo $q - 1$ the equality (17) reads

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm_0 + m_{f-1}) + \sum_{s=0}^{f-1} p^s (b_s + m_{s-1} - pms) \pmod{q-1}.$$

If $b_0 - pm_0 + m_{f-1} \in \{0, \dots, p-1\}$ we get the result. If not, we only have to check that $0 \leq b_0 - pm_0 + m_{f-1} < b_0$ (so that the iteration of the preceding procedure eventually stops). As $-pm_1 + b_1 + m_0 \geq 0$ and $b_1 \leq p-1$ we get $m_1 \leq \frac{p-1+m_0}{p}$ and, inductively, $m_{s+1} \leq \frac{p^{s+1}-1+m_0}{p^{s+1}}$. Thus

$$-pm_0 + m_{f-1} \leq -pm_0 + \frac{p^{f-1} - 1 + m_0}{p^{f-1}} < 0$$

if $m_0 \geq 1$.

For the general case, we notice that there exists unique integers m'_s such that $b_s + m_{s-1} - pms \in \{0, \dots, p-1\}$ for all $s \geq 1$ and $b_0 - m_0 \in \{0, \dots, p-1\}$. As we work modulo $q - 1$ we get

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm_0 + m_{f-1}) + \sum_{s=0}^{f-1} p^s (b_s + m_{s-1} - pms) \pmod{q-1}.$$

and we are in the previous case. \square

7.2 Two rough estimates

In this section we study some elements of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ which appear naturally in the study of the socle filtration for $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ (but the results adapt immediately for the representations R_{n+1}^\pm). In particular, we will be able to have a partial control of the action of $K_0(p^m)$ on $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ (and not only on the graded pieces of the socle filtration).

The following proposition holds for a fixed couple (m, n) of integers such that $0 \leq m \leq n$; for the $m = 0$ case we just have to replace the matrix $\begin{bmatrix} 1 & 0 \\ p^m[\lambda_m] & 1 \end{bmatrix}$ with $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ in the expressions (18) and (19). Finally we recall the definition of the $\overline{\mathbf{F}}_p$ -linear subspace $\mathfrak{W}_{(\underline{l}_m, \dots, \underline{l}_n)}$ of $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ for a given $(n+1-m)f$ -tuple $(\underline{l}_m, \dots, \underline{l}_n) \in \{0, \dots, p-1\}^{n+1-m}$, given in §4.1.2.

PROPOSITION 7.3. *Let $F_{\underline{l}_m, \dots, \underline{l}_n}^{m, n} \in \mathcal{B}$, and $N \stackrel{\text{def}}{=} N_{m, n}(\underline{l}_m, \dots, \underline{l}_n)$. For $m \leq j \leq n$ let $T_j \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{j-1}]$ be a polynomial of degree $\deg(T_j) \leq p^{j-m}$ (where, for $j \in \{0, \dots, n-1\}$, we define λ_{j+m} to be homogeneous of degree p^j), and \underline{i}_j be a f -tuple such that $\underline{i}_j \leq \underline{l}_j$. Finally, fix $M < p^f - 1$. Then the image inside $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/N - M$ of the element x defined as*

$$x \stackrel{\text{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n - \underline{i}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e] \quad (18)$$

is contained in the image inside $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/N - M$ of the subspace

$$\mathfrak{W}_{(\underline{l}_m, \dots, \underline{l}_n)}.$$

Proof. The technique of the proof is very simple: we fix $0 \leq t \leq M$ and $n \in \mathbf{N}$ such that $n(p-1) \leq t < (n+1)(p-1)$. If we write x as a suitable sum of elements $F_{\underline{l}'_m, \dots, \underline{l}'_n}^{m, n}$, the statement is proved if we

check that any such element lying in the antidiagonal $X_0 + \dots + X_{f-1} = N - t$ verifies $x'_j \leq x_j + n$ for all $j = 0, \dots, f-1$ (where, as usual, $(x_0, \dots, x_{f-1}), (x'_0, \dots, x'_{f-1})$ are the coordinates of $F_{\underline{l}_m, \dots, \underline{l}_n}^{m, n}$ via the map (5)).

