A decomposition approach for the discrete-time approximation of FBSDEs with a jump I : the Lipschitz case

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Abstract

We study the discrete-time approximation for solutions of forward-backward stochastic differential equations (FBSDEs) with a jump. In this part, we study the case of Lipschitz generators, and we refer to the second part of this work [15] for the quadratic case. Our method is based on a result given in the companion paper [14] which allows to link a FBSDE with a jump with a recursive system of Brownian FBSDEs. Then we use the classical results on discretization of Brownian FBSDEs to approximate the recursive system of FBSDEs and we recombine these approximations to get a discretization of the FBSDE with a jump. This approach allows to get a convergence rate similar to that of schemes for Brownian FBSDEs.

Keywords: discrete-time approximation, forward-backward SDE, Lipschitz generator, progressive enlargement of filtrations, decomposition in the reference filtration.

MSC classification (2000): 60H10, 60J75, 60G57, 65C99.

1 Introduction

In this paper, we study a discrete-time approximation for the solution of a forward-backward stochastic differential equation (FBSDE) with a jump of the form

$$\begin{cases} X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s + \int_0^t \beta(s, X_{s^-})dH_s \ , \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s dH_s \ , \end{cases}$$

where $H_t = \mathbb{1}_{\tau \leq t}$ and τ is a jump time, which can represent a default time in credit risk or counterparty risk. Such equations naturally appear in finance, see for example Bielecki and Jeanblanc [2], Lim and Quenez [18], Peng and Xu [21], Ankirchner *et al.* [1] for an application to exponential utility maximization and Kharroubi and Lim [14] for the hedging problem in a complete market. Our work is divided into two parts. In this part, we study the case where the generator f is Lipschitz. The case of a generator f with quadratic growth w.r.t. Z is studied in the second part [15].

For Lipschitz generators, the discrete-time approximation of FBSDEs with jumps is studied by Bouchard and Elie [4] in the case of Poissonian jumps. Their approach is based on a regularity result for the process Z, which is given by Malliavin calculus tools. This regularity result for the process Z was first proved by Zhang [23] in a Brownian framework to provide a convergence rate for the discrete-time approximation of FBSDEs.

In our case, we only assume that the random jump time τ admits a conditional density given W, which is assumed to be absolutely continuous w.r.t. the Lebesgue measure. In particular, we do not specify a particular law for τ and we do not assume that τ is independent of W as for the case of a Poisson random measure.

To the best of our knowledge, no Malliavin calculus theory has been set for such a framework. Thus, the method used by in [4] fails to provide a rate of convergence for the approximation in this context.

We therefore follow another approach, which consists in using the decomposition result given in the companion paper [14] to write the solution of a FBSDE with a jump as a combination of solutions to a recursive system of FBSDEs without jump. We then use the existing results on the discretization of Brownian FBSDEs with a Lipschitz generator, which is a well understood problem, see e.g. Chevance [5], Bouchard and Touzi [3], Zhang [23], Gobet *et al.* [8], Delarue and Menozzi [6]. Finally, we recombine the approximations of the recursive system of Brownian FBSDEs to get a discretization of the solution to the FBSDE with a jump.

We notice that our approach also allows to weaken the assumption on the forward jump coefficient. More precisely, we only assume that β is Lipschitz continuous, contrary to [4] who assumes that β is regular and the matrix $I_d + \nabla\beta$ is elliptic.

As said upper, this kind of FBSDEs with a jump appears in finance. The wide assumptions made on the jump τ allow to modelize general phenomenon as a default of an asset or simpler as a jump that can be seen as contagion from the default of another asset of the

market, see e.g. [13] for some examples. In particular, the approximation of these FBSDEs has its own interest, since it provides approximations of optimal gains and strategies of the studied investment problems. We study in this part the case of FBSDEs with Lipschitz generators, which is related to valuation in complete markets (see [14]) and the utility maximization in incomplete markets with compact investment constraints (see [18]). The study of the discretization of FBSDEs with a quadratic generator, which are related to more general investment problems (see [1]), is postponed to the second part of this work.

We choose to present our results in the case of a single jump for the sake of simplicity. We notice that they can easily be extended to the case of multiple jumps with eventually random marks, as in [14], taking values in a finite space.

The paper is organized as follows. The next section presents the framework of progressive enlargement of a Brownian filtration by a random jump, and the well posedness of FBSDEs in this context. In Section 3, we present the discrete-time schemes for the forward and backward solutions based on the decomposition given in the previous section. Finally, in Sections 4 and 5, we study the convergence rate of these schemes respectively for the forward and the backward solutions.

2 Preliminaries

2.1 Notation

Throughout this paper, W denotes a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denotes the completion of the filtration generated by W. We also consider on this space a random time τ, i.e. a nonnegative $\mathcal{G}$-measurable random variable, and we denote classically the associated jump process by H which is given by

$$H_t \;=\; \mathbb{1}_{\tau\leq t}\,, \qquad t\geq 0\,.$$

We denote by $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ the smallest right-continuous filtration for which τ is a stopping time. The global information is then defined by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of the initial filtration where $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$. This kind of enlargement was studied by Jacod, Jeulin and Yor in the 80s (see e.g. [10], [11] and [9]). We introduce some notations used throughout the paper:

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes,
- $\mathcal{PM}(\mathbb{F})$ (resp. $\mathcal{PM}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-progressively measurable subsets of $\Omega \times \mathbb{R}_+$.

We shall make, throughout the sequel, the standing assumption in the progressive enlargement of filtrations known as density assumption (see e.g. [12, 13, 14]).

(DH) There exists a positive and bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process γ such that

$$\mathbb{P}[\tau \in d\theta | \mathcal{F}_t] \;=\; \gamma_t(\theta) d\theta \;, \qquad t \geq 0 \;.$$

Using Proposition 2.1 in [14] we get that Assumption **(DH)** ensures that the process H admits an intensity.

Proposition 2.1. *The process H admits a compensator of the form $\lambda_t dt$, where the process λ is defined by*

$$\lambda_t \;=\; \frac{\gamma_t(t)}{\mathbb{P}[\tau > t | \mathcal{F}_t]} \mathbb{1}_{t \leq \tau} \;, \qquad t \geq 0 \;.$$

We impose the following assumption to the process λ:

(HBI) The process λ is bounded.

We also introduce the martingale invariance assumption known as the **(H)**-hypothesis.

