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# HIGH FREQUENCY WAVES AND THE MAXIMAL SMOOTHING EFFECT FOR NONLINEAR SCALAR CONSERVATION LAWS

STÉPHANE JUNCA

**ABSTRACT.** The article first studies the propagation of well prepared high frequency waves with small amplitude  $\varepsilon$  near constant solutions for entropy solutions of multidimensional nonlinear scalar conservation laws. Second, such oscillating solutions are used to highlight a conjecture of Lions, Perthame, Tadmor, (1994), [34], about the maximal regularizing effect for nonlinear conservation laws. For this purpose, a new definition of nonlinear flux is stated and compared to classical definitions. Then it is proved that the smoothness expected by [34] in Sobolev spaces cannot be exceeded.

**Key-words:** multidimensional conservation laws, nonlinear flux, geometric optics, Sobolev spaces, smoothing effect.

**Mathematics Subject Classification:**

Primary: 35L65, 35B65; Secondary: 35B10, 35B40, 35C20.

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## 1. INTRODUCTION

This paper deals with the maximal regularizing effects for nonlinear multidimensional scalar conservation laws. The important point to note here is the definition of nonlinear flux. Indeed there are various definitions see [18, 34, 4, 11]. In [34] they give the well known definition 1.1 below and a conjecture about the maximal smoothing effect in Sobolev spaces related to the parameter “ $\alpha$ ” from their definition. The study of periodic solutions leads to another definitions [18, 4]. We obtain new definition 3.1 for smooth flux. It generalizes the definition of [4]. For smooth flux, our definition is equivalent

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to the classical definition 1.1 and implies the strict non-linearity of [18]. Furthermore, it gives an easy way to compute the parameter “ $\alpha$ ”. Our definition shows that smoothing effects for scalar conservation laws strongly depend on the space dimension. Our new characterization of nonlinear flux comes from the study of the highest oscillations which can be propagated by the semi-group  $\mathcal{S}_t$  associated to the conservation law. Indeed properties of  $\mathcal{S}_t$  are linked to the derivatives of the flux as in [4, 11, 19].

To be more precise, we look for Sobolev bounds for entropy solutions  $u(.,.)$  of

$$(1.1) \quad \partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u) = 0,$$

where  $t \in [0, +\infty[$ ,  $\mathbf{x} \in \mathbb{R}^d$ ,  $u : [0, +\infty[ \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$  is a smooth flux function,  $\mathbf{F} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ , and the initial data is only bounded in  $L^\infty(\mathbb{R}^d, \mathbb{R})$ :

$$(1.2) \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

Let  $\mathbf{a}(u)$  be  $\mathbf{F}'(u)$ . Obviously, if  $\mathbf{F}$  is linear,  $\mathbf{a}(u) = \mathbf{a}$  a constant vector,  $u(t, \mathbf{x}) = u_0(\mathbf{x} - t\mathbf{a})$ , there is no smoothing effect. In [34] was first proved a regularizing effect if the flux  $\mathbf{F}$  is non linear. The sharp measurement of the non-linearity plays a key role in our study. Let us recall the classical definition for nonlinear flux from [34].

**Definition 1.1. [ Nonlinear Flux [34]]**

Let  $M$  be a positive constant,  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$  is said to be nonlinear on  $[-M, M]$  if there exist  $\alpha > 0$  and  $C = C_\alpha > 0$  such that for all  $\delta > 0$

$$(1.3) \quad \sup_{\tau^2 + |\xi|^2 = 1} |W_\delta(\tau, \xi)| \leq C \delta^\alpha,$$

where  $(\tau, \xi) \in S^d \subset \mathbb{R}^{d+1}$ , i.e.  $\tau^2 + |\xi|^2 = 1$ , and  $|W_\delta(\tau, \xi)|$  is the one dimensional measure of the singular set:

$$W_\delta(\tau, \xi) := \{|v| \leq M, |\tau + \mathbf{a}(v) \cdot \xi| \leq \delta\} \subset [-M, M] \text{ and } \mathbf{a} = \mathbf{F}'.$$

Indeed  $W_\delta(\tau, \xi)$  is a neighborhood of the critical value  $v$  for the symbol of the linear operator  $\mathcal{L}[v]$  in the Fourier direction  $(\tau, \xi)$  where  $\mathcal{L}[v] = \partial_t + \mathbf{a}(v) \cdot \nabla_x$ . The symbol in this direction is:  $i(\tau + \mathbf{a}(v) \cdot \xi)$ . This operator is simply related with any smooth solution  $u$  of equation (1.1) by the chain rule formula:

$$\partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u) = \partial_t u + \mathbf{a}(u) \cdot \nabla_x u = \mathcal{L}[u]u.$$

$\alpha$  is a degeneracy measurement of the operator  $\mathcal{L}$  parametrized by  $v$ .  $\alpha$  depends only on the flux  $\mathbf{F}$  and the compact set  $[-M, M]$ :  $\alpha = \alpha[\mathbf{F}, M]$ . In the sequel we denote by

$$(1.4) \quad \alpha_{\sup} = \alpha_{\sup}[\mathbf{F}, M], \quad \text{the supremum of all } \alpha \text{ satisfying (1.3).}$$

$\alpha$ , or more precisely  $\alpha_{\sup}$ , is the key parameter to describe the sharp smoothing effect for entropy solutions of nonlinear scalar conservation laws. For smooth flux the parameter  $\alpha$  always belongs to  $[0, 1]$ , for instance:  $\alpha_{\sup} = 0$  for a linear flux,  $\alpha = 1$  for strictly convex flux in dimension one. For the first time  $\alpha_{\sup}$  is characterized below in section 3. Indeed, for smooth nonlinear flux,

$\frac{1}{\alpha_{\text{sup}}}$  is always an integer greater or equal to the space dimension.

In all the sequel we assume that  $M \geq \|u_0\|_\infty$  and the flux  $\mathbf{F}$  is nonlinear on  $[-M, M]$ , so

$$(1.5) \quad \alpha_{\text{sup}} > 0.$$

If (1.5) is true then the entropy solution operator associated with the nonlinear conservation law (1.1), (1.2),

$$\begin{aligned} \mathcal{S}_t : L^\infty(\mathbb{R}^d, \mathbb{R}) &\rightarrow L^\infty(\mathbb{R}^d, \mathbb{R}) \\ u_0(\cdot) &\mapsto u(t, \cdot), \end{aligned}$$

has a regularizing effect for all  $t > 0$ , mapping  $L^\infty(\mathbb{R}^d, \mathbb{R})$  into  $W_{loc}^{s,1}(\mathbb{R}^d, \mathbb{R})$ .

In [34], they proved this regularizing effect for all  $s < \frac{\alpha}{2 + \alpha}$ .

In [39] the result is improved for all  $s < \frac{\alpha}{1 + 2\alpha}$  under a generic assumption on  $\mathbf{a}' = \mathbf{F}''$ .

P.L. Lions, B. Perthame and E. Tadmor conjectured in 1994 a better regularizing effect, see [34], (remark 3, p .180, line 14-17). In [34] they proposed an optimal bound  $s_{\text{sup}}$  for Sobolev exponents of entropy solutions:

$$(1.6) \quad s_{\text{sup}} = \alpha_{\text{sup}}.$$

That is to say that  $u$  belongs in all  $W_{loc}^{s,1}(\mathbb{R}^d, \mathbb{R})$  for all  $s < \alpha_{\text{sup}}$ .

The shocks formation implies  $s < 1$  and  $s_{\text{sup}} \leq 1$  since  $W^{1,1}$  functions do not have shock.

In one dimension ( $d=1$ ) and for strictly convex flux it is well known from Lax and Oleinik that the entropy solution becomes  $BV$ , see [33]. (1.6) is true in this case since  $u$  belongs in  $W_{loc}^{s,1}$  for all  $s < 1$ :  $s_{\text{sup}} = 1 = \alpha_{\text{sup}}$ .

A main result of the paper is to give an insight of the conjecture (1.6) by proving the inequality

$$(1.7) \quad s_{\text{sup}} \leq \alpha_{\text{sup}}.$$

Examples of family of solutions exactly bounded in  $W_{loc}^{s,1}$  with the conjectured maximal  $s = \alpha_{\text{sup}}$  and with no improvement of the Sobolev exponent in a strip  $[0, T_0] \times \mathbb{R}^d$ ,  $T_0 > 0$ , are given in this paper.

A first proof of (1.7), for some interesting examples, can be found in [16] for  $d = 1$ , and also in [11] for  $d \geq 1$ .

It will be proved that for a well chosen  $\underline{u} \in [-M, M]$ , there exists  $T_0 > 0$ , such that for all  $\rho > 0$  and for all  $0 < t < T_0$ ,  $\mathcal{S}_t(\mathcal{B}^\infty(\underline{u}, \rho))$  is not a subset of  $W_{loc}^{s,1}(\mathbb{R}^d)$  for all  $s > \alpha_{\text{sup}}$ , where  $\mathcal{B}^\infty(\underline{u}, \rho) = \{u \in L^\infty(\mathbb{R}^d, \mathbb{R}), \|u - \underline{u}\|_{L^\infty(\mathbb{R}^d, \mathbb{R})} < \rho\}$ .

High oscillating solutions of (1.1) are used for this purpose. Near a constant state and for  $L^\infty$  data, a complete study of critical geometric optics for weak entropy solutions is done in [4]. Near a smooth (non constant) solution, another features are given in [29]. Here, results of [4] are simplified and proved for particular super-critical highly oscillating classical solutions (without shocks on a strip). This allows to give proof of (1.7).

Consider the problem (1.1) with oscillating initial data:

$$(1.8) \quad u_\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon U_0 \left( \frac{\mathbf{v} \cdot \mathbf{x}}{\varepsilon^\gamma} \right),$$

where  $U_0(\theta)$  is a one periodic function w.r.t.  $\theta$ ,  $\gamma > 0$ ,  $\underline{u}$  is a constant ground state,  $\underline{u} \in [-M, M]$ ,  $\mathbf{v} \in \mathbb{R}^d$ . The case  $\gamma = 1$  is the classical geometric optics for quasilinear equations, see [35, 17, 25, 27]. In this paper we focus on *critical oscillations* when  $\gamma > 1$ .

