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# Invariant elements for $p$ -modular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$

Stefano Morra

## ABSTRACT

Let  $p$  be an odd rational prime and  $F$  a  $p$ -adic field. We give a realization of the universal  $p$ -modular representations of  $\mathbf{GL}_2(F)$  in terms of an explicit Iwasawa module. We specialize our constructions to the case  $F = \mathbf{Q}_p$ , giving a detailed description of the invariants under principal congruence subgroups of irreducible admissible  $p$ -modular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ , generalizing previous works of Breuil and Paskunas [BP]. We apply these results to the local/global compatibility of Emerton [Eme10], giving a generalization of the classical multiplicity one results for the Jacobians of modular curves with arbitrary level at  $p$ .

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## 1. Introduction

Let  $F$  be a  $p$ -adic field, with ring of integers  $\mathcal{O}_F$  and residue field  $k_F$ . This article is framed in the broad context of the  $p$ -modular Langlands correspondence, aimed to match continuous Galois representations of  $\text{Gal}(\overline{F}/F)$  over finite dimensional  $\overline{\mathbf{F}}_p$  vector spaces with certain  $\overline{\mathbf{F}}_p$  valued, smooth representations of the  $F$  points of  $p$ -adic reductive groups.

This correspondence has first been defined in the particular case of  $F = \mathbf{Q}_p$  and the group  $\mathbf{GL}_2$ , thanks to the parametrization of *supersingular* representation of  $\mathbf{GL}_2(\mathbf{Q}_p)$  (cf. [Bre03a]). It is now completely established in the wide horizon of the  $p$ -adic Langlands correspondence for  $\mathbf{GL}_2(\mathbf{Q}_p)$  (cf. [Col1], [Kis], [Pas10]) and admits a cohomological realization according to the Local/Global compatibility of Emerton [Eme10].

For other groups the situation turns out to be extremely more delicate. While  $p$ -modular Galois representations are well understood, the theory of  $p$ -modular representation of  $p$ -adic reductive

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*Keywords:* mod  $p$  Langlands correspondence, supersingular representations, socle filtration, quadratic extensions, Iwahori structure, unramified extensions.

groups is at its beginning, starting with the pionieristic works of Barthel and Livné [BL94], [BL95] and recently achieved in greater generality by Herzig [Her2]. Even for  $\mathbf{GL}_2$ , recent constructions of Breuil and Paskunas [BP] and Hu [Hu2] show a troubling proliferation of supersingular representations as soon as  $F \neq \mathbf{Q}_p$ . This phenomenon remains, at present, misunderstood.

Nevertheless, the works [BP], [Hu] highlight that the crucial point in order to understand an irreducible admissible  $p$ -modular  $\mathbf{GL}_2(F)$  representation  $\pi$  relies in a complete control of its *internal structure*, i.e. of the extensions between irreducible representations of certain congruence subgroups appearing as *subquotients* of  $\pi$ . A exhaustive study in this direction has started in [Mo1], where the author realizes the  $\mathbf{GL}_2(\mathbf{Z}_p)$ -socle filtration for irreducible admissible  $\mathbf{GL}_2(\mathbf{Q}_p)$ -representations.

In this article we pursue the investigation undertaken in [Mo1], clarifying the internal behavior of *universal*  $p$ -modular representations of  $\mathbf{GL}_2(F)$  by means of structure theorems, showing the prominent role of an explicit Iwasawa module. This enables us, in the particular case of  $F = \mathbf{Q}_p$ , to describe exhaustively the space of fixed vectors of supersingular representations under principal congruence subgroups, generalizing previous results of Breuil and Paskunas [BP]. Thanks to the Local/Global compatibility theorems of Emerton [Eme10], we are able to generalize the classical “multiplicity one” results ([Maz], [Rib], [Edi], [Kha]) in the case of modular curves whose level is highly divisible by  $p$ .

We give a more precise account of the main results appearing in this paper.

Let  $k$  be a finite extension of  $k_F$  (the “field of coefficients”): all representations are on  $k$  linear spaces. From the classification of Barthel and Livné [BL94], a supersingular representation  $\pi$  of  $\mathbf{GL}_2(F)$  is, up to twist, an irreducible admissible quotient of an explicit universal representation  $\pi(\sigma, 0)$ . The latter is defined as the cokernel of a canonical Hecke operator on the compact induction  $\text{ind}_{\mathbf{GL}_2(\mathcal{O}_F)F^\times}^{\mathbf{GL}_2(F)} \sigma$ , where  $\sigma$  is an irreducible smooth representation of  $\mathbf{GL}_2(\mathcal{O}_F)F^\times$ .

According to the works of Breuil and Paskunas [BP] and [Hu2], the representation  $\pi$  is *completely determined* by its structure as  $\mathbf{GL}_2(\mathcal{O}_F)$  and  $N$  representation, where  $N$  is the normalizer of the Iwahori subgroup  $I$  of  $\mathbf{GL}_2(\mathcal{O}_F)$ .

Our first results give a realization of the  $\mathbf{GL}_2(\mathcal{O}_F)$  and the  $N$  restriction of  $\pi(\sigma, 0)$  in terms of certain  $k[I]$ -modules  $R_{\infty,0}^-$ ,  $R_{\infty,-1}^-$ :

**THEOREM 1.1** (Corollary 3.4). *There is a canonical  $\mathbf{GL}_2(\mathcal{O}_F)F^\times$ -isomorphism  $\pi(\sigma, 0)|_{\mathbf{GL}_2(\mathcal{O}_F)F^\times} \cong R_{\infty,0} \oplus R_{\infty,-1}$  where the representations  $R_{\infty,0}$ ,  $R_{\infty,-1}$  fit in the following exact sequences of  $k[K]$ -modules:*

$$\begin{aligned} 0 \rightarrow V_1 \rightarrow \text{ind}_I^K(R_{\infty,0}^-) \rightarrow R_{\infty,0} \rightarrow 0 \\ 0 \rightarrow V_2 \rightarrow \text{ind}_I^K(R_{\infty,-1}^-) \rightarrow R_{\infty,-1} \rightarrow 0 \end{aligned}$$

for suitable subquotients  $V_1$ ,  $V_2$  of a finite parabolic induction from a smooth character of the Iwahori subgroup, depending on  $\sigma$ .

The second structure theorem clarifies a result already appearing in [Mo4] (Proposition 3.5) and is concerned with the  $N$ -restriction of the universal representation  $\pi(\sigma, 0)$ :

**THEOREM 1.2** (Propositions 3.5 and 3.6). *In the notations of Theorem 1.1, we have the following  $I$ -equivariant exact sequences*

$$\begin{aligned} 0 \rightarrow W_1 \rightarrow (R_{\infty,-1}^-)^s \oplus R_{\infty,0}^- \rightarrow R_{\infty,0}|_I \rightarrow 0 \\ 0 \rightarrow W_2 \rightarrow (R_{\infty,0}^-)^s \oplus R_{\infty,-1}^- \rightarrow R_{\infty,-1}|_I \rightarrow 0 \end{aligned}$$

where  $W_1$ ,  $W_2$  are convenient 1-dimensional  $k[I]$ -modules. Moreover, the action of the element

$\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$  on the universal representation  $\pi(\sigma, 0)$  induces the  $k[I]$ -equivariant involution

$$\begin{aligned} (R_{\infty, -1}^-)^s \oplus R_{\infty, 0}^- &\xrightarrow{\sim} \left( (R_{\infty, 0}^-)^s \oplus R_{\infty, -1}^- \right)^s \\ (v_1, v_2) &\longmapsto (v_2, v_1) \end{aligned}$$

which restricts to an isomorphism  $W_1 \xrightarrow{\sim} W_2^s$ .

Here, the notation  $(*)^s$  means that we are considering the action of  $I$  on  $*$  obtained by conjugation by the element  $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$  (which is a representative for the only nontrivial coset of  $N/I$ ).

An exhaustive control of the subquotients for the  $k[I]$ -modules  $R_{\infty, 0}^-$ ,  $R_{\infty, -1}^-$  is crucial in order to extract the most subtle properties of supersingular representations for  $\mathbf{GL}_2(F)$ . For instance, in [Hu2] Hu gives a method to detect a subquotient of  $R_{\infty, 0}^- \oplus R_{\infty, -1}^-$  which essentially *characterizes* a supersingular quotient of  $\pi(\sigma, 0)$  and the studies of Schraen on the homological properties of  $R_{\infty, 0}^-$ ,  $R_{\infty, -1}^-$  show that supersingular representations are *not* of finite presentation when  $[F : \mathbf{Q}_p] = 2$  ([Sch]).

The  $k[I]$ -modules  $R_{\infty, 0}^-$ ,  $R_{\infty, -1}^-$  admit an explicit construction, which is recalled in §3.1. They are obtained as co-limits (over  $\mathbf{N}$ ) of finitely presented modules, whose first syzygy requires a strictly increasing number of generators as soon as  $F \neq \mathbf{Q}_p$ ; in particular, they are not admissible unless  $F = \mathbf{Q}_p$ . A first study of  $R_{\infty, 0}^-$ ,  $R_{\infty, -1}^-$  has been pursued by the author in [Mo4] (by representation theoretic methods) and in [Mo5], [Mo6] (using methods from Iwasawa theory).

In the case  $F = \mathbf{Q}_p$  the behavior of  $R_{\infty, 0}^-$ ,  $R_{\infty, -1}^-$  is particularly simple:

**THEOREM 1.3** ([Mo1], Proposition 5.10). *Let  $F = \mathbf{Q}_p$ . For  $\bullet \in \{0, -1\}$  the  $k[I]$ -module  $R_{\infty, \bullet}^-$  is uniserial.*

This phenomenon, which is at the heart of the results in [Mo1], [Mo4], [AM], let us detect in greatest detail the space of invariant vectors of supersingular representations  $\pi(\sigma, 0)$  of  $\mathbf{GL}_2(\mathbf{Q}_p)$  under certain congruence subgroups.

The following result is a sharpening of the main result of [Mo1] and of [BP], Proposition 20.1:

**THEOREM 1.4** (Corollary 4.9). *Let  $t \geq 1$  and let  $K_t$  be the principal congruence subgroup of  $\mathbf{GL}_2(\mathcal{O}_F)$  of level  $p^t$ . Assume  $\sigma = \text{Sym}^r k^2$  where  $r \in \{0, \dots, p-1\}$ .*

*The space of  $K_t$  fixed vectors for the supersingular representation  $\pi(\sigma, 0)$  decomposes into the direct sum of two  $k[K]$ -modules  $(\pi(\sigma, 0))^{K_t} = (R_{\infty, 0}^-)^{K_t} \oplus (R_{\infty, -1}^-)^{K_t}$ . Each direct summand admits a  $K$ -equivariant filtration whose graded pieces are described by:*

$$(R_{\infty, 0}^-)^{K_t} : \quad \text{Sym}^r k^2 \text{---ind}_I^K \chi_{r+2} \det^{-1} \text{---ind}_I^K \chi_{r+4} \det^{-2} \text{---} \dots \text{---ind}_I^K \chi_r \text{---Sym}^{p-3-r} k^2 \otimes \det^{r+1}$$

$$(R_{\infty, -1}^-)^{K_t} : \quad \text{Sym}^{p-1-r} k^2 \otimes \det^r \text{---ind}_I^K \chi_{-r+2} \det^{r-1} \text{---ind}_I^K \chi_{-r+4} \det^{r-2} \text{---} \dots \text{---ind}_I^K \chi_{-r} \det^r \text{---Sym}^{r-2} k^2 \otimes \det$$

*where we have  $p^{t-1} - 1$  parabolic inductions in each line and the algebraic representation  $\text{Sym}^{p-3-r} k^2 \otimes \det^{r+1}$  in the first line (resp.  $\text{Sym}^{r-2} k^2 \otimes \det$  in the second line) appears only if  $p-3-r \geq 0$  (resp.  $r-2 \geq 0$ ).*

We recall that for any  $n \in \mathbf{N}$  the natural  $\mathbf{GL}_2(\mathbf{F}_p)$ -representation  $\text{Sym}^n k^2$  is viewed as a  $\mathbf{GL}_2(\mathbf{Z}_p)$ -representation by inflation and that the smooth character  $\chi_n$  of the Iwahori  $I$  is defined by  $\begin{bmatrix} a & b \\ pc & d \end{bmatrix} \mapsto a^n \pmod{p}$ . One can indeed prove that the  $\mathbf{GL}_2(\mathbf{Z}_p)$  socle filtration for  $(R_{\infty, 0}^-)^{K_t}$ ,

$(R_{\infty,-1})^{K_t}$  is obtained by the evident refinement of the filtration described by Theorem 1.4 (cf. Corollary 4.9).

The statement of Theorem 1.4 is deduced from Theorem 1.1, through a careful study of the  $K_t$  fixed vectors of the Iwasawa modules  $R_{\infty,0}^-$ ,  $R_{\infty,-1}^-$ . As the latter are uniserial, their  $K_t$  fixed vectors can be easily recovered with a direct argument on Witt vectors (cf. Proposition 4.2).

In a similar fashion, we detect the fixed vectors for the congruence subgroup  $I_t$ , which is defined as the subgroup of  $K_{t-1}$  whose elements are upper unipotent modulo  $p^t$ :

**THEOREM 1.5** (Corollary 3.9, Propositions 4.4, 4.5). *Let  $t \geq 1$  and assume  $\sigma = \text{Sym}^r k^2$  for  $r \in \{0, \dots, p-1\}$ .*

*The space of  $I_t$  fixed vectors for the supersingular representation  $\pi(\sigma, 0)$  decomposes as  $(\pi(\sigma, 0))^{I_t} = (R_{\infty,0})^{I_t} \oplus (R_{\infty,-1})^{I_t}$ . Each direct summand is a  $k[I]$ -module admitting an equivariant filtration whose graded pieces are described by:*

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & \chi_{r+2}\det^{-1} & \text{---} & \chi_{r+4}\det^{-2} & \text{---} & \dots & \text{---} & \chi_r \\
 & & / & & / & & / & & / \\
 (R_{\infty,0})^{I_t} : & \chi_r & & \oplus & & \oplus & & \oplus & \\
 & & \backslash & & \backslash & & \backslash & & \backslash \\
 & & \chi_{r-2}\det & \text{---} & \chi_{r-4}\det^2 & \text{---} & \dots & \text{---} & \chi_r
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 & & \chi_{-r+2}\det^{r-1} & \text{---} & \chi_{-r+4}\det^{r-2} & \text{---} & \dots & \text{---} & \chi_{-r}\det^r \\
 & & / & & / & & / & & / \\
 (R_{\infty,-1})^{I_t} : & \chi_{-r}\det^r & & \oplus & & \oplus & & \oplus & \\
 & & \backslash & & \backslash & & \backslash & & \backslash \\
 & & \chi_{-r-2}\det^{r+1} & \text{---} & \chi_{-r-4}\det^{r+2} & \text{---} & \dots & \text{---} & \chi_{-r}\det^r
 \end{array}
 \end{array}$$

and we have  $p^{t-1} - 1$  characters on each horizontal line.

We point out that Theorem 1.4 and 1.5 had first been proved by the author in [Mo2], essentially with the same technical tools, but the lack of the structure Theorems 1.1 and 1.2 required a considerable amount of delicate estimates on Witt vectors. Moreover, our techniques could be applied to detect the fixed vectors for irreducible admissible  $\mathbf{GL}_2(\mathbf{Q}_p)$ -representations under other congruence subgroups (see for instance [Mo3]).

As we remarked above, a precise control of  $K_t$ ,  $I_t$  invariants has global applications, thanks to the geometric realization of the  $p$ -adic Langlands correspondence by Emerton [Eme10]. Let  $\bar{\rho} : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathbf{GL}_2(k)$  be a continuous, irreducible odd Galois representation which we assume to be absolutely irreducible at  $p$ . Let  $N$  be its Artin conductor and  $k$  its level (cf. [Ser87]); up to twist, we may assume  $2 \leq k \leq p$ . Let  $Y(Np^t)$  be the modular curve (defined over  $\mathbf{Q}$ ) of level  $Np^t$  and  $\mathfrak{m}_{\bar{\rho}}$  the maximal ideal in the Hecke algebra of  $H_{\text{ét}}^1(Y(Np^t) \times_{\mathbf{Q}} \bar{\mathbf{Q}}, k)$  corresponding to  $\bar{\rho}$ .

