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# HIGH FREQUENCY WAVES AND THE MAXIMAL SMOOTHING EFFECT FOR NONLINEAR SCALAR CONSERVATION LAWS

STÉPHANE JUNCA

ABSTRACT. The article first studies the propagation of well prepared high frequency waves with small amplitude  $\varepsilon$  near constant solutions for entropy solutions of multidimensional nonlinear scalar conservation laws. Second, such oscillating solutions are used to highlight a conjecture of Lions, Perthame, Tadmor, ([23]), about the maximal regularizing effect for nonlinear conservation laws. For this purpose, a definition of smooth nonlinear flux is stated and compared to classical definitions. Then it is proved that the uniform smoothness expected by [23] in Sobolev spaces cannot be exceeded for all smooth nonlinear fluxes.

**Key-words:** multidimensional conservation laws, nonlinear smooth flux, geometric optics, Sobolev spaces, smoothing effect.

**Mathematics Subject Classification:**

Primary: 35L65, 35B65; Secondary: 35B10, 35B40, 35C20.

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## 1. INTRODUCTION

This paper deals with super critical geometric optics to highlight the maximal regularizing effect for nonlinear multidimensional scalar conservation laws.

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This effect is studied in Sobolev spaces by P.L. Lions, B. Perthame and E. Tadmor in [23]. They obtain a uniform fractional Sobolev bounds for any ball of  $L^\infty$  initial data under a non linearity condition on the flux quantified by a parameter  $\alpha \in ]0, 1]$ . They also conjectured the better Sobolev exponent  $s = \alpha$  for entropy solutions.

For the first time, the multidimensional case is investigated to bound this maximal smoothing effect. Furthermore, all smooth nonlinear fluxes are considered in this paper. We build sequences of solutions uniformly bounded in  $W_{loc}^{s,1}$  with the conjectured maximal exponent  $s = \alpha$  and with no possible improvement of the Sobolev exponent.

High frequency periodic solutions are used for this purpose. Near a constant state and for  $L^\infty$  data, geometric optics expansions with various frequencies and various phases are validated in the framework of weak entropy solutions and of  $L_{loc}^1$  convergence in [5]. Here, results of [5] are specified in  $C^1$  for a well chosen phase and proved for a particular sequence of smooth solutions (without shocks on a strip). This allows us to prove that, necessarily, the best uniform Sobolev exponent  $s$ , for entropy solutions and for positive time, satisfies  $s \leq \alpha$  for a ball of  $L^\infty$  initial data. Notice that we look for the best *uniform* Sobolev exponent for a set of solutions. The smoothness of any individual solution is not studied in this paper. This point is discussed later.

An important point to note here is the definition of nonlinear flux. In [23] they give well known Definition 2.4 below and the conjecture about the maximal smoothing effect in Sobolev spaces related to the parameter “ $\alpha$ ” from their definition. The study of periodic solutions leads to another definitions [15, 5]. We obtain Definition 5.1 for smooth flux. It generalizes the definition of [5] and it is typical in the context of stationary phase. For  $C^\infty$  flux, our definition is equivalent to classical Definition 2.4. Furthermore, Definition 5.1 gives a way to compute the parameter “ $\alpha$ ”. Our definition also shows that smoothing effects for scalar conservation laws depend on the space dimension.

The paper is organized as follows. To be more precise, the smoothing effect and the related conjecture in the Sobolev framework are recalled in Section 2. Section 3 contains the two main results, propagations of high frequency waves and the consequence for the maximal smoothing effect. Some comments and other approaches are also discussed. In Section 4 examples of highly oscillating solutions are validated under new orthogonality conditions between the flux derivatives and the phase gradient. In Section 5, these orthogonality conditions lead to a new definition of nonlinear smooth flux. The concept of flux non-linearity is clarified, characterized and compared with other classical definitions. Section 6 is devoted to get optimal Sobolev estimates on oscillating solutions built in Section 4. It is a quite technical part. Finally, Section 7 gives the super critical geometric optics expansion with the highest frequency related to the geometric structure of the nonlinear flux. The existence of this family of high frequency waves implies  $s \leq \alpha$ . But the whole conjecture  $s = \alpha$  is still an open problem.

## 2. THE SMOOTHING EFFECT IN SOBOLEV SPACES

We look for Sobolev bounds for entropy solutions  $u(.,.)$  of

$$(2.1) \quad \partial_t u + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u) = 0,$$

where  $t \in [0, +\infty[$ ,  $\mathbf{x} \in \mathbb{R}^d$ ,  $u : [0, +\infty[ \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$  is a smooth flux function,  $\mathbf{F} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ , and the initial data are only bounded in  $L^\infty(\mathbb{R}^d, \mathbb{R})$ :

$$(2.2) \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

The smoothing effect depends on the class of solutions studied. It is well known that shock occurs even with smooth initial data [10]. Let us recall definitions of solutions for initial-value problem (2.1),(2.2).

**Definition 2.1. [Weak solutions]**

We say that a function  $u \in L^\infty([0, +\infty[ \times \mathbb{R}^d)$  is a weak solution of initial-value problem (2.1),(2.2) provided

$$\int_0^{+\infty} \int_{\mathbb{R}^d} u \partial_t \phi + \mathbf{F}(u) \cdot \nabla_{\mathbf{x}} \phi \, d\mathbf{x} \, dt + \int_{\mathbb{R}^d} u_0(\mathbf{x}) \phi(0, \mathbf{x}) \, d\mathbf{x} = 0$$

for all smooth functions  $\phi$  with compact support.

This definition means that partial differential equation (2.1) is written in the sense of distribution. This class of solutions is too large to provide uniqueness and smoothing effect. We have to restrict our attention to a more physical class of solutions.

**Definition 2.2. [entropy solutions]**

We say that a function  $u \in L^\infty([0, +\infty[ \times \mathbb{R}^d)$  is an entropy solution of initial-value problem (2.1),(2.2) provided the following two conditions hold:

- $\partial_t \varphi(u) + \operatorname{div}_{\mathbf{x}} \mathbf{Q}(u) \leq 0$   
in the sense of distribution, for all convex functions  $\varphi$  where  $\mathbf{Q}' = \varphi' \mathbf{F}'$ ,  $\varphi$  is called an entropy and  $\mathbf{Q}$  its entropy-flux,
- $u \in C^0([0, +\infty[, L^1_{loc}(\mathbb{R}^d, \mathbb{R}))$  and  $\lim_{t \rightarrow 0} u(t, \mathbf{x}) = u_0(\mathbf{x})$  in  $L^1_{loc}(\mathbb{R}^d, \mathbb{R})$ .

It is classical to check that an entropy solution is a weak solution. It suffices to take  $\varphi(u) = \pm u$  and to recover the weak trace  $u_0$  at  $t = 0$  since the second condition means that  $u$  has a strong trace at  $t = 0$ . For this class of solution, Kruzkov proved in 70' the uniqueness (see for instance [10] and the references therein) and Lions, Perthame, Tadmor proved a smoothing effect in Sobolev spaces [23]. Indeed, the smoothing effect and the proof given in [23] is valid in the following larger class of solutions.

