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# Multiplicity theorems modulo $p$ for $\mathrm{GL}_2(\mathbf{Q}_p)$

Stefano Morra

## ABSTRACT

Let  $F$  a nonarchimedean local field,  $\pi, \rho$  admissible irreducible representations of  $\mathrm{GL}_n(F)$ ,  $\mathrm{GL}_{n-1}(F)$  respectively. Aizenbud, Gourvitch, Rallis and Schiffmann have recently proved ([AGRS]) that the space of  $\mathrm{Hom}_{\mathrm{GL}_{n-1}(F)}(\pi, \rho)$  is at most 1-dimensional and by the works of Waldspurger and Mœglin ([MW]) such dimension is related to a local factor of the couple  $(\pi, \rho)$ . In the case  $n = 2$  this phenomenon is known by the works of Tunnell and Saito ([Tun], [Sai]).

In this paper we describe the restriction to Cartan subgroups of irreducible mod  $p$  representations  $\pi$  of  $\mathrm{GL}_2(\mathbf{Q}_p)$ . In particular we deduce a mod  $p$  multiplicity theorem, giving the dimension of  $\mathrm{Hom}_{L^\times}(\pi, \chi)$  where  $L/\mathbf{Q}_p$  is a quadratic extension and  $\chi$  a smooth character of  $L^\times$ .

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## 1. Introduction, Notations and Preliminaries

Let  $F$  be a non-archimedean local field of characteristic 0,  $V$  a finite dimensional  $F$ -vector space endowed with a non degenerated quadratic form  $q$ . If  $E$  is a non isotropic line and  $W$  is the  $E$ -orthogonal in  $V$  we write  $G, H$  for the special orthogonal group of  $V$  and  $W$  respectively. If  $\pi, \rho$  are irreducible admissible complex representations of  $G(F), H(F)$ , the Gross-Prasad conjectures (cf. [GP1], [GP2]) predict a precise relation between  $\dim(\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \rho))$  and the epsilon factor of the pair associated to  $(\pi, \rho)$ . In this direction, Aizenbud, Gourvitch, Rallis and Schiffmann ([AGRS]) proved a “multiplicity one” result:

**THEOREM 1.1** ([AGRS], Theorem 1 and Theorem 1’). *In the previous hypothesis we have*

$$\dim(\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \rho)) \leq 1.$$

Moreover, recent works of Waldspurger and Mœglin ([W1], [W2], [W3], [MW]) proved the Gross-Prasad conjecture in a large number of cases.

Let us consider the situation for  $\mathrm{GL}_2$ . The Gross-Prasad conjecture is then a particular case of a result of Tunnell and Saito (cf. [Tun], [Sai]). More precisely, let  $\pi$  be an infinite dimensional irreducible admissible representation of  $\mathrm{GL}_2(F)$  and  $\sigma_\pi$  the associated representation of the Weil group  $W_F$  (via the local Langlands correspondence). For a quadratic extension  $L$  of  $F$  we fix an  $L^\times$  character  $\chi$  which extends the central character  $\omega_\pi$  of  $\pi$  (as usual,  $\chi$  will be considered as a character of the Weil group  $W_L$  by local class field theory). Finally, we fix an embedding  $L^\times \hookrightarrow \mathrm{GL}_2(F)$  and an additive character  $\psi$  of  $F$ , letting  $\psi_L \stackrel{\mathrm{def}}{=} \psi \circ \mathrm{tr}_{L/F}$ .

The theorem is then the following (see also [Pra1], Theorem 1.1 and remark) :

**THEOREM 1.2** (Tunnell, Saito). *In the previous hypothesis, the conditions*

- i)  $\dim(\mathrm{Hom}_L^\times(\pi|_{L^\times}, \chi)) \neq 0$
- ii)  $\varepsilon(\sigma_\pi|_{W_L} \otimes \chi^{-1}, \psi_L)\omega_\pi(-1) = 1$

*are equivalent.*

Indeed, the problem of looking for multiplicities of  $L^\times$ -characters in  $\pi|_{L^\times}$  goes back to a work of Silberger ([Sil]) and has been approached again in the works of Tunnell [Tun] and Prasad [Pra]. In particular, the Tunnell-Saito theorem appears again in [Rag], where Raghuram gives explicit sufficient conditions for a character to appear in the  $\pi|_{L^\times}$  for  $\pi$  supercuspidal.

In this paper we approach the mod  $p$ -analogue of such problems in the case  $F = \mathbf{Q}_p$ . We describe the structure of  $\pi|_{L^\times}$  where  $\pi$  is an admissible irreducible smooth representation with coefficient in  $\overline{\mathbf{F}}_p$ . We rely on the works [Mo1], [Mo2], where we gave a detailed description of the Iwahori and  $\mathrm{GL}_2(\mathbf{Z}_p)$ -structure for irreducible admissible  $\mathrm{GL}_2(\mathbf{Q}_p)$ -representations. In all what follows, we will assume  $p \geq 5$ .

The results when  $\pi$  is supersingular (§2 and §3) can be summed up in the following theorem. Recall that supersingular representations for  $\mathrm{GL}_2(\mathbf{Q}_p)$  are parametrised, up to twist, by the universal representations  $\pi(r, 0, 1)$  where  $r \in \{0, \dots, p-1\}$  (see the end of the paragraph for the precise definition of the representations  $\pi(r, 0, 1)$ ).

**THEOREM 1.3** (Corollary 2.9 and Proposition 3.5). *Let  $L/\mathbf{Q}_p$  be a quadratic extension,  $r \in \{0, \dots, p-1\}$  and consider the universal representation  $\pi(r, 0, 1)$ . Write  $\mathrm{soc}_{L^\times}^{(j)} \stackrel{\mathrm{def}}{=} \mathrm{soc}_{L^\times}^{(j)}(\pi(r, 0, 1)|_{L^\times})$  for the  $j$ -th composition factor of the  $L^\times$ -socle filtration for  $\pi(r, 0, 1)|_{L^\times}$ .*

*Then:*

- i) *if  $L/\mathbf{Q}_p$  is unramified we have an isomorphism of  $k_L^\times$ -representations*

$$\mathrm{soc}_{L^\times}^{(j)} / \mathrm{soc}_{L^\times}^{(j-1)} \cong (\oplus_{i=0}^p \eta_i(r))^2$$

*where  $\eta_i(r)$ , for  $i = 0, \dots, p$ , are the  $(p+1)$  distinct characters of  $k_L^\times$  extending the character  $x \mapsto x^r$  on  $k_{\mathbf{Q}_p}$ ;*

- ii) *if  $L/\mathbf{Q}_p$  is totally ramified we have an isomorphism of  $L^\times$ -representations*

$$\mathrm{soc}_{L^\times}^{(j)} / \mathrm{soc}_{L^\times}^{(j-1)} \cong (V)^{2-\delta_{0,j}}$$

*where  $V$  is a two dimensional vector space, endowed with the  $\mathcal{O}_L^\times$  action inflated from the  $k_{\mathbf{Q}_p}$ -character  $x \mapsto x^r$  and the action of the uniformiser given by a nontrivial involution.*

We can actually prove a stronger result, namely the fact that  $\pi(r, 0, 1)|_{L^\times}$  decomposes into a direct sum of  $2(p+1)$ -uniserial representations if  $L/\mathbf{Q}_p$  is unramified, and “almost” in a direct sum

of two uniserial representations if  $L/\mathbf{Q}_p$  is totally ramified. The reader is invited to see Corollary 2.9 and Proposition 3.5 for more details.

Suitable modifications in the proofs of the supersingular case let us deduce the structure of principal and special series (§4):

**THEOREM 1.4** (Proposition 4.2 and 4.5). *Let  $r \in \{0, \dots, p-2\}$ ,  $\lambda \in \overline{\mathbf{F}}_p^\times$  and consider the unramified principal series  $\pi_{r,\lambda} \stackrel{\text{def}}{=} \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbf{Q}_p)} un_\lambda \otimes un_{\lambda^{-1}} \omega^r$ , where  $B$  is the subgroup of upper triangular matrices,  $\omega$  is the cyclotomic character and  $un_\lambda$  is the unramified character defined by  $un_\lambda(p) = \lambda$ .*

*If  $L/\mathbf{Q}_p$  is a quadratic extension write  $\mathrm{soc}_{L^\times}^{(j)} \stackrel{\text{def}}{=} \mathrm{soc}_{L^\times}^{(j)}(\pi_{r,\lambda}|_{L^\times})$  for the  $j$ -th composition factor of the  $L^\times$ -socle filtration for  $\pi_{r,\lambda}|_{L^\times}$ . Then*

i) *if  $L/\mathbf{Q}_p$  is unramified we have an isomorphism of  $k_L^\times$ -representations*

$$\mathrm{soc}_{L^\times}^{(j)} / \mathrm{soc}_{L^\times}^{(j-1)} \cong \bigoplus_{i=0}^p \eta_i(r)$$

*where  $\eta_i(r)$ , for  $i = 0, \dots, p$ , are the  $(p+1)$  distinct characters of  $k_L^\times$  extending the character  $x \mapsto x^r$  on  $k_{\mathbf{Q}_p}$ ;*

ii) *if  $L/\mathbf{Q}_p$  is totally ramified we have an isomorphism of  $L^\times$ -representations*

$$\mathrm{soc}_{L^\times}^{(j)} / \mathrm{soc}_{L^\times}^{(j-1)} \cong V$$

*where  $V$  is a two dimensional vector space, endowed with the  $\mathcal{O}_L^\times$  action inflated from the  $k_{\mathbf{Q}_p}$ -character  $x \mapsto x^r$  and the action of the uniformiser given by a nontrivial involution.*

Once again, we can actually prove that in the unramified case the restriction  $\pi_{r,\lambda}|_{L^\times}$  decomposes into a direct sum of  $(p+1)$  uniserial representations with isomorphic Jordan-Hölder factors (cf. Proposition 4.2).

As a consequence, we get a mod  $p$  analogous of the Aizenbud-Gourevitch-Rallis-Schiffmann multiplicity one theorem:

**THEOREM 1.5** (Corollaries 2.10, 3.6, 4.3, 4.6). *Let  $L/\mathbf{Q}_p$  be a quadratic extension,  $\pi$  an infinite dimensional smooth admissible representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$  modulo  $p$  and  $\chi$  a smooth character of  $L^\times$  extending the central character  $\omega_\pi$  of  $\pi$ .*

*Then*

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

We briefly introduce the essential notation for the paper (see also [Bre]).

For a finite extension  $L$  of  $\mathbf{Q}_p$  we write  $\mathcal{O}_L$  to denote its ring of integers and  $k_L$  the residue field. We write  $[\cdot] : \mathbf{F}_p^\times \rightarrow W(\mathbf{F}_p)^\times$  for the Teichmüller character. We put as usual  $G \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbf{Q}_p)$ ,  $K \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbf{Z}_p)$  and  $Z \stackrel{\text{def}}{=} Z(G)$  for the center of  $G$ .

We fix for the rest of the paper an integer  $r \in \{0, \dots, p-1\}$ . The algebraic representation  $\sigma_r \stackrel{\text{def}}{=} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2$  (endowed with the modular action of  $\mathrm{GL}_2(\mathbf{F}_p)$ ) is seen as a representation of  $KZ$  (by inflation and making  $p$  act trivially) so that we can consider the compact induction

$$c\text{-Ind}_{KZ}^G \sigma_r.$$

In [Ba-Li], Proposition 8-(1) Barthel and Livné showed the existence of a canonical Hecke operator  $T \in \mathrm{End}_G(c\text{-Ind}_{KZ}^G \sigma_r)$  (depending on  $r$ ) which realizes an isomorphism of the  $\overline{\mathbf{F}}_p$ -algebra of endomorphisms  $\mathrm{End}_G(c\text{-Ind}_{KZ}^G \sigma_r)$  with the ring of polynomials in one variable over  $\overline{\mathbf{F}}_p$ . We then define the universal representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$  as the cokernel of the canonical operator  $T$ :

$$\pi(r, 0, 1) \stackrel{\text{def}}{=} \mathrm{coker}(T).$$

As recalled above, the computations of Breuil ([Bre]) show that all such representations are irreducible and, up to twist by a character of  $\mathbf{Q}_p$ , exhaust all the supersingular representations for  $\mathrm{GL}_2(\mathbf{Q}_p)$ .