The result is the following:

**THEOREM 1.6** (Proposition 6.1). *Let  $K(N)$  be the kernel of the map*

$$\prod_{\ell|N} \mathbf{GL}_2(\mathbf{Z}_{\ell}) \rightarrow \prod_{\ell|N} \mathbf{GL}_2(\mathbf{Z}_{\ell}/N)$$

and define

$$d \stackrel{\text{def}}{=} \dim_k \left( \bigotimes_{\ell|N} \pi(\bar{\rho}|_{\text{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)})^{K(N)} \right)$$

where  $\pi(\bar{\rho}|_{\text{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)})$  is the smooth  $\mathbf{GL}_2(\mathbf{Q}_\ell)$ -representation attached to  $\bar{\rho}|_{\text{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)}$  by the Emerton-Helm  $p$ -modular Langlands correspondence ([EH]).

If  $t \geq 1$  and  $Np^t > 4$  we have

$$\begin{aligned} \dim_k (H_{\text{ét}}^1(Y(Np^t) \times_{\mathbf{Q}} \bar{\mathbf{Q}}, k)[\mathfrak{m}_{\bar{\rho}}]) &= 2d(2p^{t-1}(p+1) - 3) && \text{if } k - 2 = 0 \\ \dim_k (H_{\text{ét}}^1(Y(Np^t) \times_{\mathbf{Q}} \bar{\mathbf{Q}}, k)[\mathfrak{m}_{\bar{\rho}}]) &= 2d(2p^{t-1}(p+1) - 4) && \text{if } k - 2 \neq 0. \end{aligned}$$

Thanks to the relation between the étale cohomology of the modular curve  $Y(Np^t)$  and the Tate module of its Jacobian, Theorem 1.6 generalizes the classical multiplicity one theorems ([Rib], [Maz]) to modular curves of arbitrary level at  $p$ . It is consistent with the results of Khare [Kha], where it is shown that the dimension of the  $\bar{\rho}$ -isotypical component of the Jacobian of  $Y(Np^t)$  tends to infinity as the level at  $p$  increases.

The organization of the paper is the following.

We start (§2) by recalling the construction of the universal representation  $\pi(\sigma, 0)$  for a  $p$ -adic field  $F$ , as well as some properties of finite parabolic induction for  $\mathbf{GL}_2(\mathbf{F}_p)$  which will be used later on to describe the  $K_t$ -invariants for irreducible admissible representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ .

Section §3 is devoted to the realization of the structure theorems for universal representations. We first refine the constructions of §2 in order to define the Iwasawa modules  $R_{\infty,0}^-$ ,  $R_{\infty,-1}^-$  (§3.1); we subsequently specialize to the case  $F = \mathbf{Q}_p$  (§3.2).

The space of invariant vectors for irreducible admissible representations is worked out in section 4. We first detect the invariants for the Iwasawa modules  $R_{\infty,0}^-$ ,  $R_{\infty,-1}^-$  (§4.1), relying crucially on the fact that such modules are unimodular (Proposition 4.2). We then use the structure theorems of section 3 to deduce the space of  $K_t$  and  $I_y$  fixed vectors for supersingular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ .

Section 5 is devoted to the case of principal and special series representations for  $\mathbf{GL}_2(\mathbf{Q}_p)$ . The results are somehow similar, but can be detected with much less efforts.

Finally, we give in §6 a precise description of the global application of Theorems 1.4, 1.5 for the multiplicity spaces of mod  $p$  cohomology of modular curves.

## 1.1 Notation

Let  $p$  be an odd prime. We consider a  $p$  adic field  $F$ , with ring of integers  $\mathcal{O}_F$ , uniformizer  $\varpi$  and residue field  $k_F$ . We assume that  $[k_F : \mathbf{F}_p] = f$  is finite and we set  $q \stackrel{\text{def}}{=} \text{Card}(k_F)$  for its cardinality. We write  $x \mapsto \bar{x}$  for the reduction morphism  $\mathcal{O}_F \rightarrow k_F$  and  $\bar{x} \mapsto [\bar{x}]$  for the Teichmüller lift  $k_F^\times \rightarrow \mathcal{O}_F^\times$  (we set  $[0] \stackrel{\text{def}}{=} 0$ ).

Consider the general linear group  $\mathbf{GL}_2$ . We fix the maximal torus  $\mathbf{T}$  of diagonal matrices and the unipotent radical  $\mathbf{U}$  of upper unipotent matrices, so that  $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{T} \ltimes \mathbf{U}$  is the Borel subgroup of upper triangular matrices. We write  $\mathcal{S}$  for the Bruhat-Tits tree associated to  $\mathbf{GL}_2(F)$  (cf. [Ser77]) and we consider the hyperspecial maximal compact subgroup  $K \stackrel{\text{def}}{=} \mathbf{GL}_2(\mathcal{O}_F)$ .

The object of study of this article are the following congruence subgroups of  $K$ :

$$K_t \stackrel{\text{def}}{=} \ker \left( K \xrightarrow{\text{red}_t} \mathbf{GL}_2(\mathcal{O}_F/(\varpi^t)) \right), \quad I_t \stackrel{\text{def}}{=} \left( \text{red}_t^{\leftarrow}(\mathbf{U}(\mathcal{O}_F/(\varpi^t))) \right) \cap K_{t-1}$$

where  $t \in \mathbf{N}$  and  $red_t$  denotes the mod  $\varpi^t$  reduction map. For notational convenience, we introduce the following objects

$$\omega \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \quad \alpha \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \quad K_0(\varpi) \stackrel{\text{def}}{=} red_t^{\leftarrow}(\mathbf{B}(k_F)).$$

Let  $E$  be a  $p$ -adic field, with ring of integers  $\mathcal{O}$  and finite residue field  $k$  (the ‘‘coefficient field’’). Up to enlarging  $E$ , we can assume that  $\text{Card}(\text{Hom}_{\mathbf{F}_p}(k_F, k)) = [k_F : \mathbf{F}_p]$ .

A representation  $\sigma$  of a subgroup  $H$  of  $\mathbf{GL}_2(\mathbf{Q}_p)$  is always understood to be smooth with coefficients in  $k$ . If  $h \in H$  will sometimes write  $\sigma(h)$  to denote the  $k$ -linear automorphism induced by the action of  $h$  on the underlying vector space of  $\sigma$ . We denote by  $(\sigma)^H$  the linear space of  $H$  fixed vectors of  $\sigma$ .

Let  $H_2 \leq H_1$  be compact open subgroups of  $K$ . For a smooth representation  $\sigma$  of  $H_2$  we write  $\text{ind}_{H_2}^{H_1} \sigma$  to denote the (compact) induction of  $\sigma$  from  $H_2$  to  $H_1$ . If  $v \in \sigma$  and  $h \in H_1$  we write  $[h, v]$  for the unique element of  $\text{ind}_{H_2}^{H_1} \sigma$  supported in  $H_2 h^{-1}$  and sending  $h$  to  $v$ . We deduce in particular the following equalities:

$$h' \cdot [h, v] = [h' h, v], \quad [h k, v] = [h, \sigma(k)v] \tag{1}$$

for any  $h' \in H_1, k \in H_2$ .

The previous construction will mainly be used when  $H_1 = K, H_2 = K_0(\varpi)$ . In this situation we define, for any  $v \in \sigma$  and  $l \in \mathbf{N}$ , the element

$$f_l(v) \stackrel{\text{def}}{=} \sum_{\lambda \in k_F} \lambda^l \begin{bmatrix} [\lambda] & 1 \\ 1 & 0 \end{bmatrix} [1, v] \in \text{ind}_{K_0(\varpi)}^K \sigma.$$

If  $Z \cong F^\times$  is the center of  $\mathbf{GL}_2(F)$  and  $\sigma$  is a representation of  $KZ$  we will similarly write  $\text{ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$  for the subspace of the full induction  $\text{Ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$  consisting of functions which are compactly supported modulo the center  $Z$  (cf. [Bre03a], §2.3). For  $g \in \mathbf{GL}_2(F), v \in \sigma$  we use the same notation  $[g, v]$  for the element of  $\text{ind}_{KZ}^{\mathbf{GL}_2(F)} \sigma$  having support in  $KZg^{-1}$  and sending  $g$  to  $v$ ; the element  $[g, v]$  verifies similar compatibility relations as in (1).

A *Serre weight* is an absolutely irreducible representation of  $K$ . Up to isomorphism they are of the form

$$\bigotimes_{\tau \in \text{Gal}(k_F/\mathbf{F}_p)} (\det^{t_\tau} \otimes_{k_F} \text{Sym}^{r_\tau} k_F^2) \otimes_{k_F, \tau} k \tag{2}$$

where  $r_\tau, t_\tau \in \{0, \dots, p-1\}$  for all  $\tau \in \text{Gal}(k_F/\mathbf{F}_p)$  and  $t_\tau < p-1$  for at least one  $\tau$ . This gives a bijective parametrisation of isomorphism classes of Serre weights by  $2f$ -tuples of integers  $r_\tau, t_\tau \in \{0, \dots, p-1\}$  such that  $t_\tau < p-1$  for some  $\tau$ .

Recall that the  $K$  representations  $\text{Sym}^{r_\tau} k_F^2$  can be identified with  $k_F[X, Y]_{r_\tau}^h$ , the homogeneous component of degree  $r_\tau$  of the monoidal algebra  $k_F[X, Y]$ . In this case, the action of  $K$  is described by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X^{r_\tau-i} Y^i \stackrel{\text{def}}{=} (\bar{a}X + \bar{c}Y)^{r_\tau-i} (\bar{b}X + \bar{d}Y)^i$$

for any  $0 \leq i \leq r_\tau$ .

We fix once for all a field homomorphism  $k_F \hookrightarrow k$ . The results of this paper do not depend on this choice.

Up to twist by a power of  $\det$ , a Serre weight has now the more concrete expression

$$\sigma_\tau \cong \bigotimes_{i=0}^{f-1} (\text{Sym}^{r_i} k^2)^{\text{Frob}^i}$$

where  $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$  and  $(\mathrm{Sym}^{r_i} k^2)^{\mathrm{Frob}^i}$  is the representation of  $K$  obtained from  $\mathrm{Sym}^{r_i} k^2$  via the homomorphism  $\mathbf{GL}_2(k_F) \rightarrow \mathbf{GL}_2(k_F)$  induced by the Frobenius  $x \mapsto x^{p^i}$  on  $k_F$ .

We will usually extend the action of  $K$  on a Serre weight to the group  $KZ$  by imposing the scalar matrix  $\varpi \in Z$  to act trivially.

A  $k$ -valued character  $\chi$  of the torus  $\mathbf{T}(k_F)$  will be considered, by inflation, as a smooth character of any subgroup of  $K_0(\varpi)$ . We will write  $\chi^s$  to denote the conjugate character of  $\chi$ , defined by

$$\chi^s(t) \stackrel{\mathrm{def}}{=} \chi(\omega t \omega)$$

for any  $t \in \mathbf{T}(k_F)$ .

Similarly, if  $\tau$  is any representation of  $K_0(\varpi)$ , we will write  $\tau^s$  to denote the conjugate representation, defined by

$$\tau^s(h) = \tau(\alpha h \alpha)$$

for any  $h \in K_0(\varpi)$ .

Finally, if  $\sigma$  is a Serre weight, we write  $\sigma^{[s]}$  for the unique Serre weight non isomorphic to  $\sigma$  and whose highest weight space affords the character  $((\sigma)^{K_0(\varpi)})^s$ . Concretely, if  $\sigma$  appears in the  $K$  socle of an induction  $\mathrm{ind}_{K_0(\varpi)}^K \chi$ , then  $\sigma^{[s]}$  appears in the cosocle of  $\mathrm{ind}_{K_0(\varpi)}^K \chi^s$ .

If  $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$  is an  $f$ -tuple we define the characters of  $\mathbf{T}(k_F)$ :

$$\chi_{\underline{r}} \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \stackrel{\mathrm{def}}{=} a^{\sum_{i=0}^{f-1} p^i r_i}, \quad \mathfrak{a} \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \stackrel{\mathrm{def}}{=} ad^{-1}.$$

If  $H \leq K$  is an open subgroup and  $\tau$  is a representation of  $H$  we write  $\{\mathrm{soc}^i(\tau)\}_{i \in \mathbf{N}}$  to denote its socle filtration (we set  $\mathrm{soc}^0(\tau) \stackrel{\mathrm{def}}{=} \mathrm{soc}(\tau)$ ). We will use the notation

$$\mathrm{soc}^1(\tau) \text{---} \mathrm{soc}^1(\tau)/\mathrm{soc}^0(\tau) \text{---} \dots \text{---} \mathrm{soc}^{n+1}(\tau)/\mathrm{soc}^n(\tau) \text{---} \dots$$

to denote the sequence of consecutive graded pieces of the socle filtration for  $\tau$  (in particular, each  $\mathrm{soc}^{i+1}(\tau)/\mathrm{soc}^i(\tau) \text{---} \mathrm{soc}^{i+2}(\tau)/\mathrm{soc}^{i+1}(\tau)$  is a non-split extension).

More generally, if  $\tau$  is an  $H$ -representation endowed with an increasing filtration  $\{\tau\}_{i \in \mathbf{N}}$  we will write

$$\mathrm{socfil}(\tau_0) \text{---} \mathrm{socfil}(\tau_1/\tau_0) \text{---} \dots \text{---} \mathrm{socfil}(\tau_{i+1}/\tau_i) \text{---} \dots$$

to mean that

- i) the socle filtration for  $\tau$  is obtained as the refined filtration induced from the socle filtration on each graded piece  $\tau_{i+1}/\tau_i$ ;
- ii) the sequence of consecutive graded pieces of the socle filtration for  $\tau$  is obtained as the juxtaposition of the sequences of the graded pieces associated to the socle filtration of each  $\tau_{i+1}/\tau_i$ .

If  $S$  is any set, and  $s_1, s_2 \in S$  we define the Kronecker delta

$$\delta_{s_1, s_2} \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if } s_1 \neq s_2 \\ 1 & \text{if } s_1 = s_2. \end{cases}$$

Moreover, for  $x \in \mathbf{Z}$ , we define  $[x] \in \{0, \dots, p-2\}$  (resp.  $\lceil x \rceil \in \{1, \dots, p-1\}$ ) by the condition  $[x] \equiv x \equiv \lceil x \rceil \pmod{p-1}$ .



## 2. Reminders on the universal representations for $\mathbf{GL}_2$

We recall here the precise definition of the universal representation of  $\mathbf{GL}_2$ . We provide an explicit construction in terms of Hecke operators and Mackey decomposition, which turns out to be useful to realize the structure theorems of §3. We end the section collecting some results on finite inductions for smooth characters of the Iwahori subgroup.

The main references are the work of Breuil [Bre03a], §2 and [Mo1], §2 and §3.

### 2.1 Construction of the Universal representation

We fix an  $f$ -tuple  $\underline{r} \in \{0, \dots, p-1\}^f$  and write  $\sigma = \sigma_{\underline{r}}$  for the associated Serre weight described in (1.1). In particular, the highest weight space of  $\sigma$  affords the character  $\chi_{\underline{r}}$ . We recall ([BL95], [Her1]) that the Hecke algebra  $\mathcal{H}_{KZ}(\sigma)$  is commutative and isomorphic to the monoidal algebra of  $\mathbf{N}$  on  $k$ :

$$\mathcal{H}_{KZ}(\sigma) \xrightarrow{\sim} k[T].$$

The Hecke operator  $T$  is supported on the double coset  $K\alpha KZ$  and completely determined as a suitable linear projection on  $\sigma$  (cf. [Her1], Theorem 1.2); it admits an explicit description in terms of the Bruhat-Tits tree of  $\mathbf{GL}_2(F)$  (cf. [Bre03a], §2.5).

The universal representation  $\pi(\sigma, 0, 1)$  for  $\mathbf{GL}_2$  is then defined by the exact sequence

$$0 \rightarrow \mathrm{ind}_{KZ}^G \sigma \xrightarrow{T} \mathrm{ind}_{KZ}^G \sigma \rightarrow \pi(\sigma, 0, 1) \rightarrow 0.$$

In the rest of this section we study the  $KZ$ -restriction of  $\pi(\sigma, 0, 1)$  in terms of its Mackey decomposition, giving a precise construction by means of a family of suitable Hecke operators.