(H) Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale

We now introduce the following spaces, where $a, b \in \mathbb{R}_+$ with $a \leq b$, and $T < \infty$ is the terminal time:

- $\mathcal{S}^\infty_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^\infty_{\mathbb{F}}[a,b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ essentially bounded:

$$\|Y\|_{\mathcal{S}^\infty[a,b]} \;:=\; \operatorname*{ess\,sup}_{t\in[a,b]} |Y_t| \;<\; \infty \;.$$

- $\mathcal{S}^2_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^2_{\mathbb{F}}[a,b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ such that

$$\|Y\|_{\mathcal{S}^2[a,b]} \;:=\; \Big(\mathbb{E}\Big[\sup_{t\in[a,b]} |Y_t|^2\Big]\Big)^{\frac{1}{2}} \;<\; \infty \;.$$

- $L^2_{\mathbb{G}}[a,b]$ (resp. $L^2_{\mathbb{F}}[a,b]$) is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_t)_{t\in[a,b]}$ such that

$$\|Z\|_{L^2[a,b]} \;:=\; \Big(\mathbb{E}\Big[\int_a^b |Z_t|^2 dt\Big]\Big)^{\frac{1}{2}} \;<\; \infty \;.$$

- $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\mu)} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |U_s|^2 \lambda_s ds\Big]\Big)^{\frac{1}{2}} \;<\; \infty \;.$$

2.2 Forward-Backward SDE with a jump

Given measurable functions $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$, $\beta : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}$ and $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, and an initial condition $x\in\mathbb{R}^d$, we study the discrete-time approximation of the solution (X,Y,Z,U) in $\mathcal{S}^2_{\mathbb{G}}[0,T]\times\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$ to the following forward-backward stochastic differential equation:

$$\begin{aligned} X_t &= x+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s+\int_0^t\beta(s,X_{s^-})dH_s\ , \quad 0\le t\le T\ , \qquad (2.1)\\ Y_t &= g(X_T)+\int_t^T f\big(s,X_s,Y_s,Z_s,U_s(1-H_s)\big)ds \\ &\qquad -\int_t^T Z_s dW_s-\int_t^T U_s dH_s\ , \quad 0\le t\le T\ , \qquad (2.2)\end{aligned}$$

when the generator of the BSDE is Lipschitz.

Remark 2.1. In BSDE (2.2), the jump component U of the unknown (Y,Z,U) appears in the generator f with an additional multiplicative term $1-H$. This ensures the equation to be well posed in $\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$. Indeed, the component U lives in $L^2(\lambda)$, thus its value on $(\tau\wedge T,T]$ is not defined, since the intensity λ vanishes on $(\tau\wedge T,T]$. We therefore introduce the term $1-H$ to kill the value of U on $(\tau\wedge T,T]$ and hence to avoid making the equation depends on it.

We first prove that the decoupled system (2.1)-(2.2) admits a solution. To this end, we introduce several assumptions on the coefficients b, σ, β, g and f. We consider the following assumption for the forward coefficients:

(HF) There exists a constant K such that the functions b, σ and β satisfy

$$|b(t,0)|+|\sigma(t,0)|+|\beta(t,0)| \ \le\ K\ ,$$

and

$$|b(t,x)-b(t,x')|+|\sigma(t,x)-\sigma(t,x')|+|\beta(t,x)-\beta(t,x')| \ \le\ K|x-x'|\ ,$$

for all $(t,x,x')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

For the backward coefficients g and f, we impose the following assumption:

(HBL) There exists a constant K such that the functions g and f satisfy

$$|f(t,x,0,0,0)|+|g(x)| \ \le\ K\ ,$$

and

$$|f(t,x,y,z,u)-f(t,x,y',z',u')| \ \le\ K\big(|y-y'|+|z-z'|+|u-u'|\big)\ ,$$

for all $(t,x,y,y',z,z',u,u')\in[0,T]\times\mathbb{R}^d\times[\mathbb{R}]^2\times[\mathbb{R}^d]^2\times[\mathbb{R}]^2$.

Following the decomposition approach of [14], we introduce the recursive system of FBSDEs associated to (2.1)-(2.2):

• Find $(X^1(\theta), Y^1(\theta), Z^1(\theta)) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times L^2_{\mathbb{F}}[\theta,T]$ such that

$$X^1_t(\theta) \;=\; x + \int_0^t b\big(s, X^1_s(\theta)\big)ds + \int_0^t \sigma\big(s, X^1_s(\theta)\big)dW_s + \beta\big(\theta, X^1_{\theta^-}(\theta)\big)\mathbb{1}_{\theta\le t}\;,\; 0\le t\le T\;, \quad (2.3)$$

$$Y^1_t\big(\theta\big) \;=\; g\big(X^1_T(\theta)\big) + \int_t^T f\big(s, X^1_s(\theta), Y^1_s(\theta), Z^1_s(\theta), 0\big)ds - \int_t^T Z^1_s(\theta)dW_s\;,\; \theta\le t\le T\;, \quad (2.4)$$

for all $\theta \in [0,T]$.

• Find $(X^0, Y^0, Z^0) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times L^2_{\mathbb{F}}[0,T]$ such that

$$X^0_t \;=\; x + \int_0^t b(s, X^0_s)ds + \int_0^t \sigma(s, X^0_s)dW_s\;,\quad 0\le t\le T\;, \quad (2.5)$$

$$Y^0_t \;=\; g(X^0_T) + \int_t^T f\big(s, X^0_s, Y^0_s, Z^0_s, Y^1_s(s) - Y^0_s\big)ds - \int_t^T Z^0_s dW_s\;,\quad 0\le t\le T\;. \quad (2.6)$$

Then, the link between FBSDE (2.1)-(2.2) and the recursive system of FBSDEs (2.5)-(2.6) and (2.3)-(2.4) is given by the following result.

Theorem 2.1. *Assume that **(HF)** and **(HBL)** hold true. Then, FBSDE* (2.1)-(2.2) *admits a unique solution* $(X,Y,Z,U) \in \mathcal{S}^2_{\mathbb{G}}[0,T] \times \mathcal{S}^\infty_{\mathbb{G}}[0,T] \times L^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ *given by*

$$\begin{cases} X_t \;=\; X^0_t\mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{\tau\le t}\;, \\ Y_t \;=\; Y^0_t\mathbb{1}_{t<\tau} + Y^1_t(\tau)\mathbb{1}_{\tau\le t}\;, \\ Z_t \;=\; Z^0_t\mathbb{1}_{t\le\tau} + Z^1_t(\tau)\mathbb{1}_{\tau<t}\;, \\ U_t \;=\; \big(Y^1_t(t) - Y^0_t\big)\mathbb{1}_{t\le\tau}\;, \end{cases} \quad (2.7)$$

where $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ *is the unique solution to FBSDE* (2.3)-(2.4) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times L^2_{\mathbb{F}}[\theta,T]$*, for* $\theta \in [0,T]$*, and* (X^0, Y^0, Z^0) *is the unique solution to FBSDE* (2.5)-(2.6) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times L^2_{\mathbb{F}}[0,T]$.

Proof.

Step 1. Solution to (2.1) under **(HF)**.