Let  $H$  be the maximal torus of  $\mathrm{GL}_2(\mathbf{F}_p)$  and  $t \in \{0, \dots, p-2\}$ . We have the characters

$$\begin{aligned} \chi_t : H &\rightarrow \overline{\mathbf{F}}_p & \mathfrak{a} : H &\rightarrow \overline{\mathbf{F}}_p \\ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} &\mapsto a^t & \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} &\mapsto ad^{-1}. \end{aligned}$$

If  $\chi$  is a character of  $H$  we will write  $\chi^s$  to denote its conjugate. Finally any character of  $H$  will be considered as a character of  $B(\mathbf{F}_p)$  or (by inflation) as a character of  $K_0(p) = I$ , without any commentary.

We recall (see [Mo1]) that the  $\mathrm{GL}_2(\mathbf{Z}_p)$ -restriction of the supersingular representation  $\pi(r, 0, 1)$  admits a decomposition

$$\pi(r, 0, 1)|_{\mathrm{GL}_2(\mathbf{Z}_p)} \xrightarrow{\sim} \underbrace{\varinjlim_{n \text{ odd}} (R_0 \oplus R_1 \cdots \oplus R_n R_{n+1})}_{\stackrel{\text{def}}{=} \pi_r} \oplus \underbrace{\varinjlim_{m \text{ even}} (R_1/R_0 \oplus R_2 \cdots \oplus R_m R_{m+1})}_{\stackrel{\text{def}}{=} \pi_{p-1-r}} \quad (1)$$

(see [Mo1], §2 for the precise definition of the representations  $R_n$ ).

For any  $n \geq 0$  (resp.  $n \geq 1$ ) we define the following elements of  $\varinjlim (\cdots \oplus R_n R_{n+1})$ .

If  $i \in \{1, \dots, r\}$  and  $(l_0, \dots, l_n) \in \{0, \dots, p-1\}^{n+1}$  is an  $(n+1)$ -tuple (resp.  $(l_1, \dots, l_n) \in \{0, \dots, p-1\}^n$  is an  $n$ -tuple) we define the following elements of  $R_{n+1}$ :

$$F_{l_0, \dots, l_n}^{(0, n)}(i) \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-i} Y^i]$$

(resp.

$$F_{l_1, \dots, l_n}^{(1, n)}(i) \stackrel{\text{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-i} Y^i]).$$

For  $i = 0$  we define similarly  $F_{l_0, \dots, l_n}^{(0, n)}(i)$  and  $F_{l_1, \dots, l_n}^{(1, n)}(i)$ , where we add the additional condition  $l_n \geq r+1$  (in particular for  $r = p-1$  such functions are not considered).

We formally define

$$\begin{aligned} F_{\emptyset}^{(0, -1)}(i) &\stackrel{\text{def}}{=} X^i Y^{r-i} \in R_0; \\ F_{\emptyset}^{(1, 0)}(i) &\stackrel{\text{def}}{=} [1, X^{r-i} Y^i] \in R_1; \\ F_{\emptyset}^{(-1, 0)} &\stackrel{\text{def}}{=} Y^r \in R_0. \end{aligned}$$

From the results of [Mo1] it follows that:

**PROPOSITION 1.6.** *An  $\overline{\mathbf{F}}_p$ -basis for the representation  $\pi_r$  is given by the elements  $F_{l_j, \dots, l_n}^{(j, n)}(i)$  described above, for  $n \geq 1$  odd,  $j \in \{0, 1\}$  and the elements  $F_{\emptyset}^{(0, -1)}(i)$ .*

*An  $\overline{\mathbf{F}}_p$ -basis for the representation  $\pi_{p-1-r}$  is given by the elements  $F_{l_j, \dots, l_n}^{(j, n)}(i)$  described above for  $n \geq 1$  even,  $j \in \{0, 1\}$ , the elements  $F_{\emptyset}^{(1, 0)}(i)$ ,  $F_{l_0}^{(0, 0)}(i)$  for  $r \geq i \geq 1$ ,  $l_0 \in \{0, \dots, p-1\}$  and the elements  $F_{l_0}^{(0, 0)}(0)$  for  $l_0 \in \{r, \dots, p-1\}$ .*

Notice that for  $j \in \{0, 1\}$  the injective map

$$F_{l_j, \dots, l_n}^{(j, n)}(i) \mapsto \sum_{s=j}^n p^{s-j} l_s + p^{n+1-j} i \in \mathbf{N}$$

provides the set of functions  $F_{l_0, \dots, l_n}^{(0, n)}(i)$  (resp.  $F_{l_1, \dots, l_n}^{(1, n)}(i)$ ) with a natural linear ordering; we will then write  $F_{l_{j-1}, \dots, l_n}^{(j, n)}(i)$  to mean the antecedent of  $F_{l_j, \dots, l_n}^{(j, n)}(i)$ .

The direct summand  $\pi_r$  (resp.  $\pi_{p-1-r}$ ) in the decomposition (1) admits a  $\mathrm{GL}_2(\mathbf{Z}_p)$ -filtration  $\{S_n(r)\}_{n \in \mathbf{N}}$  (resp.  $\{S_n(p-1-r)\}_{n \in \mathbf{N}}$ ) with respect to which the extensions between two consecutive graded pieces can be summarized as follow:

$$\mathrm{Sym}^r \overline{\mathbf{F}}_p^2 \text{---} \mathrm{Ind}_{K_0(p)}^K \langle [1, F_{r+1}^{(1,1)}(0)] \rangle \text{---} \dots \text{---} \mathrm{Ind}_{K_0(p)}^K \langle [1, F_{l_1, \dots, l_n}^{(1, n)}(i)] \rangle \text{---} \dots \quad (2)$$

(resp.

$$\mathrm{Sym}^{p-1-r} \overline{\mathbf{F}}_p^2 \otimes \det^r \text{---} \mathrm{Ind}_{K_0(p)}^K \langle [1, F_{\emptyset}^{(1,0)}(1)] \rangle \text{---} \dots \text{---} \mathrm{Ind}_{K_0(p)}^K \langle [1, F_{l_1, \dots, l_n}^{(1, n)}(i)] \rangle \text{---} \dots) \quad (3)$$

where

$$\mathrm{Ind}_{K_0(p)}^K \langle [1, F_{l_1, \dots, l_n}^{(1, n)}(i)] \rangle \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_r^s \mathfrak{a}^{i + \sum_{j=1}^n l_j}.$$

Moreover we have, for  $a, b, c, d \in \mathbf{Z}_p$ ,

$$\begin{bmatrix} 1 + pa & pb \\ pc & 1 + pd \end{bmatrix} [1, F_{l_1, \dots, l_n}^{(1, n)}(i)] = F_{l_1, \dots, l_n}^{(1, n)}(i) - \bar{c} \kappa(l_1, \dots, l_n, i) (F_{l_1-1, \dots, l_n}^{(1, n)}(i)) + y \quad (4)$$

where  $y$  is a suitable sum of functions strictly preceding  $F_{l_1-1, \dots, l_n}^{(1, n)}(i)$  in the natural ordering on  $F_{l_1, \dots, l_n}^{(1, n)}(i)$  and  $\kappa(l_1, \dots, l_n, i) \in \mathbf{F}_p^\times$ .

Let  $L$  be a quadratic extension of  $\mathbf{Q}_p$ . Fixing a  $\mathbf{Z}_p$ -base  $\mathcal{B}$  of  $\mathcal{O}_L$  gives us the embeddings groups:

$$\begin{array}{ccc} L^\times \cong \mathrm{Aut}_L(L) & \xhookrightarrow{\iota_{\mathcal{B}}} & \mathrm{GL}_2(\mathbf{Q}_p) \\ \uparrow & & \uparrow \\ \mathcal{O}_L^\times \cong \mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_L) & \xhookrightarrow{\iota_{\mathcal{B}'}} & \mathrm{GL}_2(\mathbf{Z}_p) \end{array}$$

and therefore we can study the structure of  $\pi(r, 0, 1)|_{L^\times}$ . Note that such a structure does not depend on the choice of the basis as the subgroups  $\iota_{\mathcal{B}}(L^\times)$ ,  $\iota_{\mathcal{B}'}(L^\times)$ , for  $\mathcal{B}, \mathcal{B}'$  two  $\mathbf{Z}_p$ -bases of  $\mathcal{O}_L$ , are conjugated in  $\mathrm{GL}_2(\mathbf{Q}_p)$ .

## 2. The unramified case

Throughout this section, we will assume  $L/\mathbf{Q}_p$  unramified. The main result is Proposition 2.9, which gives the  $L^\times$  structure for the representation  $\pi(r, 0, 1)$ . After an analysis of the finite case in §2.2 (which is made possible by the filtration (3)) we give the key result (lemma 2.3) which let us glue together the characters appearing between two consecutive graded pieces of the filtration (3). This will enable us to detect the socle filtration for  $\pi(r, 0, 1)|_{L^\times}$ , and an elementary observation (Lemma 2.5) gives us a full description of the extensions between two two consecutive graded pieces of the socle filtration.

### 2.1 Preliminaries and notations

Fix  $\alpha \in \mathbf{F}_{p^2}$ , a  $(p^2-1)$ -th primitive root of unity; its minimal polynomial over  $\mathbf{F}_p$  is  $X^2 - Tr(\alpha) + N(\alpha)$  where  $Tr, N$  denotes respectively the trace and norm of  $\mathbf{F}_{p^2}$  over  $\mathbf{F}_p$ . Thus, we get a  $\mathbf{Z}_p$ -basis

$\mathcal{B} \stackrel{\text{def}}{=} \{1, [\alpha]\}$  of  $\mathcal{O}_L$  and a  $\mathbf{Z}_p$ -linear isomorphism

$$\mathcal{O}_L \cong \mathbf{Z}_p \otimes \mathbf{Z}_p[\alpha].$$

We see that  $\iota_{\mathcal{B}}(p) = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$  and such element acts trivially on  $\pi(r, 0, 1)$ : in order to study  $\pi(r, 0, 1)|_{L^\times}$  we can therefore content ourself to study the restriction

$$\pi(r, 0, 1)|_{\mathcal{O}_L^\times}.$$

Finally, let  $x, y \in \mathbf{Z}_p$  be such that

$$[\alpha^2] = [-N(\alpha)] + [\alpha][Tr(\alpha)] + px + p[\alpha]y.$$

It follows, for  $a, b \in \mathbf{Z}_p$  such that  $a + [\alpha]b \in \mathcal{O}_L^\times$ , that

$$\iota_{\mathcal{B}}(a + [\alpha]b) = \begin{bmatrix} a & b[-N(\alpha)] + pxb \\ b & a + b[Tr(\alpha)] + pby \end{bmatrix}.$$

## 2.2 The finite case

Let  $l, m \in \mathbf{Z}$  be integers. We consider

$$V_{l,m} \stackrel{\text{def}}{=} \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} \chi_l^s \otimes \det^m$$

as a  $K$ -representation via the inflation map  $K \rightarrow \text{GL}_2(\mathbf{F}_p)$ . One verifies that the restriction

$$V_{l,m}|_{\mathcal{O}_L^\times}$$

is naturally isomorphic to the  $\mathcal{O}_L^\times$ -representation obtained -via the inflation map  $\mathcal{O}_L^\times \rightarrow \mathbf{F}_{p^2}^\times$ - from

$$V_{l,m}|_{\mathbf{F}_{p^2}^\times}.$$

The object of this subsection is to describe the representation  $V_{l,m}|_{\mathbf{F}_{p^2}^\times}$ .