Let  $n \in \mathbf{N}$ . We consider the anti-dominant co-weight  $\lambda_n \in X(\mathbf{T})_*$  characterized by

$$\lambda_n(\varpi) = \begin{bmatrix} 1 & 0 \\ 0 & \varpi^n \end{bmatrix}$$

and we introduce the subgroup

$$K_0(\varpi^n) \stackrel{\mathrm{def}}{=} \lambda_n(\varpi)K \cap K = \left\{ \begin{bmatrix} a & b \\ \varpi^n c & d \end{bmatrix} \in K \right\}.$$

The element  $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$  normalizes  $K_0(\varpi^n)$  and we define the  $K_0(\varpi^n)$ -representation  $\sigma^{(n)}$  as the  $K_0(\varpi^n)$  restriction of  $\sigma$  endowed with the twisted action of  $K_0(\varpi^n)$  by the element  $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$ . Explicitly,

$$\sigma^{(n)} \left( \begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \right) \cdot \underline{X}^{r-j} \underline{Y}^j \stackrel{\mathrm{def}}{=} \sigma \left( \begin{bmatrix} d & c \\ p^n b & a \end{bmatrix} \right) \underline{X}^{r-j} \underline{Y}^j.$$

Finally, we write

$$R_n(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{ind}_{K_0(\varpi^n)}^K(\sigma^{(n)}).$$

If the Serre weight  $\sigma$  is clear from the context, we set  $R_n = R_n(\sigma)$ . For notational convenience we define  $R_{-1} \stackrel{\mathrm{def}}{=} 0$ .

We have a  $K$ -equivariant isomorphism (deduced from Frobenius reciprocity)

$$\begin{aligned} R_n &\xrightarrow{\sim} k[K\lambda_n(\varpi)KZ] \otimes_{k[KZ]} \sigma \\ [1, v] &\longmapsto \lambda_n(\varpi) \otimes v \end{aligned} \tag{3}$$

which realizes the Mackey decomposition for  $\text{ind}_{KZ}^G \sigma$ :

$$(\text{ind}_{KZ}^G \sigma)|_{KZ} \xrightarrow{\sim} \bigoplus_{n \in \mathbf{N}} R_n.$$

The interpretation in terms of the tree of  $\mathbf{GL}_2$  is clear: the  $k[K]$ -module  $R_n$  maps isomorphically onto the space of elements of  $\text{ind}_{KZ}^G \sigma$  having support on the double coset  $K\lambda_n(\varpi)KZ$ . In particular, if  $\sigma$  is the trivial weight, a linear basis for  $R_n$  is parametrized by the vertices of  $\mathcal{T}$  lying at distance  $n$  from the central vertex.

The Hecke endomorphism  $T$  induces, by transport of structure, a family of  $K$ -equivariant morphisms  $T_n$  defined on the  $k[K]$ -modules  $R_n$ :  $T_n \stackrel{\text{def}}{=} T|_{R_n}$ . From the explicit description of the Hecke operator  $T$  one sees (cf. [Mo4], §2.2.1) that  $\text{Im}(T_n)$  is a sub-object of  $R_{n+1} \oplus R_{n-1}$  so that we can further consider the composition with the canonical the projections

$$T_n^\pm : R_n \xrightarrow{T_n} R_{n+1} \oplus R_{n-1} \longrightarrow R_{n\pm 1}.$$

It turns out that, for  $n \geq 1$ , the operators  $T_n^\pm$  are obtained by compact induction (from  $K_0(\varpi^n)$  to  $K$ ) from the following morphisms  $t_n^\pm$ :

$$\begin{aligned} t_n^+ : \sigma^{(n)} &\hookrightarrow \text{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)} \\ \underline{X}^{r-j} \underline{Y}^j &\mapsto \sum_{\lambda_n \in k_F} (-\lambda_n)^j \begin{bmatrix} 1 & 0 \\ \varpi^n [\lambda_n] & 1 \end{bmatrix} [1, \underline{X}^r]; \\ t_{n+1}^- : \text{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)} &\twoheadrightarrow \sigma^{(n)} \\ [1, \underline{X}^{r-j} \underline{Y}^j] &\mapsto \delta_{j,r} \underline{Y}^r. \end{aligned}$$

For  $n = 0$  we similarly have

$$\begin{aligned} T_0^+ : \sigma^{(0)} &\hookrightarrow R_1 \\ \underline{X}^{r-j} \underline{Y}^j &\mapsto \sum_{\lambda_0 \in k_F} (-\lambda_0)^{r-j} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, \underline{X}^r] + \delta_{j,0} [1, \underline{X}^r] \\ T_1^- : R_1 &\twoheadrightarrow \sigma^{(0)} \\ [1, \underline{X}^{r-j} \underline{Y}^j] &\mapsto \delta_{j,r} \underline{Y}^r. \end{aligned}$$

In particular,  $T_n^+$  (resp.  $T_n^-$ ) are monomorphisms (resp. epimorphisms).

We deduce the following exact sequence of  $K$ -representations

$$0 \rightarrow \bigoplus_{n \in \mathbf{N}} R_n \xrightarrow{\bigoplus_n T_n} \bigoplus_{n \in \mathbf{N}} R_n \rightarrow \pi(\sigma, 0, 1)|_{KZ} \rightarrow 0$$

so that, by the exactness of filtered co-limits and the definition of the Hecke operators  $T_n$  we obtain

$$\left( \lim_{\substack{\longrightarrow \\ n \text{ odd}}} \text{coker} \left( \bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1} \right) \right) \oplus \left( \lim_{\substack{\longrightarrow \\ n \text{ even}}} \text{coker} \left( \bigoplus_{j=0}^{\frac{n}{2}} T_{2j} \right) \right) \cong \pi(\sigma, 0, 1)|_{KZ} \quad (4)$$

The representations  $\text{coker} \left( \bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1} \right)$  can be described in a more expressive way as a suitable push-out of the partial Hecke operators  $T_n^\pm$ . Indeed one verifies that  $\text{coker}(T_1) = R_0 \oplus_{R_1} R_2$ , where

the push out is defined by the following co-cartesian diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{T_1^+} & R_2 \\ -T_1^- \downarrow & & \downarrow pr_2 \\ R_0 & \xrightarrow{\quad} & R_0 \oplus_{R_1} R_2. \end{array}$$

If we assume we have inductively constructed  $pr_{n-1} : R_{n-1} \rightarrow R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1}$  (where  $n \geq 3$  is odd), we define the amalgamated sum  $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$  by the following co-cartesian diagram:

$$\begin{array}{ccc} R_n & \xrightarrow{T_n^+} & R_{n+1} \\ -pr_{n-1} \circ T_n^- \downarrow & & \downarrow pr_{n+1} \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} & \xrightarrow{\quad} & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1} \end{array}$$

and, using the universal properties of push-outs and cokernels, one obtains a canonical isomorphism of  $k[K]$ -modules

$$\operatorname{coker}\left(\bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1}\right) \cong R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$$

(cf. [Mo4], Proposition 2.9) together with a commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_n & \xrightarrow{T_n^+} & R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} & \xrightarrow{\iota_n} & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \end{array} \quad (5)$$

We construct in a completely analogous fashion the amalgamated sums  $(R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$  for  $n \in 2\mathbf{N}$ , obtaining an isomorphism

$$\left(\varinjlim_{n \text{ even}} \operatorname{coker}\left(\bigoplus_{j=0}^{\frac{n}{2}} T_{2j}\right)\right) \cong (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$$

and a similar commutative diagram as in (5).

In order to lighten notations, we put

$$R_{\infty, \text{odd}} \stackrel{\text{def}}{=} \varinjlim_{n, \text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$$

(where the inductive system is defined by the natural morphisms  $\iota_n$  appearing in the diagram (5)) and, similarly,

$$R_{\infty, \text{even}} \stackrel{\text{def}}{=} \varinjlim_{n, \text{even}} (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}.$$

If we need to emphasize their dependence on the Serre weight  $\sigma$  we will write  $R_{\infty,0}(\sigma)$ ,  $R_{\infty,-1}(\sigma)$ .

**2.1.1 Induced representations for  $\mathbf{B}(\mathbf{F}_p)$ .** In this section we specialize to  $k_F = \mathbf{F}_p$  the results of [BP], §2 (see also [BS00]), which describe the structure of a  $\mathbf{GL}_2(k_F)$ -representation parabolically induced from a character of a Borel subgroup. The results here will be used to complete the computations for the  $K_t$  invariant vectors of supersingular representations for  $\mathbf{GL}_2(\mathbf{Q}_p)$ .

Let  $i, j \in \{0, \dots, p-1\}$  and let us consider the  $\mathbf{B}(\mathbf{F}_p)$ -character  $\chi_i^s \mathbf{a}^j$ . If  $e$  is a fixed linear basis for the underlying vector space associated to  $\chi_i^s \mathbf{a}^j$  and if  $l \in \mathbf{Z}$ , we recall the elements  $f_l \stackrel{\text{def}}{=} f_l(e) \in \text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j$  defined in 1.1.

The following result clarifies the relation between the elements  $f_l$  and the socle filtration for the finite parabolic induction:

**PROPOSITION 2.1.** *Let  $i, j \in \{0, \dots, p-1\}$ . Then*

- i) *for  $l \in \{0, \dots, p-1\}$ ,  $f_l$  is an  $\mathbf{T}(k_F)$ -eigenvector, whose associated eigencharacter is  $\chi_{i-2j} \det^j \mathbf{a}^{-l}$ , and the set*

$$\mathcal{B} \stackrel{\text{def}}{=} \{f_l, [1, e] \mid 0 \leq l \leq p-1, \}$$

*is a linear basis for  $\text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j$ .*

- ii) *If  $i-2j \not\equiv 0 [p-1]$  then we have a nontrivial extension*

$$0 \rightarrow \text{Sym}^{[i-2j]} k^2 \otimes \det^j \rightarrow \text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j \rightarrow \text{Sym}^{p-1-[i-2j]} k^2 \otimes \det^{i-j} \rightarrow 0.$$

*The families*

$$\{f_0, \dots, f_{[i-2j]-1}, f_{[i-2j]} + (-1)^{i-j} [1, e]\}, \quad \{f_{i-2j}, \dots, f_{p-1}\}$$

*induce a basis for the socle and the cosocle of  $\text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j$  respectively.*

- iii) *If  $i-2j \equiv 0 [p-1]$  then  $\text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j$  is semi-simple and*

$$\text{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)} \chi_i^s \mathbf{a}^j \xrightarrow{\sim} (1 \oplus \text{Sym}^{p-1} k^2) \otimes \det^j.$$

*The families*

$$\{f_0 + (-1)^j [1, e]\}, \quad \{f_0, f_1, \dots, f_{p-2}, f_{p-1} + (-1)^j [1, e]\}$$

*induce a  $k$ -basis for  $\det^j$  and  $\text{Sym}^{p-1} k^2 \otimes \det^j$  respectively.*

*Proof.* Omissis. Cf. [BP], Lemmas 2.5, 2.6, 2.7. □

We end this section with a technical remark on Witt polynomials, which, combined with Lemma 2.1, enables us to conclude the delicate computations needed to describe the  $K_t$  fixed vectors for supersingular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$  (§4.2). We recall ([AC], Chapitre 9, §1, partie 4) that if  $\mathbf{F}$  is a finite extension of  $\mathbf{F}_p$  we have the following equality in the associated ring of Witt vectors  $\mathbf{W}(\mathbf{F})$ :

$$[\mu] + [\lambda] \equiv [\lambda + \mu] + p[S_1(\lambda, \mu)] \pmod{p^2}$$

where  $\mu, \lambda \in \mathbf{F}$ ,  $[\cdot] : \mathbf{F} \rightarrow \mathbf{W}(\mathbf{F})$  is the usual Teichmüller lift and  $S_1 \in \mathbf{Z}[X, Y]$  is an homogeneous polynomial of degree  $p$ :

$$S_1(X, Y) = - \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} X^{p-s} Y^s. \quad (6)$$

An immediate manipulation gives

$$S_1(X - Y, Y) = -S_1(X, -Y). \quad (7)$$

### 3. Structure theorems for universal representations

The aim of this section is to introduce some structure theorems for the universal representation  $\pi(\sigma, 0, 1)$  of  $\mathbf{GL}_2$ . These results concern both the  $KZ$ -restriction and the  $N$ -restriction of  $\pi(\sigma, 0, 1)$

and show that the behavior of universal representations is controlled by a certain, explicit, Iwasawa module  $R_\infty^-$ . If  $F = \mathbf{Q}_p$  such module turns out to be of finite type over a suitable discrete valuation ring : this is the crucial phenomenology which let us dispose of a complete understanding of irreducible admissible representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ . If  $F$  is a nontrivial finite extension of  $\mathbf{Q}_p$  the situation is extremely delicate:  $R_\infty^-$  is defined over a complete Noetherian regular local ring of Krull dimension  $[F : \mathbf{Q}_p]$  and is not any longer of finite type (if  $F$  is of characteristic  $p$  the module is defined over a non-noetherian profinite ring, and still not of finite type, cf. [Mo6]).

We keep the notation of the previous section, in particular  $\sigma$  is a fixed Serre weight. We invite the reader to refer to [Mo4] for the omitted details.

### 3.1 Refinement of the Iwahori structure

Let  $n \in \mathbf{N}_>$ . Restriction of functions from  $K$  to  $K_0(\varpi)$  gives a  $K_0(\varpi)$ -equivariant exact sequence

$$0 \rightarrow R_n^+ \rightarrow R_n \rightarrow \text{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}(\sigma^{(n)}) \rightarrow 0$$

which is easily checked to be split, therefore realizing the Mackey decomposition for  $R_n|_{K_0(\varpi)}$ . We thus define for  $n \geq 1$

$$R_n^- \stackrel{\text{def}}{=} \text{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}(\sigma^{(n)})$$

and one verifies (cf. [Mo4], §3.1) that the partial Hecke morphisms  $T_n^\pm$  give rise to a family of  $K_0(\varpi)$ -equivariant morphisms

$$\begin{aligned} (T_n^+)^{\text{neg}} : R_n^- &\hookrightarrow R_{n+1}^-, & (T_n^+)^{\text{pos}} : R_n^+ &\hookrightarrow R_{n+1}^+ \\ (T_{n+1}^-)^{\text{neg}} : R_{n+1}^- &\twoheadrightarrow R_n^-, & (T_{n+1}^-)^{\text{pos}} : R_{n+1}^+ &\twoheadrightarrow R_n^+. \end{aligned}$$

For technical reasons we define

$$R_0^+ \stackrel{\text{def}}{=} R_0|_{K_0(\varpi)}, \quad R_0^- \stackrel{\text{def}}{=} \text{cosoc}_{K_0(\varpi)}(R_1^-), \quad R_{-1}^+ \stackrel{\text{def}}{=} \text{cosoc}_{K_0(\varpi)}(R_0^+), \quad R_{-1}^- \stackrel{\text{def}}{=} 0$$

as well as the operators

$$\begin{aligned} (T_0^+)^{\text{neg}} : R_0^- &\xrightarrow{0} R_1^-, & (T_0^+)^{\text{pos}} : R_0^+ &\hookrightarrow R_1 \twoheadrightarrow R_1^+ \\ (T_1^-)^{\text{neg}} : R_1^- &\twoheadrightarrow R_0^-, & (T_1^-)^{\text{pos}} &= T_1^-|_{R_1^+} : R_1^+ \twoheadrightarrow R_0^+ \\ (T_0^-)^{\text{neg}} : R_0^- &\xrightarrow{0} R_{-1}^-, & (T_0^-)^{\text{pos}} : R_0^+ &\twoheadrightarrow R_{-1}^+. \end{aligned}$$

We leave as an exercise to the reader to check that the morphism  $(T_0^+)^{\text{pos}}$  is injective and the amalgamated sum  $R_{-1}^+ \oplus_{R_0^+} R_1^+$  with respect to the couple  $(-(T_0^-)^{\text{pos}}, (T_0^+)^{\text{pos}})$  is canonically isomorphic to the image of the  $K_0(\varpi)$ -morphism  $R_1^+ \rightarrow R_1 \rightarrow R_1/R_0$ .