**Definition 2.3. [Solutions with bounded entropy production]**

We say that a weak solution of initial-value problem (2.1),(2.2) has a bounded entropy production provided for each entropy  $\varphi$  and entropy-flux  $\mathbf{Q}$ , there exists a signed measure  $\mu$ , locally bounded, such that

$$\partial_t \varphi(u) + \operatorname{div}_{\mathbf{x}} \mathbf{Q}(u) = \mu.$$

The uniqueness is lost for this class, but, since the fundamental paper [23], various results and interests can be found in [11, 13, 16]. Indeed, for entropy solutions, the measure is non positive,  $\mu \leq 0$  and for smooth solutions,  $\mu \equiv 0$ . The smoothing effect for the class of solutions with bounded entropy production is clearly weaker than the smoothing effect for the class of entropy solutions since the first class is larger than the second class.

Let us turn to the notion of nonlinear flux. Set  $\mathbf{a}(u) = \mathbf{F}'(u)$ . Obviously, when  $\mathbf{F}$  is linear:  $\mathbf{a}(u) = \mathbf{a}$  where  $\mathbf{a}$  is a constant vector,  $u(t, \mathbf{x}) = u_0(\mathbf{x} - t\mathbf{a})$  so that there is no smoothing effect. In [23], it was first proved a regularizing effect for *nonlinear* multidimensional flux  $\mathbf{F}$ . The sharp measurement of the non-linearity plays a key role in our study. Let us recall the classical definition for nonlinear flux from [23].

**Definition 2.4.** [ **Nonlinear flux** [23]]

Let  $M$  be a positive constant.  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^d$  is said to be nonlinear on  $[-M, M]$  if there exist  $\alpha > 0$  and  $C = C_\alpha > 0$  such that for all  $\delta > 0$

$$(2.3) \quad \sup_{\tau^2 + |\xi|^2 = 1} |W_\delta(\tau, \xi)| \leq C \delta^\alpha,$$

where  $(\tau, \xi) \in S^d \subset \mathbb{R}^{d+1}$ , i.e.  $\tau^2 + |\xi|^2 = 1$ , and  $|W_\delta(\tau, \xi)|$  is the one dimensional measure of the singular set:

$$W_\delta(\tau, \xi) := \{ |v| \leq M, |\tau + \mathbf{a}(v) \cdot \xi| \leq \delta \} \subset [-M, M] \text{ and } \mathbf{a} = \mathbf{F}'.$$

Indeed,  $W_\delta(\tau, \xi)$  is a neighborhood of the critical value  $v$  for the symbol of the linear operator  $\mathcal{L}[v]$  in the Fourier direction  $(\tau, \xi)$  where  $\mathcal{L}[v] = \partial_t + \mathbf{a}(v) \cdot \nabla_x$ . The symbol in this direction is:  $i(\tau + \mathbf{a}(v) \cdot \xi)$ . This operator is simply related to any smooth solution  $u$  of equation (2.1) by the chain rule formula:

$$\partial_t u + \operatorname{div}_x \mathbf{F}(u) = \partial_t u + \mathbf{a}(u) \cdot \nabla_x u = \mathcal{L}[u]u.$$

$\alpha$  is a degeneracy measurement of the operator  $\mathcal{L}$  parametrized by  $v$ .  $\alpha$  depends only on the flux  $\mathbf{F}$  and the compact set  $[-M, M]$ :  $\alpha = \alpha[\mathbf{F}, M]$ . In the sequel we denote by

$$(2.4) \quad \alpha_{\text{sup}} = \alpha_{\text{sup}}[\mathbf{F}, M], \quad \text{the supremum of all } \alpha \text{ satisfying (2.3).}$$

$\alpha$ , or more precisely  $\alpha_{\text{sup}}$ , is a key parameter to describe the sharp smoothing effect for entropy solutions of nonlinear scalar conservation laws. For smooth flux the parameter  $\alpha$  always belongs to  $[0, 1]$ , for instance:  $\alpha_{\text{sup}} = 0$  for a linear flux,  $\alpha = 1$  for strictly convex flux in dimension one. For the first time  $\alpha_{\text{sup}}$  is characterized below in Section 5. Indeed, for smooth nonlinear flux,  $\frac{1}{\alpha_{\text{sup}}}$  is always an integer greater or equal to the space dimension.

In the sequel we assume that  $M \geq \|u_0\|_\infty$  and the flux  $\mathbf{F}$  is nonlinear on  $[-M, M]$  so that:

$$(2.5) \quad \alpha_{\text{sup}} > 0.$$

When nonlinear condition (2.5) is true, the entropy solution operator associated with the nonlinear conservation law (2.1), (2.2),

$$\begin{aligned} \mathcal{S}_t : L^\infty(\mathbb{R}_x^d, [-M, M]) &\rightarrow L^\infty(\mathbb{R}_x^d, [-M, M]) \\ u_0(\cdot) &\mapsto u(t, \cdot), \end{aligned}$$

has a regularizing effect: mapping  $L^\infty(\mathbb{R}_x^d, [-M, M])$  into  $W_{loc}^{s,1}(\mathbb{R}_x^d, \mathbb{R})$  for all  $t > 0$ .

In [23], they proved this regularizing effect for all  $s < \frac{\alpha}{2 + \alpha}$ .

In [31] the result is improved for all  $s < \frac{\alpha}{1 + 2\alpha}$  under a generic assumption on  $\mathbf{a}' = \mathbf{F}''$ . These results are based on averaging lemmas and are still valid for solutions with bounded entropy production (Definition 2.3).

For entropy solutions, Lions, Perthame and Tadmor conjectured in 1994 a better regularizing effect, ([23], remark 3, p .180, line 14-17). In [23], they proposed an optimal bound  $s_{\text{sup}}$  for Sobolev exponents of entropy solutions:

$$(2.6) \quad s_{\text{sup}} = \alpha_{\text{sup}}.$$

That is to say,  $u(t, \cdot)$  belongs in all  $W_{loc}^{s,1}(\mathbb{R}^d, \mathbb{R})$  for all  $s < s_{\text{sup}} = \alpha_{\text{sup}}$  and for all  $t > 0$ . The shock formation implies  $s < 1$  and  $s_{\text{sup}} \leq 1$  since  $W^{1,1}$  functions do not have shock.

The main goal of the paper is to give an insight of the conjecture (2.6) by bounding the uniform Sobolev smoothing effect  $s_{\text{sup}}$  for the whole set of entropy solutions with initial data bounded by  $M$  in  $L^\infty$ :

$$(2.7) \quad s_{\text{sup}} \leq \alpha_{\text{sup}}.$$

Some previous results highlight the conjecture (2.6) or the inequality (2.7) in the one dimensional case. This is discussed in Subsection 3.2.1 for entropy solutions and for solutions with bounded entropy production. But for the multidimensional case and for all smooth fluxes, our results are new.

### 3. MAIN RESULTS

Our main results are about the propagation of high frequency waves with frequencies higher than usual [14, 29] and the consequence on the uniform maximal smoothing effect for scalar conservation laws.

Weakly nonlinear geometric optics study propagations of sequences of high frequency waves. The classical case [14, 29] deals with uniformly bounded derivatives. That is to say the sequence is bounded in  $W^{1,1}$  or in  $BV$  when shocks occur. We show that sequence of high frequencies waves, not uniformly bounded in  $W^{1,1}$ , can propagate under a stationary phase assumption on the flux, condition (3.5) below. Since the frequencies are higher than usual in geometric optics, we call this case *critical* geometric optics. A natural question is about the maximal frequencies arising in critical geometric optics. This last case is called *super critical* geometric optics.