As the group  $\mathbf{F}_{p^2}^\times$  is abelian, and  $|\mathbf{F}_{p^2}^\times|$  is coprime with  $p$  it follows that  $V_{l,m}|_{\mathbf{F}_{p^2}^\times}$  decomposes in a direct sum of characters. Moreover, Mackey decomposition gives us an isomorphism of  $\mathbf{F}_{p^2}^\times$ -representations

$$V_{l,m}|_{\mathbf{F}_{p^2}^\times} \xrightarrow{\sim} \text{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times} (\cdot)^{-l} \otimes N^{m+l}.$$

We give below the explicit description of such isomorphism.

Define the following permutation  $\sigma$  of  $\{0, \dots, p-1, \infty\}$ . For  $\lambda_0 \in \{0, \dots, p-1\}$  we set

$$\begin{aligned} \sigma(\lambda_0) &\stackrel{\text{def}}{=} -\frac{N(\alpha)}{\lambda_0 + Tr(\alpha)} \quad \text{if } \lambda_0 \neq -Tr(\alpha); \\ \sigma(\lambda_0) &\stackrel{\text{def}}{=} \infty \quad \text{if } \lambda_0 = -Tr(\alpha); \end{aligned}$$

and

$$\sigma(\infty) \stackrel{\text{def}}{=} 0$$

In other words, we are considering the projective transformation on  $\mathbf{P}^1(\mathbf{F}_p)$  associated to the matrix  $\begin{bmatrix} Tr(\alpha) & 1 \\ -N(\alpha) & 0 \end{bmatrix}$ . We moreover define a map  $x(\cdot) : \{0, \dots, p-1, \infty\} \rightarrow \mathbf{F}_p$  by

$$\begin{aligned} x(\lambda_0) &\stackrel{\text{def}}{=} \lambda_0 + Tr(\alpha) \quad \text{if } \lambda_0 \notin \{-Tr(\alpha), \infty\}; \\ x(-Tr(\alpha)) &\stackrel{\text{def}}{=} -N(\alpha); \\ x(\infty) &\stackrel{\text{def}}{=} 1. \end{aligned}$$

Recall that a  $\overline{\mathbf{F}}_p$ -basis for  $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{-l} \otimes N^{m+l}$  is described by

$$\mathcal{B} = \left\{ [\lambda_0 + \alpha, e] \text{ for } \lambda_0 \in \mathbf{F}_p; \quad [1, e] \right\}.$$

We finally consider the lemma

LEMMA 2.1. *We have an  $\mathbf{F}_{p^2}^\times$ -equivariant isomorphism defined by:*

$$\begin{aligned} V_{l,m}|_{\mathbf{F}_{p^2}^\times} &\xrightarrow{\sim} \mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^{-l} \otimes N^{m+l} \\ \left[ \begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix}, e \right] &\mapsto [\lambda_0 + \alpha, e]; \\ \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e \right] &\mapsto [\alpha, e](-1)^l. \end{aligned}$$

*Proof.* The  $\overline{\mathbf{F}}_p$ -linear morphism of the statement is clearly an isomorphism and we claim it is  $\mathbf{F}_{p^2}^\times$ -equivariant. It is enough to check the compatibility of the isomorphism with the action of  $\alpha$ , on a fixed base of  $V_{l,m}|_{\mathbf{F}_{p^2}^\times}$ . A direct computation gives

$$\begin{aligned} \begin{bmatrix} 0 & -N(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{bmatrix} \left[ \begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix}, e \right] &= \begin{cases} (N(\alpha))^m (-\sigma(\lambda_0))^l \left[ \begin{bmatrix} \sigma(\lambda_0) & 1 \\ 1 & 0 \end{bmatrix}, e \right] & \text{if } \lambda_0 \neq -\mathrm{Tr}(\alpha); \\ (N(\alpha))^m [1, e] & \text{if } \lambda_0 = -\mathrm{Tr}(\alpha); \end{cases} \\ \begin{bmatrix} 0 & -N(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{bmatrix} [1, e] &= (N(\alpha))^m (-N(\alpha))^l \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e \right] \end{aligned}$$

and

$$\alpha[\lambda_0 + \alpha, e] = (N(\alpha))^{m+l} (x(\lambda_0))^{-l} [\sigma(\lambda_0) + \alpha, e].$$

The conclusion follows.  $\square$

2.2.1 *Study of  $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^l$ .* Let  $l \in \{0, \dots, p-1\}$ . The  $\mathbf{F}_{p^2}^\times$ -representation  $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_{p^2}^\times}(\cdot)^l$  decomposes into a direct sums of  $p+1$ -characters, and these characters are precisely all the  $p+1$ -possible extensions of  $\lambda \mapsto \lambda^l$  to  $\mathbf{F}_{p^2}^\times$ .

If  $s_0, s_1 \in \{0, \dots, p-1\}$  are such that  $(s_0, s_1) \neq (p-1, p-1)$  then the  $\mathbf{F}_{p^2}^\times$ -character defined by

$$\alpha \mapsto \alpha^{s_0 + ps_1}$$

extends  $(\cdot)^l$  if and only if

$$(s_0 + s_1) + p(s_0 + s_1) \equiv l + pl \pmod{p^2 - 1}$$

that is if and only if the couple  $(s_0, s_1)$  verify one of the following relations:

$$s_0 + s_1 = l \quad s_0 + s_1 = p - 1 + l.$$

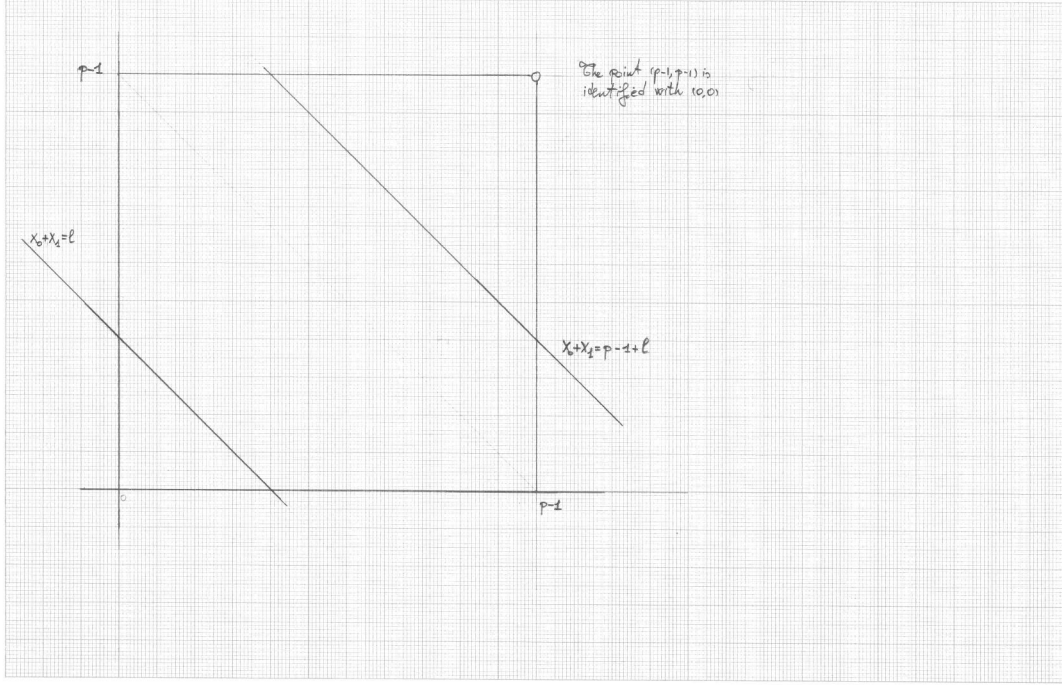
We will say that  $(s_0, s_1)$  is an *admissible couple for  $l$* . This can be effectively summed up by figure III.1.

For an admissible couple  $(s_0, s_1)$  we let

$$v^{(s_0, s_1)} \stackrel{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0, s_1)} [\lambda_0 + \alpha, e] + \mu_\infty^{(s_0, s_1)} [1, e]$$

be an eigenvector for the action of  $\mathbf{F}_{p^2}^\times$ , of associated eigencharacter  $(\cdot)^{s_0 + ps_1}$ . The scalars  $\mu_{\lambda_0}^{(s_0, s_1)}$ ,  $\mu_\infty^{(s_0, s_1)}$  admit the following description:




 FIGURE 1. The combinatoric of admissible couples for  $l$ .

LEMMA 2.2. Let  $(s_0, s_1)$  be an admissible couple for  $l$  and let  $n \in \{0, \dots, p\}$ . Then

$$\mu_{\sigma^{-n}(0)}^{(s_0, s_1)} = \mu_0^{(s_0, s_1)} \alpha^{n(s_0 + ps_1)} (x(\sigma^{-1}(0)) \cdots x(\sigma^{-n}(0)))^{-l}.$$

*Proof.* It is enough to study the action of  $\alpha$  on  $v^{(s_0, s_1)}$ . A computation gives:

$$\begin{aligned} \alpha v^{(s_0, s_1)} &= \sum_{\lambda_0 \notin \{-Tr(\alpha), \infty\}} \mu_{\lambda_0}^{(s_0, s_1)} (x(\lambda_0))^l [\sigma(\lambda_0) + \alpha, e] + \\ &\quad + \mu_{-Tr(\alpha)}^{(s_0, s_1)} (x(-Tr(\alpha)))^l [1, e] + \mu_{\infty}^{(s_0, s_1)} [\alpha, e] \end{aligned}$$

so that, assuming that  $v^{(s_0, s_1)}$  is an eigenvector of associated eigencaracter  $(\cdot)^{s_0 + ps_1}$  we get the relations:

$$\begin{aligned} \mu_{\lambda_0}^{(s_0, s_1)} (x(\lambda_0))^l &= \alpha^{s_0 + ps_1} \mu_{\sigma(\lambda_0)}^{s_0 + ps_1} \text{ if } \lambda_0 \notin \{-Tr(\alpha), \infty\}; \\ \mu_{-Tr(\alpha)}^{(s_0, s_1)} (x(-Tr(\alpha)))^l &= \alpha^{s_0 + ps_1} \mu_{\infty}^{(s_0, s_1)}; \\ \mu_{\infty}^{(s_0, s_1)} &= \alpha^{s_0 + ps_1} \mu_0. \end{aligned}$$

The result is trivial for  $n = 0$  and follows immediately for  $n = 1$  (as  $\sigma^{-1}(0) = \infty$ , and  $x(\infty) = 1$ ). The general case follows by induction.  $\square$

### 2.3 Extensions inside irreducible representations

We are now ready to study the  $\mathcal{O}_L^\times$ -restriction of the supersingular representation  $\pi(r, 0, 1)$ . To fix the ideas, we will consider the summand  $\pi_r$  appearing in the decomposition (1), and the associated filtration  $\{S_n(r)\}_{n \in \mathbb{N}}$ ; not to overload the notations, we will write  $S_n$  instead of  $S_n(r)$ .

We first give the structure of  $(\pi_r/\mathrm{Sym}^r \overline{\mathbf{F}}_p^2)|_{\mathcal{O}_L^\times}$ ; after some remarks in §2.4 we will state the main result (Corollary 2.9).