Following the procedures of section 2.1 we can construct inductive systems of amalgamated sums via the partial Hecke operators  $(T_n^\pm)^{\text{pos, neg}}$ :

$$\left\{ R_{\bullet}^* \oplus_{R_{\bullet+1}^*} \cdots \oplus_{R_n^*} R_{n+1}^* \right\}_{n \in 2\mathbf{N} + \bullet + 1} \quad (8)$$

where  $\bullet \in \{0, -1\}$ ,  $* \in \{+, -\}$ .

For  $\bullet \in \{0, -1\}$ ,  $* \in \{+, -\}$  we write

$$R_{\infty, \bullet}^* \stackrel{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N} + \bullet + 1} R_{\bullet}^* \oplus_{R_{\bullet+1}^*} \cdots \oplus_{R_n^*} R_{n+1}^*.$$

The relation between the amalgamated sums (8) and the ones defined in §2.1 is given by the following

PROPOSITION 3.1. *Let  $\bullet \in \{0, -1\}$ ,  $\ast \in \{+, -\}$  and  $n \in 2\mathbf{N} + \bullet + 1$ ,  $n \geq 2$ . We have a commutative diagram of  $K_0(\varpi)$ -representations, with exact rows*

$$\begin{array}{ccccccc}
 0 & \xrightarrow{\quad} & R_n^\ast & \xrightarrow{\quad} & R_{n+1}^\ast & \xrightarrow{\quad} & R_{n+1}^\ast/R_n^\ast \rightarrow 0 \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
 0 & \xrightarrow{\quad} & R_n & \xrightarrow{\quad} & R_{n+1} & \xrightarrow{\quad} & R_{n+1}/R_n \rightarrow 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow \\
 0 & \xrightarrow{\quad} & R_n^\ast \oplus R_{n+1}^\ast & \xrightarrow{\quad} & R_n^\ast \oplus R_{n+1}^\ast & \xrightarrow{\quad} & R_{n+1}^\ast/R_n^\ast \rightarrow 0 \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & (R_\bullet \oplus R_{\bullet+1} \cdots \oplus R_{n-2} R_{n-1})|_{K_0(\varpi)} & \xrightarrow{\quad} & (R_\bullet \oplus R_{\bullet+1} \cdots \oplus R_{n-2} R_{n+1})|_{K_0(\varpi)} & \xrightarrow{\quad} & (R_{n+1}/R_n)|_{K_0(\varpi)} & \rightarrow 0.
 \end{array}$$

*Proof.* This is proved in [Mo4], in the proof of Proposition 3.5, by induction.  $\square$

REMARK 3.2. *We write explicitly the morphisms which initialize the inductive argument of Proposition 3.1. Concerning  $R_0^+$ ,  $R_1^-$  we have the evident monomorphisms*

$$R_0^+ \xrightarrow{\sim} R_0; \quad R_{-1}^- \oplus_{R_0^-} R_1^- = R_1^- \hookrightarrow R_1;$$

concerning  $R_0^-$  we have

$$R_0^- \hookrightarrow R_0 \\ e \mapsto \underline{Y}^r.$$

Finally, it is easy to verify that the morphism  $R_{-1}^+ \oplus_{R_0^+} R_1^+ \hookrightarrow R_1/R_0$  is induced by the couple:

$$R_1^+ \rightarrow R_1/R_0; \quad R_{-1}^+ \hookrightarrow R_1/R_0; \\ e \mapsto [1, \underline{X}^r].$$

It is therefore convenient to write  $\underline{Y}^r$ ,  $\underline{X}^r$  for a linear basis for  $R_0^-$  and  $R_{-1}^+$  respectively.

We are now ready to introduce the first structure theorem for universal representations of  $\mathbf{GL}_2$ . If  $\chi$  is a smooth regular character of  $K_0(\varpi)$  we write

$$\mathrm{Rad}(\chi) \stackrel{\mathrm{def}}{=} \mathrm{Rad}\left(\mathrm{ind}_{K_0(\varpi)}^K \chi\right), \quad \mathrm{Soc}(\chi) \stackrel{\mathrm{def}}{=} \mathrm{Soc}\left(\mathrm{ind}_{K_0(\varpi)}^K \chi\right). \quad (9)$$

We have

PROPOSITION 3.3. *For any  $n \in 2\mathbf{N} + 1$ ,  $m \in 2\mathbf{N}$  we have the following exact sequence of  $k[K]$ -modules:*

$$\begin{array}{l}
 0 \rightarrow \mathrm{Rad}(\chi_r) \rightarrow \mathrm{ind}_{K_0(\varpi)}^K (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \rightarrow R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \rightarrow 0 \\
 0 \rightarrow \mathrm{Soc}(\chi_r^s) \rightarrow \mathrm{ind}_{K_0(\varpi)}^K (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_m^-} R_{m+1}^-) \rightarrow (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1} \rightarrow 0
 \end{array}$$

where we define

$$\mathrm{Rad}(\chi_r) \stackrel{\mathrm{def}}{=} \overline{St}, \quad \mathrm{Soc}(\chi_r^s) \stackrel{\mathrm{def}}{=} 1 \quad \text{if } r = \underline{0} \quad (10)$$

$$\mathrm{Rad}(\chi_r) \stackrel{\mathrm{def}}{=} 1, \quad \mathrm{Soc}(\chi_r^s) \stackrel{\mathrm{def}}{=} \overline{St} \quad \text{if } r = \underline{p-1} \quad (11)$$

*Proof.* We start proving the first exact sequence. Recall that  $R_0^-$  is isomorphic to the character  $\chi_{\underline{r}}$  and therefore

$$0 \rightarrow \text{Rad}(\chi_{\underline{r}}) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_0^-) \rightarrow R_0 \rightarrow 0$$

is true by definition. By induction we assume the statement holds true for all  $-1 \leq j \leq n-2$ ,  $j$  odd (the case  $j = -1$  being the initialization of the inductive argument).

Recall that for all  $i \in \mathbf{N}_>$  the natural  $K_0(\varpi)$ -monomorphism  $R_i^- \hookrightarrow R_i$  gives rise to a  $K$ -isomorphism  $\text{ind}_{K_0(\varpi)}^K R_i^- \xrightarrow{\sim} R_i$  and, by exactness of compact induction, we get an exact sequence

$$0 \rightarrow \text{ind}_{K_0(\varpi)}^K(R_n^-) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_{n+1}^-) \rightarrow R_{n+1}/R_n \rightarrow 0.$$

We therefore deduce from Proposition 3.1, using Frobenius reciprocity and exactness of compact induction, the following commutative diagram with exact lines

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(R_n^-) & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(R_{n+1}^-) & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & R_n & \longrightarrow & R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(\cdots \oplus_{R_{n-2}^-} R_{n-1}^-) & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(\cdots \oplus_{R_n^-} R_{n+1}^-) & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \cdots \oplus_{R_{n-2}} R_{n-1} & \longrightarrow & \cdots \oplus_{R_n} R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0.
 \end{array}$$

In particular we deduce

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(\cdots \oplus_{R_{n-2}^-} R_{n-1}^-) & \longrightarrow & \text{ind}_{K_0(\varpi)}^K(\cdots \oplus_{R_n^-} R_{n+1}^-) & \longrightarrow & R_{n+1}/R_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \cdots \oplus_{R_{n-2}} R_{n-1} & \longrightarrow & \cdots \oplus_{R_n} R_{n+1} & \longrightarrow & R_{n+1}/R_n \longrightarrow 0
 \end{array}$$

and the conclusion follows from the Snake lemma and the inductive hypothesis on the morphism  $\text{ind}_{K_0(\varpi)}^K(\cdots \oplus_{R_{n-2}^-} R_{n-1}^-) \rightarrow \cdots \oplus_{R_{n-2}} R_{n-1}$ .

The second exact sequence is proved in the evident, similar fashion, noticing that the  $K$ -subrepresentation of  $R_1/R_0$  generated by  $[1, \underline{X}^{\underline{r}}]$  is isomorphic to  $\text{coker}(\text{Soc}(\chi_{\underline{r}}^s) \rightarrow \text{ind}_{K_0(\varpi)}^K \chi_{\underline{r}}^s)$ .  $\square$

As a corollary, we deduce the first structure theorem for the universal representations of  $\mathbf{GL}_2$ :

**COROLLARY 3.4.** *We have the following exact sequences of  $k[K]$ -modules:*

$$\begin{array}{l}
 0 \rightarrow \text{Rad}(\chi_{\underline{r}}) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_{\infty,0}^-) \rightarrow R_{\infty,0} \rightarrow 0 \\
 0 \rightarrow \text{Soc}(\chi_{\underline{r}}^s) \rightarrow \text{ind}_{K_0(\varpi)}^K(R_{\infty,-1}^-) \rightarrow R_{\infty,-1} \rightarrow 0
 \end{array}$$

where  $\text{Rad}(\chi_{\underline{r}})$ ,  $\text{Soc}(\chi_{\underline{r}}^s)$  are defined by (9), (10), (11) according to the  $f$ -tuple  $\underline{r}$ .

*Proof.* The functor  $\text{ind}_{K_0(\varpi)}^K(-)$  commutes with co-limits, as it is exact and commutes with (arbitrary) co-products. Since filtered co-limits are exact the result follow from Proposition 3.3.  $\square$

We can now introduce the second structure theorem for universal representations of  $\mathbf{GL}_2$ , which gives a description of the action of the normalizer  $N$  of the Iwahori subgroup  $K_0(\varpi)$  on  $\pi(\sigma, 0, 1)$ .

We start recalling the structure theorem for the  $K_0(\varpi)$ -restriction of  $\pi(\sigma, 0, 1)$  (cf. [Mo4], Proposition 3.5).

PROPOSITION 3.5. *We have the following  $K_0(\varpi)$ -equivariant exact sequences*

$$\begin{aligned} 0 \rightarrow W_1 \rightarrow R_{\infty,0}^+ \oplus R_{\infty,0}^- &\rightarrow R_{\infty,0}|_{K_0(\varpi)} \rightarrow 0 \\ 0 \rightarrow W_2 \rightarrow R_{\infty,-1}^+ \oplus R_{\infty,-1}^- &\rightarrow R_{\infty,-1}|_{K_0(\varpi)} \rightarrow 0 \end{aligned}$$

where  $W_1, W_2$  are the 1-dimensional spaces defined by  $W_1 \stackrel{\text{def}}{=} \langle \underline{Y}^r, -\underline{Y}^r \rangle$  and  $W_2 \stackrel{\text{def}}{=} \langle \underline{X}^r, -\underline{X}^r \rangle$ .

*Proof.* This is [Mo4], Proposition 3.5 (notice that, in the notation of *loc. cit.*, the elements  $(-1)^r F_r^{(0)}(0)$  and  $-[1, \underline{X}^r]$  of  $R_1$  coincide in the quotient  $R_1/R_0$ , by the definition of the operator  $T_0$ ).  $\square$

In order to control the action of the normalizer  $N$  we are therefore left to study the action of the element  $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ . The result is the following:

PROPOSITION 3.6. *There exists two  $K_0(\varpi)$ -equivariant isomorphisms*

$$\begin{aligned} \iota_{-1} : R_{\infty,-1}^- &\xrightarrow{\sim} (R_{\infty,0}^+)^s \\ \iota_0 : R_{\infty,0}^- &\xrightarrow{\sim} (R_{\infty,-1}^+)^s \end{aligned}$$

such that

i)  $\iota_{-1}(\underline{X}^r) = \underline{Y}^r$  and  $\iota_0(\underline{Y}^r) = \underline{X}^r$ ;

ii) *The isomorphisms  $\iota_{-1}, \iota_0$  induce a commutative diagram (of  $k$ -linear spaces) with exact lines*

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_1 \oplus W_2 & \longrightarrow & R_{\infty,0}^- \oplus R_{\infty,0}^+ \oplus R_{\infty,-1}^- \oplus R_{\infty,-1}^+ & \longrightarrow & \pi(\sigma, 0, 1) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & W_1 \oplus W_2 & \longrightarrow & R_{\infty,0}^- \oplus R_{\infty,0}^+ \oplus R_{\infty,-1}^- \oplus R_{\infty,-1}^+ & \longrightarrow & \pi(\sigma, 0, 1) \longrightarrow 0 \end{array}$$

where the right vertical arrow is the automorphism induced by the action of  $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$  on  $\pi(\sigma, 0, 1)$ .

*Proof.* We start showing that, for any  $n \geq -1$ , we have a  $K_0(\varpi)$ -equivariant isomorphism

$$r_n : R_{n+1}^- \xrightarrow{\sim} (R_n^+)^s.$$

The case  $n = -1$  is trivial, as the spaces  $R_0^-, R_{-1}^+$  are 1-dimensional, affording the characters  $\chi_r$  and  $\chi_r^s$  respectively and we have  $R_0^- = \langle \underline{Y}^r \rangle$ ,  $R_{-1}^+ = \langle \underline{X}^r \rangle$  via the equivariant embeddings  $R_0^- \hookrightarrow R_0$ ,  $R_{-1}^+ \hookrightarrow (R_1/R_0)^+$  respectively (cf. Remark 3.2).

Assume now  $n \geq 0$ . Recall that for any  $j \geq 0$  we have a  $K$ -equivariant isomorphism (cf. (3)):

$$\begin{aligned} \text{ind}_{K_0(\varpi^j)}^K(\sigma^{(j)}) &\xrightarrow{\sim} k[K\lambda_j(\varpi)KZ] \otimes_{k[KZ]} \sigma \\ [1, v] &\mapsto \begin{bmatrix} 0 & 1 \\ \varpi^j & 0 \end{bmatrix} \otimes (\omega \cdot v). \end{aligned}$$

We deduce the following  $k$ -linear morphism:

$$R_{n+1}^- \hookrightarrow R_{n+1} \xrightarrow{\sim} k[K\lambda_{n+1}(\varpi)KZ] \otimes_{k[KZ]} \sigma \xrightarrow{\sim} k[K\lambda_n(\varpi)KZ] \otimes_{k[KZ]} \sigma \xrightarrow{\sim} R_n \rightarrow R_n^+$$



where the central arrow is induced by the action of  $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$  on the compact induction  $(\text{ind}_{KZ}^G \sigma)|_{KZ}$ .

As  $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$  normalizes  $K_0(\varpi)$  we deduce that the composite arrow  $r_n : R_{n+1}^- \rightarrow (R_n^+)^s$  is  $K_0(\varpi)$ -equivariant and an easy check shows that  $r_n$  is an epimorphism, hence an isomorphism by dimension reasons. As the Hecke operator  $T$  is equivariant, we deduce furthermore that the diagram

$$\begin{array}{ccc} R_{n+1}^- & \xrightarrow[r_n]{\sim} & (R_n^+)^s \\ (T_{n+1}^-)^{\text{neg}} \downarrow & & \downarrow (T_n^-)^{\text{pos}} \\ R_n^- & \xrightarrow[r_{n-1}]{\sim} & (R_{n-1}^+)^s \end{array} \quad (12)$$

commutes for all  $n \geq 0$ .

Finally, for  $n = 0$ , we have  $r_0(\underline{X}^r) = \underline{Y}^r$  (this can be checked by hand, or noticing that, by  $K_0(\varpi)$ -equivariance,  $r_0$  induces an isomorphism between the highest weight spaces of the representations  $R_1^-, R_0^+$ ).

The Proposition will be completely proved once we show that for any  $n \geq -1$  we have a  $K_0(\varpi)$ -equivariant isomorphism  $f_n : \cdots \oplus_{R_n^-} R_{n+1}^- \rightarrow (\cdots \oplus_{R_{n-1}^+} R_n^+)^s$  which verifies the prescribed conditions on the images of  $\underline{X}^r, \underline{Y}^r$  (for  $n$  even, odd respectively).

We treat the case when  $n$  is even, the other being symmetric. It is an induction on  $n$  where the case  $n = 0$  is given by  $r_0 : R_1^- \xrightarrow{\sim} (R_0^+)^s$ .

Assume  $n \geq 2$  and that we have an isomorphism  $f_{n-2}$  making the following diagram commute:

$$\begin{array}{ccc} R_n^- & \longrightarrow & R_1^- \oplus_{R_2^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^- \\ \wr \downarrow & & \downarrow \wr \\ (R_{n-1}^+)^s & \longrightarrow & (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-3}^+} R_{n-2}^+)^s \end{array}$$

(with the prescribed property on the image of the element  $\underline{X}^r \in R_1^-$ ).