One of the two following asymptotic expansions (1.9) or (1.10), is expected in  $L^1_{loc}([0, +\infty[ \times \mathbb{R}^d, \mathbb{R})$  for the entropy-solution  $u_\varepsilon$  of conservation law (1.1) with highly oscillating data (1.8) when  $\varepsilon$  goes to 0,

$$(1.9) \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + o(\varepsilon)$$

$$(1.10) \quad \text{or} \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \overline{U}_0 + o(\varepsilon),$$

where the profile  $U(t, \theta)$  satisfies a conservation law with initial data  $U_0(\theta)$ ,  $\overline{U}_0 = \int_0^1 U_0(\theta) d\theta$  and the phase  $\phi$  satisfies the eikonal equation:

$$(1.11) \quad \begin{aligned} \partial_t \phi + \mathbf{a}(\underline{u}) \cdot \nabla_{\mathbf{x}} \phi &= 0, \quad \phi(0, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x}, \\ \text{i.e.} \quad \phi(t, \mathbf{x}) &= \mathbf{v} \cdot (\mathbf{x} - t \mathbf{a}(\underline{u})). \end{aligned}$$

The behavior described by (1.9) is a lack of compactness for the semi-group  $\mathcal{S}_t$  of equation (1.1) since this semi-group propagates data uniformly bounded in  $W^{1/\gamma, 1}_{loc}$  without improving the Sobolev exponent (as we will see at the end of this paper in the section 5).

Otherwise, if  $\gamma$  is too big ( $\gamma \alpha_{\sup} > 1$  as we will see below) and the initial oscillating data are not constant, then the high oscillations are canceled for positive time. Behavior (1.10) means that a nonlinear smoothing effect is associated for the semi-group of equation (1.1).

Combining these two possible behaviors of highly oscillating solutions the article highlights the conjecture. Indeed this allow to prove that the maximal smoothing effect conjectured by Lions, Perthame and Tadmor in [34] cannot be exceeded: (1.7). Furthermore, Theorem 5.1 below shows that there exists a family of well chosen initial data in  $W^{s, 1}_{loc}$  with  $s = \alpha_{\sup}$  such that the associated solutions keep the same uniform bound without any improvement of the Sobolev exponent  $s = \alpha_{\sup}$ . Nevertheless the complete conjecture: equality (1.6), is still an open problem.

On other hand this maximal Sobolev exponent is not sufficient to get some traces for entropic solutions on sets with co-dimension-one. This seems to contradict the structure of a BV function of entropy solutions obtained by De Lellis, Otto and Westdickenberg in [14]. Indeed, this BV structure for one solution and Sobolev bounds for a set of solutions are different approach of the smoothing effects. See for instance [6, 11, 14, 15, 16, 31, 41] where the traces properties or the Sobolev exponent of entropy solutions are studied. In fact, this means that the maximal Sobolev exponent for entropy solutions does not give enough information about the fine structure for entropy solutions of conservation laws.

The paper is organized as follows. In section 2 examples of super-critical highly oscillating solutions are expounded. In section 3, the concept of flux non-linearity is clarified and characterized. Section 4 is devoted to get optimal Sobolev estimates on oscillating solutions built in section 2. Finally, the section 5 highlights conjecture (1.6).

## 2. HIGH FREQUENCY WAVES WITH SMALL AMPLITUDE

The section 2 deals with highly oscillating initial data near a constant state (1.8) only uniformly bounded in  $W^{1/\gamma,1}$  (see section 4 below). The propagation of such oscillating data is obtained under the crucial compatibility condition (2.1) below. Otherwise, if the the compatibility condition (2.1) is nowhere satisfied, the nonlinear semi-group associated to equation (1.1) cancels this too high oscillations, see Theorem 2.2. The validity or invalidity of assumption (2.1) is a key point to characterize nonlinear flux in section 3.

### Theorem 2.1. [Propagation of smooth high oscillations]

Let  $\gamma$  belongs to  $]1, +\infty[$  and let  $q$  be the integer such that  $q - 1 < \gamma \leq q$ . Assume  $\mathbf{F}$  belongs to  $C^{q+3}(\mathbb{R}, \mathbb{R}^d)$ ,  $U_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ ,  $\mathbf{v} \neq (0, \dots, 0)$  and

$$(2.1) \quad \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} = 0, \quad k = 1, \dots, q - 1$$

then there exists  $T_0 > 0$  such that, for all  $\varepsilon \in ]0, 1]$ , the solutions of conservation law (1.1) with initial oscillating data (1.8) are smooth on  $[0, T_0] \times \mathbb{R}$  and

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + \mathcal{O}(\varepsilon^{1+r}) \text{ in } L^\infty([0, T_0] \times \mathbb{R}^d),$$

where  $0 < r = \begin{cases} 1 & \text{if } \gamma = q, \\ q - \gamma & \text{else,} \end{cases}$  and the smooth profile  $U$  is uniquely determined by the Cauchy problem (2.2), (2.3),  $\phi$  is given by the eikonal equation (1.11):

$$(2.2) \quad \partial_t U + b \frac{\partial U^{q+1}}{\partial \theta} = 0, \quad b = \begin{cases} \frac{1}{(q+1)!} (\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}) & \text{if } \gamma = q, \\ 0 & \text{else.} \end{cases}$$

$$(2.3) \quad U(0, \theta) = U_0(\theta).$$

We deal with smooth solutions to compute later Sobolev bounds. Indeed the asymptotic stays valid after shocks formation and for all positive time but in  $L^1_{loc}$  instead of  $L^\infty$ , see [4].

When  $\gamma = 1$ , we do not need assumption (2.1). It is the classic case for geometric optics, see [35, 17, 25, 26, 27, 28].

In dimension  $d \geq 2$ , it is always possible to find a non trivial vector  $\mathbf{v}$  satisfying (2.1). At least for  $\gamma = 2$ , (2.1) reduces to find  $\mathbf{v} \neq 0$  such that  $\mathbf{a}'(\underline{u}) \cdot \mathbf{v} = 0$ . Thus, such singular solutions always exists in dimension greater than one. But, for genuine nonlinear one dimensional conservation law, there is never such solution. Of course, we assume  $U_0$  non constant and  $\mathbf{F}$  not a linear function near  $\underline{u}$ , else the Theorem is obvious. If  $U_0$  is constant  $u_\varepsilon$  too. If  $\mathbf{F}$  is linear on  $[\underline{u} - \delta, \underline{u} + \delta]$  for some  $\delta > 0$ , high oscillations propagate for all time when  $\varepsilon$ .

In fact Theorem 2.1 expresses a kind of degeneracy of **multidimensional** scalar conservation laws. This degeneracy (period smaller than the amplitude) appears for quasilinear systems with a linearly degenerate eigenvalue [7, 8, 9, 3], and for linear [32] and semi-linear systems [25, 26, 28].

Notice that for  $\gamma > 1$ , smooth solutions exist for larger time than it is currently known [13, 33]:  $T_\varepsilon \sim 1/|\nabla_{\mathbf{x}} u_0^\varepsilon| \sim \varepsilon^{\gamma-1}$ . Furthermore equation (2.2) is nonlinear only if  $\gamma \in \mathbb{N}$  and  $\mathbf{a}^q(\underline{u}) \cdot \mathbf{v} \neq 0$ .

**Proof :** First one performs a WKB computations with following ansatz:

$$\begin{aligned}
 (2.4) \quad u_\varepsilon(t, \mathbf{x}) &= \underline{u} + \varepsilon U_\varepsilon \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right), \\
 \mathbf{F}(u_\varepsilon) &= \sum_{k=0}^{q+1} \varepsilon^k \frac{U_\varepsilon^k}{k!} \mathbf{F}^{(k)}(\underline{u}) + \varepsilon^{q+2} \mathbf{G}_q^\varepsilon(U_\varepsilon), \\
 \mathbf{G}_q^\varepsilon(U) &= U^{q+2} \int_0^1 \frac{(1-s)^{q+1}}{(q+1)!} \mathbf{F}^{(q+2)}(\underline{u} + s\varepsilon U) ds, \\
 g_q^\varepsilon(U) &= \mathbf{v} \cdot \mathbf{G}_q^\varepsilon(U), \\
 \partial_t U_\varepsilon \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) &= \partial_t U_\varepsilon - \varepsilon^{-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon \\
 \operatorname{div}_{\mathbf{x}} \mathbf{F}(u_\varepsilon) &= \sum_{k=0}^q \varepsilon^{k+1-\gamma} \frac{\partial_\theta U_\varepsilon^{k+1}}{(k+1)!} \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} + \varepsilon^{q+2} \operatorname{div}_{\mathbf{x}} \mathbf{G}_q^\varepsilon(U_\varepsilon) \\
 &= \varepsilon^{1-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon + \varepsilon^{q+1-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{q+2-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon),
 \end{aligned}$$

where  $c_q = \frac{\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}}{(q+1)!}$ . Then simplification yields

$$(2.5) \quad \partial_t u_\varepsilon + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u_\varepsilon) = \varepsilon \left( \partial_t U_\varepsilon + \varepsilon^{q-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{1+q-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon) \right).$$

It suffices to take  $U_\varepsilon$  solution of the one dimensional scalar conservation laws with  $\psi_\varepsilon(U) = \varepsilon^{q-\gamma} c_q U^{q+1} + \varepsilon^{1+q-\gamma} \partial_\theta g_q^\varepsilon(U)$

$$(2.6) \quad \partial_t U_\varepsilon + \partial_\theta \psi_\varepsilon(U_\varepsilon) = 0, \quad U_\varepsilon(0, \theta) = U_0(\theta).$$

Notice that  $\psi_\varepsilon = O(1) \in C_{loc}^1$ . If  $\gamma \neq q$ ,  $\psi_\varepsilon$  is even smaller:  $\psi_\varepsilon = O(\varepsilon^r) \in C_{loc}^1$ . By the method of characteristics, for each  $\varepsilon$ , the existence of a smooth solution is obvious for time or order  $1/|U'_0|$ , more precisely a smooth solutions exists on a maximal interval  $T_\varepsilon$ , where a first shock locates at the time  $t = T_\varepsilon$  (which always occurs if the initial periodic data is non constant and  $c_q \neq 0$ ). Furthermore,  $\psi_\varepsilon$  depends smoothly of  $\varepsilon$  so we can take an uniform time for all  $\varepsilon \in [0, 1]$ . Which prove the existence of  $0 < T_0 < T^* = \min\{T_\varepsilon, \varepsilon \in ]0, 1]\}$  for  $u_\varepsilon$ . Indeed,  $T_\varepsilon = -1/\min\{\psi_\varepsilon''(U_0(\theta))U'_0(\theta), \theta \in [0, 1]\}$ . Let  $0 < T_0 < T^*$ , the one periodic function w.r.t.  $\theta$ ,  $U_\varepsilon$  belongs to  $C^1([0, T_0] \times \mathbb{R}/\mathbb{Z})$  and  $u_\varepsilon$  is well defined by (2.4). By construction  $u_\varepsilon$  satisfies (1.1), (1.8) and  $u_\varepsilon \in C^1([0, T_0] \times \mathbb{R}^d)$  for all  $0 < \varepsilon \leq 1$ .

There are two cases:  $\gamma$  is an integer or not.