For  $[1, F_{l_1-1, \dots, l_n}^{(1,n)}(i)] \neq Y^r$  consider the extension

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} [1, F_{l_1-1, \dots, l_n}^{(1,n)}(i)] \text{---} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} [1, F_{l_1, \dots, l_n}^{(1,n)}(i)] \quad (5)$$

where we recall that

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} [1, F_{l_1-\epsilon, \dots, l_n}^{(1,n)}(i)] \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_{r-2m+2\epsilon}^s \otimes \det^{m-\epsilon} \xrightarrow{\sim} \mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_p^\times} (\cdot)^{2m-2\epsilon-r} \otimes N^{r-n+\epsilon}$$

for  $\epsilon \in \{0, 1\}$ ,  $m \stackrel{\mathrm{def}}{=} i + \sum_{j=1}^n l_j$ . Let  $(s_0, s_1)$  an admissible couple for  $2m - r$  and let

$$v^{(s_0, s_1)}(l_1, \dots, l_n, i) \stackrel{\mathrm{def}}{=} v^{(s_0, s_1)}(F_{l_1, \dots, l_n}^{(1,n)}(i)) \in \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} [1, F_{l_1, \dots, l_n}^{(1,n)}(i)]$$

be an eigenvector for the action of  $\mathbf{F}_p^\times$ , of associated eigencharacter  $(\cdot)^{s_0+ps_1} N^{r-m}$ . We have the following result:

LEMMA 2.3. *Inside the extension (5) we have the following equality, for any  $x', y' \in \mathbf{Z}_p$ :*

$$\begin{aligned} \begin{bmatrix} 1 & p[N(\alpha)] + p^2 x' \\ p & 1 + p[\mathrm{Tr}(\alpha)] + p^2 y' \end{bmatrix} v^{(s_0, s_1)}(l_1, \dots, l_n, i) &= v^{(s_0, s_1)}(l_1, \dots, l_n, i) + \\ &+ N(\alpha) \kappa(l_1 - 1, \dots, l_n, i) v^{(s_0-1, s_1-1)}(l_1, \dots, l_n, i) \end{aligned}$$

where  $s_i - 1$  are the cyphers of the  $p$ -adic development of  $s_0 + ps_1 - (p+1) \bmod p^2 - 1$  and  $\kappa(l_1, \dots, l_n, i) \in \mathbf{F}_p^\times$ .

*Proof.* We rely crucially on the behaviour of the functions  $F_{l_1, \dots, l_n}^{(1,n)}(i)$  described in (4). Thanks to the isomorphism of Lemma 2.1 we can write:

$$v^{(s_0, s_1)}(l_1, \dots, l_n, i) = \sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0, s_1)} \left[ \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}, F_{l_1, \dots, l_n}^{(1,n)}(i) \right] + (-1)^{r-2m} \mu_{\infty}^{(s_0, s_1)} [1, F_{l_1, \dots, l_n}^{(1,n)}(i)]$$

where we may assume, without loss of generality, that  $\mu_0^{(s_0, s_1)} = 1$ ; moreover we have the matrix equality

$$\begin{bmatrix} 1 & p[N(\alpha)] + p^2 x' \\ p & 1 + p[\mathrm{Tr}(\alpha)] + p^2 y' \end{bmatrix} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + p* & p* \\ p[-\lambda_0^2 - \lambda_0 \mathrm{Tr}(\alpha) - N(\alpha)] + p^2* & 1 + p* \end{bmatrix}$$

where we do not care about the value of  $* \in \mathbf{Z}_p$  thanks to the relation given in (4). As the roots of  $P(X) \stackrel{\mathrm{def}}{=} X^2 + \mathrm{Tr}(\alpha)X + N(\alpha)$  are  $-\alpha, -\alpha^p$ , we notice that  $-P(\lambda_0) \neq 0$  and we get

$$\begin{aligned} \begin{bmatrix} 1 & p[N(\alpha)] + p^2 x' \\ p & 1 + p[\mathrm{Tr}(\alpha)] + p^2 y' \end{bmatrix} v^{(s_0, s_1)}(l_1, \dots, l_n, i) &= v^{(s_0, s_1)}(l_1, \dots, l_n, i) + \\ \kappa(l_1, \dots, l_n, i) \left( \sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0, s_1)} P(\lambda_0) \left[ \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}, F_{l_1-1, \dots, l_n}^{(1,n)}(i) \right] \right. &+ (-1)^{r-2m} \mu_{\infty}^{(s_0, s_1)} [1, F_{l_1-1, \dots, l_n}^{(1,n)}(i)] \left. \right). \end{aligned}$$

We have of course

$$\begin{aligned} P(0) &= N(\alpha); \\ (-1)^{r-2m} \mu_{\infty}^{(s_0, s_1)} &= (-1)^{r-2m} \alpha^{s_0+ps_1} = (-1)^{r-2m} \alpha^{s_0-1+p(s_1-1)} N(\alpha) = (-1)^{r-2m} \mu_{\infty}^{(s_0-1, s_1-1)} N(\alpha) \end{aligned}$$

and we are left to prove that

$$\mu_{\sigma^{-n}(0)}^{(s_0, s_1)} P(\sigma^{-n}(0)) = N(\alpha) \mu_{\sigma^{-n}(0)}^{(s_0-1, s_1-1)}$$

where  $2 \leq n \leq p$ .

This will be done by induction on  $n$ , the case  $n = 1$  being proved; for a notational convenience, we put  $P(\infty) \stackrel{\text{def}}{=} 1$ . Assume the result true for  $n - 1$ ; if  $i \stackrel{\text{def}}{=} \sigma^{-(n-1)}(0)$  we then have

$$\begin{aligned} \mu_{\sigma^{-n}(0)}^{(s_0, s_1)} P(\sigma^{-n}(0)) &= \mu_{\sigma^{-1}(i)}^{(s_0, s_1)} P(\sigma^{-1}(i)) \\ &= \mu_i^{(s_0, s_1)} \alpha^{(s_0 + ps_1)} P(\sigma^{-1}(i)) (x(\sigma^{-1}(i)))^{r-2m} \\ &= \mu_i^{(s_0-1, s_1-1)} N(\alpha) (P(i))^{-1} \alpha^{(s_0 + ps_1)} P(\sigma^{-1}(i)) (x(\sigma^{-1}(i)))^{r-2m+2} (x(\sigma^{-1}(i)))^{-2} \\ &= N(\alpha) \underbrace{\mu_i^{(s_0-1, s_1-1)} x(\sigma^{-1}(i))^{r-2m+2} \alpha^{s_0-1+ps_1-1}}_{\mu_{\sigma^{-1}(i)}^{(s_0-1, s_1-1)}} N(\alpha) P(i)^{-1} P(\sigma^{-1}(i)) (x(\sigma^{-1}(i)))^{-2}. \end{aligned}$$

To conclude the induction is then enough to show that

$$N(\alpha) (x(i))^{-2} P(i) = P(\sigma(i)).$$

A direct computation gives, for  $\sigma(i) \notin \{0, \infty\}$ :

$$P(\sigma(i)) = \left( -\frac{N(\alpha)}{i + Tr(\alpha)} \right)^2 - \frac{Tr(\alpha)N(\alpha)}{i + Tr(\alpha)} + N(\alpha) = N(\alpha) x(i)^{-2} P(i).$$

The remaining are formal: if  $\sigma(i) = \infty$  we get  $i = -Tr(\alpha)$ ,  $x(i) = -N(\alpha)$ ,  $P(\infty) = 1$  and  $P(-Tr(\alpha)) = N(\alpha)$ ; if finally  $\sigma(i) = 0$  then  $i = \infty$ ,  $x(\infty) = 1$ ,  $P(\infty) = 1$  and  $P(0) = N(\alpha)$ . This ends the inductive step and the proof is complete.  $\square$

REMARK 2.4. *It is immediate to see that the statement of Lemma 2.3 can be improved as follow: if  $a \in \mathcal{O}_L$ ,  $b \in \mathcal{O}_L^\times$  and*

$$g \equiv \begin{bmatrix} 1 + p[\bar{a}] & p[-\bar{b}N(\alpha)] \\ p[\bar{b}] & 1 + p[\bar{b}T(\alpha) + \bar{a}] \end{bmatrix} \pmod{p^2}$$

then

$$gv^{(s_0, s_1)}(l_1, \dots, l_n, i) = v^{(s_0, s_1)}(l_1, \dots, l_n, i) + \kappa(l_1, \dots, l_n, i) N(\alpha) \bar{b} v_{l_1-1, \dots, l_n, i}^{(s_0-1, s_1-1)}.$$

The next lemma let us deduce that for any  $n \geq 1$  we have a  $\mathcal{O}_L^\times$ -splitting

$$S_n/S_0 = \bigoplus_{i=0}^p \bar{S}_n(\eta_i)$$

where  $\eta_i$  are the  $(p+1)$ -characters of  $\mathbf{F}_p^\times$  extending the  $\mathbf{F}_p^\times$  character  $x \mapsto x^r$  and each representation  $\bar{S}_n(\eta_i)$  is uniserial, of length  $n$ , having Jordan-Hölder factors isomorphic to  $\eta_i$

LEMMA 2.5. *Let  $\chi_1 \neq \chi_2$  be smooth  $\mathcal{O}_L^\times$ -characters and  $T$  an uniserial  $\mathcal{O}_L^\times$ -representation, having Jordan-Hölder factors isomorphic to  $\chi_1$ . Then*

$$\text{Ext}_{\mathcal{O}_L^\times}^n(\chi_2, T) = 0 = \text{Ext}_{\mathcal{O}_L^\times}^n(T, \chi_2)$$

for all  $n \in \mathbf{N}$ .

*Proof.* As the characters  $\chi_1, \chi_2$  are distincts, it exists  $g_0 \in \mathcal{O}_L^\times$  such that  $\chi_1(g_0) \neq \chi_2(g_0)$ . As the group  $\mathcal{O}_L^\times$  is commutative, the maps

$$\begin{aligned} f_1 : \chi_1 &\rightarrow \chi_1 \\ e_{\chi_1} &\mapsto g_0 \cdot e_1 - \chi_1(g_0) e_{\chi_1} \end{aligned}$$

and

$$\begin{aligned} f_2 : \chi_2 &\rightarrow \chi_2 \\ e_{\chi_2} &\mapsto g_0 \cdot e_2 - \chi_1(g_0) e_{\chi_2} \end{aligned}$$

are  $\mathcal{O}_L^\times$ -equivariant morphisms and it is immediate to see that  $f_1$  is the zero morphism while  $f_2$  is an isomorphism.

By functoriality, the maps  $f_1, f_2$  induce natural morphisms

$$\mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_1, \chi_2) \xrightarrow{(f_1)^*} \mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_1, \chi_2) \quad \text{and} \quad \mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_1, \chi_2) \xrightarrow{(f_2)^*} \mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_1, \chi_2)$$

and we check that  $(f_1)_* = (f_2)^*$ . Thus  $\mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_1, \chi_2) = 0$ . By dévissage on the length of  $T$  we get  $\mathrm{Ext}_{\mathcal{O}_L^\times}^n(T, \chi_2) = 0$ . The result on  $\mathrm{Ext}_{\mathcal{O}_L^\times}^n(\chi_2, T)$  follows analogously.  $\square$

We deduce then

**COROLLARY 2.6.** *The  $\mathcal{O}_L^\times$ -representation  $(\pi_r/\mathrm{Sym}^r \overline{\mathbf{F}}_p^2)|_{\mathcal{O}_L^\times}$  admits a natural splitting*

$$(\pi_r/\mathrm{Sym}^r \overline{\mathbf{F}}_p^2)|_{\mathcal{O}_L^\times} \cong \bigoplus_{i=0}^p \overline{\mathcal{S}}_\infty(\eta_i)$$

where  $\eta_i$  are the  $(p+1)$ -characters of  $\mathbf{F}_{p^2}^\times$  extending the  $\mathbf{F}_p^\times$  character  $x \mapsto x^r$  and each representation  $\overline{\mathcal{S}}_\infty(\eta_i)$  is uniserial, of infinite length, having Jordan-Hölder factors isomorphic to  $\eta_i$ .