Using (12) we deduce the commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_n^- & \longrightarrow & R_{n+1}^- & \longrightarrow & R_{n+1}^-/R_n^- \longrightarrow 0 \\ & & \searrow \cong & \downarrow & \searrow \cong & \downarrow & \downarrow \cong \\ 0 & \longrightarrow & (R_{n-1}^+)^s & \longrightarrow & (R_n^+)^s & \longrightarrow & (R_n^+/R_{n-1}^+)^s \longrightarrow 0 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \cong \\ 0 & \longrightarrow & \cdots \oplus_{R_{n-2}^-} R_{n-1}^- & \longrightarrow & \cdots \oplus_{R_n^-} R_{n+1}^- & \longrightarrow & R_{n+1}^-/R_n^- \longrightarrow 0 \\ & & \searrow f_{n-2} \cong & \downarrow & \searrow f_n \exists! & \downarrow & \downarrow \cong \\ 0 & \longrightarrow & (\cdots \oplus_{R_{n-3}^+} R_{n-2}^+)^s & \longrightarrow & (\cdots \oplus_{R_{n-1}^+} R_n^+)^s & \longrightarrow & (R_{n+1}^+/R_n^+)^s \longrightarrow 0 \end{array}$$

where the morphism  $f_n$  is obtained from the universal property of  $\cdots \oplus_{R_n^-} R_{n+1}^-$  (notice also that

$(\cdots \oplus_{R_{n-1}^+} R_n^+)^s = \cdots \oplus_{(R_{n-1}^+)^s} (R_n^+)^s$ . The morphism  $f_n$  is moreover an isomorphism, verifying the prescribed property on the image of the element  $\underline{X}^r \in R_1^-$ .

This completes the inductive step and, passing to co-limits, one gets the isomorphism  $\iota_{-1}$  as in the statement.  $\square$

### 3.2 The case $F = \mathbf{Q}_p$

We specialize some of the previous constructions to the case  $F = \mathbf{Q}_p$ . By Corollary 3.4, Proposition 3.5 and Proposition 3.6 we see that the structure of the universal representation  $\pi(\sigma, 0, 1)$  depends crucially on the Iwasawa modules  $R_{\infty, \bullet}^-$ , where  $\bullet \in \{-1, 0\}$ .

It can be shown that  $R_{\infty, \bullet}^-$  is a module on the Iwasawa algebra  $k[[\mathcal{O}_F]]$ , and is *not* of finite type as soon as  $F \neq \mathbf{Q}_p$ . Moreover  $k[[\mathcal{O}_F]]$  is a complete regular noetherian local ring of dimension  $[F : \mathbf{Q}_p]$  if  $F$  is a finite extension of  $\mathbf{Q}_p$  and is not even noetherian if  $\text{char}(F) = p$  (see also [Mo5], [Mo6]).

When  $F = \mathbf{Q}_p$  the situation is much simpler :  $R_{\infty, \bullet}^-$  turns out to be a monogenous module over a discrete valuation ring.

We start recalling the following result:

**PROPOSITION 3.7.** *Let  $n \geq 0$ . The  $k[K_0(p)]$ -module  $R_{n+1}^-$  is uniserial, of dimension  $(r+1)p^n$ , and its socle filtration is described by*

$$\chi_r^s - \chi_r^s \mathbf{a} - \chi_r^s \mathbf{a}^2 - \dots - \chi_r^s \mathbf{a}^{(r+1)p^n - 1}.$$

*Proof.* This is [Mo4], Theorem 5.18. It can equally be deduced from [Mo1], Proposition 5.10 and the fact that we have a  $K_0(p^{n+1})$ -equivariant embedding  $\sigma^{(n+1)} \hookrightarrow \text{ind}_{K_0(p^{n+2})}^{K_0(p^{n+1})} \chi_r^s$ .  $\square$

The statement of Proposition 3.7 can be made more expressive.

We recall from [Mo4] that for a finite unramified extension  $F/\mathbf{Q}_p$  the module  $k[K_0(p)]$  admits a linear basis  $\mathcal{B}_{n+1}^-$  which is endowed with a partial order (cf. *op. cit.*, Lemma 2.6). The partial ordering on  $\mathcal{B}_{n+1}^-$  induces therefore a  $k$ -linear filtration on the space  $R_{n+1}^-$  and one of the main result of [Mo4] (cf. *op. cit.*, Proposition 4.10) is to show that such filtration is  $K_0(p)$ -stable.

When  $F = \mathbf{Q}_p$  such ordering is indeed *total* (this is linked with the aforementioned phenomenon that we are considering modules over a complete local noetherian regular ring of Krull dimension  $[F : \mathbf{Q}_p]$ ).

Explicitly, we have a bijection

$$\begin{aligned} \{0, \dots, p-1\}^n \times \{0, \dots, r\} &\xrightarrow{\sim} \mathcal{B}_{n+1}^- \\ (l_1, \dots, l_{n+1}) &\mapsto F_{(l_1, \dots, l_n)}^{(1, n)}(l_{n+1}) \end{aligned} \quad (13)$$

where we define the element

$$F_{(l_1, \dots, l_n)}^{(1, n)}(l_{n+1}) \stackrel{\text{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \lambda_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-l_{n+1}} Y^{l_{n+1}}] \in R_{n+1}^-.$$

The total ordering on  $\mathcal{B}_{n+1}^-$  is then induced from the order of  $\mathbf{N}$  via the *injective* map

$$\begin{aligned} \mathcal{B}_{n+1}^- &\xrightarrow{P} \mathbf{N} \\ F_{(l_1, \dots, l_n)}^{(1, n)}(l_{n+1}) &\mapsto P(F_{(l_1, \dots, l_n)}^{(1, n)}(l_{n+1})) \stackrel{\text{def}}{=} \sum_{j=0}^n p^j l_{j+1} \end{aligned}$$

(and thus coincides with the anti-lexicographical order  $\prec$  on the LHS of (13)). If  $F_1, F_2 \in \mathcal{B}_{n+1}^-$ , we write  $F_1 \prec F_2$  if  $P(F_1) < P(F_2)$ .

Since  $R_{n+1}^-$  is uniserial, it is easy to describe the amalgamated sum  $\cdots \oplus_{R_n^-} R_{n+1}^-$ :

PROPOSITION 3.8. *Let  $n \geq 1$ . The kernel of the projection map  $R_{n+1}^- \rightarrow \cdots \oplus_{R_n^-} R_{n+1}^-$  is described by:*

$$\ker(R_{n+1}^- \rightarrow \cdots \oplus_{R_n^-} R_{n+1}^-) = \begin{cases} \langle F \in \mathcal{B}_{n+1}^-, F \prec F_{r,p-1-r,\dots,p-1-r,r}^{(1,n)}(0) \rangle & \text{if } n \in 2\mathbf{N} + 1 \\ \langle F \in \mathcal{B}_{n+1}^-, F \prec F_{p-1-r,r,\dots,p-1-r,r}^{(1,n)}(0) \rangle & \text{if } n \in 2\mathbf{N} + 2. \end{cases}$$

*Proof.* We consider the case where  $n$  is odd (the other is similar). Since  $R_{n+1}^-$  is uniserial and the linear filtration on  $R_{n+1}^-$  induced by the linear order on  $\mathcal{B}_{n+1}^-$  coincides with the socle filtration, it will be enough to show that

$$\dim(\langle F \in \mathcal{B}_{n+1}^-, F \prec F_{r,p-1-r,\dots,p-1-r,r}^{(1,n)}(0) \rangle) = \dim(R_{n+1}^-) - \dim(\cdots \oplus_{R_n^-} R_{n+1}^-).$$

This is a straightforward check: indeed

$$\begin{aligned} \dim(\langle F \in \mathcal{B}_{n+1}^-, F \prec F_{r,p-1-r,\dots,p-1-r,r}^{(1,n)}(0) \rangle) &= r \left( \sum_{j=0}^{\frac{n-1}{2}} p^{2j} \right) + p(p-1-r) \left( \sum_{j=0}^{\frac{n-3}{2}} p^{2j} \right) \\ &= (p-r) \frac{p^{n-1} - 1}{p+1} + rp^{n-1} \end{aligned}$$

and

$$\dim(\cdots \oplus_{R_n^-} R_{n+1}^-) = 1 + (r+1)(-1) \sum_{j=0}^n (-p)^j = (r+1) \frac{p^{n+1} - 1}{p+1} + 1;$$

$$\dim(R_{n+1}^-) = p^n(r+1).$$

□

As  $R_{\infty,\bullet}^-$  is a co-limit of the modules  $\cdots \oplus_{R_n^-} R_{n+1}^-$ , the transition maps being monomorphisms, we deduce

COROLLARY 3.9. *For  $\bullet \in \{0, -1\}$  the  $k[K_0(p)]$ -module  $R_{\infty,\bullet}^-$  is uniserial. Its socle filtration is described by*

$$\begin{aligned} (R_{\infty,0}^-) : & \quad \chi_r^s \mathbf{a}^r - \chi_r^s \mathbf{a}^{r+1} - \chi_r^s \mathbf{a}^{r+2} - \dots \\ (R_{\infty,-1}^-) : & \quad \chi_r^s - \chi_r^s \mathbf{a} - \chi_r^s \mathbf{a}^2 - \dots \end{aligned}$$

respectively.

We write  $\{\mathcal{F}_n\}_{n \in \mathbf{N}}$  for the socle filtration for  $R_{\infty,0}^-$  (in particular,  $\mathcal{F}_0$  is the socle of  $R_{\infty,0}^-$  and  $\mathcal{F}_n$  is the  $n+1$ -dimensional sub-module of  $R_{\infty,0}^-$ ).

#### 4. Study of $K_t$ and $I_t$ invariants

In this section we assume  $F = \mathbf{Q}_p$ . The aim is to describe in detail the  $K_t$  and  $I_t$  invariants for supersingular representations  $\pi(\sigma, 0, 1)$  of  $\mathbf{GL}_2(\mathbf{Q}_p)$ .

Thanks to the structure theorems of §3 we are essentially left to understand the invariants for the Iwasawa modules  $R_{\infty,\bullet}^-$ . This is developed in section 4.1: the argument follows easily from the uniserial property of  $R_{\infty,\bullet}^-$ , but one should carefully carry out computations in order to handle some delicate  $K$ -extensions which will appear later on in section 4.2.

The invariants for the supersingular representation  $\pi(\sigma, 0, 1)$  will be then determined in §4.2, combining the results on  $R_{\infty,\bullet}^-$  with the structure theorems.

#### 4.1 Invariants for the Iwasawa modules $R_{\infty, \bullet}^-$

We are going to describe in detail the spaces of  $K_t$  invariants (resp.  $I_t$  invariants) for the  $k[K_0(p)]$ -modules  $R_{\infty, \bullet}^-$  (resp.  $R_{\infty, \bullet}^*$ ).

Recall (§1.1) that for a Serre weight  $\sigma$  we write  $\sigma^{[s]}$  for its conjugate weight. We start from the following

LEMMA 4.1. *The intertwining operator  $\pi(\sigma, 0, 1) \xrightarrow{\sim} \pi(\sigma^{[s]}, 0, 1)$  induces a  $KZ$ -isomorphism*

$$R_{\infty, 0}(\sigma) \xrightarrow{\sim} R_{\infty, -1}(\sigma^{[s]}).$$

*Proof.* We have a  $KZ$ -equivariant monomorphism

$$R_{\infty, 0}(\sigma) \hookrightarrow \pi(\sigma, 0, 1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^s, 0, 1)|_{KZ} \xrightarrow{\sim} R_{\infty, 0}(\sigma^{[s]}) \oplus R_{\infty, -1}(\sigma^{[s]}).$$

As  $R_{\infty, 0}(\sigma)$  and  $R_{\infty, 0}(\sigma^{[s]})$  have irreducible non isomorphic socles, we deduce that the composite

$$\phi_1 : R_{\infty, 0}(\sigma) \hookrightarrow \pi(\sigma, 0, 1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^{[s]}, 0, 1)|_{KZ} \twoheadrightarrow R_{\infty, -1}(\sigma^{[s]})$$

is a  $KZ$ -equivariant monomorphism. Similarly, the composite

$$\phi_2 : R_{\infty, -1}(\sigma) \hookrightarrow \pi(\sigma, 0, 1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^{[s]}, 0, 1)|_{KZ} \twoheadrightarrow R_{\infty, 0}(\sigma^{[s]})$$

is a  $KZ$ -equivariant monomorphism. As  $\phi_1 \oplus \phi_2$  coincides (by construction) with the intertwining operator  $\pi(\sigma, 0, 1) \xrightarrow{\sim} \pi(\sigma^{[s]}, 0, 1)$  via the isomorphism (4) we deduce that  $\phi_1, \phi_2$  are epimorphisms and the proof is complete.  $\square$

By virtue of Lemma 4.1 (resp. Proposition 3.6) it will be enough to study the  $K_t$  invariants (resp.  $I_t$  invariants) for the Iwasawa module  $R_{\infty, 0}^-$  (resp.  $R_{\infty, 0}^-$  and  $R_{\infty, -1}^-$ ).

Recall that  $R_{\infty, 0}^-$  is uniserial, and we denoted by  $\{\mathcal{F}_n\}_{n \in \mathbf{N}}$  its socle filtration (cf. Corollary 3.9). The  $K_t$  invariants of  $R_{\infty, 0}^-$  are then described by the following

PROPOSITION 4.2. *Let  $t \geq 1$ . We have a  $K_0(p)$ -equivariant exact sequence*

$$0 \rightarrow (R_{\infty, 0}^-)^{K_t} \rightarrow \mathcal{F}_{p^t-1} \rightarrow \chi_r^s \mathfrak{a}^{r+1} \rightarrow 0.$$

Moreover, for any lift  $e_1 \in \mathcal{F}_{p^t-1}$  of a linear basis of  $\chi_r^s \mathfrak{a}^{r+1}$  we have

$$\left( \left[ \begin{array}{cc} 1+pa & b \\ pc & 1+pd \end{array} \right] - 1 \right) \cdot e_1 = \bar{c} \kappa_{e_1} e_0$$

where  $a, b, c, d \in \mathbf{Z}_p$ ,  $\kappa_{e_1} \in k^\times$  is a suitable nonzero scalar depending only on  $e_1$  and  $e_0$  is a linear generator of  $\text{soc}(R_{\infty, 0}^-)$ .

*Proof.* As  $R_{\infty, 0}^-$  is admissible uniserial and  $K_t$  is normal in  $K_0(p)$  we deduce that  $(R_{\infty, 0}^-)^{K_t} = \mathcal{F}_{n(t)}$  where  $n(t) \in \mathbf{N}$  is defined by

$$n(t) = \max \left\{ n \in \mathbf{N}, \quad \text{s.t. } \mathcal{F}_n = (\mathcal{F}_n)^{K_t} \right\}$$

and hence we are left to prove that  $n(t) = p^{t-1} - 1$  (an elementary computation shows that the graded piece  $\mathcal{F}_{p^t-1}/\mathcal{F}_{p^t-1-1}$  affords the character  $\chi_r^s \mathfrak{a}^{r+1}$ ).

This will be a careful computation, using the properties of Witt polynomials. We remark that the cases where  $r \in \{0, p-1\}$  are slightly more delicate to verify.

We start from the following

LEMMA 4.3. Let  $K_1(p^{t+2})$  be the maximal pro- $p$  subgroup of  $K_0(p^{t+2})$ . Let  $z \stackrel{\text{def}}{=} \sum_{j=1}^t p^j[\lambda_j] \in \mathbf{Z}_p$  and  $a, b, c, d \in \mathbf{Z}_p$ . Then we have

$$\begin{bmatrix} 1 + p^t a & p^t b \\ p^t c & 1 + p^t d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}^{\kappa'}$$

for a suitable element  $\kappa' \in K_1(p^{t+2})$  and

$$z' = \sum_{j=1}^{t-1} p^j[\lambda_j] + p^t[\lambda_t + \bar{c}] + p^{t+1}[S_1(\lambda_t, \bar{c}) + r(\lambda_1)]$$

where  $S_1(\lambda_t, \bar{c})$  is the specialization of the Witt polynomial (6) and  $r \in \mathbf{F}_p[\lambda_1]$  is a linear polynomial in  $\lambda_1$  depending on  $a, b, c, d$ .