Under **(HF)** there exist unique processes $X^0 \in \mathcal{S}^2_{\mathbb{F}}[0,T]$ satisfying (2.5), and $X^1(\theta) \in \mathcal{S}^2_{\mathbb{F}}[0,T]$ satisfying (2.3) for all $\theta \in [0,T]$ such that X^1 is $\mathcal{PM}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Then, from the definition of H, we easily check that the process X defined by

$$X_t \;=\; X^0_t\mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{t\ge\tau}\;, \quad (2.8)$$

satisfies (2.1). We now check that $X \in \mathcal{S}^2_{\mathbb{G}}[0,T]$. We first notice that from **(HF)**, there exists a constant K such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X^0_t\big|^2\Big] \;\le\; K\;. \quad (2.9)$$

Then, from the definition of X^1, we have

$$\sup_{s\in[\theta,t]}\left|X_s^1(\theta)\right|^2 \;\leq\; K\Big(\left|X_\theta^0\right|^2+\left|\beta(\theta,X_\theta^0)\right|^2+\int_\theta^t\left|b(u,X_u^1(\theta))\right|^2du+\sup_{s\in[\theta,t]}\Big|\int_\theta^s\sigma(u,X_u^1(\theta))dW_u\Big|^2\Big)\,.$$

Using **(HF)** and BDG-inequality, we get

$$\mathbb{E}\Big[\sup_{s\in[\theta,t]}\left|X_s^1(\theta)\right|^2\Big] \;\leq\; K\Big(1+\int_\theta^t\mathbb{E}\Big[\sup_{u\in[\theta,s]}\left|X_u^1(\theta)|^2\right]du\Big)\,,$$

for some constant K which does not depend on θ. Applying Gronwall's lemma, we get

$$\sup_{\theta\in[0,T]}\left\|X^1(\theta)\right\|_{\mathcal{S}^2_{\mathbb{F}}[\theta,T]} \;\leq\; K\,. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we get that $X\in\mathcal{S}^2_{\mathbb{G}}[0,T]$. Moreover still using **(HF)** we get the uniqueness of a solution to (2.1) in $\mathcal{S}^2_{\mathbb{G}}[0,T]$.

Step 2. Solution to (2.2) under **(HBL)**.
To follow the decomposition approach initiated by the authors in [14], we need the generator to be predictable. To this end, we notice that in BSDE (2.2), we can replace the generator $(t,y,z,u)\mapsto f(t,X_t,y,z,u(1-H_t))$ by the predictable map $(t,y,z,u)\mapsto f(t,X_{t^-},y,z,u(1-H_{t^-}))$.

Using the decomposition (2.8), we are able to write explicitly the decompositions of the $\mathcal{G}_T$-measurable random variable $g(X_T)$ and the $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$-measurable map $(\omega,t,y,z,u)\mapsto f(t,X_{t^-}(\omega),y,z,u(1-H_{t^-}(\omega)))$ given by Lemma 2.1 in [14]:

$$\begin{aligned} g(X_T) &= g(X_T^0)\mathbb{1}_{T<\tau}+g(X_T^1(\tau))\mathbb{1}_{T\geq\tau}\,,\\ f(t,X_{t^-},y,z,(1-H_{t^-})u) &= f^0(t,y,z,u)\mathbb{1}_{t\leq\tau}+f^1(t,y,z,u,\tau)\mathbb{1}_{t>\tau}\,,\end{aligned}$$

with $f^0(t,y,z,u)=f(t,X_t^0,y,z,u)$ and $f^1(t,y,z,u,\theta)=f(t,X_{t^-}^1(\theta),y,z,0)$, for all $(t,y,z,u,\theta)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}_+$.

Suppose now that **(HBL)** holds true. Then, BSDE (2.6) admits a solution (Y^0,Z^0) and from Proposition C.1 in [14] BSDE (2.4) admits a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}([0,T])$-measurable solution (Y^1,Z^1). Using Proposition 2.1 in [17], we obtain the existence of a constant K such that

$$\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]}+\|Z^1(\theta)\|_{L^2[\theta,T]} \;\leq\; K\,,$$

for all $\theta\in[0,T]$, and

$$\|Y^0\|_{\mathcal{S}^\infty[0,T]}+\|Z^0\|_{L^2[0,T]} \;\leq\; K\,.$$

We can then apply Theorem 3.1 in [14] and we get the existence of a solution to (2.2) in $\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$.

Let (Y,Z,U) and (Y',Z',U') be two solutions to (2.2) in $\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$. Since $f(t,x,y,z,u(1-H_t))=f(t,x,y,z,0)$ for all $t\in(\tau\wedge T,T]$ and λ vanishes on $(\tau\wedge T,T]$,

we can assume w.l.o.g. that $U_t = U'_t = 0$ for $t \in (\tau \wedge T, T]$. Then, from **(HBL)**, we can apply Theorem 4.1 in [14] and we get that $Y \leq Y'$. Since Y and Y' play the same role, we obtain $Y = Y'$. Identifying the pure jump parts of Y and Y' gives $U = U'$. Finally, identifying the unbounded variation gives $Z = Z'$. □

In the following, sections we give an approximation of the solution to FBSDE (2.1)-(2.2) by studying the approximation of the solutions to the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6). For that we use the classical results of discretization in the case of Lipschitz Brownian FBSDEs.

3 Discrete-time scheme for the FBSDE

In this section, we introduce a discrete-time approximation of the solution (X, Y, Z, U) to FBSDE (2.1)-(2.2), based on its decomposition given by Theorem 2.1.

Throughout the sequel, we consider a discretization grid $\pi = \{t_0, \ldots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \ldots < t_n = T$. For $t \in [0, T]$, we denote by $\pi(t)$ (resp. $\pi_+(t)$) the largest (resp. smallest) element of π smaller (resp. larger) than t:

$$\begin{aligned} \pi(t) &:= \max\left\{ t_i \,,\ i = 0, \ldots, n \mid t_i \leq t \right\} \\ (\text{resp. } \pi_+(t) &:= \min\left\{ t_i \,,\ i = 0, \ldots, n \mid t_i \geq t \right\}) \,. \end{aligned}$$

We also denote by $|\pi|$ the mesh of π:

$$|\pi| := \max\left\{ t_{i+1} - t_i \,,\ i = 0, \ldots, n-1 \right\},$$

and by ΔW_i^π (resp. Δt_i^π) the increment of W (resp. the difference) between t_i and t_{i-1}: $\Delta W_i^\pi := W_{t_i} - W_{t_{i-1}}$ (resp. $\Delta t_i^\pi := t_i - t_{i-1}$), for $1 \leq i \leq n$.

3.1 Discrete-time scheme for X

We introduce an approximation of the process X based on the discretization of the processes X^0 and X^1.