We have an analogous result concerning the  $\mathcal{O}_L^\times$ -representation  $(\pi_{p-1-r}/\mathrm{Sym}^{p-1-r} \overline{\mathbf{F}}_p^2 \otimes \det^r)|_{\mathcal{O}_L^\times}$ .

*Proof.* We will prove the result for  $(\pi_r/\mathrm{Sym}^r \overline{\mathbf{F}}_p^2)|_{\mathcal{O}_L^\times}$ .

The natural filtration  $\{S_n\}_{n \in \mathbf{N}}$  on  $\pi_r$  induces a natural filtration  $\{\overline{S}_n\}_{n \in \mathbf{N}}$  defined by  $\overline{S}_n \stackrel{\mathrm{def}}{=} S_n/\mathrm{Sym}^r \overline{\mathbf{F}}_p^2$  for all  $n \in \mathbf{N}$ . We show that

- i) for any  $n \in \mathbf{N}$  the representation  $\overline{S}_n$  splits into a direct sum of  $(p+1)$  uniserial representations of length  $n$

$$\overline{S}_n = \bigoplus_{i=0}^p \overline{S}_n(\eta_i)$$

where the  $\eta_i$  are as in the statement and each  $\overline{S}_n(\eta_i)$  has Jordan-Hölder factors isomorphic to  $\eta_i$

- ii) the splitting of i) is compatible with the filtration, i.e.  $\overline{S}_{n-1}(\eta_i) \leq \overline{S}_n(\eta_i)$  for any  $n \in \mathbf{N}$ .

This is an induction on  $n$ , the case  $n = 1$ , i.e.  $\overline{S}_1 \cong \mathrm{Ind}_{B(\mathbf{F}_{p^2})}^{\mathrm{GL}_2(\mathbf{F}_{p^2})} \chi_r \alpha^{r+1}$ , being clear by §2.2.1. Assume i) and ii) true for  $n$ . For each  $i_0 \in \{0, \dots, p\}$  we consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=0}^p \overline{S}_n(\eta_i) & \longrightarrow & \overline{S}_{n+1} & \longrightarrow & \bigoplus_{i=0}^p \eta_i \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \bigoplus_{i=0}^p \overline{S}_n(\eta_i) & \longrightarrow & W_{i_0, n+1} & \longrightarrow & \eta_{i_0} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \bigoplus_{i \neq i_0} \overline{S}_n(\eta_i) & \longrightarrow & \overline{W}_{i_0, n+1} & \longrightarrow & \eta_{i_0} \longrightarrow 0 \end{array} \quad (6)$$

where  $W_{i_0, n+1}$  is the inverse image of  $\eta_{i_0}$  through the natural projection  $\overline{S}_{n+1} \rightarrow \bigoplus_{i=0}^p \eta_i$  and  $\overline{W}_{i_0, n+1} \stackrel{\mathrm{def}}{=} W_{i_0, n+1}/\overline{S}_n(\eta_{i_0})$ . Thanks to the Yoneda interpretation of  $\mathrm{Ext}^n$  and Lemma 2.5 the lower line of (6) splits. Therefore, by generality of  $i_0$ , we get a  $\mathcal{O}_L^\times$ -equivariant splitting

$$\overline{S}_{n+1} \cong \bigoplus_{i=0}^p \overline{S}_{n+1}(\eta_i)$$

which is compatible with the splitting  $\overline{S}_n \cong \bigoplus_{i=0}^p \overline{S}_n(\eta_i)$  coming from the inductive hypothesis and each  $\overline{S}_{n+1}(\eta_i)$  has Jordan-Hölder factors isomorphic to  $\eta_i$ . It remains to show that each  $\overline{S}_{n+1}(\eta_i)$  is uniserial but this follows from Lemma 2.3  $\square$

In terms of figure 1 the meaning of Corollary 2.6 is clear and illustrated in figure 2.

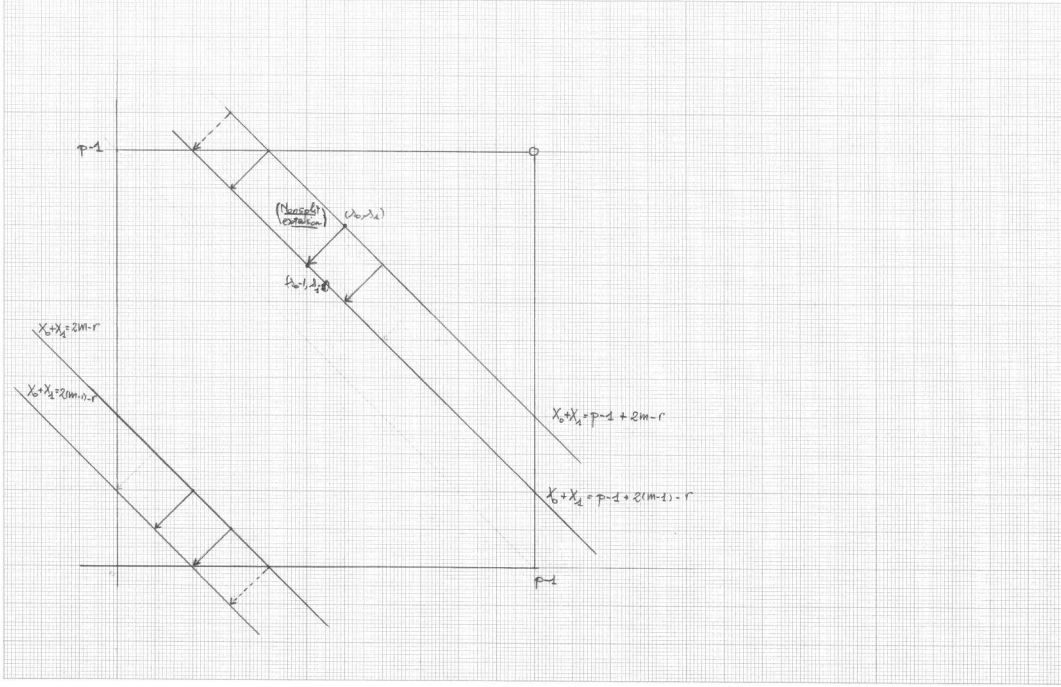


FIGURE 2. A graphic gloss of Corollary 2.6.

## 2.4 Conclusion

We are now left to treat the extensions between the first two graded pieces of the filtration (3). More precisely, for  $r \in \{1, \dots, p-2\}$  such extensions are described by

$$\text{Cosoc}(\text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} Y^r|_{B(\mathbf{F}_p)}) \longrightarrow \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, F_{r+1}^{(1)}(0)] \quad (7)$$

and

$$\text{Cosoc}(\text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, X^r]) \longrightarrow \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, X^{r-1}Y]. \quad (8)$$

respectively for the  $K$ -representations  $\varinjlim_{n \text{ odd}} (R_0 \oplus R_1 \cdots \oplus R_n R_{n+1})$  and  $\varinjlim_{m \text{ even}} (R_1/R_0 \oplus R_2 \cdots \oplus R_m R_{m+1})$ . For  $r = 0$  we have analogously

$$(\text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} e|_{B(\mathbf{F}_p)}) / \text{Sym}^{p-1} \longrightarrow \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, F_{r+1}^{(1)}(0)] \quad (9)$$

$$(\text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, e]) / \mathbf{1} \longrightarrow \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} [1, F_{0,1}^{(1,2)}(e)]. \quad (10)$$

We start with the following lemma.

LEMMA 2.7. *Let  $l, m \in \{0, \dots, p-2\}$  and let  $V \stackrel{\text{def}}{=} \text{Ind}_{B(\mathbf{F}_p)}^{\text{GL}_2(\mathbf{F}_p)} \chi_l^s \det^m$ . Then*

- i) *the  $\mathbf{F}_{p^2}^\times$ -restriction of the socle  $\text{soc}(V)|_{\mathbf{F}_{p^2}^\times}$  (resp.  $\text{St} \otimes \det^m|_{\mathbf{F}_{p^2}^\times}$  if  $l = 0$ ) decomposes as the direct sum of the characters  $(\cdot)^{s_0 + ps_1}$ , where  $(s_0, s_1)$  are the admissible couple for  $l$  lying on the line  $X_0 + X_1 = (p-1) + l$ ;*
- ii) *the  $\mathbf{F}_{p^2}^\times$ -restriction of the cosocle  $\text{cosoc}(V)|_{\mathbf{F}_{p^2}^\times}$  (resp.  $1 \otimes \det^m|_{\mathbf{F}_{p^2}^\times}$  if  $l = 0$ ) decomposes as the*

direct sum of the characters  $(\cdot)^{s_0+ps_1}$ , where the  $(s_0, s_1)$  are the admissible couple for  $l$  lying on the line  $X_0 + X_1 = l$ .

*Proof.* Up to twist by powers of  $\det$  we may assume  $V = \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_{p-1-r}^s \det^r$  for a suitable  $r \in \{1, \dots, p-1\}$ . It is now enough to show that

$$\mathrm{cosoc}(V)|_{\mathbf{F}_{p^2}^\times} \cong \mathrm{Sym}^r(\overline{\mathbf{F}}_p^2)$$

decomposes as the direct sum of the characters  $(\cdot)^{s_0+ps_1}$  where  $s_0 + ps_1 = r$ ; this will imply that  $\mathrm{soc}(V)|_{\mathbf{F}_{p^2}^\times}$  decomposes as the direct sum of the characters  $(\cdot)^{s_0+ps_1}$  where  $s_0 + ps_1 = (p-1) + r$ .

For  $r = 1$ , the action of  $\mathrm{GL}_2(\mathbf{F}_p)$  on  $\mathrm{Sym}^1(\overline{\mathbf{F}}_p) \cong \overline{\mathbf{F}}_p^2$  is the natural one. The linear automorphism  $\phi_\alpha \in \mathrm{End}(\mathrm{Sym}^1(\overline{\mathbf{F}}_p))$  associated to the action of  $\alpha \in \mathbf{F}_{p^2}^\times$  has spectrum  $\mathcal{S} = \{\alpha, \alpha^p\}$  (indeed, in a  $\overline{\mathbf{F}}_p$ -basis of  $\mathrm{Sym}^1(\overline{\mathbf{F}}_p)$ , the associated matrix is  $\phi_\alpha \stackrel{\mathrm{def}}{=} \begin{bmatrix} 0 & -N(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{bmatrix}$ ). This gives the case  $r = 1$ .

Let now  $\mathcal{B} \stackrel{\mathrm{def}}{=} \{v_1, v_2\}$  be a basis of eigenvectors for  $\phi_\alpha$ . If we define, for  $j \in \{0, \dots, r\}$ ,

$$v_j \stackrel{\mathrm{def}}{=} \bigvee_{i=1}^j v_1 \vee \bigvee_{i=j+1}^r v_2 \in \mathrm{Sym}^r(\overline{\mathbf{F}}_p^2),$$

then the family  $\mathcal{B}_r \stackrel{\mathrm{def}}{=} \{v_j, 0 \leq j \leq r\}$  is obviously a basis of eigenvectors for the automorphism  $\mathrm{Sym}(\phi_\alpha)$  such that the eigenvalue associated to  $v_j$  is  $\alpha^{j+p(r-j)}$ . This gives the result.  $\square$

Combining Propositions 2.3, 2.5 and Lemma 2.7 we get the main result

**PROPOSITION 2.8.** *Let  $r \in \{0, \dots, p-2\}$  and  $n \in \mathbf{N}$*

*The  $\mathcal{O}_L^\times$ -restriction of the representation  $S_n(r)$  admits an equivariant splitting*

$$S_n(r)|_{\mathcal{O}_L^\times} \cong \bigoplus_{i=0}^p S_n(r)(\eta_i)$$

where  $\eta_i$  are the  $(p+1)$ -characters of  $\mathbf{F}_{p^2}^\times$  extending the  $\mathbf{F}_p^\times$  character  $x \mapsto x^r$  and each representation  $\overline{S}_\infty(\eta_i)$  is uniserial having Jordan-Hölder factors isomorphic to  $\eta_i$ . Moreover, the length of  $S(r)_n(\eta_i)$  is  $n+1$  if the character  $\eta_i$  corresponds to an admissible couple  $(s_0, s_1)$  such that  $s_0 + s_1 = r$  and it is  $n$  otherwise.