*Proof.* We have

$$\begin{bmatrix} 1 & 0 \\ -z' & 1 \end{bmatrix} \begin{bmatrix} 1 + p^t a & p^t b \\ p^t c & 1 + p^t d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 + p^t(a + bz) & p^t b \\ w & 1 + p^t(d - bz') \end{bmatrix}$$

where  $w \stackrel{\text{def}}{=} -z'(1 + p^t(a + bz)) + z + p^t(c + dz)$ . Thus  $z' \equiv (z + p^t c + p^t dz)(1 + p^t a)^{-1} \pmod{p^{t+2}}$  (notice that  $p^t b z z' \equiv 0 \pmod{p^{t+2}}$ ) and we deduce

$$\begin{aligned} z' &\equiv (z + p^t c + p^t dz)(1 - p^t a) \pmod{p^{t+2}} \\ &\equiv z + p^t c + p^t dz - p^t a z - p^{2t} a c \pmod{p^{t+2}} \end{aligned}$$

(the first line is deduced noticing that  $(z + p^t c + p^t dz)p^{2t} \equiv 0 \pmod{p^{t+2}}$  and the second noticing that  $p^{2t} z \equiv 0 \pmod{p^{t+2}}$ ).

The result follows from an immediate computation on Witt vectors.  $\square$

In order to complete the proof of the Proposition we now distinguish two cases.

*Case A:  $t$  is odd.* It suffices to show that  $(R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-)^{K_t}$  is a proper sub  $k[K_0(p)]$ -module of dimension  $p^{t-1}$  sitting inside  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$  (notice that  $\dim(R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-) \geq p^{t-1} + 1$  is verified for all values of  $t \geq 1$ ,  $p \geq 3$  and  $r \in \{0, \dots, p-1\}$ ).

We recall that, for an  $t$ -tuple  $(l_1, \dots, l_t) \in \{0, \dots, p-1\}^t$ , we have

$$F_{l_1, \dots, l_t}^{(1,t)}(0) \equiv \begin{cases} 0 & \text{if } (l_1, \dots, l_t) \prec (r, p-1-r, \dots, p-1-r, r) \\ F_{r, p-1-r, \dots, p-1-r, r}^{(1,t)}(0) & \text{if } (l_1, \dots, l_t) = (r, p-1-r, \dots, p-1-r, r) \end{cases} \quad (14)$$

in  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ , by Proposition 3.8.

We again have to distinguish two situations

*Sub-case A1:  $r < p-1$ .* By the unseriality of  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$  and the compatibility between the  $K_0(p)$ -action and the linear ordering on  $\mathcal{B}_{n+1}^-$  we deduce that the  $p^{t-1} + 1$ -dimensional sub-module of  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$  is generated by the element  $F_{r, p-1-r, \dots, p-1-r, r+1}^{(1,t)}(0) \in \mathcal{B}_{t+1}^-$

If  $(l_1, \dots, l_{t-1}) \in \{0, \dots, p-1\}^{t-1}$  is a  $(t-1)$ -tuple we deduce, from Lemma 4.3 and (14), the

following equality in  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ :

$$\begin{aligned} & \left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F_{l_1, \dots, l_{t-1}, r+1}^{(1,t)}(0) = \\ &= \sum_{j=1}^{r+1} \binom{r+1}{j} (-\bar{c})^j F_{l_1, \dots, l_{t-1}, r+1-j}^{(1,t)}(0) \equiv \\ &\equiv \begin{cases} 0 & \text{if } (l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, p-1-r) \\ -(r+1)\bar{c} F_{r, p-1-r, \dots, p-1-r, r}^{(1,t)}(0) & \text{if } (l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, p-1-r) \end{cases} \end{aligned}$$

This proves the Proposition for  $t$  odd and  $r < p-1$ .

*Sub-case A2:  $r = p-1$ .*

This situation is slightly more delicate and we need to know the properties of the homogeneous degree of the Witt polynomial  $S_1(X, Y)$  defined in (6).

As in case A1 we see that the  $p^{t-1} + 1$ -dimensional sub-module of  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$  is generated by the element  $F_{r, p-1-r, \dots, p-1-r, 0}^{(1,t)}(1) \in \mathcal{B}_{t+1}^-$ .

We now have, for a  $(t-1)$ -tuple  $(l_1, \dots, l_{t-1})$

$$\begin{aligned} & \left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F_{l_1, \dots, l_{t-1}, 0}^{(1,t)}(1) = \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \\ & \cdots \sum_{\lambda_{t-1} \in \mathbf{F}_p} \lambda_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\lambda_{t-1}] & 1 \end{bmatrix} \sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t + \bar{c}] & 1 \end{bmatrix} (S_1(\lambda_t, \bar{c}) + r)[1, X^r] \end{aligned}$$

where  $S_1(\lambda_1, \bar{c}) + r$  is defined as in Lemma 4.3. Thanks to (7) we can write

$$\sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t + \bar{c}] & 1 \end{bmatrix} (S_1(\lambda_t, \bar{c}) + r)[1, X^r] = \sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t] & 1 \end{bmatrix} (-S_1(\lambda_t, -\bar{c}) + r')[1, X^r]$$

where  $r' \in \mathbf{F}_p[\lambda_1]$  is a convenient polynomial of degree 1 in  $\lambda_1$  (depending on  $a, b, c, d$ ). In particular,  $-S_1(\lambda_t, -\bar{c}) + r' = (-\bar{c})\lambda_t^{p-1} + P(\lambda_t)$  for a convenient polynomial of degree  $p-2$  in  $\lambda_t$  and hence, by (14),

$$\left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F_{l_1, \dots, l_{t-1}, 0}^{(1,t)}(1) \equiv (-\bar{c})F_{l_1, \dots, l_{t-1}, p-1}^{1,t}(0).$$

Again, we have

$$F_{l_1, \dots, l_{t-1}, p-1}^{(1,t)}(0) \equiv \begin{cases} 0 & \text{if } (l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, p-1-r) \\ -(r+1)\bar{c} F_{r, p-1-r, \dots, p-1-r, r}^{(1,t)}(0) & \text{if } (l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, p-1-r). \end{cases}$$

This let us conclude the case  $r = p-1$  (the  $K_t$  invariance of the elements  $F_{l_1, \dots, l_t}^{(1,t)}(0)$  is indeed clear).

*Case B:  $t$  is even.*

The argument are completely analogous to those of Case A the details are left to the reader. We distinguish again two situations.

*Sub-case B1:  $r > 0$ .*

We now consider the element  $F_{r, p-1-r, \dots, p-1-r, r}^{(1,t-1)}(1) \in \mathcal{B}_t^-$  as a linear generator for the  $p^{t-1}$ -dimensional sub-module of  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{t-1}^-} R_t^-$ .

As we have seen for the Case A1 we have

$$\begin{aligned} & \left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F_{l_1, \dots, l_{t-1}}^{(1, t-1)}(1) \equiv \\ & \equiv \begin{cases} 0 & \text{if } (l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, p-1-r, r) \\ \bar{c} F_{r, p-1-r, \dots, p-1-r, r}^{(1, t-1)}(0) & \text{if } (l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, p-1-r, r) \end{cases} \end{aligned}$$

*Sub-case B1:*  $r = 0$ . In this situation we have to consider the element  $F_{r, p-1-r, \dots, r, 0, 1}^{(1, t+1)}(0) \in \mathcal{B}_{t+2}^-$  as a linear generator for the  $p^{t-1}$ -dimensional sub-module of  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{t+1}^-} R_{t+2}^-$ .

A direct computation together with an argument on Witt polynomials (as in Case A2) shows that

$$\begin{aligned} & \left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F_{l_1, \dots, l_{t-1}, 0, 1}^{(1, t+1)}(0) \equiv \\ & \equiv \begin{cases} 0 & \text{if } (l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, p-1-r, r) \\ \bar{c} F_{r, p-1-r, \dots, p-1-r, r}^{(1, t+1)}(0) & \text{if } (l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, p-1-r, r) \end{cases} \end{aligned}$$

□

We turn now our attention to the analysis of  $I_t$  invariants for the modules  $R_{\infty, \bullet}^-$ . The result is the following:

PROPOSITION 4.4. *If either  $t \geq 1$  and  $p \geq 5$  or  $t \geq 2$  and  $p = 3$  we have the equalities*

$$(R_{\infty, \bullet}^-)^{I_t} = (R_{\infty, \bullet}^-)^{K_t}.$$

and  $(R_{\infty, \bullet}^-)^{I_t}$  is a  $p^{t-1}$ -dimensional sub-module of  $R_{\infty, \bullet}^-$ .

*Proof.* Using the Iwasawa decomposition and the fact that  $I_t$  is a subgroup of  $K_t$  it suffices to show that any element in  $(R_{\infty, \bullet}^-)^{K_t}$  is fixed under the action of  $\begin{bmatrix} 1 & p^{t-1} \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$ .

Again, we can check this by an explicit argument on Witt vectors. We notice that, for  $z = \sum_{j=1}^{t+1} p^j [\lambda_j]$  and  $b \in \mathbf{Z}_p$  we have

$$\begin{bmatrix} 1 & p^{t-1} b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \kappa$$

where  $\kappa \in K_1(p^{t+2})$  (the maximal pro- $p$  subgroup of  $K_0(p^{t+2})$ ) and  $z' = \sum_{j=1}^t p^j [\lambda_j] + p^{t+1} [\lambda_{t+1} - \lambda_1^2 \bar{b}]$ .

We distinguish several cases according to the values of  $t$  and  $r$ .

*Case A:  $t$  is odd.*

A direct computation gives the following equality inside  $R_{t+1}$ :

$$\left( \begin{bmatrix} 1 & p^{t-1} b \\ 0 & 1 \end{bmatrix} - 1 \right) F_{l_1, \dots, l_t}^{(1, t)}(l_{t+1}) = \begin{cases} 0 & \text{if } l_{t+1} = 0 \\ -\bar{b} F_{[l_1+2], l_2, \dots, l_t}^{(1, t)}(0) & \text{if } l_{t+1} = 1. \end{cases}$$

This proves the result for  $R_{\infty, 0}^-$  when  $t \geq 1$  is odd and  $r < p-1$ , by the description of  $(R_{\infty, 0}^-)^{K_t}$  given in case A1 in the proof of Proposition 4.2.

As far as the case  $r = p-1$  is concerned, we recall that  $F_{[l_1+2], l_2, \dots, l_{t-1}, 0}^{(1, t)}(0) \equiv 0$  inside the amalgamated sum  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$  as soon as  $t > 1$ . If  $t = 1$  we have  $F_2^{(1)}(0) \equiv 0$  as soon as  $2 < r$ . This let us deduce the required result for  $R_{\infty, 0}^-$  when  $r = p-1$  and  $t \geq 2$  (or  $t = 1$  and  $p > 3$ ), using the description of  $(R_{\infty, 0}^-)^{K_t}$  given in case A2 in the proof of Proposition 4.2.

Case B:  $t$  is even. Since

$$\left( \begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} - 1 \right) F_{l_1, \dots, l_{t-1}}^{(1, t-1)}(l_t) = 0$$

inside  $R_t^-$ , the result is clear for  $r > 0$  via the description of  $(R_{\infty, 0}^-)^{K_t}$  (cf. the case B1 in the proof of Proposition 4.2).

Concerning the case  $r = 0$  we similarly have

$$\left( \begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} - 1 \right) F_{l_1, \dots, l_{t-1}, 0, 1}^{(1, t+1)}(0) = \bar{b} F_{[l_1+2], \dots, l_{t-1}, 0, 0}^{(1, t+1)}(0)$$

which is zero in the amalgamated sum  $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{t+1}^-} R_{t+2}^-$ .

The arguments concerning the  $I_t$  invariants for the module  $R_{\infty, -1}^-$  are completely analogous and are left to the reader.  $\square$

## 4.2 Invariants for supersingular representations.

We are now in the position to determine precisely the space of  $K_t, I_t$  fixed vectors for supersingular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ .

The case of  $I_t$  fixed vectors is an immediate consequence of Proposition 3.6 and Proposition 4.4:

PROPOSITION 4.5. *Let  $t \geq 1$  and  $p \geq 5$  (or  $t \geq 2$  and  $p = 3$ ). We have a short exact sequence of  $k[K_0(p)]$ -modules*

$$0 \rightarrow W_1 \rightarrow (R_{\infty, 0}^-)^{I_t} \oplus (R_{\infty, -1}^+)^{I_t} \rightarrow (R_{\infty, 0})^{I_t} \rightarrow 0 \quad (15)$$

(where  $W_1$  is the 1-dimensional space defined in Proposition 3.5).

In particular, for any  $t \geq 1$  and  $p \geq 3$  we have

$$\dim_k(\pi(\sigma, 0, 1))^{I_t} = 2(2p^{t-1} - 1).$$

*Proof.* Applying the functor  $H^0(I_t, -)$  to the first exact sequence of Proposition 3.6 we obtain

$$0 \rightarrow (W_1)^{I_t} \rightarrow (R_{\infty, 0}^-)^{I_t} \oplus (R_{\infty, -1}^+)^{I_t} \rightarrow (R_{\infty, 0})^{I_t}$$

Since the space  $W_1$  is fixed under the action of  $I_t$ , we deduce that the last arrow is indeed surjective, yielding (15).

Thanks to Proposition 4.4 and the isomorphism  $R_{\infty, 0}^- \xrightarrow{\sim} (R_{\infty, -1}^+)^s$  (Proposition 3.6) we deduce that

$$\dim_k(R_{\infty, 0})^{I_t} = (2p^{t-1} - 1)$$

if either  $t \geq 1$  and  $p > 5$  or  $t \geq 2$  and  $p = 3$ , and hence (by generality of  $\sigma$  and Proposition 4.1)

$$\dim_k(\pi(\sigma, 0, 1))^{I_t} = 2(2p^{t-1} - 1).$$

if either  $t \geq 1$  and  $p > 5$  or  $t \geq 2$  and  $p = 3$ . The remaining case  $t = 1$  and  $p = 3$  is covered in [Bre03a] and the proof is complete.  $\square$

We turn our attention to the analysis of  $K_t$  fixed vectors for supersingular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ . We start recalling some results (cf. [Mo1]) concerning the  $KZ$ -socle filtration for the representations  $\text{ind}_{K_0(p)}^K(R_{\infty, 0}^-)$  and  $\pi(\sigma, 0, 1)$ .

The  $K_0(p)$ -socle filtration  $R_{\infty, 0}^-$  induces a  $K$ -equivariant filtration on  $\text{ind}_{K_0(p)}^K(R_{\infty, 0}^-)$  (hence on  $R_{\infty, 0}$ ); the extensions under its first graded pieces look as follow:

$$\text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r \text{---} \text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1} \text{---} \text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+2} \text{---} \text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+3} \text{---} \dots \quad (16)$$



One can show ([Mo1], Lemmas 6.8 and 6.9 or [AM]) that the  $KZ$ -equivariant filtration on  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$  obtained by the evident refinement of (16) is indeed the  $KZ$ -socle filtration for  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$ .