This is a long computation. If we expand each of the polynomials $T_{m+1}^{\underline{l}_{m+1}}, \dots, T_n^{\underline{l}_n}$, we obtain:

$$\sum_{i \in I} \beta_i \sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)} \begin{bmatrix} 1 & 0 \\ p^m [\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)} \begin{bmatrix} 1 & 0 \\ p^n [\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e] \quad (19)$$

where I is a suitable set of indices, $\beta_i \in \overline{\mathbf{F}}_p$, and the exponents $\kappa_j(i)$ (for $j \in \{m, \dots, n\}$) admit the following explicit description:⁴

$$\kappa_a = p^{\lfloor -1 \rfloor} \kappa_a^{(a+1)} + \dots + p^{\lfloor -(n-a) \rfloor} \kappa_a^{(n)} + \underline{l}_a - \underline{i}_a$$

and (for $a+1 \leq b \leq n$)

$$\kappa_a^{(b)} = \kappa_a^{(b),0} + p \kappa_a^{(b),1} + \dots + p^{f-1} \kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a apperaring in a fixed monomial of $(T_b)^{i_b^{(s)}}$.

Recall that, by the hypothesys on the T_b 's, we have

$$\kappa_m^{(b),s} + p \kappa_{m+1}^{(b),s} + \dots + p^{b-1-m} \kappa_{b-1}^{(b),s} \leq p^{b-m} i_b^{(s)}. \quad (20)$$

Thanks to Lemma 7.1, we have the following inequalities:

$$\mathfrak{s}(\kappa_m) + p \mathfrak{s}(\kappa_{m+1}) + \dots + p^{n-m} \mathfrak{s}(\kappa_n) \leq \quad (21)$$

$$\begin{aligned} &\leq (\mathfrak{s}(\underline{l}_m - \underline{i}_m) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_m^{(m+1)}) + \dots + \mathfrak{s}(p^{\lfloor -(n-m) \rfloor} \kappa_m^{(n)})) + \\ &\quad + p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_{m+1}^{(m+2)}) + \dots + \mathfrak{s}(p^{\lfloor -(n-m-1) \rfloor} \kappa_{m+1}^{(n)})) + \dots \\ &\quad \dots + p^{n-m-1} (\mathfrak{s}(\underline{l}_{n-1} - \underline{i}_{n-1}) + \mathfrak{s}(p^{\lfloor -1 \rfloor} \kappa_{n-1}^{(n)})) + p^{n-m} (\mathfrak{s}(\underline{l}_n - \underline{i}_n)) \leq \end{aligned} \quad (22)$$

$$\begin{aligned} &\leq \mathfrak{s}(\underline{l}_m - \underline{i}_m) + \sum_{s=0}^{f-1} \mathfrak{s}(\kappa_m^{(m+1),s}) + \\ &\quad p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1})) + \left(\sum_{s=0}^{f-1} (\mathfrak{s}(\kappa_m^{(m+2),s}) + p \mathfrak{s}(\kappa_{m+1}^{(m+2),s})) \right) + \dots \\ &\quad \dots + \left(\sum_{s=0}^{f-1} (\mathfrak{s}(\kappa_m^{(n),s}) + p \mathfrak{s}(\kappa_{m+1}^{(n),s}) + \dots + p^{n-m-1} \mathfrak{s}(\kappa_{n-1}^{(n),s})) \right) + p^{n-m} \mathfrak{s}(\underline{l}_n - \underline{i}_n) \leq \end{aligned} \quad (23)$$

$$\leq \mathfrak{s}(\underline{l}_m - \underline{i}_m) + p \mathfrak{s}(\underline{i}_{m+1}) + p \mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \dots + p^{n-m} \mathfrak{s}(\underline{i}_n) + p^{n-m} \mathfrak{s}(\underline{l}_n - \underline{i}_n)$$

where the inequality (23) is deduced from (20) and Lemma 7.1-d).