Similarly,

$$S_n(p-1-r)|_{\mathcal{O}_L^\times} \cong \bigoplus_{i=0}^p S_n(p-1-r)(\eta_i)$$

where, this time, the length of  $S_n(p-1-r)(\eta_i)$  is  $n+1$  if the character  $\eta_i$  corresponds to an admissible couple  $(s_0, s_1)$  such that  $s_0 + s_1 = p-1-r$  and it is  $n$  otherwise.

*Proof.* We will deal with the representations  $S_n(r)$ , the proof be analogous for  $S_n(p-1-r)$ .

We argue by induction on  $n$ . The case  $n = 0$  is clear from Lemma 2.7, as  $S(r)_0 = \mathrm{Sym}^r \overline{\mathbf{F}}_p^2 = \mathrm{Cosoc}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} Y^r|_{B(\mathbf{F}_p)})$ . The general case follows by the very same arguments of Corollary 2.6: the details are left to the reader.  $\square$

We deduce the structure of the  $L^\times$  restriction for a supersingular representation  $\pi(r, 0, 1)|_{L^\times}$ :

**COROLLARY 2.9.** *Let  $r \in \{0, \dots, p-1\}$ . The  $L^\times$ -restriction of the supersingular representation  $\pi(r, 0, 1)$  admits a splitting*

$$\pi(r, 0, 1) \cong (\bigoplus_{i=0}^p S_\infty(\eta_i))^2$$

where  $\eta_i$  are the  $(p+1)$ -characters of  $\mathbf{F}_{p^2}^\times$  extending the  $\mathbf{F}_p^\times$  character  $x \mapsto x^r$  and each  $\bar{S}_\infty(\eta_i)$  is an infinite length uniserial representation of  $L^\times$ , with a trivial action of  $p$ , having Jordan-Hölder factors isomorphic to  $\eta_i$ .

*Proof.* This is deduced from Propositions 2.8. □

Another corollary is a multiplicity result:

**COROLLARY 2.10.** *Let  $\pi$  be a supersingular representation and  $\chi : L^\times \rightarrow \bar{\mathbf{F}}_p^\times$  a smooth character of  $L^\times$  extending the central character  $\omega_\pi$  of  $\pi$ . Then*

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

*Proof.* Twisting by a power of the cyclotomic character, we can assume  $\pi \cong \pi(r, 0, 1)$  for a suitable  $r \in \{0, \dots, p-1\}$ . As  $\chi$  extends the central character of  $\pi(r, 0, 1)$  we see that  $\chi$  is unramified, and  $\chi|_{\mathcal{O}_L^\times}$  is defined as the  $\mathcal{O}_L^\times$ -inflation of an  $\mathbf{F}_{p^2}^\times$ -character extending the  $\mathbf{F}_p^\times$ -character  $x \mapsto x^r$ .

Thanks to Corollary 2.9 we can content ourselves with the space  $\mathrm{Hom}_{L^\times}(S_\infty(\bar{\chi}), \chi)$  where  $S_\infty(\bar{\chi})$  is one of the smooth  $L^\times$ -representation appearing in the decomposition of Corollary 2.9, having Jordan-Hölder factors isomorphic to  $\bar{\chi}$ .

But

$$\mathrm{Hom}_{L^\times}(S_\infty(\bar{\chi}), \chi) \cong \mathrm{Hom}_{L^\times}(S_\infty(\bar{\chi}) \otimes \chi^{-1}, \bar{\mathbf{F}}_p) = \mathrm{Hom}_{L^\times}((S_\infty(\bar{\chi}) \otimes \chi^{-1})_{L^\times}, \bar{\mathbf{F}}_p)$$

where we wrote  $(S_\infty(\bar{\chi}) \otimes \chi^{-1})_{L^\times}$  for the space of  $L^\times$  co-invariants of  $S_\infty(\bar{\chi}) \otimes \chi^{-1}$ . It will therefore be enough to show that the space of  $1+p\mathcal{O}_L$  co-invariants of  $S_\infty(\bar{\chi})$  is zero (as  $1+p\mathcal{O}_L$ , being a pro- $p$  subgroup of  $L^\times$ , acts trivially on  $\chi^{-1}$ ). By Lemma 2.3 we know that  $S_\infty(\bar{\chi})|_{1+p\mathcal{O}_L}$  is uniserial, with one dimensional Jordan-Hölder factors: it follows that  $(S_\infty(\bar{\chi}))_{1+p\mathcal{O}_L} = 0$  and the the proof is complete. □

### 3. The ramified case

We assume now  $L/\mathbf{Q}_p$  totally ramified. The structure of  $\pi(r, 0, 1)|_{L^\times}$  is much easier to deduce and actually follows almost immediately as a particular case of the results of [Mo2] (which describes the Iwahori structure of the universal representations of  $\mathrm{GL}_2(F)$  for  $F/\mathbf{Q}_p$  unramified). The main result is Proposition 3.5, giving the  $L^\times$ -structure for the supersingular representation  $\pi(r, 0, 1)$ .

#### 3.1 Preliminaries and notations

We consider the  $\mathbf{Z}_p$ -base  $\mathcal{B} \stackrel{\mathrm{def}}{=} \{\varpi, 1\}$  of  $\mathcal{O}_L$ , where  $\varpi \in \mathcal{O}_L$  is a fixed uniformiser (as remarked before the choice of a basis is influential for the statement of Proposition 3.5). Since  $\varpi^2 = p$  and  $p$  acts trivially on  $\pi(r, 0, 1)$  we see that  $\iota_{\mathcal{B}}(\varpi)$  is an involution: we can first content ourself to study the restriction

$$\pi(r, 0, 1)|_{\mathcal{O}_L^\times}.$$

We notice that

$$\iota_{\mathcal{B}}(\mathcal{O}_L^\times) = \left\{ \begin{bmatrix} a & b \\ pb & a \end{bmatrix}, \quad a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p \right\}$$

is a subgroup of the Iwahori  $I$  of  $\mathrm{GL}_2(\mathbf{Z}_p)$ . The structure of  $\pi(r, 0, 1)|_I$  is known (cf. [Mo2]) and is described by the following two propositions.

**PROPOSITION 3.1.** *Let  $r \in \{0, \dots, p-1\}$  and consider the restriction  $\pi(r, 0, 1)|_I$ . We have the*

following  $I$ -equivariant exact sequences

$$\begin{aligned} 0 \rightarrow \langle (Y^r, -Y^r) \rangle &\rightarrow \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus \varinjlim_{n \text{ odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \rightarrow \\ &\rightarrow \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \langle (F_r^{(0)}(0), [1, X^r]) \rangle &\rightarrow \varinjlim_{n \text{ even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \oplus \varinjlim_{n \text{ even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \rightarrow \\ &\rightarrow \varinjlim_{n \text{ even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) \rightarrow 0. \end{aligned}$$

*Proof.* It is a particular case of [Mo2].  $\square$

PROPOSITION 3.2. *We have an  $I$ -equivariant filtration on each of*

$$\varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+); \quad \varinjlim_{n \text{ even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \quad (11)$$

$$\varinjlim_{n \text{ odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-); \quad \varinjlim_{n \text{ even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-). \quad (12)$$

The extensions between two consecutive graded pieces for the representations (11) (resp. (12)) is described as

$$\langle F_{l_0-1, \dots, l_n}^{(0,n)}(i) \rangle \longrightarrow \langle F_{l_0, \dots, l_n}^{(0,n)}(i) \rangle \quad (13)$$

(resp.

$$\langle F_{l_0-1, \dots, l_n}^{(1,n)}(i) \rangle \longrightarrow \langle F_{l_0, \dots, l_n}^{(1,n)}(i) \rangle) \quad (14)$$

where  $\langle F_{l_0, \dots, l_n}^{(0,n)}(i) \rangle \cong \chi_r(\mathfrak{a}^{-1})^{i+\sum_{j=0}^n l_j}$  (resp.  $\langle F_{l_0-1, \dots, l_n}^{(1,n)}(i) \rangle \cong \chi_r^s \mathfrak{a}^{i+\sum_{j=1}^n l_j}$ ).

Finally, for  $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in I$  and  $p \geq 5$  we have the following equality in the extension (13) and (14):

$$\begin{bmatrix} a & b \\ pc & d \end{bmatrix} F_{l_0, \dots, l_n}^{(0,n)}(i) = \chi_r(\mathfrak{a}^{-1})^{i+\sum_{j=0}^n l_j}(g)(F_{l_0, \dots, l_n}^{(0,n)}(i) - \bar{b}\kappa_0(l_1, \dots, l_n, i)F_{l_0-1, \dots, l_n}^{(0,n)}(i));$$

$$\begin{bmatrix} a & b \\ pc & d \end{bmatrix} F_{l_1, \dots, l_n}^{(1,n)}(i) = \chi_r^s(\mathfrak{a})^{i+\sum_{j=1}^n l_j}(g)(F_{l_1, \dots, l_n}^{(1,n)}(i) - \bar{c}\kappa_1(l_1, \dots, l_n, i)F_{l_1-1, \dots, l_n}^{(1,n)}(i))$$

where  $\kappa_0(l_0, \dots, l_n, i), \kappa_1(l_1, \dots, l_n, i) \in \mathbf{F}_p^\times$ .

*Proof.* Using the Iwahori decomposition for  $I$  we get

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ pz' & 1 \end{bmatrix} \begin{bmatrix} \alpha' & 0 \\ 0 & \beta' \end{bmatrix} \begin{bmatrix} 1 & x' \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where  $\bar{\alpha} = \bar{\alpha}' = \bar{a}$ ,  $\bar{\beta} = \bar{\beta}' = \bar{d}$ ,  $\bar{x} = bd^{-1}$ ,  $\bar{x}' = cd^{-1}$ . The result follows from [Mo2].  $\square$

As we can always find  $\begin{bmatrix} a & b \\ pb & a \end{bmatrix} \in \iota_{\mathcal{B}}(\mathcal{O}_L^\times)$  with  $b \in \mathbf{Z}_p^\times$  we deduce immediately from Propositions 3.1 and 3.2 the required structure of  $\pi(r, 0, 1)|_{\mathcal{O}_L^\times}$ :



PROPOSITION 3.3. *Let  $r \in \{0, \dots, p-1\}$ ; the restriction  $\pi(r, 0, 1)|_{\mathcal{O}_L^\times}$  is described as follow. We have two exact,  $\mathcal{O}_L^\times$ -equivariant sequences:*

$$\begin{aligned} 0 \rightarrow \langle (Y^r, -Y^r) \rangle |_{\mathcal{O}_L^\times} &\rightarrow \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) |_{\mathcal{O}_L^\times} \oplus \varinjlim_{n \text{ odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) |_{\mathcal{O}_L^\times} \rightarrow \\ &\rightarrow \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) |_{\mathcal{O}_L^\times} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \langle (F_r^{(0)}(0), [1, X^r]) \rangle |_{\mathcal{O}_L^\times} &\rightarrow \varinjlim_{n \text{ even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) |_{\mathcal{O}_L^\times} \oplus \varinjlim_{n \text{ even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) |_{\mathcal{O}_L^\times} \rightarrow \\ &\rightarrow \varinjlim_{n \text{ even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) |_{\mathcal{O}_L^\times} \rightarrow 0. \end{aligned}$$