In particular, if  $e_1 \in \mathcal{F}_{p^t-1}$  is a linear generator for the socle

$$\text{soc}_{K_0(p)}\left(R_{\infty,0}^-/\mathcal{F}_{(p^t-1-1)}\right) = \chi_r^s \mathbf{a}^{r+1}$$

we see that

$$\text{soc}_K\left(\text{ind}_{K_0(p)}^K(R_{\infty,0}^-/\mathcal{F}_{(p^t-1-1)})\right) = \text{soc}_K(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^{r+1}),$$

where the finite induction of the RHS is generated under  $K$  by the image of the element  $[1, e_1]$  via the map  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \rightarrow \text{ind}_{K_0(p)}^K(R_{\infty,0}^-/\mathcal{F}_{(p^t-1-1)})$ . Notice moreover that if  $R_{\infty,t}$  denotes the image of  $\text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^t-1-1)})$  inside  $R_{\infty,0}$  via the epimorphism of Corollary 3.4, we have

$$\text{ind}_{K_0(p)}^K(R_{\infty,0}^-/\mathcal{F}_{(p^t-1-1)}) \xrightarrow{\sim} R_{\infty,0}/R_{\infty,t}.$$

We define the following element of  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$ :

$$f(e_1) \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, e_1] - \delta_{r,p-3} [1, e_1] \in \text{ind}_{K_0(p)}^K(R_{\infty,0}^-). \quad (17)$$

By Proposition 2.1 we see that

LEMMA 4.6. *Via the natural epimorphism  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \rightarrow \text{ind}_{K_0(p)}^K(R_{\infty,0}^-/\mathcal{F}_{(p^t-1-1)})$  the element  $f(e_1)$  maps to a highest weight vector for the Serre weight  $\text{Sym}^{\lfloor p-3-r \rfloor} k^2 \otimes \det^{r+1}$  appearing in  $\text{soc}_K(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^{r+1})$ , unless  $(r, p) = (p-1, 3)$  (in which case it maps to a highest weight vector for  $\overline{St} \otimes \det$ ).*

The sub-module  $R_{\infty,t}$  is formed by  $K_t$  fixed vectors. Nevertheless, for  $r \leq p-3$ , it is strictly contained in  $(R_{\infty,0})^{K_t}$ :

LEMMA 4.7. *Assume  $r \leq p-3$ . The natural morphism*

$$\left\langle \text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^t-1-1)}), f(e_1) \right\rangle_{k[K]} \hookrightarrow \text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$$

*factors through  $(R_{\infty,0})^{K_t} \hookrightarrow R_{\infty,0}$ .*

*Proof.* It suffices to show that for any  $\kappa \in K_t$  we have

$$(\kappa - 1) \cdot f(e_1) \in \ker(\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}).$$

For  $a, b, c, d \in \mathbf{Z}_p$  we have

$$\begin{bmatrix} 1 + p^t a & p^t b \\ p^t c & 1 + p^t d \end{bmatrix} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + p^t a' & p^t b' \\ p^t c' & 1 + p^t d' \end{bmatrix} \quad (18)$$

with  $\bar{c}' = \bar{b} + (\overline{a-d})\lambda_0 - \bar{c}\lambda_0^2$ .

We therefore deduce from Proposition 4.2 the following equality in  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$ :

$$\left( \begin{bmatrix} 1 + p^t a & p^t b \\ p^t c & 1 + p^t d \end{bmatrix} - 1 \right) \cdot f(e_1) = \bar{b}f_0(e_0) + (\overline{a-d})f_1(e_0) - \bar{c}(f_2(e_0) + \delta_{r,p-3}[1, e_0]).$$

where  $e_0$  is a convenient linear generator of  $\text{soc}_{K_0(p)}(R_{\infty,0}^-)$ .

Since the kernel of the epimorphism  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$  is linearly generated by the elements

$$\ker(\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}) = \langle f_0(e_0), \dots, f_{p-2-r}(e_0), f_{p-1-r}(e_0) + [1, e_0] \rangle_k$$

(cf. Proposition 2.1) the required result follows.  $\square$

We can now describe completely the  $K_t$  fixed vectors of  $R_{\infty,0}$ :

PROPOSITION 4.8. *Let  $t \geq 1$ . The space of  $K_t$  fixed vectors of  $R_{\infty,0}$  is given by*

$$(R_{\infty,0})^{K_t} = \begin{cases} \left\langle \text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^{t-1}-1)}), f(e_1) \right\rangle_{k[K_t]} & \text{if } r \leq p-3 \\ \text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^{t-1}-1)}) & \text{if } r \in \{p-2, p-1\}. \end{cases}$$

*Proof.* In order to ease notations we write

$$M \stackrel{\text{def}}{=} \begin{cases} \left\langle \text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^{t-1}-1)}), f(e_1) \right\rangle_{k[K]} & \text{if } r \leq p-3 \\ \text{ind}_{K_0(p)}^K(\mathcal{F}_{(p^{t-1}-1)}) & \text{if } r \in \{p-2, p-1\}. \end{cases}$$

Assume that  $R_{\infty,0}^{K_t}/M \neq \{0\}$ .

Then, by the description of the  $K$ -socle filtration for  $R_{\infty,0}$ , we see that

$$\text{soc}(R_{\infty,0}^{K_t}/M) = \text{soc}(R_{\infty,0}/M) = \begin{cases} \text{cosoc}(\text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1}) & \text{if } r \leq p-4 \\ \overline{St} \otimes \det^{-1} & \text{if } r = p-3 \\ \text{soc}(\text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1}) & \text{if } r \in \{p-2, p-1\}. \end{cases}$$

Moreover, the following elements of  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$

$$\begin{cases} f_{p-3-r}(e_1) & \text{if } r \leq p-3 \\ f_0(e_1) & \text{if } r \in \{p-2, p-1\} \text{ and } (r, p) \neq (p-1, 3) \\ f_0(e_1), f_0(e_1) - [1, e_1] & \text{if } (r, p) = (p-1, 3). \end{cases} \quad (19)$$

are mapped to a linear basis for the highest weight space of  $\text{soc}(R_{\infty,0}/M)$  (in the case  $(r, p) = (p-1, 3)$  then  $f_0(e_1), f_0(e_1) - [1, e_1]$  are mapped to the highest weight space of  $\overline{St} \otimes \det$ ,  $\det$  respectively).

Since we have assumed  $R_{\infty,0}^{K_t}/M \neq \{0\}$  (and  $M$  is formed by  $K_t$  fixed vectors) the elements described in (19) should be  $K_t$  fixed vectors of  $R_{\infty,0}$ . This is absurd, as we show in the following lines.

We treat first the case  $r \leq p-3$ . Thanks to (18) and Proposition 4.2 we have the following equality in  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-)$ :

$$\left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot f_{p-3-r}(e_1) = \bar{b} f_{p-3-r}(e_0) + (\overline{a-d}) f_{p-2-r}(e_0) - \bar{c} f_{p-1-r}(e_0).$$

Via Proposition 2.1 and the epimorphism of Corollary 3.4 we deduce the following equality in  $R_{\infty,0}$ :

$$\left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot f_{p-3-r}(e_1) = -\bar{c} f_{p-1-r}(e_0).$$

But  $f_{p-1-r}(e_0)$  is a linear generator for the highest weight space of  $\text{soc}(R_{\infty,0})$  (Proposition 2.1) and hence  $f_{p-3-r}(e_1)$  can not be a  $K_t$  fixed vector in  $R_{\infty,0}$ .

The case  $r \in \{p-2, p-1\}$  is completely analogous: we have

$$\left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot f_0(e_1) = \bar{b}f_0(e_0) + (\overline{a-d})f_1(e_0) - \bar{c}f_2(e_0)$$

(resp.

$$\left( \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot (f_0(e_1) - [1, e_1]) = \bar{b}f_0(e_0) + (\overline{a-d})f_1(e_0) - \bar{c}(f_2(e_0) + [1, e_0])$$

when  $(r, p) = (p-1, 3)$ ) and it is easy to see (Proposition 2.1) that the RHS is mapped to a linearly independent family inside  $\text{soc}(R_{\infty,0})$  via the epimorphism  $\text{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$ .

This completes the proof.  $\square$

As a corollary, we get the desired structure for  $K_t$  fixed vectors of supersingular representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$ :

**COROLLARY 4.9.** *Let  $t \geq 1$ . The space of  $K_t$  fixed vectors for the supersingular representation  $\pi(\sigma, 0, 1)$  decomposes into the direct sum of two  $k[K]$ -modules  $(\pi(\sigma, 0, 1))^{K_t} = (R_{\infty,0})^{K_t} \oplus (R_{\infty,-1})^{K_t}$ , whose socle filtration is respectively described by:*

$$(R_{\infty,0})^{K_t} : \text{Sym}^r k^2 \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^{r+1}) \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^{r+2}) \text{---} \dots \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^r) \text{---Sym}^{p-3-r} k^2 \otimes \det^{r+1}$$

$$(R_{\infty,-1})^{K_t} : \text{Sym}^{p-1-r} k^2 \otimes \det^r \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}) \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathbf{a}^2) \text{---} \dots \text{---socfil}(\text{ind}_{K_0(p)}^K \chi_r^s) \text{---Sym}^{r-2} k^2 \otimes \det$$

where we have  $p^{t-1} - 1$  parabolic inductions in each line and the weight  $\text{Sym}^{p-3-r} k^2 \otimes \det^{r+1}$  in the first line (resp.  $\text{Sym}^{r-2} k^2 \otimes \det$  in the second line) appears only if  $p-3-r \geq 0$  (resp.  $r-2 \geq 0$ ).

*Proof.* The statement concerning the direct summand  $(R_{\infty,0})^{K_t}$  follows immediately from Corollary 3.4 and Proposition 4.8. By the generality of  $\sigma$  and Proposition 4.1 one deduces the result for  $(R_{\infty,-1})^{K_t}$ .  $\square$

In particular, we have

**COROLLARY 4.10.** *Let  $t \geq 1$ . The dimension of  $K_t$  invariant for the supersingular representation  $\pi(\sigma, 0, 1)$  is given by*

$$\dim_k((\pi(\sigma, 0, 1))^{K_t}) = (p+1)(2p^{t-1} - 1) + \begin{cases} p-3 & \text{if } r \notin \{0, p-1\} \\ p-2 & \text{if } r \in \{0, p-1\}. \end{cases}$$

## 5. The case of principal series and the Steinberg.

In order to complete the picture concerning  $K_t$  and  $I_t$  invariants for irreducible admissible representations of  $\mathbf{GL}_2(\mathbf{Q}_p)$  we are left to treat the case of principal and special series.

Recall ([BL94], [Her2]) that the irreducible principal series for  $\mathbf{GL}_2(\mathbf{Q}_p)$  are described by the parabolic induction

$$\text{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\text{un}_\mu \otimes \omega^r \text{un}_{\mu^{-1}})$$

where  $\mu \in \bar{k}^\times$ ,  $r \in \{0, \dots, p-1\}$  and  $(\mu, r) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ , while the special series are described (up to twist) by the short exact sequence

$$0 \rightarrow 1 \rightarrow \text{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)} 1 \rightarrow \text{St} \rightarrow 0. \quad (20)$$

It is easy to see that we have  $K$ -equivariant isomorphisms (see for instance [Mo1], §10):

$$(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}}))|_K \cong \mathrm{ind}_{K_0(p^\infty)}^K \chi_r^s \cong \varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^K \chi_r^s)$$

where  $K_{p^\infty} \stackrel{\mathrm{def}}{=} \mathbf{B}(\mathbf{Z}_p)$  and the transition morphisms for the co-limit are obtained inducing the natural monomorphisms of  $K_0(p^n)$ -representations

$$\chi_r^s \hookrightarrow \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s.$$

Moreover by the Bruhat-Iwahori and Mackey decompositions, we have a  $K_0(p)$ -equivariant split exact sequence

$$0 \rightarrow (\mathrm{ind}_{K_0(p^\infty)}^K \chi_r^s)^+ \rightarrow \mathrm{ind}_{K_0(p^\infty)}^K \chi_r^s \rightarrow \mathrm{ind}_{K_0(p)}^{K_0(p)} \chi_r^s \rightarrow 0.$$

The following results are formal

LEMMA 5.1. *Let  $\mu \in \bar{k}^\times$  and  $r \in 0, \dots, p-1$ . We have a  $K$ -equivariant isomorphism*

$$(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}}))|_K \cong \mathrm{ind}_{K_0(p)}^K \left( \varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s) \right).$$

The action of  $\begin{bmatrix} 0 & 1 \\ p & 1 \end{bmatrix}$  on the principal series  $\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})$  induces an isomorphism

$$(\mathrm{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s)^s \xrightarrow{\sim} (\mathrm{ind}_{K_0(p^\infty)}^K \chi_r^s)^+.$$

*Proof.* The first isomorphism comes from the continuity and transitivity of the induction functor  $\mathrm{ind}_{K_0(p)}^K(-)$ . The second comes from a direct computation on the explicit isomorphism

$$(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}}))|_{K_0(p)} \cong \mathrm{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s \oplus (\mathrm{ind}_{K_0(p^\infty)}^K \chi_r^s)^+$$

given by Mackey decomposition (recalling that  $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$  normalizes  $K_0(p)$ ).  $\square$

The  $K_t, I_t$  fixed vectors for the co-limit  $\varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s)$  are described by the

PROPOSITION 5.2. *Let  $t \geq 1$  and  $r \in \{0, \dots, p-2\}$ . Then*

$$\left( \varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s) \right)^{K_t} = \mathrm{ind}_{K_0(p^t)}^{K_0(p)} \chi_r^s = \left( \varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s) \right)^{I_t}.$$

*Proof.* We know that  $\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s$  is uniserial for all  $n \geq 1$ , in particular the co-limit  $\varinjlim_{n \geq 1} (\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s)$  is uniserial.

It is therefore sufficient to prove the result of the statement replacing the co-limit by  $\mathrm{ind}_{K_0(p^{t+1})}^{K_0(p)} \chi_r^s$ .

In this case, we have again an explicit linear basis  $\mathcal{B}_{t+1}^-$  for the induced representation  $\mathrm{ind}_{K_0(p^{t+1})}^{K_0(p)} \chi_r^s$ , endowed with a linear ordering which is compatible with the  $K_0(p)$ -socle filtration (see [Mo1] §5 or [Mo4], §4):

$$\mathcal{B}_{t+1}^- \ni F_{l_1, \dots, l_t}^{(1, t)} \stackrel{\mathrm{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p^1[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_t \in \mathbf{F}_p} \lambda_t^{l_t} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t] & 1 \end{bmatrix} [1, e]$$

where  $(l_1, \dots, l_t) \in \{0, \dots, p-1\}^t$  and  $e$  is a linear basis for the character  $\chi_r^s$  (again  $\mathcal{B}_{t+1}^-$  is endowed with the lexicographical order).

The statement can be now directly verified, as for Proposition 4.2, 4.4, but the computations are much easier.  $\square$

We finally deduce:

PROPOSITION 5.3. *Let  $\mu \in \overline{k}^\times$  and  $r \in \{0, \dots, p-1\}$  and let  $t \geq 0$ . The  $K$ -socle filtration for the  $K_t$  invariants of the principal series  $\text{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\text{un}_\mu \otimes \omega^r \text{un}_{\mu^{-1}})$  is described by*

$$\text{socfil}(\text{ind}_{K_0(p)}^K \chi_r^s) \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}) \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^2) \text{---} \dots \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K \chi_r^s)$$

where the number of parabolic induction is  $p^{t-1}$ .

Moreover, the  $I_t$  fixed vectors for the principal series  $\text{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\text{un}_\mu \otimes \omega^r \text{un}_{\mu^{-1}})$  are described by

$$\left(\text{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\text{un}_\mu \otimes \omega^r \text{un}_{\mu^{-1}})\right)^{I_t} \cong \text{ind}_{K_0(p^t)}^{K_0(p)} \chi_r^s \oplus \left(\text{ind}_{K_0(p^t)}^{K_0(p)} \chi_r^s\right)^s.$$

The Steinberg representation verifies in particular

$$(St)^{K_t} : \quad \overline{St} \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K \mathfrak{a}) \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K \mathfrak{a}^2) \text{---} \dots \text{---} \text{socfil}(\text{ind}_{K_0(p)}^K)$$

(where the number of parabolic induction is  $p^{t-1} - 1$ ) and

$$(St)^{I_t} \cong \text{ind}_{K_0(p^t)}^{K_0(p)} 1 \oplus_1 \left(\text{ind}_{K_0(p^t)}^{K_0(p)} 1\right)^s$$

where the amalgamated sum on the RHS is defined through the natural  $K_0(p)$ -equivariant morphism  $1 \hookrightarrow \text{ind}_{K_0(p^t)}^{K_0(p)} \chi_r^s$ .

*Proof.* This is an immediate consequence of Lemma 5.1 and Proposition 5.2 (using the fact that  $K_t$  is normal in  $K$ ).

The statement concerning the Steinberg representation is clear from the exact sequence (20)  $\square$

Notice that from the uniseriality of  $\text{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s$  and Lemma 5.1 we have an isomorphism

$$\left(\text{ind}_{K_0(p^t)}^{K_0(p)} \chi_r^s\right)^s \xrightarrow{\sim} \left(\text{ind}_{K_0(p^{t-1})}^K \chi_r^s\right)^+$$

for any  $t \geq 1$  and any smooth character  $\chi_r^s$  (by a counting dimension argument).

## 6. Global applications

In this section we describe the relation between the results of §4.2 and the local/global compatibility of the  $p$ -modular Langlands correspondence recently established by Emerton. We first need to recall some of the constructions of [Eme10] (see also [Bre11]).