If we impose our function to lie on the hyperplane $X_0 + \dots + X_{f-1} = t$ we get a ‘‘control’’ on the exponents $\kappa_a^{(b),s}$. More precisely,

i) the inequality (21) give rise to the conditions:

$$\mathfrak{s}(\kappa_a) = \mathfrak{s}(\underline{l}_a - \underline{i}_a) + \mathfrak{s}(\kappa_a^{(a+1)}) + \dots + \mathfrak{s}(\kappa_a^{(n)}) - u_a(p-1)$$

for $a \in \{m, \dots, n-1\}$ and some $u_a \in \mathbf{N}$;

⁴from now on, we fix an index $i \in I$, and we put $\kappa_j \stackrel{\text{def}}{=} \kappa_j(i)$

ii) the inequality (22) give rise to the conditions:

$$\mathfrak{s}(\kappa_a^{(b)}) = \mathfrak{s}(\kappa_a^{(b),0}) + \cdots + \mathfrak{s}(\kappa_a^{(b),f-1}) - w_a^{(b)}(p-1)$$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$ and some $w_a^{(b)} \in \mathbf{N}$;

iii) the inequality (23) give rise to the conditions

$$\mathfrak{s}(\kappa_a^{(b),s}) = \kappa_a^{(b),s} - v_a^{(b),c}(p-1)$$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$, $s \in \{0, \dots, f-1\}$ and some $v_a^{(b),s} \in \mathbf{N}$;

iv) condition $t < (n+1)(p-1)$ imposes finally

$$\sum_{a=m}^{n-1} p^{a-m} u_a + \sum_{a=m}^{n-1} p^{a-m} \left(\sum_{b=a+1}^n w_a^{(b)} \right) + \sum_{a=m}^{n-1} p^{a-m} \left(\sum_{b=a+1}^n \sum_{s=0}^{f-1} v_a^{(b),s} \right) \leq n.$$

First, notice that the condition $n(p-1) < p^f - 1$ imply $k_a^{(b),s} \leq p^f - 1$ for all possible choices of a, b, s (as $\mathfrak{s}(k_a^{(b),s}) \leq \lceil k_a^{(b),s} \rceil$). If $k_a^{(b),s}(i)$, for $i \in \{0, \dots, f-1\}$, are the cyphers of the p -adic development of $\kappa_a^{(b),s}$, we then see that iii) gives the necessary condition

$$\sum_{i=1}^{f-1} \kappa_a^{(b),s}(i) \leq v_a^{(b),s}$$

(indeed, $v_a^{(b),s}$ can uniquely written as $v_a^{(b),s} = \alpha_{a,b,s}(1) + (p+1)\alpha_{a,b,s}(2) + \cdots + \alpha_{a,b,s}(f-1)(1+p + \cdots + p^{f-1})$ for suitable integers $\alpha_{a,b,s}(j)$).

Fix now $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$. Working in $\mathbf{Z}/(p^f - 1)$, we see that

$$\kappa_a^{(b),0} + \cdots + p^{f-1} \kappa_a^{(b),f-1} \equiv \sum_{j=0}^{f-1} p^j (\kappa_a^{(b),0}(j) + \kappa_a^{(b),1}(\lfloor j-1 \rfloor) + \cdots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor)).$$

