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Stéphane Junca

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HIGH FREQUENCY WAVES AND THE MAXIMAL SMOOTHING EFFECT FOR NONLINEAR SCALAR CONSERVATION LAWS

STÉPHANE JUNCA

ABSTRACT. The article first studies the propagation of well prepared high frequency waves with small amplitude ε near constant solutions for entropy solutions of multidimensional nonlinear scalar conservation laws. Second, such oscillating solutions are used to highlight a conjecture of Lions, Perthame, Tadmor, ([23]), about the maximal regularizing effect for nonlinear conservation laws. For this purpose, a new definition of smooth nonlinear flux is stated and compared to classical definitions. Then it is proved that the uniform smoothness expected by [23] in Sobolev spaces cannot be exceeded for all smooth nonlinear fluxes.

Key-words: multidimensional conservation laws, nonlinear smooth flux, geometric optics, Sobolev spaces, smoothing effect.

Mathematics Subject Classification:

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CONTENTS

1. Introduction	1
1.1. The smoothing effect in Sobolev spaces	2
1.2. Other approaches for the smoothing effect	5
2. High frequency waves with small amplitude	5
3. Characterization of nonlinear flux	10
3.1. Nonlinear smooth flux	10
3.2. Comparisons with other nonlinear flux definitions	14
4. Sobolev estimates	16
5. Super critical geometric optics	24
5.1. Propagation of highest frequency waves	24
5.2. Highlight of the Lions, Perthame, Tadmor conjecture	25
References	27

1. INTRODUCTION

This paper deals with super critical geometric optics to highlight the maximal regularizing effect for nonlinear multidimensional scalar conservation laws.

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This effect is studied in Sobolev spaces by P.L. Lions, B. Perthame and E. Tadmor in [23]. They obtain an uniform fractional Sobolev bounds for any ball of L^∞ initial data under a non linearity condition on the flux. They also conjectured a better Sobolev exponent. In this framework, we can prove that the Sobolev exponent conjectured in [23] cannot be exceeded. Indeed, we construct a sequence of smooth solutions which are exactly uniformly bounded in the Sobolev space conjectured in [23]. The uniform Sobolev estimate of this sequence blows up in all more regular Sobolev spaces.

Notice that we look for the best *uniform* Sobolev exponent for a set of solutions. The smoothness of any individual solution is not studied in this paper. This point is discussed later.

A very important point to note here is the definition of nonlinear flux. Indeed, there are various definitions ([15, 23, 5, 8]). In [23] they give well known Definition 1.1 below and a conjecture about the maximal smoothing effect in Sobolev spaces related to the parameter “ α ” from their definition. The study of periodic solutions leads to another definitions [15, 5]. We obtain new Definition 3.1 for smooth flux. It generalizes the definition of [5]. For smooth flux, our definition is equivalent to classical Definition 1.1. Definition 3.1 gives a way to compute the parameter “ α ”. This new definition also shows that smoothing effects for scalar conservation laws depend on the space dimension.

To be more precise, the smoothing effect and the related conjecture in the Sobolev framework are recalled in Subsection 1.1. Sobolev spaces are not sufficient to describe all the properties of the solutions. Some comments are given in Subsection 1.2 for other approaches. Finally the outline of the paper are given to close the introduction.

1.1. The smoothing effect in Sobolev spaces.

We look for Sobolev bounds for entropy solutions $u(.,.)$ of

$$(1.1) \quad \partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u) = 0,$$

where $t \in [0, +\infty[$, $\mathbf{x} \in \mathbb{R}^d$, $u : [0, +\infty[\times \mathbb{R}_{\mathbf{x}}^d \rightarrow \mathbb{R}$, $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$ is a smooth flux function, $\mathbf{F} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$, and the initial data are only bounded in $L^\infty(\mathbb{R}_{\mathbf{x}}^d, \mathbb{R})$:

$$(1.2) \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

Let $\mathbf{a}(u)$ be $\mathbf{F}'(u)$. Obviously, when \mathbf{F} is linear: $\mathbf{a}(u) = \mathbf{a}$ where \mathbf{a} is a constant vector, $u(t, \mathbf{x}) = u_0(\mathbf{x} - t\mathbf{a})$, there is no smoothing effect. In [23], it was first proved a regularizing effect for *nonlinear* multidimensional flux \mathbf{F} . The sharp measurement of the non-linearity plays a key role in our study. Let us recall the classical definition for nonlinear flux from [23].

Definition 1.1. [Nonlinear flux [23]]

Let M be a positive constant, $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$ is said to be nonlinear on $[-M, M]$ if there exist $\alpha > 0$ and $C = C_\alpha > 0$ such that for all $\delta > 0$

$$(1.3) \quad \sup_{\tau^2 + |\xi|^2 = 1} |W_\delta(\tau, \xi)| \leq C \delta^\alpha,$$

where $(\tau, \xi) \in S^d \subset \mathbb{R}^{d+1}$, i.e. $\tau^2 + |\xi|^2 = 1$, and $|W_\delta(\tau, \xi)|$ is the one dimensional measure of the singular set:

$$W_\delta(\tau, \xi) := \{|v| \leq M, |\tau + \mathbf{a}(v) \cdot \xi| \leq \delta\} \subset [-M, M] \text{ and } \mathbf{a} = \mathbf{F}'.$$

Indeed, $W_\delta(\tau, \xi)$ is a neighborhood of the critical value v for the symbol of the linear operator $\mathcal{L}[v]$ in the Fourier direction (τ, ξ) where $\mathcal{L}[v] = \partial_t + \mathbf{a}(v) \cdot \nabla_x$. The symbol in this direction is: $i(\tau + \mathbf{a}(v) \cdot \xi)$. This operator is simply related to any smooth solution u of equation (1.1) by the chain rule formula:

$$\partial_t u + \operatorname{div}_x \mathbf{F}(u) = \partial_t u + \mathbf{a}(u) \cdot \nabla_x u = \mathcal{L}[u]u.$$

α is a degeneracy measurement of the operator \mathcal{L} parametrized by v . α depends only on the flux \mathbf{F} and the compact set $[-M, M]$: $\alpha = \alpha[\mathbf{F}, M]$. In the sequel we denote by

$$(1.4) \quad \alpha_{\text{sup}} = \alpha_{\text{sup}}[\mathbf{F}, M], \quad \text{the supremum of all } \alpha \text{ satisfying (1.3).}$$

α , or more precisely α_{sup} , is a key parameter to describe the sharp smoothing effect for entropy solutions of nonlinear scalar conservation laws. For smooth flux the parameter α always belongs to $[0, 1]$, for instance: $\alpha_{\text{sup}} = 0$ for a linear flux, $\alpha = 1$ for strictly convex flux in dimension one. For the first time α_{sup} is characterized below in section 3. Indeed, for smooth nonlinear flux, $\frac{1}{\alpha_{\text{sup}}}$ is always an integer greater or equal to the space dimension.

In all the sequel we assume that $M \geq \|u_0\|_\infty$ and the flux \mathbf{F} is nonlinear on $[-M, M]$:

$$(1.5) \quad \alpha_{\text{sup}} > 0.$$

When nonlinear condition (1.5) is true, the entropy solution operator associated with the nonlinear conservation law (1.1), (1.2),

$$\begin{aligned} \mathcal{S}_t : L^\infty(\mathbb{R}_x^d, [-M, M]) &\rightarrow L^\infty(\mathbb{R}_x^d, [-M, M]) \\ u_0(\cdot) &\mapsto u(t, \cdot), \end{aligned}$$

has a regularizing effect: mapping $L^\infty(\mathbb{R}_x^d, [-M, M])$ into $W_{loc}^{s,1}(\mathbb{R}_x^d, \mathbb{R})$ for all $t > 0$.

In [23], they proved this regularizing effect for all $s < \frac{\alpha}{2 + \alpha}$.

In [31] the result is improved for all $s < \frac{\alpha}{1 + 2\alpha}$ under a generic assumption on $\mathbf{a}' = \mathbf{F}''$.

Lions, Perthame and Tadmor conjectured in 1994 a better regularizing effect, ([23], remark 3, p .180, line 14-17). In [23], they proposed an optimal bound s_{sup} for Sobolev exponents of entropy solutions:

$$(1.6) \quad s_{\text{sup}} = \alpha_{\text{sup}}.$$

That is to say that $u(t, \cdot)$ belongs in all $W_{loc}^{s,1}(\mathbb{R}^d, \mathbb{R})$ for all $s < s_{\text{sup}} = \alpha_{\text{sup}}$ and for all $t > 0$.

The shocks formation implies $s < 1$ and $s_{\text{sup}} \leq 1$ since $W^{1,1}$ functions do not

have shock.

A main result of the paper is to give an insight of the conjecture (1.6) by bounding the uniform Sobolev smoothing effect s_{sup} for the whole set of entropy solutions with initial data bounded by M in L^∞ :

$$(1.7) \quad s_{\text{sup}} \leq \alpha_{\text{sup}}.$$

Some results highlight the conjecture (1.6) or the inequality (1.7) in the one dimensional case. But for the multidimensional case and for all smooth fluxes, our examples are new.

One dimensional case:

In one dimension ($d=1$) and for uniformly convex flux it is well known from Lax and Oleinik that the entropy solution becomes BV , ([24, 22]). Conjecture (1.6) is true in this case since for all $t > 0$, $u(t, \cdot)$ belongs to $W_{loc}^{s,1}$ for all $s < 1$. In this case we have conjecture (1.6) which is simply: $s_{\text{sup}} = 1 = \alpha_{\text{sup}}$.

For power law flux: $F(u) = |u|^{1+p}$, De Lellis and Westdickenberg built entropy piecewise smooth solutions and proved (1.7) ([13], Proposition 3.4 p. 1085). For all one dimensional nonlinear smooth fluxes, new continuous examples are also given in [4]. Both examples are only justified for a bounded time interval.

Recently, for more general convex fluxes the regularity $s_{\text{sup}} = \alpha_{\text{sup}}$ is reached in $W^{s,1}$ ([19]) and also $W^{s,1/s}$ ([3]). The proofs need a generalized Oleinik condition.

For the class of solutions with bounded entropy production, the optimal smoothing effect is proved in [13, 16]. In [16] the result is restricted for uniform convex flux and in the one dimensional case. This class of solution is larger than the class of entropy solutions. For instance, the uniqueness of solution for initial value problem (1.1), (1.2) is not true in general. Thus the smoothing effect expected is smaller: the optimal Sobolev exponent for uniform convex flux is only $s = 1/3$, ([13, 16]), instead of $s = 1$ for entropy solutions.

Multidimensional case:

For the first time, the multidimensional case is investigated to highlight inequality (1.7). Furthermore, all smooth nonlinear fluxes are considered in this paper. Examples of family of solutions exactly uniformly bounded in $W_{loc}^{s,1}$ with the conjectured maximal exponent $s = \alpha_{\text{sup}}$ and with no improvement of the Sobolev exponent.