Critical geometric optics is expounded in Subsection 3.1. The explicit construction of the super critical optics are expounded later and completed in the

last section, Section 7. A main consequence of super critical geometric optics on smoothing effect is stated and commented in Subsection 3.2.

### 3.1. Critical geometric optics.

This subsection deals with highly oscillating initial data near a constant state:

$$(3.1) \quad u_\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon U_0 \left( \frac{\mathbf{v} \cdot \mathbf{x}}{\varepsilon^\gamma} \right),$$

where  $U_0(\theta)$  is a one periodic function w.r.t.  $\theta$ ,  $\gamma > 0$ ,  $\underline{u}$  is a constant ground state,  $\underline{u} \in [-M, M]$ ,  $\mathbf{v} \in \mathbb{R}^d$ . The case  $\gamma = 1$  is the classical geometric optics for scalar conservation laws, ([14]). In this paper we focus on *critical oscillations*, that is to say  $\gamma > 1$ .

The aim of this section is to see when such high frequency are propagated or not propagated. As we will see, it depends on new compatibility conditions between the phase and the flux (3.5).

One of the two following asymptotic expansions (3.2) or (3.3), is expected in  $L_{loc}^1([0, +\infty[ \times \mathbb{R}^d, \mathbb{R})$  for the entropy-solution  $u_\varepsilon$  of conservation law (2.1) with highly oscillating data (3.1) when  $\varepsilon$  goes to 0,

$$(3.2) \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + o(\varepsilon)$$

$$(3.3) \quad \text{or} \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \bar{U}_0 + o(\varepsilon),$$

where the profile  $U(t, \theta)$  satisfies a conservation law with initial data  $U_0(\theta)$ ,  $\bar{U}_0 = \int_0^1 U_0(\theta) d\theta$  and the phase  $\phi$  satisfies the eikonal equation:

$$(3.4) \quad \partial_t \phi + \mathbf{a}(\underline{u}) \cdot \nabla_{\mathbf{x}} \phi = 0, \quad \phi(0, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x}.$$

Thus the phase is simply a linear phase:

$$\phi(t, \mathbf{x}) = \mathbf{v} \cdot (\mathbf{x} - t \mathbf{a}(\underline{u})).$$

The propagation of such oscillating data is obtained under the crucial compatibility condition (3.5) below. On the other hand, when the the compatibility condition (3.5) is nowhere satisfied, the nonlinear semi-group associated with equation (2.1) cancels these high oscillations, see Theorem 4.1. The validity or invalidity of assumption (3.5) is a key point related to the nonlinearity of the flux (Section 5).

#### Theorem 3.1. [Propagation of smooth high oscillations]

Let  $\gamma$  belong to  $]1, +\infty[$  and let  $q$  be the integer such that  $q - 1 < \gamma \leq q$ .

Assume  $\mathbf{F}$  belongs to  $C^{q+3}(\mathbb{R}, \mathbb{R}^d)$ ,  $U_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ ,  $\mathbf{v} \neq (0, \dots, 0)$  and

$$(3.5) \quad \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} = 0, \quad k = 1, \dots, q - 1.$$

Then there exists  $T_0 > 0$  such that, for all  $\varepsilon \in ]0, 1]$ , the solutions of conservation law (2.1) with initial oscillating data (3.1) are smooth on  $[0, T_0] \times \mathbb{R}$  and

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) + \mathcal{O}(\varepsilon^{1+r}) \text{ in } C^1([0, T_0] \times \mathbb{R}^d),$$

where  $0 < r = \begin{cases} 1 & \text{if } \gamma = q, \\ q - \gamma & \text{else,} \end{cases}$

$\phi$  is given by the eikonal equation (3.4) and the smooth profile  $U$  is uniquely determined by the Cauchy problem (3.6):

$$(3.6) \quad \frac{\partial U}{\partial t} + b \frac{\partial U^{q+1}}{\partial \theta} = 0, \quad U(0, \theta) = U_0(\theta),$$

with  $b = \begin{cases} \frac{1}{(q+1)!} (\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}) & \text{if } \gamma = q, \\ 0 & \text{else.} \end{cases}$

We deal with smooth solutions to compute Sobolev bounds later. Indeed, the asymptotic stays valid after shocks formation and for all positive time but in  $L^1_{loc}$  instead of  $L^\infty$  ([5]).

When  $\gamma = 1$ , we do not need assumption (3.5). It is the classic case for geometric optics ([14, 29]).

In dimension  $d \geq 2$ , it is always possible to find a non trivial vector  $\mathbf{v}$  satisfying (3.5). For example for  $\gamma = 2$ , (3.5) is reduced to find  $\mathbf{v} \neq 0$  such that  $\mathbf{a}'(\underline{u}) \cdot \mathbf{v} = 0$ . Thus, such singular solutions always exist in dimension greater than one. But, for genuine nonlinear one dimensional conservation law, there is no such solution. Of course, we assume that  $U_0$  is a non constant function and that  $\mathbf{F}$  is a nonlinear function near  $\underline{u}$ , else the theorem is obvious. Indeed, when  $U_0$  is constant,  $u_\varepsilon$  is also constant. When  $\mathbf{F}$  is linear on  $[\underline{u} - \delta, \underline{u} + \delta]$  for some  $\delta > 0$ , high oscillations propagate for all time without any restriction of the phase and of the frequency size.

In fact, Theorem 3.1 expresses a kind of degeneracy of **multidimensional** scalar conservation laws. This degeneracy (period smaller than the amplitude) appears for quasilinear systems whit some nonlinear degeneracy (see for instance [7]).

Notice that for  $\gamma > 1$ , smooth solutions exist for larger time than it is currently known [10, 22]:  $T_\varepsilon \sim 1/|\nabla_x u_0^\varepsilon| \sim \varepsilon^{\gamma-1}$ . Furthermore, equation (3.6) is nonlinear if and only if  $\gamma \in \mathbb{N}$  and  $\mathbf{a}^q(\underline{u}) \cdot \mathbf{v} \neq 0$ .

### 3.2. Consequence on the uniform maximal smoothing effect.

We construct a sequence of smooth solutions which are exactly uniformly bounded in the Sobolev space conjectured in [23]. The uniform Sobolev estimate of this sequence blows up in all more regular Sobolev spaces. This super critical geometric optics expansion and its optimal Sobolev estimates are the goal of this paper which is completed in Section 7. Let us state the main consequence of this construction in the following theorem.

#### **Theorem 3.2. [Bound of the maximal smoothing effect]**

Let  $\mathbf{F}$  be a nonlinear flux which belongs to  $C^\infty([-M, M], \mathbb{R}^d)$ . Let  $\alpha_{sup}$  be the sharp measurement of the flux non-linearity. Then there exist a constant  $\underline{u} \in [-M, M]$ , a time  $T_0 > 0$ , and a sequence of initial data  $(u_0^\varepsilon)_{0 < \varepsilon < 1}$  such that



$\|u_0^\varepsilon - \underline{u}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$ , and the sequence of entropy solutions  $(u_\varepsilon)_{0 < \varepsilon < 1}$  associated with conservation law (2.1) satisfy the followings:

- for all  $s \leq \alpha_{\text{sup}}$ , the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is uniformly bounded in  $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d) \cap C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$ ,
- for all  $s > \alpha_{\text{sup}}$ , the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is **unbounded** in  $W_{loc}^{s,1}([0, T_0] \times \mathbb{R}^d)$  and in  $C^0([0, T_0], W_{loc}^{s,1}(\mathbb{R}^d))$ .