• *Euler scheme for* X^0. We consider the classical scheme $X^{0,\pi}$ defined by

$$\left\{ \begin{array}{rcl} X_{t_0}^{0,\pi} &=& x \,, \\ X_{t_i}^{0,\pi} &=& X_{t_{i-1}}^{0,\pi} + b(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta t_i^\pi + \sigma(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta W_i^\pi \,, \quad 1 \leq i \leq n \,. \end{array} \right. \tag{3.11}$$

• *Euler scheme for* X^1. Since the process X^1 depends on two parameters t and θ, we introduce a discretization of X^1 in these two variables. We then consider the following scheme[1]

$$\left\{ \begin{array}{rcl} X_{t_0}^{1,\pi}(t_k) &=& x + \beta(t_0, x)\delta_{k=0} \,, \quad 0 \leq k \leq N \,, \\ X_{t_i}^{1,\pi}(t_k) &=& X_{t_{i-1}}^{1,\pi}(t_k) + b(t_{i-1}, X_{t_{i-1}}^{1,\pi}(t_k))\Delta t_i^\pi + \sigma(t_{i-1}, X_{t_{i-1}}^{1,\pi}(t_k))\Delta W_i^\pi \\ && \quad + \beta(t_{i-1}, X_{t_{i-1}}^{1,\pi}(t_k))\delta_{i=k} \,, \qquad 1 \leq i, k \leq n \,. \end{array} \right. \tag{3.12}$$

[1] $\delta_{i=k} = 0$ if $i \neq k$ and $\delta_{i=k} = 1$ if $i = k$.

We are now able to provide an approximation of the process X solution to FSDE (2.1). We consider the scheme X^π defined by

$$X_t^\pi \;=\; X_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + X_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau}\;,\quad 0\leq t\leq T\;. \tag{3.13}$$

We shall denote by $\{\mathcal{F}_i^{0,\pi}\}_{0\leq i\leq n}$ (resp. $\{\mathcal{F}_i^{1,\pi}(\theta)\}_{0\leq i\leq n}$) the discrete-time filtration associated to $X^{0,\pi}$ (resp. $X^{1,\pi}$)

$$\begin{array}{rcl}\mathcal{F}_i^{0,\pi} &:=& \sigma(X_{t_j}^{0,\pi},\; j\leq i)\\ (\text{resp. } \mathcal{F}_i^{1,\pi}(\theta) &:=& \sigma(X_{t_j}^{1,\pi}(\theta),\; j\leq i))\;.\end{array}$$

3.2 Discrete-time scheme for (Y, Z)

We introduce an approximation of (Y, Z) based on the discretization of (Y^0, Z^0) and (Y^1, Z^1). To this end we introduce the backward implicit schemes on π associated to BSDEs (2.4) and (2.6). Since the system is recursively coupled, we first introduce the scheme associated to (2.4). We then use it to define the scheme associated to (2.6).

• *Backward Euler scheme for* (Y^1, Z^1). We consider the classical implicit scheme $(Y^{1,\pi}, Z^{1,\pi})$ defined by

$$\left\{\begin{array}{rcl} Y_T^{1,\pi}(\pi(\theta)) &=& g(X_T^{1,\pi}(\pi(\theta)))\;,\\ Y_{t_{i-1}}^{1,\pi}(\pi(\theta)) &=& \mathbb{E}_{i-1}^{1,\pi(\theta)}\big[Y_{t_i}^{1,\pi}(\pi(\theta))\big] + f(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)), Y_{t_{i-1}}^{1,\pi}(\pi(\theta)), Z_{t_{i-1}}^{1,\pi}(\pi(\theta)), 0)\Delta t_i^\pi\;,\\ Z_{t_{i-1}}^{1,\pi}(\pi(\theta)) &=& \dfrac{1}{\Delta t_i^\pi}\mathbb{E}_{i-1}^1\big[Y_{t_i}^{1,\pi}(\pi(\theta))\Delta W_i^\pi\big]\;,\qquad t_{i-1}\geq\pi(\theta)\;,\end{array}\right. \tag{3.14}$$

where $\mathbb{E}_i^{1,\theta} = \mathbb{E}[\;.\;|\mathcal{F}_i^{1,\pi}(\theta)]$ for $0\leq i\leq n$ and $\theta\in[0,T]$.

• *Backward Euler scheme for* (Y^0, Z^0). Since the generator of (2.6) involves the process $(Y_t^1(t))_{t\in[0,T]}$, we consider a discretization based on $Y^{1,\pi}$. We therefore consider the scheme $(Y^{0,\pi}, Z^{0,\pi})$ defined by

$$\left\{\begin{array}{rcl} Y_T^{0,\pi} &=& g(X_T^{0,\pi})\;,\\ Y_{t_{i-1}}^{0,\pi} &=& \mathbb{E}_{i-1}^0\big[Y_{t_i}^{0,\pi}\big] + \bar f^\pi(t_{i-1}, X_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^{0,\pi}, Z_{t_{i-1}}^{0,\pi})\Delta t_i^\pi\;,\\ Z_{t_{i-1}}^{0,\pi} &=& \dfrac{1}{\Delta t_i^\pi}\mathbb{E}_{i-1}^0\big[Y_{t_i}^{0,\pi}\Delta W_i^\pi\big]\;,\qquad 1\leq i\leq n\;,\end{array}\right. \tag{3.15}$$

where $\mathbb{E}_i^0 = \mathbb{E}[\;.\;|\mathcal{F}_i^{0,\pi}]$ for $0\leq i\leq n$, and $\bar f^\pi$ is defined by

$$\bar f^\pi(t,x,y,z) \;=\; f\big(t,x,y,z,Y_{\pi(t)}^{1,\pi}(\pi(t))-y\big)\;,$$

for all $(t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$.
We then consider the following scheme defined by

$$\left\{\begin{array}{rcl} Y_t^\pi &=& Y_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + Y_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau}\\ Z_t^\pi &=& Z_{\pi(t)}^{0,\pi}\mathbb{1}_{t\leq\tau} + Z_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t>\tau}\\ U_t^\pi &=& \big(Y_{\pi(t)}^{1,\pi}(\pi(t)) - Y_{\pi(t)}^{0,\pi}\big)\mathbb{1}_{t\leq\tau}\end{array}\right. \tag{3.16}$$

for $t \in [0,T]$.

4 Convergence of the scheme for the FSDE

We introduce the following assumption, which will be used to discretize X.

(HFD) There exists a constant K such that the functions b, σ and β satisfy

$$\begin{aligned} \big|b(t,x) - b(t',x)\big| + \big|\sigma(t,x) - \sigma(t',x)\big| &\leq K|t-t'|^{\frac{1}{2}} , \\ \big|\beta(t,x) - \beta(t',x)\big| + \big|\sigma(t,x) - \sigma(t',x)\big| &\leq K|t-t'| , \end{aligned}$$

for all $(t,t',x) \in [0,T] \times [0,T] \times \mathbb{R}^d$.

We now provide an error estimate of the approximation schemes for X^0 and X^1. We then use these estimates to control the error between X and X^π.