We have an  $\mathcal{O}_L^\times$ -equivariant filtration on each of

$$\varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) |_{\mathcal{O}_L^\times}; \quad \varinjlim_{n \text{ even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) |_{\mathcal{O}_L^\times} \quad (15)$$

$$\varinjlim_{n \text{ odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) |_{\mathcal{O}_L^\times}; \quad \varinjlim_{n \text{ even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) |_{\mathcal{O}_L^\times} \quad (16)$$

such that the extension determined by two consecutive graded pieces of the representations in (15) and (16) is nonsplit and admits a description

$$\langle F_{l_0-1, \dots, l_n}^{(0,n)}(i) \rangle \longrightarrow \langle F_{l_0, \dots, l_n}^{(0,n)}(i) \rangle \quad \text{where} \quad \langle F_{l_0, \dots, l_n}^{(0,n)}(i) \rangle \cong \chi_r(\mathfrak{a}^{-1})^{i + \sum_{j=0}^n l_j} |_{\mathbf{F}_p^\times};$$

$$\langle F_{l_0-1, \dots, l_n}^{(1,n)}(i) \rangle \longrightarrow \langle F_{l_0, \dots, l_n}^{(1,n)}(i) \rangle \quad \text{where} \quad \langle F_{l_1, \dots, l_n}^{(1,n)}(i) \rangle \cong \chi_r^s(\mathfrak{a})^{i + \sum_{j=1}^n l_j} |_{\mathbf{F}_p^\times}.$$

### 3.2 Conclusion

We are now left to study the involution  $\iota_{\mathcal{B}}(\varpi) = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$ . As

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \begin{bmatrix} p^{n+1} & \sum_{j=0}^n p^j [\lambda_j] \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sum_{j=0}^n p^{j+1} [\lambda_j] & p^{n+2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sum_{j=1}^n p^j [\lambda_j] & p^{n+1} \end{bmatrix} = p \begin{bmatrix} p^n & \sum_{j=0}^{n-1} p^j [\lambda_{j+1}] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the action of  $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$  is deduced from [Mo1] -Proposition 3.4:

PROPOSITION 3.4. *For  $i \in \{0, 1\}$  consider a function*

$$F_{l_0, \dots, l_n}^{(0,n)}(i) \in \varinjlim_{n \text{ odd}} ((R_i/R_{i-1})^+ \oplus_{R_{i+1}^+} \cdots \oplus_{R_{n+i}^+} R_{n+i+1}^+)$$

(resp.

$$F_{l_1, \dots, l_n}^{(1,n)}(i) \in \varinjlim_{n \text{ odd}} ((R_i/R_{i-1})^- \oplus_{R_{i+1}^-} \cdots \oplus_{R_{n+i}^-} R_{n+i+1}^-).$$

Then we have

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F_{l_0, \dots, l_n}^{(0,n)}(i) = F_{l'_1, \dots, l'_{n+1}}^{(1,n+1)}(i) \quad (\text{resp.} \quad \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F_{l_1, \dots, l_n}^{(1,n)}(i) = F_{l'_0, \dots, l'_{n-1}}^{(0,n-1)}(i))$$

where  $l'_j = l_{j-1}$  for all  $j \in \{1, \dots, n+1\}$  (resp.  $l'_j = l_{j+1}$  for all  $j \in \{0, \dots, n-1\}$ ).

We can sum up the previous result, getting the  $L^\times$  structure for  $\pi(r, 0, 1)|_{L^\times}$ :

**PROPOSITION 3.5.** *Keep the notations of §3. There exists two  $L^\times$  sub-representations  $U(0), U(r) \leq \pi(r, 0, 1)|_{L^\times}$ , each of them being moreover  $K_0(p^0)$ -stable, such that we have an exact sequence*

$$0 \rightarrow U_0 \rightarrow U(0) \oplus U(r) \rightarrow \pi(r, 0, 1)|_{L^\times} \rightarrow 0.$$

For  $\epsilon_2 \in \{0, r\}$  the  $L^\times$ -representation  $U(\epsilon_2)$  admits an  $L^\times$  (and  $K_0(p)$ ) equivariant filtration  $\{U(\epsilon_2)_j\}_{j \in \mathbf{N}}$  such that for all  $j \geq 0$  the space  $U_j(\epsilon_2)/U_{j-1}(\epsilon_2)$  is two dimensional, admitting a basis  $\mathcal{U}(\epsilon_2)_j = \{v_{j,1}(\epsilon_2), v_{j,s}(\epsilon_2)\}$ , and

i) for  $\epsilon_1 \in \{1, s\}$ ,  $K_0(p)$  acts on  $v_{j,\epsilon_1}(\epsilon_2)$  by the character  $(\chi_r \alpha^{-j-\epsilon_2})^{\epsilon_1}$  and the uniformiser  $\varpi$  by the involution  $\varpi v_{j,1}(\epsilon_2) = v_{j,s}(\epsilon_2)$ ;

ii) the  $L^\times$  extension

$$0 \rightarrow U_j(\epsilon_2)/U_{j-1}(\epsilon_2) \rightarrow U_{j+1}(\epsilon_2)/U_{j-1}(\epsilon_2) \rightarrow U_{j+1}(\epsilon_2)/U_j(\epsilon_2) \rightarrow 0$$

is non split.

Moreover,  $U_0(0) = U_0(r) = U_0 = \langle F_\emptyset^{(0,-1)}(0), F_r^{(0,0)}(0) \rangle_{\overline{\mathbf{F}}_p}$ .

*Proof.* Define  $U(0) = \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus \varinjlim_{n \text{ even}} ((R_1/R_0)^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-)$  and, for  $n \in \mathbf{N}$  odd and  $j = \sum_{i=0}^{n+1} p^i l_i$ , we set  $v_{j,1}(0) = F_{l_0, \dots, l_n}^{(0,n)}(l_{n+1})$  and  $v_{j,s}(0) = F_{l_0, \dots, l_n}^{(1,n+1)}(l_{n+1})$  (with the obvious conventions on the  $(n+1)$ -tuple  $(l_0, \dots, l_n)$ ). We define  $U(r)$  and  $v_{j,1}(r), v_{j,s}(r)$  in the evifend, similar way. The statement follows now from Propositions 3.3 and 3.4.  $\square$

Again, we can deduce a multiplicity result.

**COROLLARY 3.6.** *Let  $\pi$  be a supersingular representation and  $\chi$  be a smooth character of  $L^\times$  extending the central character  $\omega_\pi$  of  $\pi$ .*

*Then*

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

*Proof.* Up to twist by a suitable power of the cyclotomic character, we can assume  $\pi \cong \pi(r, 0, 1)$  for a convenient  $r \in \{0, \dots, p-1\}$ . By Proposition 3.5 we are left to prove that, for  $\epsilon_2 \in \{0, r\}$ ,

$$\mathrm{Hom}_{L^\times}(U(\epsilon_2) \otimes \chi^{-1}, \overline{\mathbf{F}}_p) = 0$$

and it will be enough to prove that the space of  $1 + \varpi \mathbf{Z}_p$  co-invariants of  $U(\epsilon_2)$  is zero ( $\chi^{-1}$  acts trivially on  $1 + \varpi \mathbf{Z}_p$ ). We recall (proof of Proposition 3.5) that

$$U(0)|_{1+\varpi \mathbf{Z}_p} = \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+)|_{1+\varpi \mathbf{Z}_p} \oplus \varinjlim_{n \text{ even}} ((R_1/R_0)^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-)|_{1+\varpi \mathbf{Z}_p} \quad (17)$$

Studing the action of an element of the form  $1 + \varpi[\mu]$  we deduce from Proposition 3.2 that each direct summand in (17) is uniserial; thus the  $1 + \varpi \mathbf{Z}_p$  co-invariants for  $U(0)$  is zero. The result for  $U(r)$  is deduces in the same manner and the proof is complete.  $\square$

#### 4. The case of Principal and special series

Fix  $\lambda \in \overline{\mathbf{F}}_p^\times$  and  $r \in \{0, \dots, p-2\}$ . We write  $un_\lambda : \mathbf{Q}_p^\times \rightarrow \overline{\mathbf{F}}_p^\times$  for the smooth unramified character defined by  $un_\lambda(p) = \lambda$  and  $\omega$  for the mod  $p$  cyclotomic character. We finally consider the unramified principal series  $\pi_{r,\lambda} \stackrel{\text{def}}{=} \mathrm{Ind}_B^G un_\lambda \otimes un_{\lambda^{-1}\omega^r}$ .

The aim of this section is to describe the  $L^\times$ -restriction of  $\pi_{\lambda,r}$ . The details of the proofs will be left to the reader, as they are close to the arguments of the supersingular case.

We recall (cf. [Mo2], Theorem 4.13) that the representation  $\pi_{r,\lambda}$  admits a  $\overline{\mathbf{F}}_p$ -linear basis  $\mathcal{B}_{r,\lambda}$  of discrete Fourier transform. Precisely, if  $n \in \mathbf{N}$  and  $(l_0, \dots, l_n) \in \{0, \dots, p-1\}^{n+1}$  (resp.  $(l_1, \dots, l_n) \in \{0, \dots, p-1\}^n$ ) we write  $F_{l_0, \dots, l_n, 0, \dots, 0, \dots}^{(0, \infty)}$  (resp.  $F_{l_1, \dots, l_n, 0, \dots, 0, \dots}^{(1, \infty)}$ ) for the following element of  $\text{Ind}_{B \cap K}^K \chi_r^s$ :

$$F_{l_0, \dots, l_n, 0, \dots, 0, \dots}^{(0, \infty)} = \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, F_{0, \dots, 0, \dots}^{(n+1, \infty)}]$$

(resp.

$$F_{l_1, \dots, l_n, 0, \dots, 0, \dots}^{(1, \infty)} = \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \sum_{\lambda_2 \in \mathbf{F}_p} \lambda_2^{l_2} \begin{bmatrix} 1 & 0 \\ p^2[\lambda_2] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, F_{0, \dots, 0, \dots}^{(n+1, \infty)}]).$$

where  $F_{0, \dots, 0, \dots}^{(n+1, \infty)}$  is the function supported on  $K_0(p^{n+1})$  and such that  $F_{0, \dots, 0, \dots}^{(n+1, \infty)}(g) = \chi_r^s(g)$  for  $g \in K_0(p^{n+1})$  (recall that restriction of functions to  $K$  provides us with a  $K$ -equivariant morphism  $\pi_{r,\lambda}|_K \cong \text{Ind}_{K \cap B}^K \chi_r^s$ ).

Again, for  $j \in \{0, 1\}$  the map

$$P_j : F_{l_j, \dots, l_n, \dots}^{(j, \infty)} \mapsto \sum_{i=j}^{\infty} p^{i-j} l_i \in \mathbf{N} \quad (18)$$

provides the set of functions

$$\mathcal{B}_{r,\lambda}^{(j)} \stackrel{\text{def}}{=} \left\{ F_{l_j, \dots, l_n, \dots}^{(j, \infty)}, \quad (l_j, \dots, l_n, \dots) \in \{0, \dots, p-1\}^{(\mathbf{N})} \right\}$$

with a natural, linear, ordering. We then have the following, crucial result (see [Mo2], Proposition 4.2; see also [Mo1], Lemma 5.12)

**PROPOSITION 4.1.** *Let  $a, b, c, d \in \mathbf{Z}_p$  and consider  $g = \begin{bmatrix} 1+pa & b \\ pc & 1+pd \end{bmatrix}$ . Let  $j \in \{0, 1\}$ . For any  $n$ -tuple  $(l_j, \dots, l_n, \dots) \in \{0, \dots, p-1\}^{(\mathbf{N})}$  we have*

$$\begin{bmatrix} 1+pa & pb \\ pc & 1+pd \end{bmatrix} F_{l_j, \dots, l_n, \dots}^{(j, \infty)} = F_{l_j, \dots, l_n, \dots}^{(j, \infty)} - (\bar{c}\delta_{j,1} + \bar{b}\delta_{j,0})\kappa(l_j, \dots, l_n, \dots)(F_{l_{j-1}, \dots, l_n, \dots}^{(j, \infty)}) + y$$

where  $y$  is a suitable sum of functions strictly preceding  $F_{l_{j-1}, \dots, l_n, \dots}^{(j, \infty)}$  in the natural ordering and  $\kappa(l_j, \dots, l_n, \dots) \in \mathbf{F}_p^\times$ .