Theorem 3.2 gives the upper bound (2.7) for the  $W^{s,1}$ -regularizing effect :  $s_{\text{sup}} \leq \alpha_{\text{sup}}$ . Indeed, let us denote  $\mathcal{S}_t$  the semi-group associated with conservation law (2.1) and  $\mathcal{B}^\infty(\underline{u}, \rho) = \{u \in L^\infty(\mathbb{R}^d, \mathbb{R}), \|u - \underline{u}\|_{L^\infty(\mathbb{R}^d, \mathbb{R})} < \rho\}$ . Theorem 3.2 proves that for a well chosen  $\underline{u} \in [-M, M]$ , there exists  $T_0 > 0$ , such that for all  $\rho > 0$  and for all  $0 < t < T_0$ ,  $\mathcal{S}_t(\mathcal{B}^\infty(\underline{u}, \rho))$  is not a bounded subset of  $W_{loc}^{s,1}(\mathbb{R}_x^d)$  for all  $s > \alpha_{\text{sup}}$ . This result yields several remarks.

*Remark 3.1. Optimality for large dimension or small nonlinear degeneracy.*

By Theorem 5.1,  $\alpha_{\text{sup}} \leq \frac{1}{d}$ , so, for larger dimension,  $\alpha_{\text{sup}}$  is smaller. Using a better lower bound of  $s_{\text{sup}}$  from [31] we have:

$$\frac{\alpha_{\text{sup}}}{1 + 2\alpha_{\text{sup}}} \leq s_{\text{sup}} \leq \alpha_{\text{sup}} \leq \frac{1}{d}.$$

Thus for large dimension ( $d \gg 1$ ) or small nonlinear degeneracy ( $\alpha_{\text{sup}} \ll 1$ ) we have asymptotically:

$$s_{\text{sup}} \sim \alpha_{\text{sup}}.$$

*Remark 3.2. In  $W^{s,p}$ ,  $1 < p < +\infty$ , our geometric optics expansion shows that  $s_{\text{sup}}^p \leq \alpha_{\text{sup}}$  by Theorem 6.1, where  $s_{\text{sup}}^p$  denotes the maximal uniform smoothing effect in  $W^{s,p}$ . In other words our example is not related to the parameter  $p$ . Indeed, our geometric optics expansion is bounded in  $W^{s,\infty}$ . Other examples show the importance of the parameter  $p$  in Subsection 3.2.1.*

Sobolev spaces are not sufficient to describe all the properties of the solutions. Some comments are given in Subsection 3.2.1 for other approaches.

### 3.2.1. Other approaches for the smoothing effect.

The maximal Sobolev exponent is not sufficient to characterize the smoothing effect. Other relevant ways are presented by one-sided Oleinik condition,  $BV$ , generalized characteristics, other oscillating solutions,  $BV^s$ , trace properties. First, we comment other approaches on the one dimensional case, where the optimal smoothing effect is not yet completely understood. Second, we briefly discuss the multidimensional case where there are only few results.

#### One dimensional case:

- $\alpha = 1$ ,  $s = 1$ , entropy solutions.

In one dimension ( $d=1$ ) and for uniformly convex flux, it is well known from Lax and Oleinik that the entropy solution becomes  $BV$ ,

([24, 22]). Conjecture (2.6) is true in this case since for all  $t > 0$ ,  $u(t, \cdot)$  belongs to  $W_{loc}^{s,1}$  for all  $s < 1$ .

- $\alpha = 1$ ,  $s = 1/3$ : for solutions with bounded entropy production.  
For Burgers flux,  $F(u) = u^2$ , De Lellis and Westdickenberg built piecewise constant solutions with bounded entropy production to show that  $s \leq 1/3$  [13]. Recently, for uniform convex flux ( $\alpha = 1$ ), the optimal Sobolev exponent  $s = 1/3$  is reached in [16].
- $0 < \alpha < 1$ ,  $s = \alpha$ : entropy solutions.  
For power law flux,  $F(u) = |u|^{1+p}$ ,  $p \geq 1$ , De Lellis and Westdickenberg also built piecewise smooth entropy solutions and proved that  $s \leq \alpha$  ([13], Proposition 3.4 p. 1085). For all nonlinear smooth fluxes, new continuous examples are also given in [4]. Both examples are only justified for a bounded time interval before waves interact.  
For nonlinear degenerate convex fluxes the regularity  $s_{\text{sup}} = \alpha_{\text{sup}}$  is reached in  $W^{s,1}$  ([19]) and also  $W^{s,1/s}$  ([3]).  
For this purpose, fractional  $BV$  spaces, which are called  $BV^s$ , are introduced in [3].  $BV^s$  functions have a structure similar to the one of  $BV$  maps, for all  $s \in ]0, 1]$ .  $BV^s$  spaces seem to be the natural spaces to capture the optimal regularizing effect for one dimensional scalar conservation laws.
- One-sided Oleinik condition and its generalizations:  
In the 50', Oleinik ([24]) obtained her one-sided Lipschitz condition which ensures the uniqueness and the  $BV$  regularity of the entropy solution. This is the first basis and the proof of  $s = \alpha$  for one dimensional uniformly convex flux ( $\alpha = 1$ ). Dafermos ([9, 10]), with his generalized characteristics, handled convex and some degenerate convex fluxes. Hoff extended this one-sided condition in several space variables ([17]) but restricted to vectorial fluxes which are scalar convex fluxes after a change of space variables. A generalized Oleinik condition, only validated in the one dimensional case, is the key assumption to prove the best  $W^{s,1}$  smoothing effect in [19] ( $0 < s = \alpha < 1$ ). The maximal  $W^{s,p}$  smoothing effect is proved in [3] with a one-sided Hölder condition and  $BV^s$  spaces. For a recent generalization of Oleinik condition for a flux with one inflection point, we refer the reader to [20]. To conclude, the one-sided Oleinik condition is essentially specific to the one dimensional case with some convexity on the flux.

#### Multidimensional case:

In the 90', the kinetic formulation of conservation laws ([23]) gave another approach for other authors. This approach is also valid for solutions with bounded entropy production. In this framework, some suitable averaging of nonlinear expressions of the solution are  $BV$  ([6]). Some trace properties were

first obtained in [34, 11, 12, 8]. These structure of a  $BV$  function for solutions, without being  $BV$ , cannot be given by Sobolev regularity. Neither  $BV$  nor  $W^{s,p}$  is the perfect space to explain properties of solutions with only  $L^\infty$  initial data.

#### 4. HIGH FREQUENCY WAVES WITH SMALL AMPLITUDE

In this section we prove Theorem 3.1 about the propagation of oscillations with not uniformly bounded derivatives, that is to say space or time derivatives unbounded with respect to the parameter  $\varepsilon$ . We also show the optimality of the assumption (3.5) in Theorem 4.1.