$q = \gamma$ : From (2.5) and (1.1) we get

$$\begin{aligned} \partial_t U_\varepsilon + \partial_\theta (c_q U_\varepsilon^{q+1} + \varepsilon g_q^\varepsilon(U_\varepsilon)) &= 0, & \partial_t U + c_q \partial_\theta U^{q+1} &= 0, \\ U_\varepsilon(0, \theta) &= U_0(\theta), & U(0, \theta) &= U_0(\theta). \end{aligned}$$

The classical method of characteristics, gives  $C^1$  characteristics,  $C^1$  solutions and

$$\|U_\varepsilon - U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon),$$

where

$$\|U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = \|U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_t U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_\theta U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)}.$$

integer  $q > \gamma$ : The proof is similar except the term  $\varepsilon^r c_q \partial_\theta (c_q U^{q+1})$  becomes a remainder, with  $r = q - \gamma$  and  $U(t, \theta) = U_0(\theta)$ , thus

$$\|U_\varepsilon(\cdot, \cdot) - U_0(\cdot)\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon^r).$$

□

If condition (2.1) is violated, oscillations are immediately canceled.

**Theorem 2.2. [Cancellation of high oscillations, [4]]**

Let  $\mathbf{F}$  belongs to  $C^{q+1}$  and  $U_0 \in L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ , where  $q - 1 < \gamma \leq q$  is defined in Theorem 2.1. If for some  $0 < j < q$

$$(2.7) \quad \mathbf{a}^{(j)}(\underline{u}) \cdot \mathbf{v} \neq 0$$

then the solutions  $u_\varepsilon$  of conservation law (1.1) with initial oscillating data (1.8) for  $\varepsilon \in ]0, 1]$  satisfy when  $\varepsilon \rightarrow 0$

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \overline{U}_0 + o(\varepsilon) \quad \text{in } L_{loc}^1([0, +\infty[ \times \mathbb{R}^d).$$

Obviously the interesting case is when  $U_0$  is non constant. In this context, when  $U_0$  is smooth and non constant the first time when a shock occurs  $T_\varepsilon \rightarrow 0$  when  $\varepsilon \rightarrow 0$ . Thus solutions are weak entropy solutions.

The proof is in the spirit of [4] and uses averaging lemmas. Another proof is possible with another argument: the decay of periodic solutions for large time, see [5, 10, 12, 13, 18, 22] about this decay.



**Proof :** For non constant initial data it is impossible to avoid shock waves on any fixed strip  $[0, T_0] \times \mathbb{R}^d$  with  $T_0 > 0$  as in the previous proof of Theorem 2.1 since the time span of smooth solutions is  $\varepsilon^\beta$  where  $\beta = \gamma - j > 0$ .

First, with a change of space variable  $\mathbf{x} \leftrightarrow \mathbf{x} - t\mathbf{a}(\underline{u})$ , assume that  $\mathbf{a}(\underline{u}) = 0$ . Let  $W_\varepsilon(t, \theta) = U_\varepsilon(t, \varepsilon^j \theta)$  where  $U_\varepsilon$  is defined in (2.4). Then  $W_\varepsilon$  satisfies the one dimensional conservation laws:

$$(2.8) \quad \partial_t W_\varepsilon + \partial_\theta (c_j W_\varepsilon^{j+1} + \varepsilon g_j^\varepsilon(W_\varepsilon)) = 0, \quad W_\varepsilon(0, \theta) = U_0(\varepsilon^{-\beta} \theta),$$

Since  $W_\varepsilon(0, \cdot)$  converges weakly towards  $\overline{U}_0$ , and  $W_\varepsilon$  is relatively compact in  $L_{loc}^1$  thanks to averaging lemmas,  $W_\varepsilon$  converges towards the unique entropy solution  $W$  of

$$\partial_t W + c_j \partial_\theta W^{j+1} = 0, \quad W(0, \theta) = \overline{U}_0.$$

That is to say that  $W(t, \theta) \equiv \overline{U}_0$ . Then  $v_\varepsilon(t, \mathbf{x}) = W_\varepsilon(t, \varepsilon^{-\beta} \mathbf{v} \cdot \mathbf{x})$  converges towards  $\overline{U}_0$  in  $L_{loc}^1$  which concludes the proof.  $\square$

### 3. CHARACTERIZATION OF NONLINEAR FLUX

The regularizing effect given in [34] is related with the sharp exponent  $\alpha = \alpha_{\text{sup}}$  quantifying precisely the non-linearity of the flux. There are some examples where  $\alpha$  is computed in dimension two in [34, 39] and some remarks in [2, 23, 24]. For the first time, we obtain “the sharp  $\alpha$ ” for all smooth flux in Theorem 3.1.

Another genuine nonlinear definition for the flux depending on the space dimension  $d$ , related to weakly nonlinear geometric optics, is given in [4]. We generalize the definition from [4]. Thanks to Theorem 3.1 below, this new definition is equivalent to the classical definition 1.1.

Furthermore the definition 3.1 implies the strict nonlinearity defined in [18]. This will be proved at the end of this section with other related results.

**Definition 3.1. [Nonlinear flux]**

*Let the flux  $\mathbf{F}$  belongs to  $C^\infty(\mathbb{R}, \mathbb{R}^d)$  and  $I = [c, d]$ ,  $c < d$ . The flux is said to be nonlinear on  $I$  if, for all  $u \in I$ , there exists  $m \in \mathbb{N}^*$  such that*

$$(3.1) \quad \text{rank}\{\mathbf{a}'(u), \dots, \mathbf{a}^{(m)}(u)\} = d.$$

*Furthermore, the flux is said to be genuine nonlinear if  $m = d$  is enough in (3.1) for all  $u \in I$ .*

As usual, the non-linearity is a matter of the second derivatives of  $\mathbf{F}$ :  $\mathbf{F}'' = \mathbf{a}'$ . Notice that  $m \geq d$ , thus the genuine nonlinear case is the strongest nonlinear case. The genuine nonlinear case was first stated in [4], see condition (2.8) and Lemma 2.5 p. 447 therein. The genuine nonlinear condition  $\det(\mathbf{a}', \dots, \mathbf{a}^{(d)}) \neq 0$  for multidimensional conservations laws was also in [11], see condition (16) p. 84 therein.

The simplest example of velocity  $\mathbf{a}$  associated with a genuine nonlinear flux  $\mathbf{F}$  is given in [4, 11, 2]:

$$\mathbf{a}(u) = (u, u^2, \dots, u^d) \quad \text{with } \mathbf{F}(u) = \left( \frac{u^2}{2}, \dots, \frac{u^{d+1}}{d+1} \right).$$

The definition 3.1 is more explicit with following integers with  $I = [-M, M]$ .

$$(3.2) \quad d_{\mathbf{F}}[u] = \inf\{k \geq 1, \text{rank}\{\mathbf{F}''(u), \dots, \mathbf{F}^{(k+1)}(u)\} = d\} \geq d,$$

$$(3.3) \quad d_{\mathbf{F}} = \sup_{|u| \leq M} d_{\mathbf{F}}[u] \in \{d, d+1, \dots\} \cup \{+\infty\}.$$

Indeed the definition 3.1 states that the flux is genuine nonlinear when  $d_{\mathbf{F}}$  reaches its minimal value,  $d_{\mathbf{F}} = d$ .

Conversely, if the flux  $\mathbf{F}$  is linear, then  $\mathbf{a}$  is a constant vector in  $\mathbb{R}^d$  and  $d_{\mathbf{F}}$  reaches its maximal value,  $d_{\mathbf{F}} = +\infty$ .

Between  $d_{\mathbf{F}} = d$  and  $d_{\mathbf{F}} = +\infty$ , there is a large variety of nonlinear flux.

The following theorem gives the optimal  $\alpha$ , (1.3), for smooth flux.

**Theorem 3.1. [Sharp measurement of the flux non-linearity]**

Let  $\mathbf{F}$  be a smooth flux,  $\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$ , the measurement of the flux non-linearity  $\alpha_{\text{sup}}$  is given by

$$\alpha_{\text{sup}} = \frac{1}{d_{\mathbf{F}}} \leq \frac{1}{d}.$$

Furthermore, if  $\alpha_{\text{sup}} > 0$  there exists  $\underline{u} \in [-M, M]$  such that  $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$ .

A similar result for the genuine nonlinear case:  $d_{\mathbf{F}} = d$ , can be found in [2].

This Theorem is a powerful tool to compute the parameter  $\alpha_{\text{sup}}$ , for instance:

- if  $F(u) = (\cos(u), \sin(u))$  then  $F$  is genuine nonlinear and  $\alpha_{\text{sup}} = 1/2$  since  $\det(F''(u), F'''(u)) = 1$ .
- if  $F$  is polynomial with degree less or equal to the space dimension  $d$  then  $\alpha_{\text{sup}} = 0$  and  $F$  does not satisfy definition 3.1.

It is well known that the ‘‘Burgers multi-D’’ flux  $F(u) = (u^2, \dots, u^2)$  is not nonlinear when  $d \geq 2$ . The sequence of oscillations with large amplitude  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  given by  $u_\varepsilon(t, \mathbf{x}) = \sin\left(\frac{x_1 - x_2}{\varepsilon}\right)$  gives us global smooth solutions while the sequence  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  blows up in any  $W_{loc}^{s,1}$  for all  $s > 0$ .

- If  $F$  is polynomial such that  $\deg(F_i) \geq 2$  for all  $i$  and all degrees are distinct, then  $F$  is nonlinear and  $\alpha_{\text{sup}} = \frac{1}{\max_i \deg(F'_i)}$ .

For smooth Flux the optimal  $\alpha$  is the inverse of an integer. Not all value of  $\alpha$  are possible for  $\mathbf{F} \in C^\infty$ . With less smooth flux, other values of  $\alpha$  are possible, see [34, 39, 24].

A geometric approach, like Morse theory is possible to prove Theorem 3.1, see a suggestion in [20]. We choose another approach similar to some proofs of phase stationary lemmas, see [38, 2, 24].

The proof of Theorem 3.1 needs some lemmas. First we recall a Lemma from [2, 30] giving the optimal  $\alpha$  for functions on  $\mathbb{R}$ .

Let  $\varphi \in C^\infty([c, d], \mathbb{R})$  and  $v \in [c, d]$ , the multiplicity of  $\varphi$  on  $v$  is defined by

$$m_\varphi[v] = \inf\{k \in \mathbb{N}, \varphi^{(k)}(v) \neq 0\} \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}.$$

It means that if  $k = m_\varphi$  then  $\varphi^{(k)}(v) \neq 0$  and  $\varphi^{(j)}(v) = 0$  for  $j = 0, 1, \dots, k-1$ . For instance  $m_\varphi[v] = 0$  means  $\varphi(v) \neq 0$ ;  $m_\varphi[v] = 1$  means  $\varphi(v) = 0, \varphi'(v) \neq 0$  and  $m_\varphi[v] = +\infty$  means  $\varphi^{(j)}(v) = 0$  for all  $j \in \mathbb{N}$ .