Using Lemma 7.2 we see that condition ii) let us deduce the p -adic expansion of $\kappa_a^{(b)}$:

$$\begin{aligned} \kappa_a^{(b)}(j) &= \kappa_a^{(b),0}(j) + \cdots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor) - p\alpha_a^{(b)}(j) + \beta_a^{(b)}(j) \\ &= \kappa_a^{(b),j}(0) + \rho_a^{(b)}(j) - p\alpha_a^{(b)}(j) \end{aligned} \quad (24)$$

where the integers $\alpha_a^{(b)}(j)$, $\beta_a^{(b)}(j)$ verify

$$\sum_{j=0}^{f-1} \alpha_a^{(b)}(j) = \sum_{j=0}^{f-1} \beta_a^{(b)}(j) = w_a^{(b)}$$

and

$$\rho_a^{(b)}(j) = \sum_{s \in \{0, \dots, f-1\} \setminus \{j\}_a} \kappa_a^{(b),s}(\lfloor j-s \rfloor) + \beta_a^{(b)}(j) \leq \sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)}.$$

Similarly, condition i) let us deduce the p -adic development of κ_a :

$$\begin{aligned} \kappa_a(j) &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^n \kappa_a^b(\lfloor j+b-a \rfloor) - pA_a(j) + B_a(j) \\ &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^n \kappa_a^{(b),\lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p \left(\sum_{b=a+1}^n \alpha_a^{(b)}(\lfloor j+b-a \rfloor) + A_a(j) \right) \end{aligned}$$

where the integers $A_a(j)$, $B_a(j)$, $\mathfrak{R}_a(j)$ verify

$$\sum_{j=0}^{f-1} A_a(j) = \sum_{j=0}^{f-1} B_a(j) = u_a$$

and

$$\mathfrak{R}_a(j) = \sum_{b=a+1}^n \rho_a^{(b)}(\lfloor j+b-a \rfloor) + B_a(j) \leq u_a + \sum_{b=a+1}^n \left(\sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)} \right).$$

We finally have all the ingredients to give the rough estimate of the statement. We fix a “coordinate” j . A straightforward but tedious computation gives

$$\begin{aligned} \sum_{a=m}^n p^{a-m} \kappa_a(j) &= \sum_{a=m}^n p^{a-m} (l_a^{(j)} - i_a^{(j)}) + \sum_{b=a+1}^n \kappa_a^{(b), \lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p \mathfrak{A}_a(j) \\ &= x_j - \sum_{a=m}^n i_a^{\lfloor j+a-m \rfloor} + \sum_{b=m+1}^n \sum_{a=m}^{b-1} p^{a-m} \kappa_a^{(b), \lfloor j+b-m \rfloor} + \sum_{a=m}^{n-1} p^{a-m} \mathfrak{R}_a(j) - p \left(\sum_{a=m}^{n-1} \mathfrak{A}_a(j) \right). \end{aligned}$$

The conclusion follows as

$$\sum_{a=m}^{n-1} p^{a-m} \mathfrak{R}_a(j) \leq \sum_{a=m}^{n-1} p^{a-m} \left(u_a + \sum_{b=a+1}^n w_a^{(b)} \right) + \sum_{b=a+1}^n \sum_{s=0}^{f-1} v_a^{(b),s} \leq n$$

and

$$\sum_{a=m}^{b-1} \kappa_a^{(b),s}(0) \leq p^{b-m} i_b^{(s)}$$

for any $b \in \{m+1, \dots, n\}$ and $s \in \{0, \dots, f-1\}$. \square

The following rough estimate will help us to understand the action of $\begin{bmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. of $\begin{bmatrix} 1 & 0 \\ p\mathcal{O}_F & 1 \end{bmatrix}$) on the representations in §4.1 (resp. §4.2). Apparently, the result is unsatisfactory if we want to describe the K -socle filtration for the representations $\pi(x, \lambda, 1)$, unless we impose some conditions, depending on p , on the residue degree f (we expect a condition of the form $f \leq \frac{p+1}{2}$).