**Proof of Theorem 3.1:** First one performs a WKB computations with following ansatz:

$$(4.1) \quad u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U_\varepsilon \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right).$$

Notice that we use the exact profile  $U_\varepsilon$  for the proof as in [21]. It is a method to sharply control the difference between the exact solution and the geometric optics expansion:  $U_\varepsilon$  and  $U$ .

The Taylor expansion of the flux and the remainder are:

$$\begin{aligned} \mathbf{F}(u_\varepsilon) &= \sum_{k=0}^{q+1} \varepsilon^k \frac{U_\varepsilon^k}{k!} \mathbf{F}^{(k)}(\underline{u}) + \varepsilon^{q+2} \mathbf{G}_q^\varepsilon(U_\varepsilon), \\ \mathbf{G}_q^\varepsilon(U) &= U^{q+2} \int_0^1 \frac{(1-s)^{q+1}}{(q+1)!} \mathbf{F}^{(q+2)}(\underline{u} + s\varepsilon U) ds, \\ g_q^\varepsilon(U) &= \mathbf{v} \cdot \mathbf{G}_q^\varepsilon(U). \end{aligned}$$

We now compute the partial derivatives with respect to time and space variables:

$$\begin{aligned} \partial_t U_\varepsilon \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right) &= \partial_t U_\varepsilon - \varepsilon^{-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon \\ \operatorname{div}_{\mathbf{x}} \mathbf{F}(u_\varepsilon) &= \sum_{k=0}^q \varepsilon^{k+1-\gamma} \frac{\partial_\theta U_\varepsilon^{k+1}}{(k+1)!} \mathbf{a}^{(k)}(\underline{u}) \cdot \mathbf{v} + \varepsilon^{q+2} \operatorname{div}_{\mathbf{x}} \mathbf{G}_q^\varepsilon(U_\varepsilon) \\ &= \varepsilon^{1-\gamma} (\mathbf{a}(\underline{u}) \cdot \mathbf{v}) \partial_\theta U_\varepsilon + \varepsilon^{q+1-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{q+2-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon), \end{aligned}$$

where  $c_q = \frac{\mathbf{a}^{(q)}(\underline{u}) \cdot \mathbf{v}}{(q+1)!}$ . Then simplification yields

$$(4.2) \quad \partial_t u_\varepsilon + \operatorname{div}_{\mathbf{x}} \mathbf{F}(u_\varepsilon) = \varepsilon \left( \partial_t U_\varepsilon + \varepsilon^{q-\gamma} c_q \partial_\theta U_\varepsilon^{q+1} + \varepsilon^{1+q-\gamma} \partial_\theta g_q^\varepsilon(U_\varepsilon) \right).$$

It suffices to take  $U_\varepsilon$  to be the solution of the one dimensional scalar conservation laws with  $\psi_\varepsilon(U) = \varepsilon^{q-\gamma} c_q U^{q+1} + \varepsilon^{1+q-\gamma} g_q^\varepsilon(U)$

$$(4.3) \quad \partial_t U_\varepsilon + \partial_\theta \psi_\varepsilon(U_\varepsilon) = 0, \quad U_\varepsilon(0, \theta) = U_0(\theta).$$

Notice that  $\psi_\varepsilon = O(1) \in C_{loc}^2$ . For  $\gamma < q$ ,  $\psi_\varepsilon$  is even smaller:  $\psi_\varepsilon = O(\varepsilon^r) \in C_{loc}^2$ . That is enough to prove the existence of a sequence of smooth oscillating

solutions on the same strip.

Uniform life span for smooth solutions  $(U_\varepsilon)_{0 < \varepsilon \leq 1}$ :

We use the method of characteristics with  $\psi'_\varepsilon(U) = \frac{d}{dU}\psi_\varepsilon(U)$ :

$$\frac{d}{dt}\Theta(t, \theta) = \psi'_\varepsilon(U_\varepsilon(t, \Theta(t, \theta))), \quad \Theta(0, \theta) = \theta.$$

Since  $U_\varepsilon$  is constant along the characteristics,  $\Theta(t, \theta) = \theta + t\psi'_\varepsilon(U_0(\theta))$ . As long as the map  $\theta \rightarrow \Theta(t, \theta)$  is not decreasing no shock occurs.

$$\frac{\partial}{\partial \theta}\Theta(t, \theta) = 1 + t\psi''_\varepsilon(U_0(\theta))\frac{d}{d\theta}U_0(\theta)$$

The first shock appears at the time  $T_\varepsilon$  when the right hand side vanishes. Let  $m_0 = \sup_{[0,1]}|U_0| > 0$ ,  $d_0 = \sup_{[0,1]}\left|\frac{d}{d\theta}U_0\right|$ ,  $m = \sup_{0 < \varepsilon \leq 1}\sup_{|U - u| \leq m_0}|\psi''_\varepsilon(U)|$ , then,

$$1/T_\varepsilon = \sup_{[0,1]}\left(-\psi''_\varepsilon(U_0(\theta))\frac{d}{d\theta}U_0(\theta)\right) \leq m d_0.$$

Of course, for constant initial data ( $d_0 = 0$ ), no shock occurs, the solution is constant and  $T_\varepsilon = +\infty$ . In the case  $m d_0 \neq 0$ ,  $T_\varepsilon$  is finite but  $0 < \inf_{0 < \varepsilon \leq 1}T_\varepsilon$  since  $T_\varepsilon \geq 1/(m d_0)$  for all  $0 < \varepsilon \leq 1$ .

This gives the existence of a positive time  $T_0 < T^* = \inf\{T_\varepsilon, \varepsilon \in ]0, 1]\}$  such that  $U_\varepsilon \in C^1([0, T_0] \times \mathbb{R}/\mathbb{Z})$ . Thus  $u_\varepsilon$ , which is well defined by (4.1), belongs to  $C^1([0, T_0] \times \mathbb{R}^d)$  for all  $0 < \varepsilon \leq 1$ .

Now we prove the  $C^1$  convergence of the geometric optics expansion. There are two cases:  $\gamma$  is an integer or not.

$q = \gamma$ : From (4.2) and (2.1) we get

$$\begin{aligned} \partial_t U_\varepsilon + \partial_\theta (c_q U_\varepsilon^{q+1} + \varepsilon g_q^\varepsilon(U_\varepsilon)) &= 0, & \partial_t U + c_q \partial_\theta U^{q+1} &= 0, \\ U_\varepsilon(0, \theta) &= U_0(\theta), & U(0, \theta) &= U_0(\theta). \end{aligned}$$

The method of characteristics gives  $C^1$  characteristics,  $C^1$  solutions and

$$\|U_\varepsilon - U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon),$$

where

$$\|U\|_{C^1([0, T_0] \times \mathbb{R}^d)} = \|U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_t U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)} + \|\partial_\theta U\|_{L^\infty([0, T_0] \times \mathbb{R}^d)}.$$

integer  $q > \gamma$ : The proof is similar except that the term  $\varepsilon^r c_q \partial_\theta (c_q U^{q+1})$  becomes a remainder, with  $r = q - \gamma$  and  $U(t, \theta) = U_0(\theta)$ , thus

$$\|U_\varepsilon(\cdot, \cdot) - U_0(\cdot)\|_{C^1([0, T_0] \times \mathbb{R}^d)} = O(\varepsilon^r),$$

which concludes the proof.  $\square$

When condition (3.5) is violated, oscillations are immediately canceled.