Set the multiplicity of  $\varphi$  on  $[c, d]$  by

$$m_\varphi = \sup_{v \in [c, d]} m_\varphi[v] \in \overline{\mathbb{N}}.$$

**Lemma 3.1** ([2, 30]). *Let  $\varphi \in C^\infty([c, d], \mathbb{R})$  with  $c < d$ , and*

$$Z(\varphi, \varepsilon) = \{v \in [c, d], |\varphi(v)| \leq \varepsilon\}.$$

*If  $0 < m_\varphi < +\infty$  then there exists  $C > 1$  such that, for all  $\varepsilon \in ]0, 1]$ ,*

$$(3.4) \quad C^{-1}\varepsilon^\alpha \leq \text{meas}(Z(\varphi, \varepsilon)) \leq C\varepsilon^\alpha \quad \text{with} \quad \alpha = \frac{1}{m_\varphi}.$$

The case  $m_\varphi = 0$  is not interesting since there is no zero in this situation. Quantity  $m_\varphi$  is positive simply means that the set  $Z(\varphi, 0)$  of roots of  $\varphi$  is not empty.

**Proof.** Since any root of  $\varphi$  has a finite multiplicity, the compact set  $Z(\varphi, 0)$  is discrete and then finite:  $Z(\varphi, 0) = \{z_1, \dots, z_\nu\}$ . For each  $z_i$  and  $h > 0$ , let  $V_i(h)$  be  $]z_i - h, z_i + h[ \cap [c, d]$ . For any  $0 < h < |b - a|$ , we have

$$h \leq \text{meas}(V_i(h)) \leq 2h.$$

For any root  $z_i$ , there exists  $h_i \in ]0, |b - a|]$ ,  $A_i > 0$  and  $\delta_i > 0$  such that

$$(3.5) \quad \delta_i |h|^{k_i} \leq |\varphi(z_i + h)| \leq A_i |h|^{k_i} \quad \text{for all } h \in V_i(h_i),$$

with  $k_i = m_\varphi[z_i]$ . This is a direct consequence of Taylor-Lagrange formula.

Let  $V$  be  $\bigcup_i V_i(h_i)$  and  $\varepsilon_0 = \min(1, \min_{v \in [c, d] \setminus V} |\varphi(v)|)$ . By the continuity of

$\varphi$  on the compact set  $[c, d] \setminus V$ ,  $\varepsilon_0$  is positive. Then for all  $0 < \varepsilon < \varepsilon_0$ , we have  $Z(\varphi, \varepsilon) \subset V$ . If  $\varepsilon \geq |\varphi(z_i + h)|$  for  $|h| < h_i$ , then from (3.5), we have  $(\varepsilon/\delta_i)^{1/k_i} \geq |h|$ . This last inequality implies for  $0 < \varepsilon < \varepsilon_0 \leq 1$  that  $Z(\varphi, \varepsilon)$  is a subset of  $\bigcup_i V_i((\varepsilon/\delta_i)^{1/k_i})$  and then

$$\text{meas}(Z(\varphi, \varepsilon)) \leq 2 \sum_{i=1}^{\nu} (\varepsilon/\delta_i)^{1/k_i} \leq \left( 2 \sum_{i=1}^{\nu} \delta_i^{-1/k_i} \right) \varepsilon^{1/m_\varphi}.$$

It gives inequality (3.4). To obtain the optimality of  $\alpha$ , let  $z_j$  be a root of  $\varphi$  with maximal multiplicity i.e.  $m_\varphi[z_j] = m_\varphi = k$ . Again from (3.5),  $V_j((\varepsilon/A_j)^{1/k})$  is a subset of  $Z(\varphi, \varepsilon)$  for all  $\varepsilon \in ]0, \varepsilon_0]$ . Then we have  $(\varepsilon/A_j)^{1/k} \leq \text{meas}(Z(\varphi, \varepsilon))$ , which is enough to get the optimality of  $\alpha = 1/k$  and concludes the proof.  $\square$

We recall two useful lemmas from [2] with their proofs to be self-contained. These proofs follow a proof of E. Stein about oscillatory integrals [38].

**Lemma 3.2.** [2] *Let  $k \geq 1$ ,  $I$  an interval of  $\mathbb{R}$ ,  $\phi \in C^k(I, \mathbb{R})$ .*

$$\begin{aligned} \text{If} \quad & 1 \leq |\phi^{(k)}(v)|, \quad \text{for all } v \in I, \\ \text{then} \quad & \text{measure}\{v \in I, |\phi(v)| \leq \varepsilon\} \leq \bar{c}_k \varepsilon^{1/k}, \end{aligned}$$

where  $\bar{c}_k$  is a constant independent of  $\phi, I$ .

Notice that the result is independent of  $I$ . In particular, let  $Z(\phi, \varepsilon)$  be  $\{v \in I, |\phi(v)| \leq \varepsilon\}$ .  $Z(\phi, \varepsilon)$  is always bounded. Furthermore the number or roots of  $\phi$  is finite. The proof gives at most  $2^k - 1$  roots.

**Proof :** Since the result is independent of interval  $I$  and of  $\phi^{(k-1)}(0)$  sign, let us suppose that  $I = \mathbb{R}$  with  $|\phi^{(k)}(v)| \geq 1 > 0$  on  $\mathbb{R}$ , and  $\phi^{(k-1)}(0) \leq 0$ .

We first treat the case  $k = 1$ . If  $\phi'(v)$  stays positive, we have  $\phi(0) + v \leq \phi(v)$  for  $0 \leq v$  and since  $\phi(0) \leq 0$ , there exists a unique  $c \geq 0$  such that  $\phi(c) = 0$ . In the other case,  $\phi'(v)$  stays negative, and we find a unique  $c \leq 0$  such that  $\phi(c) = 0$ . Then  $|\phi(v)| \geq |v - c|$  for all  $v$ , and  $|\phi(v)| \leq \varepsilon$  implies  $|v - c| \leq \varepsilon$  i.e.  $Z(\phi, \varepsilon) \subset [c - \varepsilon, c + \varepsilon]$ . So the lemma is proved for  $k = 1$  with  $\bar{c}_1 = 2$ .

We now prove the Lemma by induction on  $k$ . Let us suppose that the case  $k$  is known. As for  $k = 1$ , there exists a unique  $c$  such that  $\phi^{(k)}(c) = 0$ . Thus for all  $v$  we have  $|\phi^{(k)}(v)| \geq |v - c|$ . Let  $\eta > 0$  and set  $W = Z(\phi, \varepsilon) \cap [c - \eta, c + \eta]$ ,  $U = Z(\phi, \varepsilon) \cap (]-\infty, c - \eta] \cup ]c + \eta, +\infty[)$ . We have  $\text{meas}(W) \leq 2\eta$  and by our inductive hypothesis, since  $|\phi^{(k)}(v)| \geq |v - c| \geq \eta$  on  $U$ ,  $\text{meas}(U) \leq \bar{c}_k (\varepsilon / \eta)^{1/k}$ . Now the relation  $Z(\phi, \varepsilon) = W \cup U$  gives  $\text{meas}(Z(\phi, \varepsilon)) \leq \inf_{\eta > 0} (2\eta + \bar{c}_k (\varepsilon / \eta)^{1/k})$  which implies by a simple computation of the minimum that  $\text{meas}(Z(\phi, \varepsilon)) \leq \bar{c}_{k+1} (\varepsilon)^{1/(k+1)}$ , where  $\bar{c}_{k+1} = 2^{1/(k+1)} (k+1) k^{1/(k+1)-1} \bar{c}_k^{1-1/(k+1)}$  which concludes the proof.  $\square$

The previous lemma is generalized with parameters in a compact set.

**Lemma 3.3** ([2]). *Let  $P$  be a compact set of parameters,  $k$  a positive integer,  $A > 0$ ,  $V = [-A, A]$ ,  $K = V \times P$ ,  $\phi(v; p) \in C^0(P, C^k(V, \mathbb{R}))$ , such that, for all  $(v, p)$  in the compact  $K$ , we have*

$$\sum_{j=1}^k \left| \frac{\partial^j \phi}{\partial v^j} \right| (v; p) > 0.$$

*Let  $Z(\phi(\cdot; p), \varepsilon) = \{v \in V, |\phi(v; p)| \leq \varepsilon\}$ , then there exists a constant  $C$  such that*

$$\sup_{p \in P} \text{meas}(Z(\phi(\cdot; p), \varepsilon)) \leq C \varepsilon^{1/k}.$$

**Proof.** Since  $K$  is a compact set, we can choose  $0 < \delta \leq 1$  such that, everywhere on  $K$ , we have  $0 < 2\delta < \frac{1}{k} \sum_{i=1}^k \left| \frac{\partial^i \phi}{\partial v^i} \right| (v; p)$ .

For each  $(v; p) \in K$ , there exists an integer  $i \in \{1, \dots, k\}$ , a number  $r > 0$  and an open set  $O_p$  with  $p \in O_p \subset P$  such that  $|\partial_v^i \phi| > \delta$  on  $U(v, p) = ]v - r, v + r[ \times O_p$ . Therefore, we have

$$\text{meas}(Z(\phi(\cdot; p), \varepsilon) \cap ]v - r, v + r[) \leq \bar{c}_i(\varepsilon/\delta)^{1/i} \leq \bar{c} \varepsilon^{1/k} / \delta$$

using Lemma 3.2, where  $\bar{c} = \max_{i=1, \dots, k} \bar{c}_i$ .

By compactness of  $K$ , there exists a finite number of such sets  $U_j = U(v_j, p_j)$  such that  $K \subset \bigcup_{j=1}^{\nu} U_j$ . Thus, for each  $p$ ,  $Z(\phi(\cdot; p), \varepsilon)$  intersects at most  $\nu$  intervals  $]v_j - r_j, v_j + r_j[$  where Lemma 3.2 is applied. This allows to write  $\text{meas}(Z(\phi(\cdot; p), \varepsilon)) \leq \nu c \varepsilon^{1/k} / \delta$  for all  $p$  and to conclude the proof.  $\square$

Now we are able to use the key integer  $d_{\mathbf{F}}$ .

**Lemma 3.4.** *If  $\mathbf{F}$  is a nonlinear flux on  $I$  in the sense of definition 3.1 then  $d_{\mathbf{F}}$  is finite and there exists  $\underline{u} \in I$  such that  $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$ .*

**Proof** Let  $u$  be fixed in  $I$ . Then there exists,  $1 \leq j_1 < j_2 < \dots < j_d = d_{\mathbf{F}}[u]$  such that  $\text{rank}\{\mathbf{a}^{(j_1)}(u), \dots, \mathbf{a}^{(j_d)}(u)\} = d$  by definition of  $d_{\mathbf{F}}[u]$ . So the continuous function  $g(v) = \det(\mathbf{a}^{(j_1)}(v), \dots, \mathbf{a}^{(j_d)}(v))$  does not vanish at  $v = u$ . By continuity, this is still true on an open set  $J$  with  $u \in J$ . Since  $j_d = d_{\mathbf{F}}[u]$ , we have  $d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}[u]$  for all  $v \in J$ . Thus  $v \mapsto d_{\mathbf{F}}[v]$  is upper semi-continuous and the result follows immediately on the compact set  $I$ .  $\square$

**Proof of Theorem 3.1.** There are two steps.

step 1:  $\alpha_{\text{sup}} \geq \frac{1}{d_{\mathbf{F}}}$ .