We finally recall that the Steinberg representation is defined (up to isomorphism and up to twist) by the exact sequence

$$0 \rightarrow 1 \rightarrow \text{Ind}_B^G 1 \rightarrow \text{St} \rightarrow 0. \quad (19)$$

If we restrict the sequence (19) to  $K$ , we get a precise description of  $\text{St}$  in terms of the basis  $\mathcal{B}$  (see [Mo2], Lemma 4.14):

$$0 \rightarrow \langle F_{0, \dots, 0, \dots}^{(0, \infty)}, F_{0, \dots, 0}^{(1, \infty)} \rangle \rightarrow (\text{Ind}_{B \cap K}^K 1)^+ \oplus \text{Ind}_{B \cap K}^K 1 \rightarrow \text{St} \rightarrow 0 \quad (20)$$

where  $(\text{Ind}_{B \cap K}^K 1)^+$  (resp.  $\text{Ind}_{B \cap K}^K 1$ ) is the subspace of  $\text{Ind}_{B \cap K}^K 1$  generated by the elements of  $\mathcal{B}_{1,1}^{(0)}$  (resp.  $\mathcal{B}_{1,1}^{(1)}$ ).

#### 4.1 The unramified case

In this section, as for §2, we assume  $L/\mathbf{Q}_p$  unramified. We notice that the action of  $p \in \mathbf{L}^\times$  is trivial on  $\pi_{r,\lambda}$ : as we did for the supersingular case we can content ourselves to the study of  $\pi_{r,\lambda}|_{\mathcal{O}_L^\times}$ . Thanks to [Mol], Proposition 6.10 we have a natural  $\mathrm{GL}_2(\mathbf{Z}_p)$ -equivariant filtration  $\{S_n(r, \lambda)\}_{n \in \mathbf{N}}$  on  $\pi_{r,\lambda}|_{\mathcal{O}_L^\times}$ ; the extensions between the graded pieces are non split and can be summarized as follow:

$$\mathrm{Ind}_{K_0(p)}^K F_{0,\dots,0,\dots}^{(1,\infty)} \text{---} \mathrm{Ind}_{K_0(p)}^K F_{1,\dots,0,\dots}^{(1,\infty)} \text{---} \dots \text{---} \mathrm{Ind}_{K_0(p)}^K F_{l_1,\dots,l_n,\dots}^{(1,\infty)} \text{---} \dots \quad (21)$$

where

$$\mathrm{Ind}_{K_0(p)}^K F_{l_1,\dots,l_n,\dots}^{(1,\infty)} \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_r^s \mathfrak{a}^{\sum_{i=1}^\infty l_i}.$$

For the Steinberg case, the graded pieces of the natural filtration  $\{\overline{S}_n(1, 1)\}_{n \in \mathbf{N}}$  (the image of the filtration  $\{S_n(1, 1)\}_{n \in \mathbf{N}}$  on  $\pi_{1,1}$  via the epimorphism  $\pi_{1,1} \rightarrow \mathrm{St}$ ) look as follow:

$$\mathrm{Ind}_{K_0(p)}^K F_{0,\dots,0,\dots}^{(1,\infty)} / \langle F_{0,\dots,0,\dots}^{(0,\infty)} + F_{0,\dots,0,\dots}^{(1,\infty)} \rangle \text{---} \mathrm{Ind}_{K_0(p)}^K F_{1,\dots,0,\dots}^{(1,\infty)} \text{---} \dots \text{---} \mathrm{Ind}_{K_0(p)}^K F_{l_1,\dots,l_n,\dots}^{(1,\infty)} \text{---} \dots$$

Proposition 4.1 shows that the argument of Lemma 2.3 holds true. Together with the filtration (21) (and Lemma 2.7 for the Steinberg representation) we can deduce the structure of  $L^\times$ -restriction of the unramified principal series  $\pi_{r,\lambda}$ :

**PROPOSITION 4.2.** *The  $L^\times$ -restriction of the principal unramified series  $\pi_{r,\lambda}|_{L^\times}$  (resp. the Steinberg representation) admits a splitting*

$$\pi_{r,\lambda}|_{L^\times} \cong \bigoplus_{i=0}^p S_\infty(\eta_i)$$

where  $\eta_i$  are the  $(p+1)$  character of  $\mathbf{F}_p^\times$  extending the  $\mathbf{F}_p^\times$  character  $x \mapsto x^r$  and each  $S_\infty(\eta_i)$  is an infinite length  $L^\times$  uniserial representation, with a trivial action of the uniformiser  $p \in L^\times$ , having Jordan-Hölder factors isomorphic to  $\eta_i$ .

*Proof.* It is analogous to the proof of Corollary 2.6 and the details are left to the reader. The result for the Steinberg case follows as for Proposition 2.8, using Lemma 2.7 and the the filtration (22).  $\square$

As a corollary we get

**COROLLARY 4.3.** *Let  $\pi$  be a special series or a principal series and let  $\chi$  be a smooth  $L^\times$ -character extending the action of the central character  $\omega_\pi$  of  $\pi$ . Then*

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

*Proof.* It is analogous to the proof of Corollary 2.10 and left to the reader.  $\square$

#### 4.2 The ramified case

Assume now  $L/\mathbf{Q}_p$  totally ramified. Then  $L^\times$  embeds in the normaliser  $N$  of  $K_0(p)$  in  $\mathrm{GL}_2(\mathbf{Q}_p)$ . Recall that  $N$  is realised as the semi-direct product

$$N \cong K_0(p) \rtimes \left\langle \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \right\rangle.$$

It is easy to check (and we leave it as an exercise to the reader) that the action of  $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$  on the elements of the basis  $\mathcal{B}_{r,\lambda}$  is described as

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F_{l_0,\dots,l_n,\dots}^{(0,\infty)} = \lambda^{-1} F_{(l_0,\dots,l_n,\dots)}^{(1,\infty)}, \quad \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F_{l_1,\dots,l_n,\dots}^{(1,\infty)} = \lambda F_{l_1,\dots,l_n,\dots}^{(0,\infty)} \quad (22)$$

for any  $n$ -tuples  $(l_0, \dots, l_n, \dots), (l_1, \dots, l_n, \dots) \in \{0, \dots, p-1\}^{(\mathbf{N})}$ . Moreover, the Iwahori structure of  $\pi_{r,\lambda}$  is explicit and described by the following Proposition (cf. [Mo2], Theorem 4.13):

PROPOSITION 4.4. We have a  $K_0(p)$ -equivariant splitting

$$\pi_{r,\lambda} = (\text{Ind}_{B \cap K}^K \chi_r^s)^+ \oplus \text{Ind}_{B \cap K}^{K_0(p)} \chi_r^s$$

where we write  $(\text{Ind}_{B \cap K}^K \chi_r^s)^+$  (resp.  $\text{Ind}_{B \cap K}^{K_0(p)} \chi_r^s$ ) for the subspace of  $\pi_{r,\lambda}|_K$  generated by the elements of  $\mathcal{B}_{r,\lambda}^{(0)}$  (resp.  $\mathcal{B}_{r,\lambda}^{(1)}$ ).

Moreover we have a natural  $K_0(p)$ -equivariant filtration on  $(\text{Ind}_{B \cap K}^K \chi_r^s)^+$  (resp. on  $\text{Ind}_{B \cap K}^{K_0(p)} \chi_r^s$ ) induced from the natural linear order on the elements of  $\mathcal{B}_{r,\lambda}^{(0)}$  (resp.  $\mathcal{B}_{r,\lambda}^{(1)}$ ). Such filtration is described by the following sequence of nonsplit extensions

$$\langle F_{0,\dots,0,\dots}^{(0,\infty)} \rangle \longrightarrow \langle F_{0,\dots,0,\dots}^{(0,\infty)} \rangle \longrightarrow \dots \longrightarrow \langle F_{l_0,\dots,l_n,\dots}^{(0,\infty)} \rangle \longrightarrow \dots$$

(resp.

$$\langle F_{0,\dots,0,\dots}^{(1,\infty)} \rangle \longrightarrow \langle F_{0,\dots,0,\dots}^{(1,\infty)} \rangle \longrightarrow \dots \longrightarrow \langle F_{l_1,\dots,l_n,\dots}^{(1,\infty)} \rangle \longrightarrow \dots).$$

where  $\langle F_{l_0,\dots,l_n,\dots}^{(0,\infty)} \rangle \cong \chi_r \mathfrak{a}^{-\sum_{i=0}^{\infty} l_i}$  (resp.  $\langle F_{l_1,\dots,l_n,\dots}^{(1,\infty)} \rangle \cong \chi_r^s \mathfrak{a}^{\sum_{i=1}^{\infty} l_i}$ ).

We can now deduce the structure of  $\pi_{r,\lambda}|_{L^\times}$ :

PROPOSITION 4.5. Let  $r \in \{0, \dots, p-2\}$  and  $\lambda \in \overline{\mathbf{F}}_p^\times$ .

The  $L^\times$ -restriction of the unramified principal series  $\pi_{r,\lambda}$  admits an  $L^\times$ -equivariant (and actually  $N$ -equivariant) filtration  $\{S_n(r, \lambda)\}_{n \in \mathbf{N}}$  such that:

- i) Each graded piece  $S_n(r, \lambda)/S_{n-1}(r, \lambda)$  is a two dimensional vector space;
- ii) We have a natural linear basis  $\mathcal{S}_n(r, \lambda) \stackrel{\text{def}}{=} \{v_{n,1}, v_{n,-1}\}$  for  $S_n(r, \lambda)/S_{n-1}(r, \lambda)$  such that  $gv_{n,i} = (\chi_r \mathfrak{a}^{-r-n})^i$  for  $i \in \{1, -1\}$  and  $g \in K_0(p)$  and  $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} v_{n,i} = \lambda^{-i} v_{n,-i}$  for  $i \in \{1, -1\}$ .
- iii) For all  $n \in \mathbf{N}$  the extension

$$0 \rightarrow S_n(r, \lambda)/S_{n-1}(r, \lambda) \rightarrow S_{n+1}(r, \lambda)/S_{n-1}(r, \lambda) \rightarrow S_{n+1}(r, \lambda)/S_n(r, \lambda) \rightarrow$$

is nonsplit.

We an analogous result concerning the Steinberg representation  $\pi_{1,1}/\langle F_{0,\dots,0,\dots}^{(0,\infty)} + F_{0,\dots,0,\dots}^{(1,\infty)} \rangle$ , with the only difference that  $S_0(1, 1)$  is the trivial representation of  $N$ .

*Proof.* We define  $S_n(r, \lambda)$  as the linear space generated by the functions  $F_{l_j,\dots,l_n,\dots}^{(j,\infty)}$  such that  $\sum_{i=j}^{\infty} p^{i-j} l_i \leq n$ , where  $j \in \{0, 1\}$ . The statement is then deduced from Proposition 4.4 and from (22).  $\square$

Again, we deduce a multiplicity statement

COROLLARY 4.6. Let  $\pi$  be an principal or a special series and let  $\chi$  be a smooth  $L^\times$  character extending the central character  $\omega_\pi$  of  $\pi$ .

Then

$$\text{Hom}_{L^\times}(\pi|_{L^\times}, \chi) = 0.$$

*Proof.* It is analogous to the proof of Corollary 3.6 and left to the reader.  $\square$

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