**Theorem 4.1. [Cancellation of high oscillations, [5]]**

Let  $\mathbf{F}$  belong to  $C^{q+2}$  and  $U_0 \in L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ , where  $q - 1 < \gamma \leq q$  where  $q$  is defined in Theorem 3.1. If for some  $0 < j < q$

$$(4.4) \quad \mathbf{a}^{(j)}(\underline{u}) \cdot \mathbf{v} \neq 0$$

then the solutions  $u_\varepsilon$  of conservation law (2.1) with initial oscillating data (3.1) for  $\varepsilon \in ]0, 1]$  satisfy when  $\varepsilon \rightarrow 0$

$$u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \bar{U}_0 + o(\varepsilon) \quad \text{in } L^1_{loc}([0, +\infty[ \times \mathbb{R}^d).$$

Obviously the interesting case is when  $U_0$  is non constant. When  $U_0$  is smooth and non constant the first time when a shock occurs  $T_\varepsilon$  converges towards 0 when  $\varepsilon \rightarrow 0$ . Thus solutions are weak entropy solutions.

The proof is in the spirit of [5] and uses averaging lemmas (see [27] and the references given there). The proof is briefly expounded to be self-contained.

**Proof :** For non constant initial data it is impossible to avoid shock waves on any fixed strip  $[0, T_0] \times \mathbb{R}^d$  with  $T_0 > 0$  as in the previous proof of Theorem 3.1 since the time span of smooth solutions is  $\varepsilon^\beta$  where  $\beta = \gamma - j > 0$ .

First, with a change of space variable  $\mathbf{x} \leftrightarrow \mathbf{x} - t \cdot \mathbf{a}(\underline{u})$ , we can assume that  $\mathbf{a}(\underline{u}) = 0$ .

The WKB computations use the following ansatz:  $u_\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon v_\varepsilon(t, \mathbf{x})$  where  $v_\varepsilon(t, \mathbf{x}) = W_\varepsilon(t, \varepsilon^{-j} \phi(t, \mathbf{x}))$ . Indeed, the condition (4.4) leads to such ansatz as we can see in the WKB computations of the proof of Theorem 3.1. Then  $W_\varepsilon$  satisfies the one dimensional nonlinear conservation laws:

$$(4.5) \quad \partial_t W_\varepsilon + \partial_\theta (c_j W_\varepsilon^{j+1} + \varepsilon g_j^\varepsilon(W_\varepsilon)) = 0, \quad W_\varepsilon(0, \theta) = U_0(\varepsilon^{-\beta} \theta), \quad c_j \neq 0.$$

$W_\varepsilon(0, \cdot)$  converges weakly towards  $\bar{U}_0$ . As in [5],  $W_\varepsilon$  is relatively compact in  $L^1_{loc}$  thanks to averaging lemmas. Then  $W_\varepsilon$  converges towards the unique entropy solution  $W$  of

$$\partial_t W + c_j \partial_\theta W^{j+1} = 0, \quad W(0, \theta) = \bar{U}_0.$$

That is to say that  $W(t, \theta) \equiv \bar{U}_0$ . Then  $v_\varepsilon(t, \mathbf{x})$  converges towards  $\bar{U}_0$  in  $L^1_{loc}$  which completes the proof.  $\square$

## 5. CHARACTERIZATION OF NONLINEAR FLUX

The flux nonlinearity is characterized by the parameter  $\alpha$  in Definition 2.4, the Lions-Perthame-Tadmor definition of nonlinear flux. The smoothing effect depends only on the best  $\alpha = \alpha_{\text{sup}}$ . The understanding of the parameter  $\alpha_{\text{sup}}$  is a key step to the comprehension of the regularity of entropy solutions. Unfortunately, there are only few examples where  $\alpha_{\text{sup}}$  is computed in dimension 2 ([23, 31]) and there are some remarks in [18, 19, 2].

For the first time, for all nonlinear  $C^\infty$  fluxes and for all dimensions we characterize the fundamental parameter  $\alpha_{\text{sup}}$ . For this purpose we state Definition 5.1 of smooth nonlinear flux. This definition is related to the critical

geometric optics expansion given in Theorem 3.1. Notice that Definition 5.1 is typical for stationary phase methods. It is less common in the context of conservation laws.

Let us emphasize on two important consequences of Definition 5.1.

- The parameter  $\alpha_{\text{sup}}$  is explicitly characterized with the flux derivatives in Theorem 5.1.
- The super critical geometric optics expansion is built to highlight the uniform maximal smoothing effect in Theorem 3.2.

We explain Definition 5.1 in Subsection 5.1. For  $C^\infty$  flux, we prove that Definition 5.1 is equivalent to Definition 2.4 by Theorem 5.1. We compare our new definition with some other classical definitions in Subsection 5.2. We show that all definitions of nonlinear flux are equivalent for analytical flux.

### 5.1. Nonlinear smooth flux.

We introduce a definition of nonlinear  $C^\infty$  flux related to critical geometric optics expansions. When the compatibility conditions (3.5) are satisfied in Theorem 3.1, high frequency waves are smooth solutions of the conservation law (2.1). Furthermore, these conditions are optimal thanks to Theorem 4.1. What is the highest frequency waves solutions of the conservation law (2.1) which can be propagated as in Theorem 3.1? That is to say, what is the critical geometric optics for the conservation law (2.1)? Of course, the answer depends on the flux.

Indeed, near the constant state  $\underline{u}$  we can propagate waves with frequency  $\varepsilon^{-m}$ ,  $m > 1$ , if the set  $\{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m-1)}(\underline{u})\}^\perp$  is not reduced to  $\{0\}$ . Thus the maximal  $m$  occurs when  $\{0\} = \{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m)}(\underline{u})\}^\perp$  and  $\{0\} \neq \{\mathbf{a}'(\underline{u}), \mathbf{a}''(\underline{u}), \dots, \mathbf{a}^{(m-1)}(\underline{u})\}^\perp$ . We now can write the following definition.

**Definition 5.1. [Nonlinear smooth flux]**

Let the flux  $\mathbf{F}$  belong to  $C^\infty(\mathbb{R}, \mathbb{R}^d)$  and  $I = [-M, M]$ . The flux is said to be **nonlinear** on  $I$  if, for all  $u \in I$ , there exists  $m \in \mathbb{N}^*$  such that

$$(5.1) \quad \text{rank}\{\mathbf{a}'(u), \dots, \mathbf{a}^{(m)}(u)\} = d.$$

Furthermore, the flux is said to be **genuine nonlinear** if  $m = d$  is enough in (5.1) for all  $u \in I$ .

In fact, the non-linearity is a matter of the second derivatives of  $\mathbf{F}$ ,  $\mathbf{a}' = \mathbf{F}''$ . Notice that  $m \geq d$ . We need at least  $d$  vectors in (5.1) to span the space  $\mathbb{R}^d$ . Thus the genuine nonlinear case is the strongest nonlinear case.