High frequency periodic solutions of (1.1) are used for this purpose Near a constant state and for L^∞ data, geometric optics expansions with various frequencies and various phases are validated in the framework of weak entropy solutions and of L_{loc}^1 convergence in [5]. Here, results of [5] are specified in C^1 for a well chosen phase and proved for a particular sequence of smooth

solutions (without shocks on a strip). This allows to give a proof of inequality (1.7) for the ball of L^∞ initial data: $L^\infty(\mathbb{R}^d, [-M, M])$

1.2. Other approaches for the smoothing effect.

The maximal Sobolev exponent is not sufficient to get all the properties of entropy solutions. Other relevant ways are indicated.

In the 50', Oleinik ([24]) obtained her famous one-sided Lipschitz condition. This condition ensures the uniqueness and the BV regularity of the entropy solution. This is the first basis and the proof of conjecture (1.6) for one dimensional uniformly convex flux. Dafermos ([9, 10]), with his generalized characteristics, handled convex and some non convex fluxes. Hoff extended this one-sided condition in several space variables ([17]) but related to a convex assumption on the flux. The generalized Oleinik condition is the key assumption to prove the best $W^{s,1}$ smoothing effect in [19]. The maximal $W^{s,p}$ smoothing effect is proved in [3] with a one-sided Holder condition and fractional BV spaces, see remark 5.4 below. For a recent generalization of Oleinik condition for a flux with one inflection point, we refer the reader to [20].

In the 90', the kinetic formulation of conservation laws ([23]) gave another approach. It began in 2000 ([6]). Some trace properties were obtained in [34, 11, 12, 8]. These structure of a BV function for solutions cannot be given by Sobolev regularity. These results are indeed valid for solutions with bounded entropy production. Thus this method necessary misses some other properties of entropy solutions.

The paper is organized as follows. In section 2 examples of highly oscillating solutions are expounded under new orthogonality conditions between the flux derivatives and the phase gradient. In section 3, these orthogonality conditions lead to a new definition of nonlinear smooth flux. The concept of flux non-linearity is clarified, characterized and compared with other classical definitions. Section 4 is devoted to get optimal Sobolev estimates on oscillating solutions built in section 2. It is a quite technical part. Finally, the section 5 gives the super critical geometric optics expansion with the highest frequency related to the geometric structure of the nonlinear flux. This family of high frequency waves highlights inequality (1.7). But the conjecture (1.6) is still an open problem.

2. HIGH FREQUENCY WAVES WITH SMALL AMPLITUDE

The section 2 deals with highly oscillating initial data near a constant state:

$$(2.1) \quad u_\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon U_0 \left(\frac{\mathbf{v} \cdot \mathbf{x}}{\varepsilon \gamma} \right),$$

where $U_0(\theta)$ is a one periodic function w.r.t. θ , $\gamma > 0$, \underline{u} is a constant ground state, $\underline{u} \in [-M, M]$, $\mathbf{v} \in \mathbb{R}^d$. The case $\gamma = 1$ is the classical geometric optics for

scalar conservation laws, ([14]). In this paper we focus on *critical oscillations* when $\gamma > 1$.

The aim of this section is to understand when such high frequency are propagated or not propagated. As we will see, it depends on new compatibility conditions between the phase and the flux (2.5).

One of the two following asymptotic expansions (2.2) or (2.3), is expected in $L^1_{loc}([0, +\infty[\times \mathbb{R}^d, \mathbb{R})$ for the entropy-solution u_ε of conservation law (1.1) with highly oscillating data (2.1) when ε goes to 0,

$$(2.2) \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + o(\varepsilon)$$

$$(2.3) \quad \text{or} \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \bar{U}_0 + o(\varepsilon),$$

where the profile $U(t, \theta)$ satisfies a conservation law with initial data $U_0(\theta)$, $\bar{U}_0 = \int_0^1 U_0(\theta) d\theta$ and the phase ϕ satisfies the eikonal equation:

$$(2.4) \quad \partial_t \phi + \mathbf{a}(\underline{u}) \cdot \nabla_{\mathbf{x}} \phi = 0, \quad \phi(0, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x}.$$

Thus the phase is simply a linear phase:

$$\phi(t, \mathbf{x}) = \mathbf{v} \cdot (\mathbf{x} - t \mathbf{a}(\underline{u})).$$

The propagation of such oscillating data is obtained under the crucial compatibility condition (2.5) below. Otherwise, when the the compatibility condition (2.5) is nowhere satisfied, the nonlinear semi-group associated with equation (1.1) cancels these too high oscillations, see Theorem 2.2. The validity or invalidity of assumption (2.5) is a key point related to the nonlinearity of the flux (section 3).

Theorem 2.1. [Propagation of smooth high oscillations]

Let γ belong to $]1, +\infty[$ and let q be the integer such that $q - 1 < \gamma \leq q$. Assume \mathbf{F} belongs to $C^{q+3}(\mathbb{R}, \mathbb{R}^d)$, $U_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $\mathbf{v} \neq (0, \dots, 0)$ and

$$(2.5) \quad \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} = 0, \quad k = 1, \dots, q - 1$$

then there exists $T_0 > 0$ such that, for all $\varepsilon \in]0, 1]$, the solutions of conservation law (1.1) with initial oscillating data (2.1) are smooth on $[0, T_0] \times \mathbb{R}$ and

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + \mathcal{O}(\varepsilon^{1+r}) \text{ in } C^1([0, T_0] \times \mathbb{R}^d),$$

where $0 < r = \begin{cases} 1 & \text{if } \gamma = q, \\ q - \gamma & \text{else,} \end{cases}$ and the smooth profile U is uniquely determined by the Cauchy problem (2.6), ϕ is given by the eikonal equation (2.4):

$$(2.6) \quad \frac{\partial U}{\partial t} + b \frac{\partial U^{q+1}}{\partial \theta} = 0, \quad U(0, \theta) = U_0(\theta),$$

$$\text{with } b = \begin{cases} \frac{1}{(q+1)!} (\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}) & \text{if } \gamma = q, \\ 0 & \text{else.} \end{cases}.$$

We deal with smooth solutions to compute later Sobolev bounds. Indeed, the asymptotic stays valid after shocks formation and for all positive time but in L^1_{loc} instead of L^∞ ([5]).

When $\gamma = 1$, we do not need assumption (2.5). It is the classic case for geometric optics ([14, 29]).

In dimension $d \geq 2$, it is always possible to find a non trivial vector \mathbf{v} satisfying (2.5). At least for $\gamma = 2$, (2.5) is reduced to find $\mathbf{v} \neq 0$ such that $\mathbf{a}'(\underline{u}) \cdot \mathbf{v} = 0$. Thus, such singular solutions always exist in dimension greater than one. But, for genuine nonlinear one dimensional conservation law, there is never such solution. Of course, we assume U_0 be a non constant function and \mathbf{F} be a nonlinear function near \underline{u} , else the theorem is obvious. Indeed, when U_0 is constant, u_ε is also constant. When \mathbf{F} is linear on $[\underline{u} - \delta, \underline{u} + \delta]$ for some $\delta > 0$, high oscillations propagate for all time without any restriction of the phase and of the frequency size.

In fact, Theorem 2.1 expresses a kind of degeneracy of **multidimensional** scalar conservation laws. This degeneracy (period smaller than the amplitude) appears for quasilinear systems with some nonlinear degenerescence (see for instance [7]).

Notice that for $\gamma > 1$, smooth solutions exist for larger time than it is currently known [10, 22]: $T_\varepsilon \sim 1/|\nabla_{\mathbf{x}} u_0^\varepsilon| \sim \varepsilon^{\gamma-1}$. Furthermore, equation (2.6) is nonlinear if and only if $\gamma \in \mathbb{N}$ and $\mathbf{a}^q(\underline{u}) \cdot \mathbf{v} \neq 0$.

Proof : First one performs a WKB computations with following ansatz:

$$(2.7) \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U_\varepsilon \left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right).$$

Notice that we use for the proof the exact profile U_ε as in [21]. It is a method to sharply control the difference between the exact solution and the geometric optics expansion: U_ε and U .

The Taylor expansion of the flux and the remainder are:

$$\begin{aligned} \mathbf{F}(u_\varepsilon) &= \sum_{k=0}^{q+1} \varepsilon^k \frac{U_\varepsilon^k}{k!} \mathbf{F}^{(k)}(\underline{u}) + \varepsilon^{q+2} \mathbf{G}_q^\varepsilon(U_\varepsilon), \\ \mathbf{G}_q^\varepsilon(U) &= U^{q+2} \int_0^1 \frac{(1-s)^{q+1}}{(q+1)!} \mathbf{F}^{(q+2)}(\underline{u} + s\varepsilon U) ds, \\ g_q^\varepsilon(U) &= \mathbf{v} \cdot \mathbf{G}_q^\varepsilon(U). \end{aligned}$$

We now compute the partial derivatives with respect to time and space variables:

$$\partial_t U_\varepsilon \left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) = \partial_t U_\varepsilon - \varepsilon^{-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon$$

$$\begin{aligned}\operatorname{div}_{\mathbf{x}}\mathbf{F}(u_\varepsilon) &= \sum_{k=0}^q \varepsilon^{k+1-\gamma} \frac{\partial_\theta U_\varepsilon^{k+1}}{(k+1)!} \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} + \varepsilon^{q+2} \operatorname{div}_{\mathbf{x}} \mathbf{G}_q^\varepsilon(U_\varepsilon) \\ &= \varepsilon^{1-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon + \varepsilon^{q+1-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{q+2-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon),\end{aligned}$$

where $c_q = \frac{\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}}{(q+1)!}$. Then simplification yields

$$(2.8) \quad \partial_t u_\varepsilon + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u_\varepsilon) = \varepsilon \left(\partial_t U_\varepsilon + \varepsilon^{q-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{1+q-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon) \right).$$

It suffices to take U_ε solution of the one dimensional scalar conservation laws with $\psi_\varepsilon(U) = \varepsilon^{q-\gamma} c_q U^{q+1} + \varepsilon^{1+q-\gamma} g_q^\varepsilon(U)$

$$(2.9) \quad \partial_t U_\varepsilon + \partial_\theta \psi_\varepsilon(U_\varepsilon) = 0, \quad U_\varepsilon(0, \theta) = U_0(\theta).$$

Notice that $\psi_\varepsilon = O(1) \in C_{loc}^2$. For $\gamma < q$, ψ_ε is even smaller: $\psi_\varepsilon = O(\varepsilon^r) \in C_{loc}^2$. That is enough to prove the existence of a sequence of smooth oscillating solutions on the same strip.