Set  $\phi(v; \tau, \xi) = \tau + \mathbf{a}(v) \cdot \xi$  with  $\tau^2 + |\xi|^2 = 1$ .  $\tau$  and  $\xi$  are fixed. Since  $\phi(\cdot; \tau, 0) = \tau$  has no roots, we can assume that  $\xi \neq 0_{\mathbb{R}^d}$ . For  $j \geq 1$  we have  $\partial_v^j \phi(v; \tau, \xi) = \mathbf{a}^{(j)}(v) \cdot \xi$ . By definition of  $d_{\mathbf{F}}[v]$  there exists  $j \leq d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}$  such that  $\partial_v^j \phi(v; \tau, \xi) \neq 0$ . Thus, we have when  $\xi \neq 0$

$$(3.6) \quad \sum_{j=1}^{d_{\mathbf{F}}} |\partial_v^j \phi(v; \tau, \xi)| > 0.$$

When  $\xi = 0$ , we have  $\tau = \pm 1$  since  $\tau^2 + |\xi|^2 = 1$ . The function  $\phi(v; \pm 1, 0) = \pm 1 \neq 0$ . By continuity of this function there exists an open neighborhood  $V$  of  $(1, 0_{\mathbb{R}^d})$  such the function does not vanish on  $\bar{V}$ . Set  $P$  be the complementary set of  $V$  in the unit sphere of  $\mathbb{R}^{d+1}$ .  $P$  is compact and (3.6) is true on  $P$ . Now we can use lemma 3.3 to conclude the first step.

step 2:  $\alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}}$ .

Take  $\underline{u}$  from lemma 3.4. Then there exists  $\xi \neq 0$  such that  $\partial_v^j \phi(v; \tau, \xi) = 0$  for  $1 \leq j < d_{\mathbf{F}}$  and  $\partial_v^{d_{\mathbf{F}}} \phi(v; \tau, \xi) \neq 0$  for  $j = d_{\mathbf{F}}$ . For such  $\xi \neq 0$ , we choose  $\tau$  such that  $\varphi(v) = \phi(v; \tau, \xi)$  vanishes at  $v = \underline{u}$ . Now, by lemma 3.1, the second

step is proved.

Finally  $\frac{1}{d_{\mathbf{F}}} \leq \alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}}$  and the proof is complete with lemma 3.4.  $\square$

There are more general definitions of nonlinear flux [18, 34]. But the precise smoothing effect seems to be related with definition 1.1 or 3.1 and the parameter  $\alpha_{\text{sup}}$  or equivalently  $d_{\mathbf{F}}$ . Nevertheless, let us compare theses general definitions with new definition 3.1. It can be useful for other applications.

In [34] there is a more general definition of nonlinear flux.

**Definition 3.2. [General Nonlinear Flux [34]]** *A flux  $\mathbf{F}$ , differentiable on  $[-M, M]$  is said to be nonlinear if the degeneracy set*

$$W(\tau, \xi) = \{v \mid \tau + \mathbf{F}'(v) \cdot \xi = 0\}$$

*has null Lebesgue measure for all  $(\tau, \xi)$  on the sphere.*

This definition is of a great importance since this condition implies the compactness of the semi-group  $\mathcal{S}_t$  associated to the conservation law (1.1).

**Proposition 3.1.** *Let  $\mathbf{F}$  be a smooth flux in  $C^\infty$ . If  $\mathbf{F}$  satisfies definition 3.1 then  $\mathbf{F}$  is nonlinear for definition 3.2 but the converse is wrong.*

**Proof :** Lemma 3.4 and Theorem 5.1 show that nonlinearity of definition 3.1 implies nonlinearity of definition 1.1 and then of definition 1.1. But we can give a direct proof from Lemma 2.5 and remark (2.3) p. 447 in [4], see also [11] p. 84.

Notice that  $W(\tau, 0) = \emptyset$  since  $\tau = \pm 1$ . So we assume that  $\xi \neq 0$ . Set  $\phi(v) = \tau + \mathbf{F}'(v) \cdot \xi$ . Since  $\phi^k(v) = \mathbf{F}^{k+1}(v) \cdot \xi$ , for any  $v$ , there exists  $k > 0$  such that  $\phi^k(v) \neq 0$  by the definition 3.1. So the roots of  $\phi$  are isolated and the set  $W(\tau, \xi)$  is finite.

Conversely the counter-example  $\mathbf{F}'(u) = \exp(-1/u^2)(u, u^2, \dots, u^d)$  does not satisfies definition 3.1 since  $d_{\mathbf{F}}[0] = +\infty$ .

But  $\mathbf{F}$  satisfies definition 3.2. Indeed, with  $h(v) = \tau \exp(1/v^2) + \xi \cdot (v, v^2, \dots, v^d)$ , the set  $W(\tau, \xi) - \{0\}$  is the set of roots of  $h(\cdot)$ . If  $\tau = 0$ , we deal with the genuine nonlinear flux of definition 4.1 and the degeneracy set  $W(\tau, \xi)$  is a null set. Indeed it is finite. If  $\tau \neq 0$ ,  $h(\cdot)$  is analytic and non trivial on  $\mathbb{R}^*$ . Consequently  $W(\tau, \xi)$  is countable and also a null set which concludes the proof.  $\square$

Engquist and E in [18] gave another definition of strictly nonlinear flux generalizing Tartar [40].

**Definition 3.3. [Strictly Nonlinear Flux [18]]**

*Let  $M$  be a positive constant, and  $\mathbf{F} : [-M, M] \rightarrow \mathbb{R}^d$  be a function twice differentiable on  $[-M, M]$ .*

*$F$  is said to be strictly nonlinear on  $[-M, M]$  if for any sub-interval  $I$  of  $[-M, M]$ , the functions  $F_1'', \dots, F^{(d)}$  are linearly independent on  $I$ , i.e., for any constant vector  $\xi$ , if  $\xi \cdot \mathbf{F}''(u) = 0$  for all  $u \in I$  then  $\xi = 0$ .*

**Proposition 3.2.** *Let  $\mathbf{F}$  be a  $C^\infty([-M, M], \mathbb{R}^d)$  flux. If  $\mathbf{F}$  satisfies definition 3.1 then  $\mathbf{F}$  satisfies definition 3.3 but the converse is wrong.*

*Proof.* If  $\xi \cdot \mathbf{F}'' = 0$  on a open sub-interval  $I$ . Let  $u$  belongs in  $I$ . Hence  $\xi \cdot \mathbf{F}^{(k)}(u) = 0$  for all  $k \geq 2$ . But  $\mathbf{F}$  satisfies definition 3.1. It follows that  $\xi = 0$ .

Conversely take a flux  $\mathbf{F}$  such that  $\mathbf{F}''(u) = \exp(-1/u^2)(1, u, \dots, u^{d-1})$ . Obviously  $\mathbf{F}$  satisfies definition 3.3. But  $\mathbf{F}$  does not satisfies definition 3.1 since  $d_{\mathbf{F}}[0] = +\infty$ .  $\square$

If the flux is an analytic function, the situation is simpler.

**Proposition 3.3** (Analytic nonlinear flux). *If the flux is an analytic function all previous definitions: 1.1, 3.1, 3.2, 3.3 are equivalent.*

*Proof.* Again we use definition 3.1. There are two cases.

- (1) If  $\mathbf{F}$  is nonlinear for definition 3.1. By Theorem 5.1, Propositions 3.1 and 3.2,  $\mathbf{F}$  is nonlinear for other definitions.
- (2) If  $\mathbf{F}$  is not nonlinear for definition 3.1. By Theorem 5.1,  $\mathbf{F}$  does not satisfies definition 1.1.

Let  $u$  be fixed. There exists an hyperplane  $H$  such that all derivatives  $\mathbf{F}^{(k)}(u) \in H$  for all  $k \geq 2$ , i.e. there exists  $\xi \neq 0$  such that  $\xi \cdot \mathbf{F}^{(k)}(u) = 0$  for all  $k \geq 2$ . Using the power series expansion of  $\mathbf{F}''$  near  $u$  we see that  $\mathbf{F}''$  stays in  $H$  near  $u$ . And by the unique analytic extension of  $\mathbf{F}''$ ,  $\mathbf{F}''$  stays always in  $H$ , i.e.  $\xi \cdot \mathbf{F}'' = 0$  everywhere. Thus  $\mathbf{F}$  does not satisfies definition 3.3.

Integrating the relation  $\xi \cdot \mathbf{F}'' = 0$  we have  $\tau + \xi \cdot \mathbf{F}' = 0$  for some constant  $\tau$ . Dividing the relation by  $\sqrt{\tau^2 + |\xi|^2}$  we can assume that  $\tau^2 + |\xi|^2 = 1$ . Hence  $\mathbf{F}$  does not satisfies definition 3.2.

We incidentally check that definition 3.2 implies definition 3.3.  $\square$

#### 4. SOBOLEV ESTIMATES

In this section, uniform and optimal Sobolev exponents of the family of highly oscillating solutions from Theorem 2.1 are investigated.

**Theorem 4.1. [Sobolev exponent for highly oscillating solutions]**

*Let  $u_\varepsilon$  be the  $C^1([0, T_0] \times \mathbb{R}^d)$  oscillating solutions given in Theorem 2.1. For all  $1 \leq p < +\infty$ , the family  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is bounded in*

$$C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})) \cap W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R}) \quad \text{with } s = \frac{1}{\gamma}.$$

*The Sobolev exponent  $s = \frac{1}{\gamma}$  is **optimal** if  $U_0$  is a non constant function.*

*That is to say that for all  $s > 1/\gamma$  the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is unbounded in  $C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}))$  and in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R})$ .*

It is proved below that  $u_\varepsilon$  has order of  $\varepsilon^{1-s\gamma}$  in  $W_{loc}^{s,p}$  for any  $s \in [0, 1[$ . The case  $p = 1$  is the most important, since  $L^1$  norm plays an important role for conservation laws. These estimates are indeed the estimate of the initial data propagated by the semi-group  $\mathcal{S}_t$ , see [34] for  $p = 1$  and also [37] for  $TV(|u_\varepsilon - \underline{u}|^s)$ . A key point is there is no improvement of the Sobolev exponent of the family of initial data.