The genuine nonlinear case was first stated in [5] (condition (2.8) and Lemma 2.5 p. 447 therein). The genuine nonlinear condition in the  $d$  dimensional case

$$(5.2) \quad \det(\mathbf{a}'(u), \mathbf{a}''(u), \dots, \mathbf{a}^{(d)}(u)) \neq 0, \quad \forall u \in I,$$

was also in [8], see condition (16) p. 84 therein. The simplest example of genuine nonlinear flux  $\mathbf{F}$  with the velocity  $\mathbf{a}$  was given in [5, 8, 2]:

$$\mathbf{a}(u) = (u, u^2, \dots, u^d) \quad \text{with } \mathbf{F}(u) = \left( \frac{u^2}{2}, \dots, \frac{u^{d+1}}{d+1} \right).$$

Definition 5.1 is a generalization of the genuine nonlinear condition (5.2). Definition 5.1 is more explicit with following integers with  $I = [-M, M]$ .

$$(5.3) \quad d_{\mathbf{F}}[u] = \inf\{k \geq 1, \text{rank}\{\mathbf{F}''(u), \dots, \mathbf{F}^{(k+1)}(u)\} = d\} \geq d,$$

$$(5.4) \quad d_{\mathbf{F}} = \sup_{|u| \leq M} d_{\mathbf{F}}[u] \in \{d, d+1, \dots\} \cup \{+\infty\}.$$

Indeed, Definition 5.1 states that the flux is genuine nonlinear when  $d_{\mathbf{F}}$  reaches its minimal value,  $d_{\mathbf{F}} = d$ .

Conversely, when the flux  $\mathbf{F}$  is linear,  $\mathbf{a}$  is a constant vector in  $\mathbb{R}^d$  and  $d_{\mathbf{F}}$  reaches its maximal value,  $d_{\mathbf{F}} = +\infty$ .

Between  $d_{\mathbf{F}} = d$  and  $d_{\mathbf{F}} = +\infty$ , there is a large variety of nonlinear flux.

The following theorem gives the optimal parameter  $\alpha$  (2.3) for smooth flux. Notice that this theorem is essentially a corollary of the uniform estimates given by Stein [30] for stationary phase lemmas, see also [19].

**Theorem 5.1. [Sharp measurement of the flux non-linearity ]**

Let  $\mathbf{F}$  be a smooth flux,  $\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$ . Then, the measurement of the flux non-linearity  $\alpha_{\text{sup}}$  is given by

$$\alpha_{\text{sup}} = \frac{1}{d_{\mathbf{F}}} \leq \frac{1}{d}.$$

Furthermore, when  $\alpha_{\text{sup}} > 0$  there exists  $\underline{u} \in [-M, M]$  such that  $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$ .

A similar result for the genuine nonlinear case,  $d_{\mathbf{F}} = d$ , can be found in [2].

This theorem is a powerful tool to compute the parameter  $\alpha_{\text{sup}}$ , for instance:

- $F(u) = (\cos(u), \sin(u))$  is genuine nonlinear flux,  $\alpha_{\text{sup}} = 1/2$  since  $\det(F''(u), F'''(u)) = 1$ .
- When  $F$  is polynomial with degree less or equal to the space dimension  $d$ ,  $\alpha_{\text{sup}} = 0$  and  $F$  does not satisfy Definition 5.1.
- It is well known that the ‘‘Burgers multi-D’’ flux  $F(u) = (u^2, \dots, u^2)$  is not nonlinear when  $d \geq 2$ . Let us explain this fact by two arguments: the explicit computation of  $\alpha_{\text{sup}}$  and a sequence of high frequencies waves solutions of (2.1).
  - $\mathbf{a}''(u) \equiv 0$  so  $d_{\mathbf{F}} = +\infty$  and Theorem 5.1 yields  $\alpha_{\text{sup}} = 0$ .
  - The sequence of oscillations with large amplitude  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  given by  $u_\varepsilon(t, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}) = \sin\left(\frac{x_1 - x_2}{\varepsilon}\right)$  blows up in any  $W_{loc}^{s,1}$ ,  $s > 0$ : for all  $t$ ,  $\sup_{0 < \varepsilon \leq 1} \|u_\varepsilon(t, \cdot)\|_{W^{s,1}([0,1]^d, \mathbb{R})} = +\infty$ . But the sequence of initial data is uniformly bounded in  $L^\infty$ ,  $\|u_0^\varepsilon\|_{L^\infty} = 1$ . Thus there is no improvement of the uniform initial Sobolev bounds.

- When  $F$  is polynomial such that  $\deg(F_i) = 1+i$ ,  $F$  is genuine nonlinear:  

$$\alpha_{\text{sup}} = \frac{1}{d}.$$

*Remark 5.1.* For smooth Flux  $\alpha_{\text{sup}}$  is the inverse of an integer. Not all real value of  $\alpha_{\text{sup}}$  in  $[0, 1]$  are possible for  $\mathbf{F} \in C^\infty$ . With less smooth flux, all other values of  $\alpha_{\text{sup}}$  are possible ([23, 13, 31, 19, 4, 3]).

For sake of completeness, we give a proof of Theorem 5.1 related to stationary phase lemmas ([30, 19, 2]). We mainly follow Stein [30]. Notice that, when Definition 2.4 is simplified by fixing  $\tau = 0$ , a nice proof can be found in [19].

Our proof needs many lemmas. First we recall Lemma 1 p. 125 from [2] giving the optimal  $\alpha$  for real functions.

**Lemma 5.1** ([2]). *Let  $\varphi \in C^\infty([-M, M], \mathbb{R})$ ,*

$$\begin{aligned} m_\varphi[v] &= \inf\{k \in \mathbb{N}, \varphi^{(k)}(v) \neq 0\} && \in \bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}, \\ m_\varphi &= \sup_{|v| \leq M} m_\varphi[v] && \in \bar{\mathbb{N}}, \\ Z(\varphi, \varepsilon) &= \{v \in [-M, M], |\varphi(v)| \leq \varepsilon\}. \end{aligned}$$

*If  $0 < m_\varphi < +\infty$  then there exists  $C > 1$  dependent of the function  $\phi$  such that, for all  $\varepsilon \in ]0, 1]$ ,*

$$(5.5) \quad C^{-1}\varepsilon^\alpha \leq \text{meas}(Z(\varphi, \varepsilon)) \leq C\varepsilon^\alpha \quad \text{with} \quad \alpha = \frac{1}{m_\varphi}.$$

To compute the measure of  $Z(\varphi, \varepsilon)$  with a different assumption, we adapt a proof of E. Stein about stationary phase method [30]. The main point in the following lemma is that the constant does not depend on the function  $\phi$ . Indeed, the condition  $1 \leq |\phi^{(k)}(v)|$  is stronger than the condition  $m_\varphi = k$ . The following lemma is fundamental to prove Theorem 5.1.