Uniform life span for smooth solutions $(U_\varepsilon)_{0 < \varepsilon \leq 1}$:

We use the method of characteristics with $\psi'_\varepsilon(U) = \frac{d}{dU} \psi_\varepsilon(U)$:

$$\frac{d}{dt} \Theta(t, \theta) = \psi'_\varepsilon(U_\varepsilon(t, \Theta(t, \theta))), \quad \Theta(0, \theta) = \theta.$$

Since U_ε is constant along the characteristics, $\Theta(t, \theta) = \theta + t \psi'_\varepsilon(U_0(\theta))$. As soon as the map $\theta \rightarrow \Theta(t, \theta)$ is not decreasing no shock occurs.

$$\frac{\partial}{\partial \theta} \Theta(t, \theta) = 1 + t \psi''_\varepsilon(U_0(\theta)) \frac{d}{d\theta} U_0(\theta)$$

The first shock appears at the time T_ε when the right hand side vanishes. Let $m_0 = \sup_{[0,1]} |U_0| > 0$, $d_0 = \sup_{[0,1]} \left| \frac{d}{d\theta} U_0 \right|$, $m = \sup_{0 < \varepsilon \leq 1} \sup_{|U - \underline{u}| \leq m_0} |\psi''_\varepsilon(U)|$,

$$1/T_\varepsilon = \sup_{[0,1]} \left(-\psi''_\varepsilon(U_0(\theta)) \frac{d}{d\theta} U_0(\theta) \right) \leq m d_0.$$

Of course, for constant initial data ($d_0 = 0$), no shock occurs, the solution is constant and $T_\varepsilon = +\infty$. In general $m d_0 \neq 0$, T_ε is finite but $0 < \inf_{0 < \varepsilon \leq 1} T_\varepsilon$ since $T_\varepsilon \geq 1/(m d_0)$ for all $0 < \varepsilon \leq 1$.

This gives the existence of a positive time $T_0 < T^* = \inf\{T_\varepsilon, \varepsilon \in]0, 1]\}$ such that $U_\varepsilon \in C^1([0, T_0] \times \mathbb{R}/\mathbb{Z})$. Thus u_ε , which is well defined by (2.7), belongs to $C^1([0, T_0] \times \mathbb{R}^d)$ for all $0 < \varepsilon \leq 1$.

Now we prove the C^1 convergence of the geometric optics expansion. There are two cases: γ is an integer or not.

$q = \gamma$: From (2.8) and (1.1) we get

$$\begin{aligned}\partial_t U_\varepsilon + \partial_\theta (c_q U_\varepsilon^{q+1} + \varepsilon g_q^\varepsilon(U_\varepsilon)) &= 0, & \partial_t U + c_q \partial_\theta U^{q+1} &= 0, \\ U_\varepsilon(0, \theta) &= U_0(\theta), & U(0, \theta) &= U_0(\theta).\end{aligned}$$

The method of characteristics gives C^1 characteristics, C^1 solutions and

$$\|U_\varepsilon - U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon),$$

where

$$\|U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = \|U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_t U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_\theta U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)}.$$

integer $q > \gamma$: The proof is similar except the term $\varepsilon^r c_q \partial_\theta (c_q U^{q+1})$ becomes a remainder, with $r = q - \gamma$ and $U(t, \theta) = U_0(\theta)$, thus

$$\|U_\varepsilon(\cdot, \cdot) - U_0(\cdot)\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon^r),$$

which concludes the proof. \square

When condition (2.5) is violated, oscillations are immediately canceled.

Theorem 2.2. [Cancellation of high oscillations, [5]]

Let \mathbf{F} belong to C^{q+2} and $U_0 \in L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, where $q - 1 < \gamma \leq q$ where q is defined in Theorem 2.1. If for some $0 < j < q$

$$(2.10) \quad \mathbf{a}^{(j)}(\underline{u}) \cdot \mathbf{v} \neq 0$$

then the solutions u_ε of conservation law (1.1) with initial oscillating data (2.1) for $\varepsilon \in]0, 1]$ satisfy when $\varepsilon \rightarrow 0$

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \overline{U}_0 + o(\varepsilon) \quad \text{in } L^1_{loc}([0, +\infty[\times \mathbb{R}^d).$$

Obviously the interesting case is when U_0 is non constant. In this context, when U_0 is smooth and non constant the first time when a shock occurs $T_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$. Thus solutions are weak entropy solutions.

The proof is in the spirit of [5] and uses averaging lemmas (see [27] and the references given there). The proof is briefly expounded to be self-contained.

Proof : For non constant initial data it is impossible to avoid shock waves on any fixed strip $[0, T_0] \times \mathbb{R}^d$ with $T_0 > 0$ as in the previous proof of Theorem 2.1 since the time span of smooth solutions is ε^β where $\beta = \gamma - j > 0$.

First, with a change of space variable $\mathbf{x} \leftrightarrow \mathbf{x} - t \cdot \mathbf{a}(\underline{u})$, we can assume that $\mathbf{a}(\underline{u}) = 0$.

The WKB computations use the following ansatz: $u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon v_\varepsilon(t, \mathbf{x})$ where $v_\varepsilon(t, \mathbf{x}) = W_\varepsilon(t, \varepsilon^{-j} \phi(t, \mathbf{x}))$. Indeed, the condition (2.10) leads to such ansatz as we can see in the WKB computations of the proof of Theorem 2.1. Then W_ε satisfies the one dimensional nonlinear conservation laws:

$$(2.11) \quad \partial_t W_\varepsilon + \partial_\theta (c_j W_\varepsilon^{j+1} + \varepsilon g_j^\varepsilon(W_\varepsilon)) = 0, \quad W_\varepsilon(0, \theta) = U_0(\varepsilon^{-\beta} \theta), \quad c_j \neq 0.$$

$W_\varepsilon(0, \cdot)$ converges weakly towards \overline{U}_0 . As in [5], W_ε is relatively compact in L^1_{loc} thanks to averaging lemmas. Then W_ε converges towards the unique entropy solution W of

$$\partial_t W + c_j \partial_\theta W^{j+1} = 0, \quad W(0, \theta) = \overline{U}_0.$$

That is to say that $W(t, \theta) \equiv \bar{U}_0$. Then $v_\varepsilon(t, \mathbf{x})$ converges towards \bar{U}_0 in L^1_{loc} which concludes the proof. \square

3. CHARACTERIZATION OF NONLINEAR FLUX

The flux nonlinearity is characterized by the parameter α in Definition 1.1, the Lions-Perthame-Tadmor definition of nonlinear flux. The smoothing effect depends only on the best $\alpha = \alpha_{\text{sup}}$. The understanding of the parameter α_{sup} is a key step to the comprehension of the regularity of entropy solutions. Unfortunately, there are only few examples where α_{sup} is computed in dimension 2 ([23, 31]) and there are some remarks in [18, 19, 2].

For the first time, for all smooth fluxes and for all dimensions we characterize the fundamental parameter α_{sup} . For this purpose we state Definition 3.1 of smooth nonlinear flux. This new definition is related to the critical geometric optics expansion given in Section 2. Let us emphasize on three important consequences of Definition 3.1.

- The parameter α_{sup} is explicitly characterized with the flux derivatives in Theorem 3.1.
- The super critical geometric optics expansion is built in Theorem 5.1.
- The uniform maximal smoothing effect is highlighted in section 5.2.

We explain this new definition in the subsection 3.1. We compare our new definition with some other classical definitions in subsection 3.2. We prove that all definitions of nonlinear flux are equivalent for analytical flux.

3.1. Nonlinear smooth flux.

We introduce a new definition of nonlinear C^∞ flux related to critical geometric optics expansions. When the compatibility conditions (2.5) are satisfied in Theorem 2.1, very high frequency waves are smooth solutions of the conservation law (1.1). Furthermore, these conditions are optimal thanks to Theorem 2.2. What is the highest frequency waves as in Theorem 2.1 solutions of (1.1)? Indeed, near the constant state \underline{u} we can propagate waves with frequency ε^{-m} , $m > 1$, if the set $\{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m-1)}(\underline{u})\}^\perp$ is not reduced to $\{0\}$. Thus the maximal m occurs when $\{0\} = \{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m)}(\underline{u})\}^\perp$ and $\{0\} \neq \{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m-1)}(\underline{u})\}^\perp$. We now can write the following definition.

Definition 3.1. [Nonlinear smooth flux]

Let the flux \mathbf{F} belong to $C^\infty(\mathbb{R}, \mathbb{R}^d)$ and $I = [-M, M]$. The flux is said to be **nonlinear** on I if, for all $u \in I$, there exists $m \in \mathbb{N}^*$ such that

$$(3.1) \quad \text{rank}\{\mathbf{a}'(u), \dots, \mathbf{a}^{(m)}(u)\} = d.$$

Furthermore, the flux is said to be **genuine nonlinear** if $m = d$ is enough in (3.1) for all $u \in I$.

As usual, the non-linearity is a matter of the second derivatives of \mathbf{F} , $\mathbf{a}' = \mathbf{F}''$. Notice that $m \geq d$. We need at least d vectors in (3.1) to span the space \mathbb{R}^d . Thus the genuine nonlinear case is the strongest nonlinear case.

The genuine nonlinear case was first stated in [5] (condition (2.8) and Lemma 2.5 p. 447 therein). The genuine nonlinear condition in the d dimensional case

$$(3.2) \quad \det(\mathbf{a}'(u), \mathbf{a}''(u), \dots, \mathbf{a}^{(d)}(u)) \neq 0, \quad \forall u \in I,$$

was also in [8], see condition (16) p. 84 therein. The simplest example of genuine nonlinear flux \mathbf{F} with the velocity \mathbf{a} was given in [5, 8, 2]:

$$\mathbf{a}(u) = (u, u^2, \dots, u^d) \quad \text{with } \mathbf{F}(u) = \left(\frac{u^2}{2}, \dots, \frac{u^{d+1}}{d+1} \right).$$

Definition 3.1 is a generalization of the genuine nonlinear condition (3.2). Definition 3.1 is more explicit with following integers with $I = [-M, M]$.

$$(3.3) \quad d_{\mathbf{F}}[u] = \inf\{k \geq 1, \text{rank}\{\mathbf{F}''(u), \dots, \mathbf{F}^{(k+1)}(u)\} = d\} \geq d,$$

$$(3.4) \quad d_{\mathbf{F}} = \sup_{|u| \leq M} d_{\mathbf{F}}[u] \in \{d, d+1, \dots\} \cup \{+\infty\}.$$

Indeed, Definition 3.1 states that the flux is genuine nonlinear when $d_{\mathbf{F}}$ reaches its minimal value, $d_{\mathbf{F}} = d$.

Conversely, when the flux \mathbf{F} is linear, \mathbf{a} is a constant vector in \mathbb{R}^d and $d_{\mathbf{F}}$ reaches its maximal value, $d_{\mathbf{F}} = +\infty$.

Between $d_{\mathbf{F}} = d$ and $d_{\mathbf{F}} = +\infty$, there is a large variety of nonlinear flux.

The following theorem gives the optimal parameter α (1.3) for smooth flux.

Theorem 3.1. [Sharp measurement of the flux non-linearity]

Let \mathbf{F} be a smooth flux, $\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$, the measurement of the flux non-linearity α_{sup} is given by

$$\alpha_{\text{sup}} = \frac{1}{d_{\mathbf{F}}} \leq \frac{1}{d}.$$

Furthermore, when $\alpha_{\text{sup}} > 0$ there exists $\underline{u} \in [-M, M]$ such that $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$.