4.1 Error estimate for X^0 and X^1

Under **(HF)** and **(HFD)**, the upper bound of the error between X^0 and its Euler scheme $X^{0,\pi}$ is well understood, see e.g. [16], and we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X^0_t - X^{0,\pi}_{\pi(t)}\big|^2 \Big] \leq K|\pi| , \tag{4.17}$$

for some constant K which does not depend on π.

The next result provides an upper bound for the error between X^1 and its Euler scheme $X^{1,\pi}$ defined by (3.12).

Theorem 4.2. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\sup_{\theta\in[0,T]} \mathbb{E}\Big[\sup_{t\in[\theta,T]} \big|X^1_t(\theta) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2 \Big] \leq K|\pi| ,$$

for a constant K which does not depend on π.

Proof. Fix $\theta \in [0,T]$, we then have

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\in[\theta,T]} \big|X^1_t(\theta) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2 \Big] &\leq 2\mathbb{E}\Big[\sup_{t\in[\theta,T]} \big|X^1_t(\theta) - X^1_t(\pi(\theta))\big|^2 \Big] \\ &\quad + 2\mathbb{E}\Big[\sup_{t\in[\theta,T]} \big|X^1_t(\pi(\theta)) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2 \Big] . \end{aligned} \tag{4.18}$$

We study separately the two terms of right hand side.

Since $\pi(\theta) \le \theta \le t$, $X^1_s(\pi(\theta)) = X^0_s$ for all $s \in [0, \pi(\theta))$, and $X^1_s(\theta) = X^0_s$ for all $s \in [0, \theta)$, we have

$$\begin{aligned} X^1_t(\theta) - X^1_t(\pi(\theta)) &= \int_{\pi(\theta)}^{\theta} b(s, X^0_s)ds + \int_{\pi(\theta)}^{\theta} \sigma(s, X^0_s)dW_s + \beta(\theta, X^0_\theta) + \int_{\theta}^{t} b\big(s, X^1_s(\theta)\big)ds \\ &\quad + \int_{\theta}^{t} \sigma\big(s, X^1_s(\theta)\big)dW_s - \beta(\pi(\theta), X^0_{\pi(\theta)}) - \int_{\pi(\theta)}^{\theta} b\big(s, X^1_s(\pi(\theta))\big)ds \\ &\quad - \int_{\pi(\theta)}^{\theta} \sigma\big(s, X^1_s(\pi(\theta))\big)dW_s - \int_{\theta}^{t} b\big(s, X^1_s(\pi(\theta))\big)ds \\ &\quad - \int_{\theta}^{t} \sigma\big(s, X^1_s(\pi(\theta))\big)dW_s \ , \end{aligned}$$

for all $t \in [\theta, T]$. Hence, there exists a constant K such that

$$\begin{aligned} \big|X^1_t(\theta) - X^1_t(\pi(\theta))\big|^2 &\le K\Big\{\Big|\int_{\pi(\theta)}^{\theta} b(s, X^0_s)ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta} \sigma(s, X^0_s)dW_s\Big|^2 \\ &\quad + \Big|\int_{\pi(\theta)}^{\theta} b(s, X^1_s(\pi(\theta)))ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta} \sigma(s, X^1_s(\pi(\theta)))dW_s\Big|^2 \\ &\quad + \big|\beta\big(\theta, X^0_\theta\big) - \beta\big(\pi(\theta), X^0_{\pi(\theta)}\big)\big|^2 + \int_{\theta}^{t} \Big|b\big(s, X^1_s(\theta)\big) - b\big(s, X^1_s(\pi(\theta))\big)\Big|^2 ds \\ &\quad + \Big|\int_{\theta}^{t} \Big(\sigma\big(s, X^1_s(\theta)\big) - \sigma\big(s, X^1_s(\pi(\theta))\big)\Big)dW_s\Big|^2\Big\} \ . \end{aligned} \tag{4.19}$$

From **(HF)** and **(HFD)**, we have

$$\mathbb{E}\big|\beta\big(\theta, X^0_\theta\big) - \beta\big(\pi(\theta), X^0_{\pi(\theta)}\big)\big|^2 \le K\big(|\pi| + \mathbb{E}|X^0_\theta - X^0_{\pi(\theta)}|^2\big) \ ,$$

which gives with (4.17)

$$\mathbb{E}\big|\beta\big(\theta, X^0_\theta\big) - \beta\big(\pi(\theta), X^0_{\pi(\theta)}\big)\big|^2 \le K|\pi| \ .$$

We have also from **(HF)**

$$\mathbb{E}\Big[\Big|\int_{\pi(\theta)}^{\theta} b(s, X^0_s)ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta} \sigma(s, X^0_s)dW_s\Big|^2\Big] \le K|\pi| \ ,$$

and with (2.10)

$$\mathbb{E}\Big[\Big|\int_{\pi(\theta)}^{\theta} b(s, X^1_s(\pi(\theta)))ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta} \sigma(s, X^1_s(\pi(\theta)))dW_s\Big|^2\Big] \le K|\pi| \ ,$$

where K does not depend on θ.

Combining these inequalities with (4.19), **(HF)** and BDG-inequality, we get

$$\mathbb{E}\Big[\sup_{u\in[\theta,t]} \big|X^1_u(\theta) - X^1_u(\pi(\theta))\big|^2\Big] \le K\Big(\int_{\theta}^{t} \mathbb{E}\Big[\sup_{u\in[\theta,s]} \big|X^1_u(\theta) - X^1_u(\pi(\theta))\big|^2\Big]ds + |\pi|\Big) \ .$$

Applying Gronwall's lemma, we get

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\theta)-X_t^1(\pi(\theta))\big|^2\Big] \;\leq\; K|\pi|\;, \tag{4.20}$$

where K does not depend on θ.
To find an upper bound for the term $\mathbb{E}[\sup_{t\in[\theta,T]}|X_t^1(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2]$ we introduce the scheme $\tilde{X}_\cdot^\pi(\pi(\theta))$ defined by

$$\begin{cases} \tilde{X}^\pi_{\pi(\theta)}(\pi(\theta)) \;=\; X^1_{\pi(\theta)}(\pi(\theta))\;, \\ \tilde{X}^\pi_{t_i}(\pi(\theta)) \;=\; \tilde{X}^\pi_{t_{i-1}}(\pi(\theta))+b\big(t_{i-1},\tilde{X}^\pi_{t_{i-1}}(\pi(\theta))\big)\Delta t_i^\pi+\sigma\big(t_{i-1},\tilde{X}^\pi_{t_{i-1}}(\pi(\theta))\big)\Delta W_i^\pi\;,\; t_i>\pi(\theta)\;. \end{cases}$$

We have the inequality

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;\leq\; & 2\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))\big|^2\Big] \\ & +2\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\;. \end{aligned}\tag{4.21}$$