The basic idea of the proof is that the sequence of exact solutions  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  and the sequence of approximate oscillating solution given by  $\underline{u} + \varepsilon U\left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma}\right)$  have similar bounds in Sobolev spaces.

Following semi-norms parametrized by  $Q = Q_d(\mathbf{x}_0, A) = \mathbf{x}_0 + ] - A, A[^d$ , where  $A > 0$ ,  $\mathbf{x}_0 \in \mathbb{R}^d$ , are used to estimate fractional derivatives in  $W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})$ , see [1].

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))}^p = \int_{Q_d(\mathbf{x}_0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|V(\mathbf{x}) - V(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+sp}} d\mathbf{x} d\mathbf{y}.$$

Following classical definitions are used in this section.

**Definition 4.1.** [ Estimates in  $W_{loc}^{s,p}(\mathbb{R}^d)$  ]

(i)  $u$  is said to be bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0,$$

$$\|u\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} = \|u\|_{L^p(Q_d(\mathbf{x}_0, A))} + |u|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C.$$

(ii)  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is said to be bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0, \forall \varepsilon \in ]0, 1], \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C.$$

(iii) Let  $\beta \geq 0$ ,  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  has order of  $\varepsilon^{-\beta}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ , denoted by

$$u_\varepsilon \simeq \varepsilon^{-\beta},$$

$$\text{if } \forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 1, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \in ]0, \varepsilon_0],$$

$$C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C \varepsilon^{-\beta}.$$

As usual if  $u$  is bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  then for any cube  $Q$ ,  $u$  belongs to  $W^{s,p}(Q)$ . In the same way if  $u_\varepsilon \simeq \varepsilon^{-\beta}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$  then for any cube  $Q$  there exists a constant  $C \geq 1$  and  $\varepsilon_0 \in ]0, 1]$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ ,  $C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q)} \leq C \varepsilon^{-\beta}$ .

Since solutions of (1.1) are bounded in  $L^\infty$ , the key point is to focus on fractional derivatives. For convenience  $|\mathbf{x}| = |x_1| + \dots + |x_d|$  and semi-norms

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))}^p = \int_{Q_d(0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})|^p}{|\mathbf{h}|^{d+sp}} d\mathbf{x} d\mathbf{h},$$

are also used. Notice that

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A/2))} \leq |V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq |V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, 2A))}.$$



Furthermore  $|V|_{\dot{W}^{s,p}(Q_1(\mathbf{x}_0,A))} = |V|_{\dot{W}^{s,p}(Q_1(\mathbf{x}_0,A))}$  when  $V$  is periodic with period  $A$  (or  $A/2$ ). Thus, these semi-norms can be useful to estimate bounds in  $W_{loc}^{s,1}$ .

The proof of Theorem 4.1 needs some lemmas, so this proof is postponed at the end of section 4.

**Lemma 4.1. [Highly oscillating periodic function on  $\mathbb{R}$ ]**

Let  $v$  belongs to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ , and for all  $0 < \varepsilon \leq 1$ ,

$$V_\varepsilon(\theta) = v(\varepsilon^{-\gamma}\theta).$$

If  $v(\cdot)$  is a non constant periodic function then

$$V_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}).$$

The same remains true if  $V_\varepsilon(\theta) = v_\varepsilon(\varepsilon^{-\gamma}\theta)$ ,  $v_\varepsilon$  is one periodic,  $v_\varepsilon \rightarrow v$  in  $C^1$ .

Notice that the magnitude of  $V_\varepsilon$  in  $W_{loc}^{s,p}$  is independent of  $p$ .

Notice also that if  $v_\varepsilon \rightarrow v$  in  $W_{loc}^{s,p}$  then  $v_\varepsilon(\varepsilon^{-\gamma}\theta) \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}(\mathbb{R})$ .

**Proof :** In all the sequel one sets  $x_0 = 0$  in definition 4.1 since computations are invariant under translation.

First the  $L_{loc}^1$  norm is easily bounded, see [4]. Let  $A > 1/2$ ,  $X = \varepsilon^{-\gamma}x$ ,  $B_\varepsilon = \varepsilon^{-\gamma}A$ ,  $N_\varepsilon$  the integer such that  $N_\varepsilon \leq 2B_\varepsilon < N_\varepsilon + 1$  so  $2A - 1 \leq 2A - \varepsilon^\gamma \leq \varepsilon^\gamma N_\varepsilon \leq 2A$ .

$$\begin{aligned} \|V_\varepsilon\|_{L^p([-A,A])}^p &= \int_{-A}^A |V_\varepsilon(x)|^p dx = \varepsilon^{-\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \\ &= \varepsilon^{-\gamma} \left( \sum_{k=1}^{N_\varepsilon} \int_{-B_\varepsilon+k-1}^{-B_\varepsilon+k} |v(X)|^p dX + \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \right) \\ &= \varepsilon^{-\gamma} N_\varepsilon \int_0^1 |v(X)|^p dX + \varepsilon^{-\gamma} \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX. \end{aligned}$$

Finally one has

$$(4.1) \quad \|V_\varepsilon\|_{L^p([-A,A])} \leq (2A+1)^{1/p} \|v\|_{L^p([0,1])},$$

$$(4.2) \quad \|V_\varepsilon\|_{L^p([-A,A])} \geq (2A-1)^{1/p} \|v\|_{L^p([0,1])}$$

$$\|V_\varepsilon\|_{L^p([-A,A])} \sim (2A)^{1/p} \|v\|_{L^p([0,1])} \quad \text{when } \varepsilon \rightarrow 0.$$

$|V_\varepsilon|_{\dot{W}^{s,p}([-A,A])}$  is computed with the same notations and  $H = \varepsilon^{-\gamma}h$ ,

$$|V_\varepsilon|_{\dot{W}^{s,p}([-A,A])}^p = \varepsilon^{(1-sp)\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \int_{-B_\varepsilon}^{B_\varepsilon} \frac{|v(X+H) - v(X)|^p}{|H|^{1+sp}} dX dH.$$

Let  $Var(\cdot)$  be the one periodic function bounded in  $L^\infty$  by  $2^{p-1} \|v\|_{L^p([0,1])}^p$ ,

$$Var(H) = \int_0^1 |v(X+H) - v(X)|^p dX.$$

Notice that  $Var \equiv 0$  if and only if  $v$  is constant a.e.

Using one periodicity of  $v$  with respect to  $X$  yields as in (4.1)

$$\begin{aligned} |V_\varepsilon|_{\dot{\tilde{W}}^{s,p}([-A,A])}^p &= \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \left( \varepsilon^\gamma \int_{-B_\varepsilon}^{B_\varepsilon} |v(X+H) - v(X)|^p dX \right) \frac{dH}{|H|^{1+sp}}, \\ &\leq \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} ((2A+1)Var(H)) \frac{dH}{|H|^{1+sp}} \leq \varepsilon^{-sp\gamma} (2A+1) D_\infty^p, \\ D_B^p = (D_B)^p &= \int_{-B}^{+B} Var(H) \frac{dH}{|H|^{1+sp}}. \end{aligned}$$

Notice that  $D_B$  is a true constant related to the fractional derivative of  $v$  since for  $B = 1/2$ ,  $D_{1/2} = |v|_{\dot{\tilde{W}}^{s,p}([-1/2,1/2])}$  and for  $B = \infty$  the integral converges.

The lower bound is obtained in the same way and finally one has

$$\begin{aligned} |V_\varepsilon|_{\dot{\tilde{W}}^{s,p}([-A,A])} &\leq \varepsilon^{-s\gamma} (2A+1)^{1/p} D_\infty, \\ |V_\varepsilon|_{\dot{\tilde{W}}^{s,p}([-A,A])} &\geq \varepsilon^{-s\gamma} (2A-1)^{1/p} D_1, \\ |V_\varepsilon|_{\dot{\tilde{W}}^{s,p}([-A,A])} &\sim \varepsilon^{-s\gamma} (2A)^{1/p} D_\infty. \end{aligned}$$

Notice also that  $D_B > 0$  for  $B > 1/2$ . Otherwise  $D_B = 0$  implies  $Var \equiv 0$  a.e. which implies  $v$  is a constant function on  $[x_0 - 2B, x_0 + 2B]$  and on  $\mathbb{R}$  by periodicity.

A key point in this paper is the lower bound to get sharp estimates. Since  $D_B$  is non decreasing with respect to  $B$ , the previous lower bound of  $V_\varepsilon$  in  $W^{s,p}$  implies the following lower bound

$$|V_\varepsilon|_{\dot{\tilde{W}}^{s,p}([-A,A])} \geq \varepsilon^{-s\gamma} (2A-1)^{1/p} |v|_{\dot{\tilde{W}}^{s,p}([-1/2,1/2])}.$$

With more work, similar estimates are still valid for  $|V_\varepsilon|_{\dot{W}^{s,1}([-A,A])}$ , see lemmas in [4] about triangular changes of variables for oscillatory integrals. But it is enough for our purpose.

Same computations when  $v$  is replaced by  $v_\varepsilon$  are still valid which conclude the proof.  $\square$

The following Lemma is useful to check that  $W^{s,1}$  semi-norms of  $V : \mathbb{R} \mapsto \mathbb{R}$  and  $W : \mathbb{R}^d \mapsto \mathbb{R}$  where  $W(x_1, \dots, x_d) = V(x_1)$  have the same order.

**Lemma 4.2.** *Let  $d \geq 2$ ,  $s > 0$ ,  $A > 0$ ,  $h_1 > 0$ ,*

$$(4.3) \quad \mu_{d,s}(h_1) = \int_0^A \cdots \int_0^A \frac{h_1^{1+s}}{(h_1 + h_2 + \cdots + h_d)^{d+s}} dh_2 \cdots dh_d,$$

*then there exists two positive numbers  $c_{d,s}, C_{d,s}$  such that*

$$(4.4) \quad 0 < c_{d,s} \leq \mu_{d,s}(h_1) \leq C_{d,s} < +\infty, \quad \forall A > 0, \quad \forall h_1 \in ]0, A].$$

*Inequalities (4.4) are still valid for  $h_1 \in ]0, 2A]$  with other constants:*

$$0 < \tilde{c}_{d,s} \leq \mu_{d,s}(h_1) \leq \tilde{C}_{d,s} < +\infty.$$

The constants  $c_{d,s}$  and  $C_{d,s}$  are independent of  $A > 0$ . Notice that there is a singularity for  $\mu_{d,s}$  at  $h_1 = 0$  since  $\mu_{d,s}(0) = 0$  and  $\mu_{d,s} > 0$  on  $]0, A]$ .

**Proof :** It seems that  $\mu_{d,s}(h_1)$  is depending on  $A$ ,  $\mu_{d,s}(h_1) = \mu_{d,s}^A(h_1)$ . But by homogeneity the problem is reduced to the case  $A = 1$  with the change of variable  $h_i = t_i A$ ,  $0 < t_i < A$ .