A similar result for the genuine nonlinear case: $d_{\mathbf{F}} = d$, can be found in [2].

This theorem is a powerful tool to compute the parameter α_{sup} , for instance:

- $F(u) = (\cos(u), \sin(u))$ is genuine nonlinear flux, $\alpha_{\text{sup}} = 1/2$ since $\det(F''(u), F'''(u)) = 1$.
- When F is polynomial with degree less or equal to the space dimension d , $\alpha_{\text{sup}} = 0$ and F does not satisfy Definition 3.1.
- It is well known that the ‘‘Burgers multi-D’’ flux $F(u) = (u^2, \dots, u^2)$ is not nonlinear when $d \geq 2$. Let us explain this fact by two arguments: the explicit computation of α_{sup} and a sequence of high frequencies waves solutions of (1.1).
 - $\mathbf{a}''(u) \equiv 0$ so $d_{\mathbf{F}} = +\infty$ and Theorem 3.1 yields $\alpha_{\text{sup}} = 0$.

- The sequence of oscillations with large amplitude $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ given by $u_\varepsilon(t, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) = \sin\left(\frac{x_1 - x_2}{\varepsilon}\right)$ blows up in any $W_{loc}^{s,1}$, $s > 0$: for all t , $\sup_{0 < \varepsilon \leq 1} \|u_\varepsilon(t, \cdot)\|_{W^{s,1}([0,1]^d, \mathbb{R})} = +\infty$. But the sequence of initial data is uniformly bounded in L^∞ , $\|u_0^\varepsilon\|_{L^\infty} = 1$. Thus there is no improvement of the uniform initial Sobolev bounds.
- When F is polynomial such that $\deg(F_i) = 1+i$, F is genuine nonlinear:

$$\alpha_{\text{sup}} = \frac{1}{d}.$$

Remark 3.1. For smooth Flux α_{sup} is the inverse of an integer. Not all real value of α_{sup} in $[0, 1]$ are possible for $\mathbf{F} \in C^\infty$. With less smooth flux, all other values of α_{sup} are possible ([23, 13, 31, 19, 4, 3]).

We now begin the proof of Theorem 3.1 related to some proofs of phase stationary lemmas ([30, 19, 2]). The proof needs many lemmas. First we recall Lemma 1 p. 125 from [2] giving the optimal α for real functions.

Lemma 3.1 ([2]). *Let $\varphi \in C^\infty([-M, M], \mathbb{R})$,*

$$\begin{aligned} m_\varphi[v] &= \inf\{k \in \mathbb{N}, \varphi^{(k)}(v) \neq 0\} & \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}, \\ m_\varphi &= \sup_{|v| \leq M} m_\varphi[v] & \in \overline{\mathbb{N}}, \\ Z(\varphi, \varepsilon) &= \{v \in [-M, M], |\varphi(v)| \leq \varepsilon\}. \end{aligned}$$

If $0 < m_\varphi < +\infty$ then there exists $C > 1$ dependent of the function ϕ such that, for all $\varepsilon \in]0, 1]$,

$$(3.5) \quad C^{-1} \varepsilon^\alpha \leq \text{meas}(Z(\varphi, \varepsilon)) \leq C \varepsilon^\alpha \quad \text{with} \quad \alpha = \frac{1}{m_\varphi}.$$

To compute the measure of $Z(\varphi, \varepsilon)$ with a different assumption, we adapt a proof of E. Stein about stationary phase method [30]. The main point in the following lemma is that the constant does not depend on the function ϕ . Indeed, the condition $1 \leq |\phi^{(k)}(v)|$ is stronger than the condition $m_\varphi = k$. The following lemma is fundamental to prove Theorem 3.1.

Lemma 3.2. [2] *Let $k \geq 1$, I an interval of \mathbb{R} , $\phi \in C^k(I, \mathbb{R})$.*

$$\begin{aligned} \text{If} & & 1 &\leq |\phi^{(k)}(v)|, & \text{for all } v \in I, \\ \text{then} & \text{measure}\{v \in I, |\phi(v)| \leq \varepsilon\} &\leq &\bar{c}_k \varepsilon^{1/k}, \end{aligned}$$

where \bar{c}_k are constant independent of ϕ .

Proof: Since the result is independent of the interval I and the constant sign of the derivative $\phi^{(k)}$ on the interval, let us suppose that $I = \mathbb{R}$ and $\phi^{(k)}(v) \geq 1$ for all $v \in \mathbb{R}$. Thus we have for all $v \geq u$: $\phi^{(k-1)}(v) - \phi^{(k-1)}(u) \geq v - u$. This inequality shows that the function $\phi^{(k-1)}$ admits an unique root. Assume $\phi^{(k-1)}(0) = 0$ without loss of generality.

With these assumptions we prove the lemma when $k = 1$. Since $|\phi(v)| \geq |v|$ for all v , we have $Z(\phi, I, \varepsilon) = \{v \in I, |\phi(v)| \leq \varepsilon\} \subset [-\varepsilon, \varepsilon]$. So the lemma is proved for $k = 1$ with $\bar{c}_1 = 2$.

We now prove the Lemma by induction on $k > 1$. We have for all v , $|\phi^{(k-1)}(v)| \geq |v|$. Let $\eta > 0$. Notice that $\text{meas}(Z(\phi, [-\eta, \eta], \varepsilon)) \leq 2\eta$. Let ψ be the function ϕ/η . Notice that $\psi^{(k-1)}(v) \geq 1$ on $]\eta, +\infty[$. By our inductive hypothesis on ψ we have $\text{meas}(Z(\psi,]\eta, +\infty[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon)^{1/(k-1)}$, so $\text{meas}(Z(\phi,]\eta, +\infty[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)}$.

A similar argument yields $\text{meas}(Z(\phi,]-\infty, -\eta[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)}$. These previous three bounds gives $\text{meas}(Z(\phi, \mathbb{R}, \varepsilon)) \leq g(\eta) = 2(\eta + \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)})$. This last inequality is valid for all $\eta > 0$. It suffices to minimize the function g on $]0, +\infty[$. A computation of the minimum yields $\text{meas}(Z(\phi, \mathbb{R}, \varepsilon)) \leq \bar{c}_k \varepsilon^{1/k}$, where $\bar{c}_k = 4(\bar{c}_{k-1}/(k-1))^{(k-1)/k}$ which concludes the proof. \square

The previous lemma is generalized with parameters in a compact set, see Lemma 4 p. 127 in [2].

Lemma 3.3 ([2]). *Let P be a compact set of parameters, k a positive integer, $A > 0$, $V = [-A, A]$, $K = V \times P$, $\phi(v; p) \in C^0(P, C^k(V, \mathbb{R}))$, such that, for all (v, p) in the compact K , we have*

$$\sum_{j=1}^k \left| \frac{\partial^j \phi}{\partial v^j} \right| (v; p) > 0.$$

Let $Z(\phi(\cdot; p), \varepsilon) = \{v \in V, |\phi(v; p)| \leq \varepsilon\}$, then there exists a constant C such that

$$\sup_{p \in P} \text{meas}(Z(\phi(\cdot; p), \varepsilon)) \leq C \varepsilon^{1/k}.$$

We now turn to the key integer $d_{\mathbf{F}}$.

Lemma 3.4. *If \mathbf{F} is a nonlinear flux on I in the sense of Definition 3.1 then $d_{\mathbf{F}}$ is finite and there exists $\underline{u} \in I$ such that $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$.*

Proof Let u be fixed in I . Then there exists, $1 \leq j_1 < j_2 < \dots < j_d = d_{\mathbf{F}}[u]$ such that $\text{rank}\{\mathbf{a}^{(j_1)}(u), \dots, \mathbf{a}^{(j_d)}(u)\} = d$ by the definition of $d_{\mathbf{F}}[u]$. So the continuous function $g(v) = \det(\mathbf{a}^{(j_1)}(v), \dots, \mathbf{a}^{(j_d)}(v))$ does not vanish at $v = u$. By continuity, this is still true on an open set J with $u \in J$. Since $j_d = d_{\mathbf{F}}[u]$, we have $d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}[u]$ for all $v \in J$. Thus $v \mapsto d_{\mathbf{F}}[v]$ is upper semi-continuous and the result follows immediately on the compact set I . \square

Now we are able to prove Theorem 3.1.

Proof of Theorem 3.1. There are two steps.

step 1: $\alpha_{\text{sup}} \geq \frac{1}{d_{\mathbf{F}}}$.

Set $\phi(v; \tau, \xi) = \tau + \mathbf{a}(v) \cdot \xi$ with $\tau^2 + |\xi|^2 = 1$. τ and ξ are fixed. Since $\phi(\cdot; \tau, 0) = \tau$ has no roots, we can assume that $\xi \neq 0_{\mathbb{R}^d}$. For $j \geq 1$ we have $\partial_{v_j}^j \phi(v; \tau, \xi) = \mathbf{a}^{(j)}(v) \cdot \xi$. By definition of $d_{\mathbf{F}}[v]$ there exists $j \leq d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}$ such

that $\partial_v^j \phi(v; \tau, \xi) \neq 0$. Thus, we have when $\xi \neq 0$

$$(3.6) \quad \sum_{j=1}^{d_{\mathbf{F}}} |\partial_v^j \phi(v; \tau, \xi)| > 0.$$

When $\xi = 0$, we have $\tau = \pm 1$ since $\tau^2 + |\xi|^2 = 1$. The function $\phi(v; \pm 1, 0) = \pm 1 \neq 0$. By continuity of this function there exists an open neighborhood V of $(1, 0_{\mathbb{R}^d})$ such the function does not vanish on \overline{V} . Set P be the complementary set of V in the unit sphere of \mathbb{R}^{d+1} . P is compact and (3.6) is true on P . Now we can use Lemma 3.3 to conclude the first step.

$$\text{step 2: } \alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}}.$$

Take \underline{u} from Lemma 3.4. Then there exists $\xi \neq 0$ such that $\partial_v^j \phi(v; \tau, \xi) = 0$ for $1 \leq j < d_{\mathbf{F}}$ and $\partial_v^{d_{\mathbf{F}}} \phi(v; \tau, \xi) \neq 0$ for $j = d_{\mathbf{F}}$. For such $\xi \neq 0$, we choose τ such that $\varphi(v) = \phi(v; \tau, \xi)$ vanishes at $v = \underline{u}$. Now, by Lemma 3.1, the second step is proved.

Finally $\frac{1}{d_{\mathbf{F}}} \leq \alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}}$ and the proof is complete with Lemma 3.4. \square

3.2. Comparisons with other nonlinear flux definitions.

There are more general definitions of nonlinear flux [15, 23]. But the precise smoothing is related to Definition 1.1 or Definition 3.1 and the parameter α_{sup} or equivalently $d_{\mathbf{F}}$. Let us compare these definitions with Definition 3.1. It can be useful for other applications.

In [23], there is a more general definition of nonlinear flux.