We first notice that under **(HF)** the initial value of $\tilde{X}^\pi(\pi(\theta))$ satisfies

$$|\tilde{X}^\pi_{\pi(\theta)}(\pi(\theta))|=|X^1_{\pi(\theta)}(\pi(\theta))|=\big|X^0_{\pi(\theta)}+\beta\big(\pi(\theta),X^0_{\pi(\theta)}\big)\big| \;\leq\; C\big(1+\sup_{t\in[0,T]}|X_t^0|\big)\;. \tag{4.22}$$

Hence, it is bounded by a square integrable random variable which does not depend on θ. Moreover $\tilde{X}^\pi(\pi(\theta))$ is the Euler scheme of $X^1(\pi(\theta))$ on $[\pi(\theta),T]$. Therefore, we have under **(HF)** and **(HFD)** (see e.g. [16])

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1\big(\pi(\theta)\big)-\tilde{X}_{\pi(t)}^{\pi}\big(\pi(\theta)\big)\big|^2\Big] \;\leq\; K\Big(1+\mathbb{E}\Big[\big|X^1_{\pi(\theta)}(\pi(\theta))\big|^2\Big]\Big)|\pi|\;,$$

for some constant K which neither depends on π nor on θ. From (4.22), we get

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1\big(\pi(\theta)\big)-\tilde{X}_{\pi(t)}^{\pi}\big(\pi(\theta)\big)\big|^2\Big] \;\leq\; K\Big(1+\|X^0\|^2_{\mathcal{S}^2[0,T]}\Big)|\pi|\;. \tag{4.23}$$

We now study the term $\mathbb{E}\big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\big]$. We first notice that we have the following identity

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;=\; \mathbb{E}\Big[\sup_{t\in[\pi(\theta),T]}\big|\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\;.$$

Hence we can work with the second term. From the definition of $\tilde{X}^\pi$ and $X^{1,\pi}$, there exists a constant K such that

$$\begin{aligned}&\sup_{u\in[\pi(\theta),t]}\big|\tilde{X}_{\pi(u)}^{\pi}(\pi(\theta))-X_{\pi(u)}^{1,\pi}(\pi(\theta))\big|^2 \;\leq \\ &K\Big(\big|X^1_{\pi(\theta)}(\pi(\theta))-X^{1,\pi}_{\pi(\theta)}(\pi(\theta))\big|^2+\int_{\pi(\theta)}^{\pi(t)}\big|b(\pi(s),\tilde{X}^\pi_{\pi(s)}(\pi(\theta)))-b(\pi(s),X^{1,\pi}_{\pi(s)}(\pi(\theta)))\big|^2ds \\ &\qquad+\sup_{u\in[\pi(\theta),t]}\Big|\int_{\pi(\theta)}^{\pi(u)}\big(\sigma(\pi(s),\tilde{X}^\pi_{\pi(s)}(\pi(\theta)))-\sigma(\pi(s),X^{1,\pi}_{\pi(s)}(\pi(\theta)))\big)dW_s\Big|^2\Big)\;.\end{aligned}$$

Then, using **(HF)** and BDG-inequality, we get

$$\mathbb{E}\Big[\sup_{u\in[\pi(\theta),t]}\big|\tilde{X}^{\pi}_{\pi(u)}(\pi(\theta))-X^{1,\pi}_{\pi(u)}(\pi(\theta))\big|^2\Big]\leq K\Big(\mathbb{E}\big|X^{1}_{\pi(\theta)}(\pi(\theta))-X^{1,\pi}_{\pi(\theta)}(\pi(\theta))\big|^2 + \int_{\pi(\theta)}^{t}\mathbb{E}\Big[\sup_{u\in[\pi(\theta),s]}\big|\tilde{X}^{\pi}_{\pi(u)}(\pi(\theta))-X^{1,\pi}_{\pi(u)}(\pi(\theta))\big|^2\Big]ds\Big).$$

Using (4.20) with $\pi(\theta)$ instead of θ and Gronwall's lemma, we get

$$\mathbb{E}\Big[\sup_{t\in[\pi(\theta),T]}\big|\tilde{X}^{\pi}_{\pi(t)}(\pi(\theta))-X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\Big] \;\leq\; K|\pi|\,. \tag{4.24}$$

Combining (4.18), (4.20), (4.21), (4.23) and (4.24), we get the result. □

4.2 Error estimate for the FSDE with a jump

We are now able to provide an estimate of the error approximation of the process X by its scheme X^{π} defined by (3.13).

Theorem 4.3. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t-X^{\pi}_t\big|^2\Big] \;\leq\; K|\pi|\,,$$

for a constant K which does not depend on π.

Proof. Fix $t\in[0,T]$. From the definition of X^{π} we then have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t-X^{\pi}_t\big|^2\Big] &\leq \mathbb{E}\Big[\sup_{t\in[0,\tau]}\big|X^0_t-X^{0,\pi}_{\pi(t)}\big|^2\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X^1_t(\tau)-X^{1,\pi}_{\pi(t)}(\pi(\tau))\big|^2\mathbb{1}_{t\geq\tau}\Big]\\ &\leq \mathbb{E}\Big[\sup_{t\in[0,T]}\big|X^0_t-X^{0,\pi}_{\pi(t)}\big|^2\Big]+\int_0^T\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X^1_t(\tau)-X^{1,\pi}_{\pi(t)}(\pi(\tau))\big|^2\gamma_T(\theta)\Big]d\theta.\end{aligned}$$

Using (4.17) and **(DH)**, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t-X^{\pi}_t\big|^2\Big] \;\leq\; K\Big(|\pi|+\sup_{\theta\in[0,T]}\mathbb{E}\Big[\sup_{s\in[\theta,T]}\big|X^1_s(\theta)-X^{1,\pi}_{\pi(s)}(\pi(\theta))\big|^2\Big]\Big)\,.$$

From Theorem 4.2, we get

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t-X^{\pi}_t\big|^2\Big] \;\leq\; K|\pi|\,.$$

□

5 Convergence of the scheme for the BSDE

To provide error estimates for the Euler scheme of the BSDE, we need an additional regularity property for the coefficients g and f. We then introduce the following assumption.

(HBLD) There exists a constant K such that the functions g and f satisfy

$$\big|g(x)-g(x')\big|+\big|f(t,x,y,z,u)-f(t',x',y,z,u)\big| \;\leq\; K\big(|x-x'|+|t-t'|^{\frac{1}{2}}\big)\,,$$

for all $(t,t',x,x',y,z,u)\in[0,T]^2\times[\mathbb{R}^d]^2\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$.

We are now ready to provide error estimates of the approximation schemes for (Y^0,Z^0) and (Y^1,Z^1), and then for (Y,Z).

5.1 Error estimates for the recursive system of BSDEs

We first state an estimate of the approximation error for (Y^1,Z^1).