PROPOSITION 7.4. *Let $1 \leq m \leq n$ be integers and consider $F_{l_m, \dots, l_n}^{m,n} \in \mathcal{B}$; let $N \stackrel{\text{def}}{=} N_{m,n}(l_m, \dots, l_n)$. For $2m \leq j \leq n$ let $V_j \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{j-1}]$ be a pseudo-homogeneous polynomial of degree $\deg(V_j) \leq p^j - p^m(p^m - 2)$ and i_j be a f -tuple such that $i_j \leq l_j$. Finally, fix $M < p^m - 2$ and define $V_j \stackrel{\text{def}}{=} 1$, $i_j = \underline{0}$ for $m \leq j \leq 2m - 1$.*

The image inside $\text{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/N - M$ of the element x defined as

$$x \stackrel{\text{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{l_j - i_j} (V_{j+1}^{\frac{1}{p^{j+1}}})^{i_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j [\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{l_n - i_n} \begin{bmatrix} 1 & 0 \\ p^n [\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

coincides with the image of $F_{l_m, \dots, l_n}^{(m,n)}$.

Proof. The idea of the proof is completely analogous of that of Proposition 7.3 the main difference being that here we are *not* able to give an estimate of the coordinates of the points appearing in the development of x .

As in 7.3 we consider an element appearing in the development of x :

$$\sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e].$$

The exponents κ_a (for $a \in \{m, \dots, n\}$) admit the following explicit description:

$$\kappa_a = p^{\lfloor -1 \rfloor} \kappa_a^{(a+1)} + \cdots + p^{\lfloor n-a \rfloor} \kappa_a^{(n)} + \underline{l}_a - \underline{i}_a$$

and (for $a+1 \leq b \leq n$)

$$\kappa_a^{(b)} = \kappa_a^{(b),0} + p\kappa_a^{(b),1} + \cdots + p^{f-1}\kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a appearing in a fixed monomial of $(V_b)^{i_b^{(s)}}$.

As each V_a is pseudo-homogeneous, for each triple (a, b, s) we have

$$\kappa_a^{(b),s} = \beta_a^{(b),s}(1) + \cdots + p^{f-1}\beta_a^{(b),s}(f)$$

where the integers $\beta_a^{(b),s}(j)$ verify

$$\sum_{j=1}^f \beta_m^{(b),s}(j) + p \left(\sum_{j=1}^f \beta_{m+1}^{(b),s}(j) \right) + \cdots + p^{b-m-1} \sum_{j=1}^f \beta_{b-1}^{(b),s}(j) \leq (p^{b-m} - (p^m - 2))i_b^{(s)}.$$

As for the inequalities (21), (22), (23), we use Lemma 7.1 to obtain

$$\sum_{a=m}^n p^{a-m} \mathfrak{s}(\kappa_a) \leq N - (p^m - 2) \left(\sum_{a=2m}^n \mathfrak{s}(\underline{i}_a) \right)$$

and the conclusion follows. \square

We state an analogous result in the case $m = 0$.

PROPOSITION 7.5. *Let $n \geq 0$ and $F_{\underline{l}_0, \dots, \underline{l}_{n+1}}^{(0,n)} \in \mathcal{B}_{n+1}^+$; let $N \stackrel{\text{def}}{=} N_{0,n+1}(\underline{l}_0, \dots, \underline{l}_{n+1})$. For $1 \leq h \leq n+1$ let $V_h \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{h-1}]$ be a pseudo homogeneous polynomial of degree $p^h - (p-2)$ and $\underline{i}_h \leq \underline{l}_h$ be an f -tuple. We finally fix $M \in \{0, \dots, p-3\}$ and put $\underline{i}_0 \stackrel{\text{def}}{=} \underline{0}$, $V_{n+2} \stackrel{\text{def}}{=} 1$.*

The image inside $(\text{Ind}_{K_0(p^{n+2})}^K 1)^+ / N - M$ of the element

$$x \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0 - \underline{i}_0} (V_1^{\frac{1}{p}})^{\underline{i}_1} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{j=1}^{n+1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, e]$$

coincides with the image of $F_{\underline{l}_0, \dots, \underline{l}_{n+1}}^{(0,n)}$.

Proof. The proof is completely analogous to the proof of Proposition 7.4 and is left to the reader (see [Mo1] for details). \square

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