Definition 3.2. [General Nonlinear Flux [23]] *A flux \mathbf{F} , differentiable on $[-M, M]$ is said to be nonlinear if the degeneracy set*

$$W(\tau, \xi) = \{|v| \leq M, \tau + \mathbf{F}'(v) \cdot \xi = 0\}$$

has null Lebesgue measure for all (τ, ξ) on the sphere.

This definition is of a great importance since this condition implies the compactness of the semi-group \mathcal{S}_t associated with the conservation law (1.1).

Proposition 3.1. *Let \mathbf{F} be a smooth flux in C^∞ . Assume \mathbf{F} satisfy Definition 3.1 then \mathbf{F} is nonlinear for Definition 3.2 but the converse can be wrong.*

Proof : Lemma 3.4 and Theorem 5.1 show that nonlinearity of Definition 3.1 implies nonlinearity of Definition 1.1 and then of Definition 1.1. But we can give a direct proof from Lemma 2.5 and remark (2.3) p. 447 in [5], (see also [8] p. 84).

Notice that $W(\tau, 0) = \emptyset$ since $\tau = \pm 1$. So we assume that $\xi \neq 0$. Set $\phi(v) = \tau + \mathbf{F}'(v) \cdot \xi$. Since $\phi^{(k)}(v) = \mathbf{F}^{(k+1)}(v) \cdot \xi$, for any v , there exists $k > 0$ such that $\phi^{(k)}(v) \neq 0$ by Definition 3.1. So the roots of ϕ are isolated and the set $W(\tau, \xi)$ is finite.

Conversely the counter-example $\mathbf{F}'(u) = \exp(-1/u^2)(u, u^2, \dots, u^d)$ does not satisfies Definition 3.1 since $d_{\mathbf{F}}[0] = +\infty$.

But \mathbf{F} satisfies Definition 3.2. Indeed, with $h(v) = \tau \exp(1/v^2) + \xi \cdot (v, v^2, \dots, v^d)$, the set $W(\tau, \xi) - \{0\}$ is the set of roots of $h(\cdot)$. If $\tau = 0$, we deal with the genuine nonlinear flux from Definition 4.1 and the degeneracy set $W(\tau, \xi)$ is a null set. Indeed, it is finite. If $\tau \neq 0$, $h(\cdot)$ is analytic and non trivial on \mathbb{R}^* . Consequently $W(\tau, \xi)$ is countable and also a null set which concludes the proof. \square

Engquist and E in [15] gave another definition of strictly nonlinear flux generalizing Tartar [32].

Definition 3.3. [Strictly Nonlinear Flux [15]]

Let M be a positive constant, and $\mathbf{F} : [-M, M] \rightarrow \mathbb{R}^d$ be a function twice differentiable on $[-M, M]$.

F is said to be strictly nonlinear on $[-M, M]$ if for any sub-interval I of $[-M, M]$, the functions F_1'', \dots, F_d'' are linearly independent on I , i.e., for any constant vector ξ , if $\xi \cdot \mathbf{F}''(u) = 0$ for all $u \in I$ then $\xi = 0$.

Proposition 3.2. Let \mathbf{F} be a $C^\infty([-M, M], \mathbb{R}^d)$ flux. Assume \mathbf{F} satisfying Definition 3.1, then \mathbf{F} satisfies Definition 3.3 but the converse is wrong.

Proof. Assume $\xi \cdot \mathbf{F}'' = 0$ on a open sub-interval I . Let u belong in I . Hence $\xi \cdot \mathbf{F}^{(k)}(u) = 0$ for all $k \geq 2$. But \mathbf{F} satisfies Definition 3.1. It follows that $\xi = 0$.

Conversely take a flux \mathbf{F} such that $\mathbf{F}''(u) = \exp(-1/u^2)(1, u, \dots, u^{d-1})$. Obviously \mathbf{F} satisfies Definition 3.3. But \mathbf{F} does not satisfies Definition 3.1 since $d_{\mathbf{F}}[0] = +\infty$. \square

In the same way, if \mathbf{F} satisfies Definition 3.2 then \mathbf{F} satisfies Definition 3.3. For analytic flux, the situation is simpler.

Proposition 3.3 (Analytic nonlinear flux). Assume the flux to be an analytic function. All previous Definitions 1.1, 3.1, 3.2, 3.3 are equivalent.

Proof. Again we use Definition 3.1. There are two cases.

- (1) If \mathbf{F} is nonlinear for Definition 3.1. By Theorem 5.1, Propositions 3.1 and 3.2, \mathbf{F} is nonlinear for other definitions.
- (2) If \mathbf{F} is not nonlinear for Definition 3.1. By Theorem 5.1, \mathbf{F} does not satisfy Definition 1.1.

Let u be fixed. There exists an hyperplane H such that all derivatives $\mathbf{F}^{(k)}(u) \in H$ for all $k \geq 2$, i.e. there exists $\xi \neq 0$ such that $\xi \cdot \mathbf{F}^{(k)}(u) = 0$ for all $k \geq 2$. Using the power series expansion of \mathbf{F}'' near u we see that \mathbf{F}'' stays in H near u . And by the unique analytic extension of \mathbf{F}'' , \mathbf{F}'' stays always in H , i.e. $\xi \cdot \mathbf{F}'' = 0$ everywhere. Thus \mathbf{F} does not satisfies Definition 3.3.

Integrating the relation $\xi \cdot \mathbf{F}'' = 0$ we have $\tau + \xi \cdot \mathbf{F}' = 0$ for some constant τ . Dividing the relation by $\sqrt{\tau^2 + |\xi|^2}$ we can assume that $\tau^2 + |\xi|^2 = 1$. Hence \mathbf{F} does not satisfies Definition 3.2.

We incidentally check that Definition 3.2 implies Definition 3.3.

□

For less smooth flux we refer to the works of E. Yu. Panov ([25, 26]).

4. SOBOLEV ESTIMATES

In this section, uniform and optimal Sobolev exponents of the family of highly oscillating solutions from Theorem 2.1 are investigated.

Theorem 4.1. [Sobolev exponent for highly oscillating solutions]

Let u_ε be the $C^1([0, T_0] \times \mathbb{R}^d)$ oscillating solutions given in Theorem 2.1.

For all $1 \leq p < +\infty$, the family $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ is uniformly bounded in

$$C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})) \cap W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R}) \quad \text{with } s = \frac{1}{\gamma}.$$

Furthermore, if U_0 is a non constant function, then for all $s > 1/\gamma$ the sequence $(u_\varepsilon)_{0 < \varepsilon < 1}$ is unbounded in $C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}))$ and in $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R})$.

The Theorem means that the Sobolev exponent $s = \frac{1}{\gamma}$ is **optimal**. It is easily seen that the sequence $(u_\varepsilon)_{0 < \varepsilon}$ is uniformly bounded in $W_{loc}^{1/\gamma, p}$ by interpolation (see remark 4.1 below). The difficult part of the theorem is the optimality. That is to say the sequence is unbounded for too large s . For this purpose we need to get lower bound of Sobolev norms. Unfortunately, interpolation theory only gives upper bounds. Thus we use the intrinsic norm. It is rather elementary but quite long to achieve such lower bounds. All this section is essentially devoted to compute these lower bounds to highlight the conjecture about the maximal smoothing effect in the next section.

Indeed, it is proved below that u_ε has order of $\varepsilon^{1-s\gamma}$ in $W_{loc}^{s,p}$ for any $s \in [0, 1[$. The case $p = 1$ is the most important, since L^1 norm plays an important role for conservation laws. The Sobolev estimates of the initial data are propagated by the semi-group \mathcal{S}_t , (see [23] for $p = 1$ and also [28] for $TV(|u_\varepsilon - \underline{u}|^s)$). A key point is there is no improvement of the Sobolev exponent of the family of initial data.

The basic idea of the proof is that the sequence of exact solutions $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ and the sequence of approximate oscillating solution given by $\underline{u} + \varepsilon U \left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right)$ have similar bounds in Sobolev spaces.

We use the $W^{s,p}$ intrinsic semi-norm instead the interpolation theory as we explained before. More precisely, following semi-norms parametrized by $Q = Q_d(\mathbf{x}_0, A) = \mathbf{x}_0 +] - A, A[^d$, where $A > 0$, $\mathbf{x}_0 \in \mathbb{R}^d$, are used to estimate fractional derivatives in $W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})$ ([1]).

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))}^p = \int_{Q_d(\mathbf{x}_0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|V(\mathbf{x}) - V(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+sp}} d\mathbf{x}d\mathbf{y}.$$

Following classical Definitions are used in this section.

Definition 4.1. [Estimates in $W_{loc}^{s,p}(\mathbb{R}^d)$]

(i) u is said to be bounded in $W_{loc}^{s,p}(\mathbb{R}^d)$ if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0,$$

$$\|u\|_{W^{s,p}(Q_d(\mathbf{x}_0,A))} = \|u\|_{L^p(Q_d(\mathbf{x}_0,A))} + |u|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0,A))} \leq C.$$

(ii) $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ is said to be bounded in $W_{loc}^{s,p}(\mathbb{R}^d)$ if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0, \forall \varepsilon \in]0, 1], \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0,A))} \leq C.$$

(iii) Let $\beta \geq 0$, $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ has order of $\varepsilon^{-\beta}$ in $W_{loc}^{s,p}(\mathbb{R}^d)$, denoted by

$$u_\varepsilon \simeq \varepsilon^{-\beta},$$

$$\text{if } \forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 1, \exists \varepsilon_0 \in]0, 1], \forall \varepsilon \in]0, \varepsilon_0],$$

$$C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0,A))} \leq C \varepsilon^{-\beta}.$$

As usual if u is bounded in $W_{loc}^{s,p}(\mathbb{R}^d)$ then for any cube Q , u belongs to $W^{s,p}(Q)$. By the same way if $u_\varepsilon \simeq \varepsilon^{-\beta}$ in $W_{loc}^{s,p}(\mathbb{R}^d)$ then for any cube Q there exists a constant $C \geq 1$ and $\varepsilon_0 \in]0, 1]$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q)} \leq C \varepsilon^{-\beta}$.

Since solutions of (1.1) are bounded in L^∞ , the key point is to focus on fractional derivatives. For convenience $|\mathbf{x}| = |x_1| + \dots + |x_d|$ and semi-norms

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0,A))}^p = \int_{Q_d(0,A)} \int_{Q_d(\mathbf{x}_0,A)} \frac{|V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})|^p}{|\mathbf{h}|^{d+sp}} d\mathbf{x}d\mathbf{h},$$

are also used. Notice that

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0,A/2))} \leq |V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0,A))} \leq |V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0,2A))}.$$

Furthermore, $|V|_{\dot{W}^{s,p}(Q_1(\mathbf{x}_0,A))} = |V|_{\dot{W}^{s,p}(Q_1(\mathbf{x}_0,A))}$ when V is periodic with period A (or $A/2$). Thus, these semi-norms can be useful to estimate bounds in $W_{loc}^{s,1}$.