Now  $\mu_{d,s}(t_1) = \mu_{d,s}^1(t_1) = \mu_{d,s}^A(h_1)$  is computed explicitly .

Let  $\mu_{d,s}(h_1, B)$  be  $\int_0^1 \cdots \int_0^1 \frac{t_1^{1+s}}{(t_1 + t_2 + \cdots + t_d + B)^{d+s}} dt_2 \cdots dt_d$  for  $d > 1$ ,  $B \geq 0$ . Notice that  $\mu_{d,s}(t_1) = \mu_{d,s}(t_1, 0)$ .

For  $d = 1$ , set  $\mu_1(t_1, B) = \frac{t_1^{1+s}}{(t_1 + B)^{1+s}}$ ,  $\mu_1(t_1) = \mu_1(t_1, 0) = 1$ . The identity

$$\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} = (j + s)^{-1} (B^{-(j+s)} - (B + 1)^{-(j+s)}),$$

yields  $(j + s)\mu_{1+j}(t_1, B) = \mu_j(t_1, B) - \mu_j(t_1, B + 1)$ , and proceeding by induction

with the notations  $\gamma_{d,s} = \frac{1}{(d - 1 + s) \cdots (1 + s)}$ ,  $C_n^k = \frac{n!}{k!(n - k)!}$ ,

$$\mu_{d,s}(t_1, B) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \mu_1(t_1, B + k).$$

Hence, for  $B = 0$ ,

$$\mu_{d,s}(t_1) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \frac{t_1^{1+s}}{(t_1 + k)^{1+s}},$$

which gives  $\mu_{d,s}(0+) = \gamma_{d,s} > 0$ . Now,  $\mu_{d,s}(\cdot)$  belongs in  $C^0([0, 1], \mathbb{R}^+)$ ,  $\mu_{d,s}(\cdot)$  is positive on  $]0, 1]$  with a positive right limit at  $t_1 = 0$ , thus positive constants stated in the lemma exist.

For instance when  $d = 2$ ,  $C_{2,s}$  is  $\gamma_{2,s} = 1/(1 + s)$  and  $c_{2,s} = (1 - 2^{-(1+s)})/(1 + s)$ , since  $\mu_2$  is decreasing.

Notice that  $C_{d,s} \leq \gamma_{d,s}$  for all  $d \geq 2$ . It suffices to proceed by induction with this inequality  $\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} \leq (j + s)^{-1} B^{-(j+s)}$ . But  $\gamma_{d,s}$  is the right limit of  $\mu_{d,s}$  at  $t_1 = 0$ . Then  $C_{d,s} = \gamma_{d,s}$  which concludes the proof for  $h_1 \in ]0, A]$ . On  $]0, 2A]$  it suffices to take  $0 < \tilde{c}_{d,s} = \inf_{]0, 2]} \mu_{d,s}$  and  $+\infty > \tilde{C}_{d,s} = \sup_{]0, 2]} \mu_{d,s}$ .  $\square$

Our examples of oscillating solutions reduce to the following key example.

**Lemma 4.3. [Example of highly periodic oscillations on  $[0, T] \times \mathbb{R}$ ]**

Let  $T, \gamma$  be positive. If  $U$  belongs to  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  and non constant then  $V_\varepsilon(t, x) = U(t, \varepsilon^{-\gamma} x) \simeq \varepsilon^{-s\gamma}$  in  $C^0([0, T], W_{loc}^{s,p}(\mathbb{R})) \cap W_{loc}^{s,p}([0, T] \times \mathbb{R})$ .

**Proof :** First the fractional derivative w.r.t;  $\mathbf{x}$  is estimate. Second the whole fractional derivative in  $(t, \mathbf{x})$  is obtained.

Bounds in  $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$ : There exists  $t_0 \in ]0, T[$  such that  $\theta \mapsto U(t_0, \theta)$

is non constant since  $U$  is non constant and continuous on  $[0, T] \times \mathbb{R}/\mathbb{Z}$ . For  $t_0$  fixed the sharp estimate is a consequence of lemma 4.1. For another  $t$ , we get the same order  $\varepsilon^{-s\gamma}$  or  $\varepsilon^0 = 1$ . Finally, constants involving in this estimate depend continuously of  $t$  so the bound in  $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$  is obtained. Since  $U \in C^1$  this bound is automatically in  $C^0([0, T], W_{loc}^{s,p}(\mathbb{R}))$ .

Bounds in  $W_{loc}^{s,p}([0, T] \times \mathbb{R})$ : The only problem is to estimate for  $x_0 \in \mathbb{R}$ ,  $t_0 \in ]0, T[$  and  $\min(t_0, T - t_0) > A > 0$ , the quadruple integral

$$\begin{aligned} IA &= |V_\varepsilon|_{\widetilde{W_{loc}^{s,p}([t_0-A, t_0+A] \times [x_0-A, x_0+A])}}^p \\ &= \int_{t_0-A}^{t_0+A} \int_{x_0-A}^{x_0+A} \int_{-A}^A \int_{-A}^A \frac{|U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt. \end{aligned}$$

*Upper bound of  $IA$ :*

Let  $Num$  be the numerator of the previous fraction,  $Q$  be  $U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)$ ,  $R$  be  $U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}(x+\xi))$  then  $Num = |Q+R|^p \leq 2^{p-1}(|Q|^p + |R|^p)$ .

Previous inequality implies  $IA \leq 2^{p-1}(IQ + IR)$  with obvious notations.

$$\begin{aligned} IQ &= \int \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &= \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{|\xi|^{1+sp}} \mu_{2,sp}(\xi) d\xi dx dt, \end{aligned}$$

with  $\mu_{2,sp}(\cdot)$  is defined in lemma 4.2. Using lemmas 4.1, 4.2 yields  $IQ \simeq \varepsilon^{-s\gamma}$ .  $IR$  is easily bounded since

$$\begin{aligned} IR &= \int \int \int \int \frac{|U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}(x+\xi))|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &\leq \int \int \int \int \frac{\|\partial_t U\|_{L^\infty}^p |\tau|^p}{(|\tau| + |\xi|)^{2+sp}} d\tau d\xi dx dt \\ &\leq 8A^2 \|\partial_t U\|_{L^\infty}^p \int_0^A |\tau|^{p(1-s)-1} \mu_{2,sp}(\tau) d\tau, \end{aligned}$$

which is finite, so  $IA \leq IQ + IR = \mathcal{O}(\varepsilon^{-s\gamma})$ .

*Lower bound of  $IA$ :*

We again use notations  $Q$ ,  $R$ ,  $Num$ . By a convex inequality, the numerator satisfies:  $Num = |Q+R|^p \geq |Q|^p - p|Q|^{p-1}|R| = |Q|^p - \mathcal{O}(|\tau||Q|^{p-1})$  since  $R = \mathcal{O}(\tau)$ . Then  $IA \geq IQ - \mathcal{O}(IS)$ , where  $IQ$  has order of  $\varepsilon^{-s\gamma}$ . The term  $IS$  has a lower order as we can find after the following similar computations as in the proof of lemma 4.1. Notice first that for all positive numbers  $A$ ,  $b$ ,  $\int_0^A \frac{\tau}{(\tau+b)^{2+\beta}} d\tau \leq \frac{C}{2b^\beta}$  where  $\beta > 0$  and  $C = 2 \int_0^{+\infty} \frac{\tau}{(\tau+1)^{2+\beta}} d\tau < +\infty$ . Now integrating on  $\tau$  yields

$$IS = \int \int \int \int \frac{|\tau||Q|^{p-1}}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt \leq C \int \int \int \frac{|Q|^{p-1}}{|\xi|^{sp}} d\xi dx dt.$$

We set  $\eta = \varepsilon^\gamma$ ,  $X = x/\eta$ ,  $\Xi = \xi/\eta$ , the previous inequality becomes

$$IS \leq C\eta^{2-sp} \int_0^T \int_{A/\eta}^{A/\eta} \int_{A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi dX dt.$$

We now focus on the integral with respect to  $\Xi$  and remark that  $Q = O(1)$  and also  $Q = O(\Xi)$  since  $U$  is  $C^1$ .

$$\begin{aligned} \int_{A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi &= \int_{|\Xi| < 1} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} \frac{O(|\Xi|^{p-1})}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{O(1)}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} O(|\Xi|^{p(1-s)-1}) d\Xi + O(g(\eta)) = O(1) + O(g(\eta)), \end{aligned}$$

where  $g(\eta) = \eta^{sp-1}$  if  $sp \neq 1$ , else  $g(\eta) = \ln(\eta)$ .

To bound  $IS$ , we notice that the integral  $\eta \int_{A/\eta}^{A/\eta} dX$  is bounded by periodicity and we can take the supremum with respect  $t$  on  $[0, T]$ . So  $IS = O(1)$  if  $sp \neq 1$  else  $IS = O(\ln(\eta))$  which is enough to have a lower order than  $IQ$ .

In conclusion, the bounds of  $IA$  yield  $V_\varepsilon \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R})$ .  $\square$

**Lemma 4.4. [Example of highly periodic oscillations on  $\mathbb{R}^d$ ]**

Let  $v$  belongs to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ ,  $\psi(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + b$  where  $\mathbf{v} \in \mathbb{R}^d$ ,  $b \in \mathbb{R}$  and  $0 < \varepsilon < 1$ ,

$$W_\varepsilon(\mathbf{x}) = v(\varepsilon^{-\gamma}\psi(\mathbf{x})).$$

If  $v$  is a non constant periodic function and  $\nabla\psi \neq 0$ , then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}).$$

Furthermore if functions  $v_\varepsilon$  are one periodic function for all  $\varepsilon \in ]0, 1]$ , which converge towards  $v$  in  $C^1$  and  $W_\varepsilon(\mathbf{x}) = v_\varepsilon(\varepsilon^{-\gamma}\psi(\mathbf{x}))$  the conclusion holds true.

**Proof :** The expounded proof has three steps. Let  $M$  be a  $d \times d$  non-degenerate matrix and  $B \in \mathbb{R}^d$  such that  $X_1 = \psi(\mathbf{x})$  where  $X = (X_1, \dots, X_d) = M\mathbf{x} + B$ .  $M$  exists since  $\mathbf{v} \neq 0$ .