Proposition 5.2. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{\theta\in[0,T]}\Big\{\sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y_t^1(\theta)-Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z_s^1(\theta)-Z_{\pi(s)}^{1,\pi}(\pi(\theta))\big|^2ds\Big]\Big\}\;\leq\;K|\pi|\,,$$

for some constant K which does not depend on π.

Proof. Fix $\theta\in[0,T]$ and $t\in[\theta,T]$. We then have

$$\begin{aligned}\mathbb{E}\Big[\big|Y_t^1(\theta)-Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;&\leq\; 2\mathbb{E}\Big[\big|Y_t^1(\theta)-Y_t^1(\pi(\theta))\big|^2\Big]\\ &\quad+2\mathbb{E}\Big[\big|Y_t^1(\pi(\theta))-Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\,. \qquad (5.25)\end{aligned}$$

We study separately the two terms of right hand side.

Define $\delta Y_t^1(\theta)=Y_t^1(\theta)-Y_t^1(\pi(\theta))$ and $\delta Z_t^1(\theta)=Z_t^1(\theta)-Z_t^1(\pi(\theta))$. Applying Itô's formula, we get

$$\begin{aligned}|\delta Y_T^1(\theta)|^2-|\delta Y_t^1(\theta)|^2 \;&=\\ 2\int_t^T\delta Y_s^1(\theta)\Big[f\big(s,X_s^1(\pi(\theta)),Y_s^1(\pi(\theta)),Z_s^1(\pi(\theta)),0\big)-f\big(s,X_s^1(\theta),Y_s^1(\theta),Z_s^1(\theta),0\big)\Big]ds&\\ +2\int_t^T\delta Y_s^1(\theta)\delta Z_s^1(\theta)dW_s+\int_t^T|\delta Z_s^1(\theta)|^2ds\,.&\end{aligned}$$

From **(HBL)** and **(HBLD)**, there exists a constant K such that

$$\begin{aligned}\mathbb{E}\big[|\delta Y_t^1(\theta)|^2\big] \;&\leq K\Big(\mathbb{E}\big[\big|X_T^1(\theta)-X_T^1(\pi(\theta))\big|^2\big]+\mathbb{E}\Big[\int_t^T|\delta Y_s^1(\theta)|\big|X_s^1(\theta)-X_s^1(\pi(\theta))\big|ds\Big]\\ &\quad+\mathbb{E}\Big[\int_t^T|\delta Y_s^1(\theta)|^2ds\Big]+\mathbb{E}\Big[\int_t^T|\delta Y_s^1(\theta)||\delta Z_s^1(\theta)|ds\Big]\Big)-\mathbb{E}\Big[\int_t^T|\delta Z_s^1(\theta)|^2ds\Big]\,.\end{aligned}$$

Using the inequality $2ab \le a^2/\eta + \eta b^2$ for $a, b \in \mathbb{R}$ and $\eta > 0$, we can see that there exists a constant K such that

$$\begin{aligned}\mathbb{E}\big[|\delta Y_t^1(\theta)|^2\big] + \mathbb{E}\Big[\int_t^T |\delta Z_s^1(\theta)|^2 ds\Big] &\le K\Big(\mathbb{E}\big[\big|X_T^1(\theta) - X_T^1(\pi(\theta))\big|^2\big] + \int_t^T \mathbb{E}\big[|\delta Y_s^1(\theta)|^2\big] ds \\ &\quad + \mathbb{E}\Big[\int_t^T \big|X_s^1(\theta) - X_s^1(\pi(\theta))\big|^2 ds\Big]\Big) . \qquad (5.26)\end{aligned}$$

From Theorem 3.12 and Gronwall's lemma, we get

$$\mathbb{E}\big[\big|Y_t^1(\theta) - Y_t^1(\pi(\theta))\big|^2\big] \le K|\pi| , \qquad (5.27)$$

for some constant K which does not depend on π.

We now study the second term of the right hand side of (5.25). Since it is the classical error estimate of an Euler approximation, we get from [23]

$$\begin{aligned}\mathbb{E}\Big[\big|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] &\le K\Big(1 + \mathbb{E}\big[|X_{\pi(\theta)}^1(\pi(\theta))|^2\big]\Big)|\pi| \\ &\le K\Big(1 + \mathbb{E}\Big[\big|X_{\pi(\theta)}^0 + \beta\big(\pi(\theta), X_{\pi(\theta)}^0\big)\big|^2\Big]\Big)|\pi| ,\end{aligned}$$

for some constant K which neither depends on π nor on θ. Since β satisfies **(HF)** and **(HFD)**, we get

$$\mathbb{E}\Big[\big|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \le K\Big(1 + \|X^0\|_{\mathcal{S}^2[0,T]}^2\Big)|\pi| .$$

This last inequality with (5.25) and (5.27) give

$$\sup_{\theta\in[0,T]} \sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \le K|\pi| .$$

We now turn to the error on the term $Z^1(\theta)$. We first use the inequality

$$\begin{aligned}\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] &\le 2\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \\ &\quad + 2\mathbb{E}\Big[\int_\theta^T \big|\delta Z_t^1(\theta)\big|^2 dt\Big] . \qquad (5.28)\end{aligned}$$

Using (5.26) and (5.27) with $t = \theta$, we get

$$\mathbb{E}\Big[\int_\theta^T \big|\delta Z_s^1(\theta)\big|^2 ds\Big] \le K|\pi| , \qquad (5.29)$$

for some constant K which neither depends on π nor on θ. The other term in the right hand side of (5.28) is the classical error in an approximation of BSDE. Therefore, we have from [23]

$$\begin{aligned}\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] &\le K\Big(1 + \mathbb{E}\big|X_{\pi(\theta)}^1(\pi(\theta))\big|^2\Big)|\pi| \\ &\le K\Big(1 + \mathbb{E}\big|X_{\pi(\theta)}^0 + \beta(\pi(\theta), X_{\pi(\theta)}^0)\big|^2\Big)|\pi| ,\end{aligned}$$

for some constant K which neither depends on π nor on θ. Since β is Lipschitz continuous we get

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \;\leq\; K|\pi| \,. \tag{5.30}$$

Combining (5.28), (5.29) and (5.30), we get

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \;\leq\; K|\pi| \,,$$

for some constant K which neither depends on π nor on θ. □

We now turn to the estimation of the error between (Y^0, Z^0) and its Euler scheme (3.15). Since BSDE (2.6) involves Y^1, we need to introduce an intermediary quantity to control the error due to the approximation of Y^1 by $Y^{1,\pi}$. We therefore consider $(\tilde{Y}^\pi, \tilde{Z}^\pi) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times L^2_{\mathbb{F}}[0,T]$ solution to

$$\tilde{Y}_t^\pi \;=\; g(X_T^0) + \int_t^T \bar{f}^\pi(s, X_s^0, \tilde{Y}_s^\pi, \tilde{Z}_s^\pi)ds - \int_t^T \tilde{Z}_s^\pi dW_s \,, \quad 0 \leq t \leq T \,, \tag{5.31}$$

where we recall that $\bar{f}^\pi$ is defined by

$$\bar{f}^\pi(t,x,y,z) \;=\; f\big(t,x,y,z, Y_{\pi(t)}^{1,\pi}(\pi(t)) - y\big) \,,$$

for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

Proposition 5.3. *Under* ***(HF)***, ***(HFD)***, ***(HBL)*** *and* ***(HBLD)***, *we have the following estimate*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - \tilde{Y}_t^\pi\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t^0 - \tilde{Z}_t^\pi\big|^2 dt\Big] \;\leq\; K|\pi| \,,$$

for some constant K which does not depend on π.