The simplest example of high frequency oscillating functions with optimal estimates in Sobolev spaces is investigated in the following Lemma. The remainder of the section is devoted to get the same estimates for the the family of highly oscillating solutions from Theorem 2.1.

Lemma 4.1. [Highly oscillating periodic function on \mathbb{R}]

Let v belong to $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$, $\gamma > 0$, and for all $0 < \varepsilon \leq 1$,

$$V_\varepsilon(\theta) = v(\varepsilon^{-\gamma}\theta).$$

If $v(\cdot)$ is a non constant periodic function then

$$V_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}).$$

Furthermore, if $V_\varepsilon(\theta) = v_\varepsilon(\varepsilon^{-\gamma}\theta)$, v_ε is one periodic, and $v_\varepsilon \rightarrow v$ in C^1 then $V_\varepsilon \simeq \varepsilon^{-s\gamma}$ in $W_{loc}^{s,p}(\mathbb{R})$.

Notice that the magnitude of V_ε in $W_{loc}^{s,p}$ is independent of p .
 Notice also that if $v_\varepsilon \rightarrow v$ in $W_{loc}^{s,p}$ then $v_\varepsilon(\varepsilon^{-\gamma}\theta) \simeq \varepsilon^{-s\gamma}$ in $W_{loc}^{s,p}(\mathbb{R})$.

Proof : In all the sequel one sets $x_0 = 0$ in Definition 4.1 since computations are invariant under translation.

First the L_{loc}^1 norm is easily bounded in [5]. Let $A > 1/2$, $X = \varepsilon^{-\gamma}x$, $B_\varepsilon = \varepsilon^{-\gamma}A$, N_ε the integer such that $N_\varepsilon \leq 2B_\varepsilon < N_\varepsilon + 1$ so $2A - 1 \leq 2A - \varepsilon^\gamma \leq \varepsilon^\gamma N_\varepsilon \leq 2A$.

$$\begin{aligned} \|V_\varepsilon\|_{L^p([-A,A])}^p &= \int_{-A}^A |V_\varepsilon(x)|^p dx = \varepsilon^{-\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \\ &= \varepsilon^{-\gamma} \left(\sum_{k=1}^{N_\varepsilon} \int_{-B_\varepsilon+k-1}^{-B_\varepsilon+k} |v(X)|^p dX + \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \right) \\ &= \varepsilon^{-\gamma} N_\varepsilon \int_0^1 |v(X)|^p dX + \varepsilon^{-\gamma} \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX. \end{aligned}$$

Finally one has

$$(4.1) \quad \|V_\varepsilon\|_{L^p([-A,A])} \leq (2A+1)^{1/p} \|v\|_{L^p([0,1])},$$

$$(4.2) \quad \begin{aligned} \|V_\varepsilon\|_{L^p([-A,A])} &\geq (2A-1)^{1/p} \|v\|_{L^p([0,1])} \\ \|V_\varepsilon\|_{L^p([-A,A])} &\sim (2A)^{1/p} \|v\|_{L^p([0,1])} \quad \text{when } \varepsilon \rightarrow 0. \end{aligned}$$

$|V_\varepsilon|_{\widetilde{W}^{s,p}([-A,A])}$ is computed with the same notations and $H = \varepsilon^{-\gamma}h$,

$$|V_\varepsilon|_{\widetilde{W}^{s,p}([-A,A])}^p = \varepsilon^{(1-sp)\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \int_{-B_\varepsilon}^{B_\varepsilon} \frac{|v(X+H) - v(X)|^p}{|H|^{1+sp}} dX dH.$$

Let $Var(\cdot)$ be the one periodic function bounded in L^∞ by $2^p \|v\|_{L^p([0,1])}^p$,

$$Var(H) = \int_0^1 |v(X+H) - v(X)|^p dX.$$

Notice that $Var \equiv 0$ if and only if v is constant a.e.

Using one periodicity of v with respect to X yields as in (4.1)

$$\begin{aligned} |V_\varepsilon|_{\widetilde{W}^{s,p}([-A,A])}^p &= \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \left(\varepsilon^\gamma \int_{-B_\varepsilon}^{B_\varepsilon} |v(X+H) - v(X)|^p dX \right) \frac{dH}{|H|^{1+sp}}, \\ &\leq \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} ((2A+1)Var(H)) \frac{dH}{|H|^{1+sp}} \leq \varepsilon^{-sp\gamma} (2A+1) D_\infty^p, \\ D_B^p = (D_B)^p &= \int_{-B}^{+B} Var(H) \frac{dH}{|H|^{1+sp}}. \end{aligned}$$

Notice that D_B is a true constant related to the fractional derivative of v since for $B = 1/2$, $D_{1/2} = |v|_{\widetilde{W}^{s,p}([-1/2,1/2])}$ and for $B = \infty$ the integral converges.

The lower bound is obtained by the same way and finally one has

$$\begin{aligned} |V_\varepsilon|_{\dot{\widetilde{W}}^{s,p}([-A,A])} &\leq \varepsilon^{-s\gamma}(2A+1)^{1/p}D_\infty, \\ |V_\varepsilon|_{\dot{\widetilde{W}}^{s,p}([-A,A])} &\geq \varepsilon^{-s\gamma}(2A-1)^{1/p}D_1, \\ |V_\varepsilon|_{\dot{\widetilde{W}}^{s,p}([-A,A])} &\sim \varepsilon^{-s\gamma}(2A)^{1/p}D_\infty. \end{aligned}$$

Notice also that $D_B > 0$ for $B > 1/2$. Otherwise $D_B = 0$ implies $Var \equiv 0$ a.e. which implies v is a constant function on $[x_0 - 2B, x_0 + 2B]$ and on \mathbb{R} by periodicity.

A key point in this paper is the lower bound to get sharp estimates. Since D_B is non decreasing with respect to B , the previous lower bound of V_ε in $W^{s,p}$ implies the following lower bound

$$|V_\varepsilon|_{\dot{\widetilde{W}}^{s,p}([-A,A])} \geq \varepsilon^{-s\gamma}(2A-1)^{1/p}|v|_{\dot{\widetilde{W}}^{s,p}([-1/2,1/2])}.$$

With more work, similar estimates are still valid for $|V_\varepsilon|_{\dot{W}^{s,1}([-A,A])}$, see lemmas in [5] about triangular changes of variables for oscillatory integrals. But it is enough for our purpose.

Same computations when v is replaced by v_ε are still valid which conclude the proof. \square

The following lemma is useful to check that $W^{s,1}$ semi-norms of $V : \mathbb{R} \mapsto \mathbb{R}$ and $W : \mathbb{R}^d \mapsto \mathbb{R}$ where $W(x_1, \dots, x_d) = V(x_1)$ have the same order.

Lemma 4.2. *Let $d \geq 2$, $s > 0$, $A > 0$, $h_1 > 0$,*

$$(4.3) \quad \mu_{d,s}(h_1) = \int_0^A \cdots \int_0^A \frac{h_1^{1+s}}{(h_1 + h_2 + \cdots + h_d)^{d+s}} dh_2 \cdots dh_d,$$

then there exists two positive numbers $c_{d,s}, C_{d,s}$ such that

$$(4.4) \quad 0 < c_{d,s} \leq \mu_{d,s}(h_1) \leq C_{d,s} < +\infty, \quad \forall A > 0, \quad \forall h_1 \in]0, A].$$

Inequalities (4.4) are still valid for $h_1 \in]0, 2A]$ with other constants:

$$0 < \widetilde{c}_{d,s} \leq \mu_{d,s}(h_1) \leq \widetilde{C}_{d,s} < +\infty.$$

The constants $c_{d,s}$ and $C_{d,s}$ are independent of $A > 0$. Notice that there is a singularity for $\mu_{d,s}$ at $h_1 = 0$ since $\mu_{d,s}(0) = 0$ and $\mu_{d,s} > 0$ on $]0, A]$.

Proof : It seems that $\mu_{d,s}(h_1)$ is depending on A , $\mu_{d,s}(h_1) = \mu_{d,s}^A(h_1)$. But by homogeneity the problem is reduced to the case $A = 1$ with the change of variable $h_i = t_i A$, $0 < t_i < 1$.

Now $\mu_{d,s}(t_1) = \mu_{d,s}^1(t_1) = \mu_{d,s}^A(h_1)$ is computed explicitly .

Let $\mu_{d,s}(h_1, B)$ be $\int_0^1 \cdots \int_0^1 \frac{t_1^{1+s}}{(t_1 + t_2 + \cdots + t_d + B)^{d+s}} dt_2 \cdots dt_d$ for $d > 1$, $B \geq 0$. Notice that $\mu_{d,s}(t_1) = \mu_{d,s}(t_1, 0)$.

For $d = 1$, set $\mu_1(t_1, B) = \frac{t_1^{1+s}}{(t_1 + B)^{1+s}}$, $\mu_1(t_1) = \mu_1(t_1, 0) = 1$. The identity

$$\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} = (j + s)^{-1} (B^{-(j+s)} - (B + 1)^{-(j+s)}),$$

yields $(j + s)\mu_{1+j}(t_1, B) = \mu_j(t_1, B) - \mu_j(t_1, B + 1)$, and proceeding by induction with the notations $\gamma_{d,s} = \frac{1}{(d - 1 + s) \cdots (1 + s)}$, $C_n^k = \frac{n!}{k!(n - k)!}$,

$$\mu_{d,s}(t_1, B) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \mu_1(t_1, B + k).$$

Hence, for $B = 0$,

$$\mu_{d,s}(t_1) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \frac{t_1^{1+s}}{(t_1 + k)^{1+s}},$$

which gives $\mu_{d,s}(0+) = \gamma_{d,s} > 0$. Now, $\mu_{d,s}(\cdot)$ belongs in $C^0([0, 1], \mathbb{R}^+)$, $\mu_{d,s}(\cdot)$ is positive on $]0, 1]$ with a positive right limit at $t_1 = 0$, thus positive constants stated in the lemma exist.

For instance when $d = 2$, $C_{2,s}$ is $\gamma_{2,s} = 1/(1 + s)$ and $c_{2,s} = (1 - 2^{-(1+s)})/(1 + s)$, since μ_2 is decreasing.

Notice that $C_{d,s} \leq \gamma_{d,s}$ for all $d \geq 2$. It suffices to proceed by induction with this inequality $\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} \leq (j + s)^{-1} B^{-(j+s)}$. But $\gamma_{d,s}$ is the right limit of $\mu_{d,s}$ at $t_1 = 0$. Then $C_{d,s} = \gamma_{d,s}$ which concludes the proof for $h_1 \in]0, A]$. On $]0, 2A]$ it suffices to take $0 < \tilde{c}_{d,s} = \inf_{]0, 2]} \mu_{d,s}$ and $+\infty > \tilde{C}_{d,s} = \sup_{]0, 2]} \mu_{d,s}$. \square

Our examples of oscillating solutions is related to the following key example. For instance V_ε defined by $u_\varepsilon = \varepsilon V_\varepsilon$ where u_ε is the solution of $\partial_t(u_\varepsilon) + \partial_x |u_\varepsilon|^{1+\gamma} = 0$, $u_\varepsilon(0, x) = 0 + \varepsilon U(0, \varepsilon^{-\gamma} x)$ satisfies the assumption of the lemma on a bounded time interval ([4]).