Let  $\mathbf{A}_f$  be the ring of finite adeles of  $\mathbf{Q}$ ,  $G_{\mathbf{Q}}$  be the absolute Galois group of  $\mathbf{Q}$  and write  $G_{\mathbf{Q}_\ell}$  for its decomposition group at  $\ell$ .

For a compact open subgroup  $K_f$  of the adelic group  $\mathbf{GL}_2(\mathbf{A}_f)$  we write  $Y(K_f)$  to denote the modular curve (defined over  $\mathbf{Q}$ ) whose complex points are

$$Y(K_f)(\mathbf{C}) = \mathbf{GL}_2(\mathbf{Q}) \backslash ((\mathbf{C} \setminus \mathbf{R}) \times \mathbf{GL}_2(\mathbf{A}_f) / K_f).$$

For  $A \in \{\mathcal{O}, k\}$  we consider the first étale cohomology group

$$H^1(K_f)_A \stackrel{\text{def}}{=} H_{\text{ét}}^1(Y(K_f)_{\overline{\mathbf{Q}}}, A)$$

where  $Y(K_f)_{\overline{\mathbf{Q}}}$  is the base change of  $Y(K_f)$  to  $\overline{\mathbf{Q}}$ .

For a fixed compact open subgroup  $K^p$  of  $\mathbf{GL}_2(\mathbf{A}_f^p)$  we introduce the following modules, endowed with commuting actions of  $G_{\mathbf{Q}}$  and  $\mathbf{GL}_2(\mathbf{Q}_p)$ :

$$H^1(K^p)_k \stackrel{\text{def}}{=} \varinjlim_{K_p} H^1(K_p K^p)_k, \quad \text{and} \quad \widehat{H}^1(K^p)_{\mathcal{O}} \stackrel{\text{def}}{=} \left( \varinjlim_{K_p} H^1(K_p K^p)_{\mathcal{O}} \right)^{\wedge}$$

where  $K_p$  runs over the compact open subgroups of  $\mathbf{GL}_2(\mathbf{Q}_p)$  and the hat  $\wedge$  denotes the  $p$ -adic completion of the  $\mathcal{O}$ -module  $\varinjlim_{K_p} H^1(K_p K^p)_{\mathcal{O}}$ .

Let  $\Sigma_0$  be a finite set of non-archimidean places of  $\mathbf{Q}$ , not containing  $p$  and let  $\Sigma \stackrel{\text{def}}{=} \Sigma_0 \cup \{p\}$ . We will be interested in compact open subgroups of  $\mathbf{GL}_2(\mathbf{A}_f^p)$  of the form  $K_{\Sigma_0} K_0^{\Sigma}$ , where  $K_{\Sigma_0}$  is a compact open subgroup of  $G_{\Sigma_0} \stackrel{\text{def}}{=} \prod_{\ell \in \Sigma_0} \mathbf{GL}_2(\mathbf{Q}_{\ell})$  and  $K_0^{\Sigma} \stackrel{\text{def}}{=} \prod_{\ell \notin \Sigma} \mathbf{GL}_2(\mathbf{Z}_{\ell})$ ; we will write for short

$$H^1(K_{\Sigma_0})_k \stackrel{\text{def}}{=} H^1(K_{\Sigma_0} K_0^{\Sigma})_k, \quad \text{and} \quad \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}} \stackrel{\text{def}}{=} \widehat{H}^1(K_{\Sigma_0} K_0^{\Sigma})_{\mathcal{O}}.$$

For a compact open subgroup  $K_p$  in  $\mathbf{GL}_2(\mathbf{Q}_p)$  we write  $\mathbf{T}(K_p K_{\Sigma_0} K_0^{\Sigma})$  for the sub  $\mathcal{O}$  algebra of

$$\text{End}_{\mathcal{O}[G_{\mathbf{Q}}]}(\widehat{H}^1(K_p K_{\Sigma_0} K_0^{\Sigma})_{\mathcal{O}})$$

generated by the Hecke operators  $T_{\ell}$ ,  $S_{\ell l}$  for those primes  $\ell \notin \Sigma$ .

If  $K'_p \leq K_p$  are compact open in  $\mathbf{GL}_2(\mathbf{Q}_p)$  we have a (surjective) transition homomorphism  $\mathbf{T}(K'_p K_{\Sigma_0} K_0^{\Sigma}) \rightarrow \mathbf{T}(K_p K_{\Sigma_0} K_0^{\Sigma})$ , which is compatible, in the evident sense, with the actions on the étale cohomologies. We deduce a  $G_{\mathbf{Q}} \times \mathbf{GL}_2(\mathbf{Q}_p)$  equivariant action of

$$\mathbf{T}(K_{\Sigma_0}) \stackrel{\text{def}}{=} \varprojlim_{K_p} \mathbf{T}(K_p K_{\Sigma_0} K_0^{\Sigma})$$

on the module  $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}}$ , hence ([Eme10], (5.1.2)) on  $H^1(K_{\Sigma_0})_k$ .

By construction, the action of  $\mathbf{T}(K_{\Sigma_0})$  on the sub-module  $\widehat{H}^1(K_p K_{\Sigma_0} K_0^{\Sigma})_{\mathcal{O}}$  (resp.  $H^1(K_p K_{\Sigma_0} K_0^{\Sigma})_k$ ) factors through the surjection  $\mathbf{T}(K_{\Sigma_0}) \rightarrow \mathbf{T}(K_p K_{\Sigma_0} K_0^{\Sigma})$ .

Let  $\bar{\rho} : G_{\overline{\mathbf{Q}}} \rightarrow \mathbf{GL}_2(k)$  be a continuous, absolutely irreducible Galois representation. We assume moreover that  $\bar{\rho}$  is modular and we define  $\Sigma_0$  to be the set of primes dividing the Artin conductor of  $\bar{\rho}$  ([Ser87], §1.2).

We recall that a compact open subgroup  $K_{\Sigma_0}$  of  $G_{\Sigma_0}$  is an *allowable level* for  $\bar{\rho}$  if there exists a maximal ideal  $\mathfrak{m}$  of  $\mathbf{T}(K_{\Sigma_0})$ , having residue field  $k$  and such that

$$T_{\ell} \equiv \text{tr}(\bar{\rho}(\text{Frob}_{\ell})) \pmod{\mathfrak{m}}, \quad S_{\ell} \equiv \ell^{-1} \det(\bar{\rho}(\text{Frob}_{\ell})) \pmod{\mathfrak{m}}.$$

Since  $\bar{\rho}$  is modular we deduce from the level part of Serre conjecture that any compact open subgroup in the  $\Sigma_0$ -component of  $\ker(\mathbf{GL}_2(\widehat{\mathbf{Z}}) \rightarrow \mathbf{GL}_2(\widehat{\mathbf{Z}}/(N)))$  is an allowable level for  $\bar{\rho}$ .

If  $K_{\Sigma_0}$  is allowable and  $\mathfrak{m}$  is a maximal ideal associated to  $\bar{\rho}$  in the previous sense, we consider the following  $\mathfrak{m}$ -adic completion:

$$\mathbf{T}(K_{\Sigma_0})_{\bar{\rho}} \stackrel{\text{def}}{=} \mathbf{T}(K_{\Sigma_0})_{\mathfrak{m}}, \quad \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}, \bar{\rho}} \stackrel{\text{def}}{=} (\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}})_{\mathfrak{m}}, \quad H^1(K_{\Sigma_0})_{k, \bar{\rho}} \stackrel{\text{def}}{=} (H^1(K_{\Sigma_0})_k)_{\mathfrak{m}}.$$

The action of the completed Hecke algebra  $\mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$  on the  $G_{\overline{\mathbf{Q}}} \times \mathbf{GL}_2(\mathbf{Q}_p)$ -modules  $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}, \bar{\rho}}$ ,  $H^1(K_{\Sigma_0})_{k, \bar{\rho}}$  is equivariant. Moreover for an inclusion of allowable levels  $K'_{\Sigma_0} \leq K_{\Sigma_0}$  we dispose of a surjective transition homomorphism  $\mathbf{T}(K'_{\Sigma_0})_{\bar{\rho}} \rightarrow \mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$  which is compatible, in the evident sense, with the actions on the completed étale cohomologies.

Therefore, the co-limit  $\widehat{H}^1_{\mathcal{O}, \bar{\rho}, \Sigma} \stackrel{\text{def}}{=} \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}, \bar{\rho}}$  is naturally a module over the  $\mathcal{O}$ -algebra

$\mathbf{T}_{\bar{\rho}, \Sigma} \stackrel{\text{def}}{=} \varprojlim_{K_{\Sigma_0}} \mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$  and the same holds for the co-limit  $H_{k, \bar{\rho}, \Sigma}^1 \stackrel{\text{def}}{=} \varinjlim_{K_{\Sigma_0}} H^1(K_{\Sigma_0})_{k, \bar{\rho}}$  ([Eme10], (5.3.4)).

The modules  $\widehat{H}_{\mathcal{O}, \bar{\rho}, \Sigma}^1, H_{k, \bar{\rho}, \Sigma}^1$  are furthermore endowed with a linear action of  $G_{\overline{\mathbf{Q}}} \times \mathbf{GL}_2(\mathbf{Q}_p) \times G_{\Sigma_0}$  which turns out to be  $\mathbf{T}_{\bar{\rho}, \Sigma}$ -linear. Notice again that, by construction, the action of  $\mathbf{T}_{\bar{\rho}, \Sigma}$  on the sub-module  $\widehat{H}^1(K_{\Sigma_0})_{\mathcal{O}, \bar{\rho}}$  (resp.  $H^1(K_{\Sigma_0})_{k, \bar{\rho}}$ ) factors through the surjection of local  $\mathcal{O}$ -algebras  $\mathbf{T}_{\bar{\rho}, \Sigma} \rightarrow \mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$ .

We can now introduce a local/global application of the results in section 4.2.

**PROPOSITION 6.1.** *Let  $p \geq 3$  and  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathbf{GL}_2(k)$  be an odd, continuous, absolutely irreducible Galois representation such that  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$  is absolutely irreducible. Let  $\Sigma_0$  be the set of primes dividing the Artin conductor of  $\bar{\rho}$  and let  $k$  be the minimal weight associated to  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$  (cf. [Ser87], §2.2).*

Let  $K_{\Sigma_0}$  be an allowable level for  $\bar{\rho}$  and define

$$d \stackrel{\text{def}}{=} \dim_k \left( \bigotimes_{\ell \in \Sigma_0} \pi(\bar{\rho}|_{G_{\mathbf{Q}_\ell}}) \right)^{K_{\Sigma_0}} \quad (21)$$

where  $\pi(\bar{\rho}|_{G_{\mathbf{Q}_\ell}})$  is the smooth  $p$ -modular representations of  $\mathbf{GL}_2(\mathbf{Q}_\ell)$  attached to  $\bar{\rho}|_{G_{\mathbf{Q}_\ell}}$  via the  $p$ -modular Langlands correspondence of Emerton-Helm ([EH]).

Then, if either  $t \geq 1$  and  $p \geq 5$  or  $t \geq 2$  and  $p = 3$  we have

$$\begin{aligned} \dim_k (H_{\acute{e}t}^1(Y(K_t K_{\Sigma_0} K_0^\Sigma)_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]) &= 2d(2p^{t-1}(p+1) - 3) && \text{if } k - 2 \equiv 0 \pmod{p+1} \\ \dim_k (H_{\acute{e}t}^1(Y(K_t K_{\Sigma_0} K_0^\Sigma)_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]) &= 2d(2p^{t-1}(p+1) - 4) && \text{if } k - 2 \not\equiv 0 \pmod{p+1} \\ \dim_k (H_{\acute{e}t}^1(Y(I_t K_{\Sigma_0} K_0^\Sigma)_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]) &= 4d(2p^{t-1} - 1). \end{aligned}$$

where  $K_0^\Sigma \stackrel{\text{def}}{=} \prod_{\ell \notin (\Sigma_0 \cup \{p\})} \mathbf{GL}_2(\mathbf{Z}_\ell)$  and  $\mathfrak{m}$  is a maximal ideal associated to  $\bar{\rho}$  in the Hecke algebra  $\mathbf{T}(K_{\Sigma_0})$ .

*Proof.* Let  $\mathfrak{m}$  be a maximal ideal of the Hecke algebra  $\mathbf{T}(K_{\Sigma_0})$  associated to  $\bar{\rho}$ . We will use the same notation  $\mathfrak{m}$  for the maximal ideals of the local  $\mathcal{O}$ -algebras  $\mathbf{T}_{\bar{\rho}, \Sigma}, \mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$ .

Assume that either  $t \geq 1$  and  $p \geq 5$  or  $t \geq 2$  and  $p = 3$ . Then for  $K_p \in \{K_t, I_t\}$ , the congruence subgroup  $K_p \prod_{\ell \neq p} \mathbf{GL}_2(\mathbf{Z}_\ell)$  is neat in the sense of [Eme10], Definition 5.3.7 (the proof of Lemme 2 (1) in §5.5 of Carayol's article [Car] applies line to line).

We can therefore use [Eme10] Lemma 5.3.8 (2) and the equivariance of the Hecke action on the cohomology spaces to obtain

$$(H_{k, \bar{\rho}, \Sigma}^1)^{K_p K_{\Sigma_0}}[\mathfrak{m}] = H^1(K_p K_{\Sigma_0} K_0^\Sigma)_{k, \bar{\rho}}[\mathfrak{m}] = H^1(K_p K_{\Sigma_0} K_0^\Sigma)_k[\mathfrak{m}]$$

(where  $\mathfrak{m}$  is seen as an ideal of  $\mathbf{T}_{\bar{\rho}, \Sigma}, \mathbf{T}(K_{\Sigma_0})_{\bar{\rho}}$  or  $\mathbf{T}(K_{\Sigma_0})$  thanks to the compatibility of the Hecke action on the sub-modules of  $H_{k, \bar{\rho}, \Sigma}^1, H^1(K_p K_{\Sigma_0} K_0^\Sigma)_k$ ).

Let  $\overline{k-2} \in \{0, \dots, p\}$  be defined by  $\overline{k-2} \equiv k-2 \pmod{p+1}$  (we know from [Ser87] that  $\overline{k-2} < p-1$ ). From the proof of Proposition 6.1.20 in [Eme10] we have an equivariant isomorphism

$$H_{k, \bar{\rho}, \Sigma}^1[\mathfrak{m}] \cong \bar{\rho} \otimes \pi_p \otimes \pi_{\Sigma_0}(\bar{\rho})$$

where  $\pi_{\Sigma_0}(\bar{\rho}) \stackrel{\text{def}}{=} \otimes_{\ell \in \Sigma_0} \pi(\bar{\rho}|_{G_{\mathbf{Q}_\ell}})$  and  $\pi_p$  is a supersingular representation whose  $KZ$ -socle contains, up to twist, the weight  $\sigma \stackrel{\text{def}}{=} \text{Sym}^{\overline{k-2}} k^2$  (more precisely,  $\pi_p$  is, up to twist, the supersingular representation attached to  $\bar{\rho}|_{G_{\mathbf{Q}_p}}$  in [Bre03a]).

We deduce

$$H^1(K_p K_{\Sigma_0} K_0^\Sigma)_k[\mathfrak{m}] \cong \bar{\rho} \otimes (\pi_p)^{K_p} \otimes (\pi_{\Sigma_0}(\bar{\rho}))^{K_{\Sigma_0}}$$

and the result follows from Proposition 4.5, Corollary 4.10 and the definition of  $d$ , noticing that  $\pi_p \cong \pi(\sigma, 0, 1)$  up to twist.  $\square$

REMARK 6.2. By the level part of the refined Serre conjecture ([Ser87]) one expects the subgroup

$$K_{1,\Sigma_0}(N) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \prod_{\ell \in \Sigma_0} \mathbf{GL}_2(\mathbf{Z}_\ell) \mid c \equiv d - 1 \equiv 0 \pmod{N} \right\}$$

to be an allowable level for which  $d = 1$  in (21), at least if the semi-simplifications  $\bar{\rho}|_{G_{\mathbf{Q}_\ell}}^{\text{ss}}$  are not twists of  $1 \oplus |\cdot|$ .

In the classical  $\ell$ -adic correspondence this is indeed the compatibility between the Artin and adelic conductor but in the  $\ell$ -modular case, such compatibility does not seem to appear in the current literature.

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