Proof. Applying Itô's formula, we get

$$\begin{aligned}\mathbb{E}\big[|Y_t^0 - \tilde{Y}_t^\pi|^2\big] \;&=\; 2\mathbb{E}\Big[\int_t^T (Y_s^0 - \tilde{Y}_s^\pi)\big(f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0) - \bar{f}^\pi(s, X_s^0, \tilde{Y}_s^\pi, \tilde{Z}_s^\pi)\big)ds\Big] \\ &\quad -\mathbb{E}\int_t^T |Z_s^0 - \tilde{Z}_s^\pi|^2 ds \,.\end{aligned}$$

for $t \in [0,T]$. Using the Lipschitz condition on f, we can see that there exists a constant K such that

$$\begin{aligned}&\mathbb{E}\big[\big|Y_t^0 - \tilde{Y}_t^\pi\big|^2\big] + \mathbb{E}\Big[\int_t^T |Z_s^0 - \tilde{Z}_s^\pi|^2 ds\Big] \;\leq \\ &K\Big(\int_t^T \mathbb{E}\big[\big|Y_s^0 - \tilde{Y}_s^\pi\big|^2\big]ds + \int_t^T \mathbb{E}\big[\big|Y_s^1(s) - Y_{\pi(s)}^{1,\pi}(\pi(s))\big|^2\big]ds\Big) \,,\end{aligned} \tag{5.32}$$

for all $t \in [0,T]$. Proposition 5.2 and Gronwall's inequality imply that there exists a constant K such that

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t^0 - \tilde{Y}_t^\pi|^2\big] \;\leq\; K|\pi| \,. \tag{5.33}$$

Combining (5.32) and (5.33), we get

$$\mathbb{E}\Big[\int_0^T |Z_t^0 - \tilde{Z}_t^\pi|^2 dt\Big] \;\leq\; K|\pi| \,.$$

□

Using the classical error estimates of the approximation scheme for a Brownian BSDE (see e.g. [23]), we get

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|\tilde{Y}_t^\pi - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|\tilde{Z}_t^\pi - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \;\leq\; K|\pi| \,,$$

for some constant K which does not depend on π.

Applying Proposition 5.3 and the last inequality, we get the following estimate for the error of the backward equation (2.6).

Proposition 5.4. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \;\leq\; K|\pi| \,,$$

for some constant K which does not depend on π.

5.2 Error estimate for the BSDE with a jump

We now give an error estimate of the approximation scheme for the BSDE with a jump.

Theorem 5.4. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following error estimate for the approximation scheme*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t - Y_t^\pi\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2 \lambda_t dt\Big] \;\leq\; K|\pi| \,,$$

for some constant K which does not depend on π.

Proof.
Step 1. *Error for the variable Y.* Fix $t \in [0,T]$. From Theorem 2.1 and (3.16), we have

$$\mathbb{E}\Big[\big|Y_t - Y_t^\pi\big|^2\Big] \;=\; \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2 \mathbb{1}_{t<\tau}\Big] + \mathbb{E}\Big[\big|Y_t^1(\tau) - Y_{\pi(t)}^{1,\pi}(\pi(\tau))\big|^2 \mathbb{1}_{t\geq\tau}\Big] \,.$$

Using **(DH)**, we get

$$\begin{aligned}\mathbb{E}\Big[\big|Y_t - Y_t^\pi\big|^2\Big] &\leq \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \int_0^T \mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 \mathbb{1}_{t\geq\theta}\gamma_T(\theta)\Big]d\theta \\ &\leq K\Big(\mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \sup_{\theta\in[0,T]}\sup_{s\in[\theta,T]}\mathbb{E}\Big[\big|Y_s^1(\theta) - Y_{\pi(s)}^{1,\pi}(\pi(\theta))\big|^2\Big]\Big)\,.\end{aligned}$$

Using Propositions 5.2 and 5.4, and since t is arbitrary chosen in $[0,T]$, we get

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\big|Y_t - Y_t^\pi\big|^2\Big] \leq K|\pi|\,,$$

for some constant K which does not depend on π.

Step 2. *Error estimate for the variable Z.* From (2.7) and the definition of Z^π, we have

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] = \mathbb{E}\Big[\int_0^{T\wedge\tau}\big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t^1(\tau) - Z_{\pi(t)}^{1,\pi}(\pi(\tau))\big|^2\mathbb{1}_{t>\tau}dt\Big]\,.$$

Using **(DH)**, we get

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] &= \int_0^T\int_0^\theta \mathbb{E}\Big[\big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2\gamma_T(\theta)\Big]dtd\theta \\ &\quad + \int_0^T\int_\theta^T \mathbb{E}\Big[\big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\gamma_T(\theta)\Big]dtd\theta\,. \\ &\leq K\Big(\mathbb{E}\Big[\int_0^T\big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \sup_{\theta\in[0,T]}\int_\theta^T \mathbb{E}\Big[\big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]dt\Big)\,.\end{aligned}$$

From Propositions 5.2 and 5.4, we get

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] \leq K|\pi|\,,$$

for some constant K which does not depend on π.

Step 3. *Error estimate for the variable U.* From (2.7) and the definition of U^π, we have

$$\mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2\lambda_t dt\Big] \leq K\mathbb{E}\Big[\int_0^T \big(|Y_t^1(t) - Y_{\pi(t)}^{1,\pi}(\pi(t))|^2 + |Y_t^0 - Y_{\pi(t)}^{0,\pi}(\pi(t))|^2\big)\lambda_t dt\Big]\,.$$

Using **(HBI)**, we get

$$\mathbb{E}\Big[\int_0^T\big|U_t - U_t^\pi\big|^2\lambda_t dt\Big] \leq K\Big(\sup_{\theta\in[0,T]}\sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] + \sup_{t\in[0,T]}\mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big]\Big)\,.$$

Combining this last inequality with Propositions 5.2 and 5.4, we get the result. □

Remark 5.2. Our decomposition approach allows us to suppose that the jump coefficient β is only Lipschitz continuous. We do not need to impose any regularity condition on β and any ellipticity assumption on $I_d + \nabla\beta$ as done in [4] in the case where μ is a Poisson random measure independent of W.

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