Lemma 4.3. [Example of highly periodic oscillations on $[0, T] \times \mathbb{R}$]

Let T, γ be positive. If U belongs to $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$ and non constant, then $V_\varepsilon(t, x) = U(t, \varepsilon^{-\gamma} x) \simeq \varepsilon^{-s\gamma}$ in $C^0([0, T], W_{loc}^{s,p}(\mathbb{R})) \cap W_{loc}^{s,p}([0, T] \times \mathbb{R})$.

Remark 4.1. Notice that the upper bound is quite easy to get. It directly follows from the fact that $W^{s,p}$ is an interpolated space of exponent $\theta = s$ between $L^p = W^{0,p}$ and $W^{1,p}$, [33]. But we also want a lower bound to obtain an optimal estimate. This is a very crucial point in our study. For this purpose we use the intrinsic semi-norm in the proofs. The computations are elementary but long.

The same remark is still valid for all the next lemmas in this section.

Proof : First the fractional derivative w.r.t. \mathbf{x} is estimated. Second the whole fractional derivative in (t, \mathbf{x}) is obtained.

Bounds in $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$: There exists $t_0 \in]0, T[$ such that $\theta \mapsto U(t_0, \theta)$ is non constant since U is non constant and continuous on $[0, T] \times \mathbb{R}/\mathbb{Z}$. For t_0 fixed the sharp estimate is a consequence of Lemma 4.1. For another t , we get the same order $\varepsilon^{-s\gamma}$ or $\varepsilon^0 = 1$. Finally, constants involving in this estimate depend continuously of t so the bound in $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$ is obtained. Since $U \in C^1$ this bound is automatically in $C^0([0, T], W_{loc}^{s,p}(\mathbb{R}))$.

Bounds in $W_{loc}^{s,p}([0, T[\times \mathbb{R})$: The only problem is to estimate for $x_0 \in \mathbb{R}$, $t_0 \in]0, T[$ and $\min(t_0, T - t_0) > A > 0$, the quadruple integral

$$\begin{aligned} IA &= |V_\varepsilon|_{W_{loc}^{s,p}([t_0-A, t_0+A] \times [x_0-A, x_0+A])}^p \\ &= \int_{t_0-A}^{t_0+A} \int_{x_0-A}^{x_0+A} \int_{-A}^A \int_{-A}^A \frac{|U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt. \end{aligned}$$

Upper bound of IA:

Let Num be the numerator of the previous fraction, Q be $U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)$, R be $U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}(x+\xi))$ then $Num = |Q+R|^p \leq 2^{p-1}(|Q|^p + |R|^p)$.

Previous inequality implies $IA \leq 2^{p-1}(IQ + IR)$ with obvious notations.

$$\begin{aligned} IQ &= \int \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &= \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{|\xi|^{1+sp}} \mu_{2,sp}(\xi) d\xi dx dt, \end{aligned}$$

with $\mu_{2,sp}(\cdot)$ is defined in Lemma 4.2. Using Lemmas 4.1, 4.2 yield $IQ \simeq \varepsilon^{-s\gamma}$. IR is easily bounded since

$$\begin{aligned} IR &= \int \int \int \int \frac{|U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}(x+\xi))|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &\leq \int \int \int \int \frac{\|\partial_t U\|_{L^\infty}^p |\tau|^p}{(|\tau| + |\xi|)^{2+sp}} d\tau d\xi dx dt \\ &\leq 8A^2 \|\partial_t U\|_{L^\infty}^p \int_0^A |\tau|^{p(1-s)-1} \mu_{2,sp}(\tau) d\tau, \end{aligned}$$

which is finite, so $IA \leq IQ + IR = \mathcal{O}(\varepsilon^{-sp\gamma})$.

Lower bound of IA:

We again use notations Q , R , Num . By a convex inequality, the numerator satisfies: $Num = |Q+R|^p \geq |Q|^p - p|Q|^{p-1}|R| = |Q|^p - O(|\tau||Q|^{p-1})$ since $R = O(\tau)$. Then $IA \geq IQ - O(IS)$, where IQ has order of $\varepsilon^{-sp\gamma}$. The term IS has a lower order as we can find after the following similar computations as in the proof of Lemma 4.1. Notice first that for all positive numbers A , b ,

$$\int_0^A \frac{\tau}{(\tau+b)^{2+\beta}} d\tau \leq \frac{C}{2b^\beta} \text{ where } \beta > 0 \text{ and } C = 2 \int_0^{+\infty} \frac{\tau}{(\tau+1)^{2+\beta}} d\tau < +\infty. \text{ Now}$$

integrating on τ yields

$$IS = \int \int \int \int \frac{|\tau||Q|^{p-1}}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt \leq C \int \int \int \frac{|Q|^{p-1}}{|\xi|^{sp}} d\xi dx dt.$$

We set $\eta = \varepsilon^\gamma$, $X = x/\eta$, $\Xi = \xi/\eta$, the previous inequality becomes

$$IS \leq C\eta^{2-sp} \int_0^T \int_{A/\eta}^{A/\eta} \int_{A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi dX dt.$$

We now focus on the integral with respect to Ξ and remark that $Q = O(1)$ and also $Q = O(\Xi)$ since U is C^1 .

$$\begin{aligned} \int_{-A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi &= \int_{|\Xi| < 1} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} \frac{O(|\Xi|^{p-1})}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{O(1)}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} O(|\Xi|^{p(1-s)-1}) d\Xi + O(g(\eta)) = O(1) + O(g(\eta)), \end{aligned}$$

where $g(\eta) = \eta^{sp-1}$ if $sp \neq 1$, else $g(\eta) = \ln(\eta)$.

To bound IS , we notice that the integral $\eta \int_{-A/\eta}^{A/\eta} dX$ is bounded by periodicity and we can take the supremum with respect t on $[0, T]$. So $IS = O(1)$ if $sp \neq 1$ else $IS = O(\ln(\eta))$ which is enough to have a lower order than IQ .

In conclusion, the bounds of IA yield $V_\varepsilon \simeq \varepsilon^{-s\gamma}$ in $W_{loc}^{s,p}([0, T] \times \mathbb{R})$. \square

Now, we estimates the Sobolev norm for the multidimensional case with one phase.

Lemma 4.4. [Example of highly periodic oscillations on \mathbb{R}^d]

Let v belong to $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$, $\gamma > 0$, $\psi(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + b$ where $\mathbf{v} \in \mathbb{R}^d$, $b \in \mathbb{R}$ and $0 < \varepsilon < 1$,

$$W_\varepsilon(\mathbf{x}) = v(\varepsilon^{-\gamma}\psi(\mathbf{x})).$$

If v is a non constant periodic function and $\nabla\psi \neq 0$, then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}).$$

Furthermore, when functions v_ε are one periodic function for all $\varepsilon \in]0, 1]$, which converge towards v in C^1 and $W_\varepsilon(\mathbf{x}) = v_\varepsilon(\varepsilon^{-\gamma}\psi(\mathbf{x}))$, the conclusion holds true.

Proof : The expounded proof has three steps. Let M be a $d \times d$ non-degenerate matrix and $B \in \mathbb{R}^d$ such that $X_1 = \psi(\mathbf{x})$ where $X = (X_1, \dots, X_d) = M\mathbf{x} + B$. M exists since $\mathbf{v} \neq 0$.

Step 1: When $W(\mathbf{x}) = U(M\mathbf{x} + b)$ since $\det M \neq 0$, W and U are the same order in $W_{loc}^{s,p}$. More precisely, fix following positive constants $m_0 = |\det M| > 0$, $m_1 = \||M\mathbf{x}\| = \sup\{|M\mathbf{x}|, |\mathbf{x}| = 1\} > 0$, $m_{-1} = \||M^{-1}\| > 0$, $0 < r <$

R such that $Q_d(X_0, r) \subset MQ_d(\mathbf{x}_0, 1) \subset Q_d(X_0, R)$ where $X_0 = M\mathbf{x}_0 + B$. Performing the change of variables $X = M\mathbf{x} + B$, $Y = M\mathbf{y} + B$ yields for any $\mathbf{x}_0 \in \mathbb{R}^d$ and any $A > 0$

$$\begin{aligned} m_0^{-1} \|U\|_{L^p(Q_d(X_0, rA))} &\leq \|W\|_{L^p(Q_d(\mathbf{x}_0, A))} \leq m_0^{-1} \|U\|_{L^p(Q_d(X_0, RA))}, \\ \frac{m_0^{-2}}{m_{-1}^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, rA))} &\leq |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq \frac{m_0^{-2}}{m_1^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, RA))}. \end{aligned}$$

Step 2: Assume $\psi(\mathbf{x}) = x_1$, i.e. $W(\mathbf{x}) = W(x_1, \dots, x_d) = w(x_1)$, $x_0 = \psi(\mathbf{x}_0)$, then W in $W_{loc}^{s,p}(\mathbb{R}^d)$ and w in $W_{loc}^{s,p}(\mathbb{R})$ have the same order. More precisely, elementary computations yield

$$\begin{aligned} \|W\|_{L^1(Q_d(\mathbf{x}_0, A))} &= (2A)^{d-1} \|w\|_{L^1(Q_1(x_0, A))}, \\ |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &\leq (2A)^{d-1} C_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))} \\ &\geq (2A)^{d-1} c_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))}. \end{aligned}$$

The two last inequalities and constants come from Lemma 4.2 since

$$\begin{aligned} |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &= \int_{Q_d(0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|w(x_1 + h_1) - w(x_1)|}{|\mathbf{h}|^{d+sp}} d\mathbf{x} d\mathbf{h} \\ &= (2A)^{d-1} \int_{-A}^A \int_{x_0-A}^{x_0+A} \frac{|w(x_1 + h_1) - w(x_1)|}{|h_1|^{1+sp}} \mu_{d,sp}(h_1) dx_1 dh_1. \end{aligned}$$

Step 3: By step 1, $W_\varepsilon(\mathbf{x}) = V_\varepsilon(\varepsilon^{-\gamma}\psi(\mathbf{x})) \simeq V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$ in $W_{loc}^{s,p}(\mathbb{R}^d)$, by step 2, $\mathbf{x} \mapsto V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$ and $x_1 \mapsto V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$ have the same order in $W_{loc}^{s,p}(\mathbb{R}^d)$ and $W_{loc}^{s,p}(\mathbb{R})$. Finally we have by Lemma 4.1 $W_\varepsilon \simeq \varepsilon^{-s\gamma}$ in $W_{loc}^{s,p}(\mathbb{R}^d)$. \square

It is the last step to estimate the Sobolev norm for the multidimensional case before proving Theorem 4.1.