Step 1: If  $W(\mathbf{x}) = U(M\mathbf{x} + b)$  since  $\det M \neq 0$ ,  $W$  and  $U$  are the same order in  $W_{loc}^{s,p}$ . More precisely, fix following positive constants  $m_0 = |\det M| > 0$ ,  $m_1 = |||M||| = \sup\{|M\mathbf{x}|, |\mathbf{x}| = 1\} > 0$ ,  $m_{-1} = |||M^{-1}||| > 0$ ,  $0 < r < R$  such that  $Q_d(X_0, r) \subset MQ_d(\mathbf{x}_0, 1) \subset Q_d(X_0, R)$  where  $X_0 = M\mathbf{x}_0 + B$ . Performing the change of variables  $X = M\mathbf{x} + B$ ,  $Y = M\mathbf{y} + B$  yields for any  $\mathbf{x}_0 \in \mathbb{R}^d$  and any  $A > 0$

$$\begin{aligned} m_0^{-1} \|U\|_{L^p(Q_d(X_0, rA))} &\leq \|W\|_{L^p(Q_d(\mathbf{x}_0, A))} \leq m_0^{-1} \|U\|_{L^p(Q_d(X_0, RA))}, \\ \frac{m_0^{-2}}{m_{-1}^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, rA))} &\leq |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq \frac{m_0^{-2}}{m_1^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, RA))}. \end{aligned}$$

Step 2: assume  $\psi(\mathbf{x}) = x_1$ , i.e.  $W(\mathbf{x}) = W(x_1, \dots, x_d) = w(x_1)$ ,  $x_0 = \psi(\mathbf{x}_0)$ , then  $W$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$  and  $w$  in  $W_{loc}^{s,p}(\mathbb{R})$  have the same order. More precisely, elementary computations yield

$$\begin{aligned} \|W\|_{L^1(Q_d(\mathbf{x}_0, A))} &= (2A)^{d-1} \|w\|_{L^1(Q_1(x_0, A))}, \\ |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &\leq (2A)^{d-1} C_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))} \\ &\geq (2A)^{d-1} c_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))}. \end{aligned}$$

The two last inequalities and constants come from lemma 4.2 since

$$\begin{aligned} |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &= \int_{Q_d(0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|w(x_1 + h_1) - w(x_1)|}{|\mathbf{h}|^{d+sp}} d\mathbf{x} d\mathbf{h} \\ &= (2A)^{d-1} \int_{-A}^A \int_{x_0-A}^{x_0+A} \frac{|w(x_1 + h_1) - w(x_1)|}{|h_1|^{1+sp}} \mu_{d,sp}(h_1) dx_1 dh_1. \end{aligned}$$

Step 3: with step 1,  $W_\varepsilon(\mathbf{x}) = V_\varepsilon(\varepsilon^{-\gamma} \psi(\mathbf{x})) \simeq V_\varepsilon(\varepsilon^{-\gamma} \mathbf{x}_1)$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ , with step 2,  $\mathbf{x} \mapsto V_\varepsilon(\varepsilon^{-\gamma} \mathbf{x}_1)$  and  $x_1 \mapsto V_\varepsilon(\varepsilon^{-\gamma} \mathbf{x}_1)$  have the same order in  $W_{loc}^{s,p}(\mathbb{R}^d)$  and  $W_{loc}^{s,p}(\mathbb{R})$ . Finally we have with lemma 4.1  $W_\varepsilon \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ .  $\square$

**Lemma 4.5. [Example of highly periodic oscillations on  $[0, T] \times \mathbb{R}^d$ ]**  
Let  $U$  belongs to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ ,  $\varphi(t, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + b t$  where  $\mathbf{v} \in \mathbb{R}^d$ ,  $b \in \mathbb{R}$  and  $0 < \varepsilon < 1$ ,

$$W_\varepsilon(\mathbf{x}) = U(t, \varepsilon^{-\gamma} \varphi(t, \mathbf{x})).$$

If  $U$  is a non constant function in  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  and  $\mathbf{v} \neq 0_{\mathbb{R}^d}$ , then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \text{ in } W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R}).$$

Furthermore if  $U_\varepsilon$  belongs to  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  for all  $\varepsilon \in ]0, 1]$  converging towards  $U$  in  $C^1$  and  $W_\varepsilon(\mathbf{x}) = U_\varepsilon(t, \varepsilon^{-\gamma} \varphi(t, \mathbf{x}))$  the conclusion holds true.

**Proof :** We proceed as in the previous proofs. First with a linear change of variable  $(t, \mathbf{x}) \mapsto (t, \mathbf{y})$  with  $\mathbf{y}_1 = \varphi(t, \mathbf{x})$ .  $W_\varepsilon$  has the same estimates than  $V_\varepsilon = U(t, \varepsilon^{-\gamma} y_1)$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$ . Notice that the change of variable depends on  $t$  varying in the compact set  $[0, T]$ . So we have uniform estimates of positive constants  $m_0, m_1, m_{-1}$  used in the proof of lemma 4.4. Now, the estimates of  $V_\varepsilon$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$  and in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}, \mathbb{R})$  have the same order since

$$\begin{aligned} &\int_{-A}^A \dots \int_{-A}^A \frac{dh_0 dh_1 \dots dh_d}{(|h_0| + |h_1| + \dots + |h_d|)^{1+d+sp}} \\ &= \int_{-A}^A \dots \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \frac{(|h_0| + |h_1|)^{1+(sp+1)} dh_2 \dots dh_d}{(|h_0| + |h_1| + \dots + |h_d|)^{d+(sp+1)}} \\ &= \int_{-A}^A \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \mu_{2, (sp+1)}(|h_0| + |h_1|) \end{aligned}$$

where  $h_0$  plays the role of time. From the bounds to  $\mu_{2,(sp+1)}(|h_0| + |h_1|)$  on  $]0, 2A]$  see lemma 4.2, we can conclude with lemma 4.3

With a smooth extension of  $U$  on  $[-\delta, T + \delta] \times \mathbb{R}/\mathbb{Z}$ , for a small positive  $\delta$ , we obtain estimates in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$ .  $\square$

**Proof of Theorem 4.1:** Bounds  $L^\infty([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d))$ : Such bounds give bounds in  $C^0([0, T_0], W_{loc}^{s,p})$  since  $u_\varepsilon$  is in  $C^1$ .

For  $t = 0$ , it is only an application of lemma 4.4. The profile  $U(t, \cdot)$  is non constant for each  $t$ , else  $U_0$  must be constant by the method of characteristics. And the estimates are uniform.

Bounds in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$  The semi-norms  $|\cdot|_{\dot{W}^{s,p}(Q_{d+1}(\mathbf{y}_0, A))}$ , where  $\mathbf{y}_0 = (t_0, \mathbf{x}_0)$ , needs some precautions to use on  $[0, T_0] \times \mathbb{R}^d$ .  $\mathbf{y}_0$  must be such that  $0 < t_0 < T_0$  and  $A < \min(t_0, T_0 - t_0)$ . Furthermore, only  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$  smoothness can be estimate. Indeed  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is bounded in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ . To prove this, let us use the following trick. By the methods of characteristics the family of solutions  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  exists on a maximal time interval  $] - \delta, T_1[$ , with  $0 < \delta < T_0 < T_1$ . Notice that solutions exist for negative time since the initial data is smooth. Now estimates in  $W_{loc}^{s,p}([ - \delta, T_1] \times \mathbb{R}^d)$  will be obtained which is sufficient to get smoothness in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ . Now using lemma 4.4 completes the proof.  $\square$

## 5. HIGHLIGHTS ABOUT A LIONS, PERTHAME, TADMOR CONJECTURE

In [34], the authors use a kinetic formulation of conservation law (1.1) to use averaging lemmas. With only initial data uniformly bounded in  $L^\infty$ , they obtain a smoothing effect in  $W_{loc}^{s,1}$  and prove that  $s_{\sup} \geq \frac{\alpha_{\sup}}{2 + \alpha_{\sup}}$ .

They also conjectured the best Sobolev exponent  $s_{\sup} = \alpha_{\sup}$ .

Recently, this result is improved in [39] with a generic assumption on  $\mathbf{a}'$ :  $s_{\sup} \geq \frac{\alpha_{\sup}}{2\alpha_{\sup} + 1}$ .

The following result gives us the conjectured upper bound of  $s_{\sup}$ .

### Theorem 5.1. [Bound of the maximal smoothing effect]

Let  $\mathbf{F}$  be a nonlinear flux which belongs to  $C^\infty([-M, M], \mathbb{R}^d)$ . Let  $\alpha_{\sup}$  be the sharp measurement of flux non-linearity. Then there exist a constant  $\underline{u} \in [-M, M]$ , a time  $T_0 > 0$ , and a sequence of initial data  $(u_0^\varepsilon)_{0 < \varepsilon < 1}$  such that  $\|u_0^\varepsilon - \underline{u}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$ , and the sequence of entropy solutions  $(u_\varepsilon)_{0 < \varepsilon < 1}$  associated to conservation law (1.1) satisfying:

- for all  $s \leq \alpha_{\sup}$ , the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is uniformly bounded in  $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d) \cap C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$ ,
- for all  $s > \alpha_{\sup}$ , the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is **unbounded** in  $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d)$  and in  $C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$ .

The theorem states a bound for the  $W^{s,1}$ -regularizing effect, precisely (1.7):  $s_{\text{sup}} \leq \alpha_{\text{sup}}$ , which is a part of Lions-Perthame-Tadmor's conjecture (1.6).

For large dimension,  $\alpha_{\text{sup}}$  converges towards 0 since  $\alpha_{\text{sup}} \leq 1/d$  by Theorem 3.1. So combining (1.7) with the lower bound in [39] we have when  $d \rightarrow +\infty$ ,  $\alpha_{\text{sup}} \sim \frac{\alpha_{\text{sup}}}{2\alpha_{\text{sup}} + 1} \leq s_{\text{sup}} \leq \alpha_{\text{sup}}$  and then  $s_{\text{sup}} \sim \alpha_{\text{sup}}$ . In this case, in an asymptotic sense, the conjecture (1.6) becomes true for large dimension.

We also obtain  $s_{\text{sup}} \leq \alpha_{\text{sup}}$  in  $W^{s,p}$  for all  $p \in [1, +\infty[$  thanks to Theorem 4.1. But Lions, Perthame and Tadmor had conjectured a smaller Sobolev exponent in [34], precisely:  $s_{\text{sup}} = \frac{\alpha_{\text{sup}}}{p}$ .

**Proof :** The proof is a direct consequence of previous theorems. By Theorem 3.1, there exists  $\underline{u} \in [-M, M]$  such that  $\alpha = \frac{1}{d_{\mathbf{F}}[\underline{u}]}$ . Let  $U_0$  be a non constant smooth periodic function such that:  $-M \leq \underline{u} + U_0(\theta) \leq M$  for all  $\theta$ .

Let  $\mathbf{v} \in \mathbb{R}^d$  such that  $\mathbf{a}^k(\underline{u}) \cdot \mathbf{v} = 0$  and  $\mathbf{v} \neq 0$  for  $k = 1, \dots, d_{\mathbf{F}}[\underline{u}] - 1$ . Such  $\mathbf{v}$  exists by definition of  $d_{\mathbf{F}}[\underline{u}]$ .

Now, let  $(u_\varepsilon)$  be the family of smooth solutions given by Theorem 2.1. Theorem 4.1 is the desired conclusion.  $\square$

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