Lemma 4.5. [Example of highly periodic oscillations on $[0, T] \times \mathbb{R}^d$] Let U belong to $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$, $\gamma > 0$, $\varphi(t, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + bt$ where $\mathbf{v} \in \mathbb{R}^d$, $b \in \mathbb{R}$ and $0 < \varepsilon < 1$,

$$W_\varepsilon(\mathbf{x}) = U(t, \varepsilon^{-\gamma}\varphi(t, \mathbf{x})).$$

If U is a non constant function in $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\mathbf{v} \neq 0_{\mathbb{R}^d}$, then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R}).$$

Furthermore, when U_ε belongs to $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$ for all $\varepsilon \in]0, 1]$ converging towards U in C^1 and $W_\varepsilon(\mathbf{x}) = U_\varepsilon(t, \varepsilon^{-\gamma}\varphi(t, \mathbf{x}))$, the conclusion holds true.

Proof : We proceed as in the previous proofs. First with a linear change of variable $(t, \mathbf{x}) \mapsto (t, \mathbf{y})$ with $\mathbf{y}_1 = \varphi(t, \mathbf{x})$. W_ε has the same estimates than $V_\varepsilon = U(t, \varepsilon^{-\gamma}\mathbf{y}_1)$ in $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$. Notice that the change of variable depends on t varying in the compact set $[0, T]$. So we have uniform estimates of positive constants m_0, m_1, m_{-1} used in the proof of Lemma 4.4.

Now, the estimates of V_ε in $W_{loc}^{s,p}([0, T[\times\mathbb{R}^d, \mathbb{R})$ and in $W_{loc}^{s,p}([0, T[\times\mathbb{R}, \mathbb{R})$ have the same order since

$$\begin{aligned} & \int_{-A}^A \cdots \int_{-A}^A \frac{dh_0 dh_1 \cdots dh_d}{(|h_0| + |h_1| + \cdots + |h_d|)^{1+d+sp}} \\ = & \int_{-A}^A \cdots \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \frac{(|h_0| + |h_1|)^{1+(sp+1)} dh_2 \cdots dh_d}{(|h_0| + |h_1| + \cdots + |h_d|)^{d+(sp+1)}} \\ = & \int_{-A}^A \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \mu_{2,(sp+1)}(|h_0| + |h_1|) \end{aligned}$$

where h_0 plays the role of time. From the bounds of $\mu_{2,(sp+1)}(|h_0| + |h_1|)$ on $]0, 2A]$ see Lemma 4.2, we can conclude with Lemma 4.3

With a smooth extension of U on $[-\delta, T + \delta] \times \mathbb{R}/\mathbb{Z}$, for a small positive δ , we obtain estimates in $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$. \square

We are now able to prove the Theorem by using Lemma 4.4 and the method of characteristics.

Proof of Theorem 4.1: Bounds $L^\infty([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d))$: Such bounds give bounds in $C^0([0, T_0], W_{loc}^{s,p})$ since u_ε is in C^1 .

For $t = 0$, it is only an application of Lemma 4.4. The profile $U(t, \cdot)$ is non constant for each t , else U_0 must be constant by the method of characteristics. And the estimates are uniform.

Bounds in $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ The semi-norms $|\cdot|_{\tilde{W}^{s,p}(Q_{d+1}(\mathbf{y}_0, A))}$, where $\mathbf{y}_0 = (t_0, \mathbf{x}_0)$, needs some precautions to use on $[0, T_0] \times \mathbb{R}^d$. \mathbf{y}_0 must be such that $0 < t_0 < T_0$ and $A < \min(t_0, T_0 - t_0)$. Furthermore, only $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ smoothness can be estimate. Indeed, $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ is bounded in $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$. To prove this, let us use the following trick. By the methods of characteristics the family of solutions $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ exists on a maximal time interval $] - \delta, T_1[$, with $0 < \delta < T_0 < T_1$. Notice that solutions exist for negative time since the initial data is smooth. Now estimates in $W_{loc}^{s,p}([- \delta, T_1] \times \mathbb{R}^d)$ will be obtained which is sufficient to get smoothness in $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$. Now using lemma 4.4 completes the proof. \square

5. SUPER CRITICAL GEOMETRIC OPTICS

Now we can exhibit the supercritical geometric optics and some implications about the maximal smoothing effect for solutions of conservation laws with L^∞ initial data.

5.1. Propagation of highest frequency waves.

In Theorem 2.1 we saw that the frequencies of waves are related to an orthogonality condition between the phase gradient and the derivatives of the

flux. Theorem 3.1 tell us where the flux reach is maximal degeneracy and which direction the phase gradient has to be chosen. Thus we can build a geometric optics expansion with the highest frequencies. The uniform Sobolev estimates of such family of oscillating solutions highlight the conjecture about the maximal smoothing effect below.

Theorem 5.1. [Bound of the maximal smoothing effect]

Let \mathbf{F} be a nonlinear flux which belongs to $C^\infty([-M, M], \mathbb{R}^d)$. Let α_{sup} be the sharp measurement of the flux non-linearity. Then there exist a constant $\underline{u} \in [-M, M]$, a time $T_0 > 0$, and a sequence of initial data $(u_0^\varepsilon)_{0 < \varepsilon < 1}$ such that $\|u_0^\varepsilon - \underline{u}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$, and the sequence of entropy solutions $(u_\varepsilon)_{0 < \varepsilon < 1}$ associated with conservation law (1.1) satisfying:

- for all $s \leq \alpha_{\text{sup}}$, the sequence $(u_\varepsilon)_{0 < \varepsilon < 1}$ is uniformly bounded in $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d) \cap C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$,
- for all $s > \alpha_{\text{sup}}$, the sequence $(u_\varepsilon)_{0 < \varepsilon < 1}$ is **unbounded** in $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d)$ and in $C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$.

Proof : The proof is a consequence of previous theorems. By Theorem 3.1, there exists $\underline{u} \in [-M, M]$ such that $\alpha = \frac{1}{d_{\mathbf{F}}[\underline{u}]}$. Let U_0 be a non constant smooth periodic function such that: $-M \leq \underline{u} + U_0(\theta) \leq M$ for all θ .

Let $\mathbf{v} \in \mathbb{R}^d$ such that $\mathbf{a}^k(\underline{u}) \cdot \mathbf{v} = 0$ and $\mathbf{v} \neq 0$ for $k = 1, \dots, d_{\mathbf{F}}[\underline{u}] - 1$. Such \mathbf{v} exists by Definition of $d_{\mathbf{F}}[\underline{u}]$.

Now, let (u_ε) be the family of smooth solutions given by Theorem 2.1. Theorem 4.1 is the desired conclusion. \square

5.2. Highlight of the Lions, Perthame, Tadmor conjecture.

Let us recall the introductory section 1.1 and use the notations therein. In [23], the authors obtained a kinetic formulation of conservation law (1.1) and used averaging lemmas. With only initial data uniformly bounded in L^∞ , they proved an *uniform* smoothing effect in $W_{loc}^{s,1}$ for all positive time. Thus the best uniform smoothing effect in Sobolev spaces is at least ([23]):

$$\frac{\alpha_{\text{sup}}}{2 + \alpha_{\text{sup}}} \leq s_{\text{sup}}.$$

Theorem 5.1 gives an upper bound for the $W^{s,1}$ -regularizing effect (1.7):

$$s_{\text{sup}} \leq \alpha_{\text{sup}},$$

Indeed, let us denote $\mathcal{B}^\infty(\underline{u}, \rho) = \{u \in L^\infty(\mathbb{R}^d, \mathbb{R}), \|u - \underline{u}\|_{L^\infty(\mathbb{R}^d, \mathbb{R})} < \rho\}$ and \mathcal{S}_t the semi-group associated with conservation law (1.1). Theorem 5.1 proves that for a well chosen $\underline{u} \in [-M, M]$, there exists $T_0 > 0$, such that for all $\rho > 0$ and for all $0 < t < T_0$, $\mathcal{S}_t(\mathcal{B}^\infty(\underline{u}, \rho))$ is not a *bounded* subset of $W_{loc}^{s,1}(\mathbb{R}_x^d)$ for all $s > \alpha_{\text{sup}}$.

This result yields some remarks.

Remark 5.1. Optimality for large dimension or large nonlinear degeneracy.

By Theorem 3.1, $\alpha_{\text{sup}} \leq \frac{1}{d}$, so for large dimension α_{sup} is small. Using a better lower bound of s_{sup} from [31] we have:

$$\frac{\alpha_{\text{sup}}}{1 + 2 \alpha_{\text{sup}}} \leq s_{\text{sup}} \leq \alpha_{\text{sup}} \leq \frac{1}{d}.$$

Thus for large dimension ($d \gg 1$) or large nonlinear degeneracy ($\alpha_{\text{sup}} \ll 1$) we have asymptotically the right $s_{\text{sup}} \sim \alpha_{\text{sup}}$.

Remark 5.2. In $W^{s,p}$, $1 < p < +\infty$, our geometric optics expansion shows that $s_{\text{sup}}^p \leq \alpha_{\text{sup}}$ by Theorem 4.1, where s_{sup}^p denotes the maximal uniform smoothing effect in $W^{s,p}$. In other words our example is not related to the parameter p . Other examples show the importance of the parameter p in remark 5.3.

Remark 5.3. Critical entropy solutions in the one dimensional case.

In [13, 4] special initial data $u_0(x) \in W^{s,p}(\mathbb{R})$ are built, in [13] a piecewise smooth initial data and in [4] a continuous oscillating initial data. The entropy solution $u(t, x)$ preserves this smoothness at least on a bounded time interval, indeed, before the waves interactions. The regularity $s < \alpha_{\text{sup}}$ can be choose as close as we want to α_{sup} . Furthermore the parameter p is related to the parameter s : $p = \frac{1}{s}$. Indeed, we cannot expect a greater parameter p since $W^{s,p}(\mathbb{R}) \subset C^0(\mathbb{R})$ for $p > \frac{1}{s}$. It would be interesting to construct such solutions with the almost critical regularity in the multidimensional case.

Remark 5.4. Fractional BV spaces.

Recently in [4, 3], for the one dimensional case and for all nonlinear degenerate convex fluxes, conjecture (1.6) is reached. Furthermore, entropy solutions satisfy a new one-sided Holder condition. For this purpose, new spaces are introduced in the framework of conservation laws: the fractional BV spaces BV^s . BV^s functions have a structure similar to the one of BV maps, for all $s \in]0, 1]$. BV^s spaces seems to be natural spaces to capture the regularizing effect for one dimensional scalar conservation laws.

This last promising remark concludes this paper.

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(Stéphane Junca) LABORATOIRE J.A. DIEUDONNÉ UNIVERSITÉ DE NICE SPOHIA-ANTIPOLIS,
UMR CNRS 7351,PARC VALROSE, 06108, NICE, FRANCE.

(Stéphane Junca) TEAM COFFEE, INRIA SOPHIA-ANTIPOLIS MÉDITERRANÉE, 2004
ROUTES DES LUCIOLES -BP 93, 06902 SOPHIA-ANTIPOLIS, FRANCE
E-mail address: junca@unice.fr