**Lemma 5.2.** [2] *Let  $k \geq 1$ ,  $I$  an interval of  $\mathbb{R}$ ,  $\phi \in C^k(I, \mathbb{R})$ .*

$$\begin{aligned} \text{If} & & & 1 \leq |\phi^{(k)}(v)|, & \text{for all } v \in I, \\ \text{then} & \text{measure}\{v \in I, |\phi(v)| \leq \varepsilon\} & \leq & \bar{c}_k \varepsilon^{1/k}, \end{aligned}$$

*where  $\bar{c}_k$  are constant independent of  $\phi$ .*

**Proof:** Since the result is independent of the interval  $I$  and the constant sign of the derivative  $\phi^{(k)}$  on the interval, let us suppose that  $I = \mathbb{R}$  and  $\phi^{(k)}(v) \geq 1$  for all  $v \in \mathbb{R}$ . Thus we have for all  $v \geq u$ :  $\phi^{(k-1)}(v) - \phi^{(k-1)}(u) \geq v - u$ . This inequality shows that the function  $\phi^{(k-1)}$  admits an unique root. Assume  $\phi^{(k-1)}(0) = 0$  without loss of generality.

With these assumptions we prove the lemma when  $k = 1$ . Since  $|\phi(v)| \geq |v|$  for all  $v$ , we have  $Z(\phi, I, \varepsilon) = \{v \in I, |\phi(v)| \leq \varepsilon\} \subset [-\varepsilon, \varepsilon]$ . So the lemma is proved for  $k = 1$  with  $\bar{c}_1 = 2$ .

We now prove the Lemma by induction on  $k > 1$ . We have for all  $v$ ,  $|\phi^{(k-1)}(v)| \geq |v|$ . Let  $\eta > 0$ . Notice that  $\text{meas}(Z(\phi, [-\eta, \eta], \varepsilon)) \leq 2\eta$ . Let



$\psi$  be the function  $\phi/\eta$ . Notice that  $\psi^{(k-1)}(v) \geq 1$  on  $]\eta, +\infty[$ . By our inductive hypothesis on  $\psi$  we have  $\text{meas}(Z(\psi, ]\eta, +\infty[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon)^{1/(k-1)}$ , so  $\text{meas}(Z(\phi, ]\eta, +\infty[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)}$ .

A similar argument yields  $\text{meas}(Z(\phi, ]-\infty, -\eta[, \varepsilon)) \leq \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)}$ . These previous three bounds give  $\text{meas}(Z(\phi, \mathbb{R}, \varepsilon)) \leq g(\eta) = 2(\eta + \bar{c}_{k-1}(\varepsilon/\eta)^{1/(k-1)})$ . This last inequality is valid for all  $\eta > 0$ . It suffices to minimize the function  $g$  on  $]0, +\infty[$ . A computation of the minimum yields  $\text{meas}(Z(\phi, \mathbb{R}, \varepsilon)) \leq \bar{c}_k \varepsilon^{1/k}$ , where  $\bar{c}_k = 4(\bar{c}_{k-1}/(k-1))^{(k-1)/k}$  which concludes the proof.  $\square$

The previous lemma is generalized to parameters in a compact set, see Lemma 4 p. 127 in [2].

**Lemma 5.3** ([2]). *Let  $P$  be a compact set of parameters,  $k$  a positive integer,  $A > 0$ ,  $V = [-A, A]$ ,  $K = V \times P$ ,  $\phi(v; p) \in C^0(P, C^k(V, \mathbb{R}))$ , such that, for all  $(v, p)$  in the compact  $K$ , we have*

$$\sum_{j=1}^k \left| \frac{\partial^j \phi}{\partial v^j} \right| (v; p) > 0.$$

Let  $Z(\phi(\cdot; p), \varepsilon) = \{v \in V, |\phi(v; p)| \leq \varepsilon\}$ . Then, there exists a constant  $C$  such that

$$\sup_{p \in P} \text{meas}(Z(\phi(\cdot; p), \varepsilon)) \leq C \varepsilon^{1/k}.$$

We now turn to the key integer  $d_{\mathbf{F}}$ .

**Lemma 5.4.** *If  $\mathbf{F}$  is a nonlinear flux on  $I$  in the sense of Definition 5.1 then  $d_{\mathbf{F}}$  is finite and there exists  $\underline{u} \in I$  such that  $d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$ .*

**Proof.** Let  $u$  be fixed in  $I$ . Then there exist,  $1 \leq j_1 < j_2 < \dots < j_d = d_{\mathbf{F}}[u]$  such that  $\text{rank}\{\mathbf{a}^{(j_1)}(u), \dots, \mathbf{a}^{(j_d)}(u)\} = d$  by the definition of  $d_{\mathbf{F}}[u]$ . So the continuous function  $g(v) = \det(\mathbf{a}^{(j_1)}(v), \dots, \mathbf{a}^{(j_d)}(v))$  does not vanish at  $v = u$ . By continuity, this is still true on an open set  $J$  with  $u \in J$ . Since  $j_d = d_{\mathbf{F}}[u]$ , we have  $d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}[u]$  for all  $v \in J$ . Thus  $v \mapsto d_{\mathbf{F}}[v]$  is upper semi-continuous and the result follows immediately on the compact set  $I$ .  $\square$

Now we are able to prove Theorem 5.1.

**Proof of Theorem 5.1.** There are two steps.

$$\text{step 1: } \alpha_{\text{sup}} \geq \frac{1}{d_{\mathbf{F}}}.$$

Set  $\phi(v; \tau, \xi) = \tau + \mathbf{a}(v) \cdot \xi$  with  $\tau^2 + |\xi|^2 = 1$ . Since  $\phi(\cdot; \tau, 0) = \tau$  has no root, we can assume that  $\xi \neq 0_{\mathbb{R}^d}$ . For  $j \geq 1$  we have  $\partial_v^j \phi(v; \tau, \xi) = \mathbf{a}^{(j)}(v) \cdot \xi$ . By definition of  $d_{\mathbf{F}}[v]$  there exists  $j \leq d_{\mathbf{F}}[v] \leq d_{\mathbf{F}}$  such that  $\partial_v^j \phi(v; \tau, \xi) \neq 0$ . Thus, we have when  $\xi \neq 0$

$$(5.6) \quad \sum_{j=1}^{d_{\mathbf{F}}} |\partial_v^j \phi(v; \tau, \xi)| > 0.$$

When  $\xi = 0$ , we have  $\tau = \pm 1$  since  $\tau^2 + |\xi|^2 = 1$ . The function  $\phi(v; \pm 1, 0) = \pm 1 \neq 0$ . By continuity of this function there exists an open neighborhood  $V$  of  $(1, 0_{\mathbb{R}^d})$  such the function does not vanish on  $\bar{V}$ . Set  $P$  be the complementary set of  $V$  in the unit sphere of  $\mathbb{R}^{d+1}$ .  $P$  is compact and (5.6) is true on  $P$ . Now we can use Lemma 5.3 to conclude the first step.

$$\text{step 2: } \alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}}.$$

This inequality is already stated in [18, 19]. Let us give a proof to be self-contained.  $\underline{u}$  is defined in Lemma 5.4. Then, by the definition of  $\underline{u}$ , there exists  $\xi^* \neq 0$  such that  $\partial_v^j \phi(\underline{u}; \tau, \xi^*) \neq 0$  for  $1 \leq j < d_{\mathbf{F}}$  and  $\partial_v^j \phi(\underline{u}; \tau, \xi^*) = 0$  for  $j = d_{\mathbf{F}}$ . Notice that the previous derivatives are independent of  $\tau$ . Let  $\tau^*$  be  $-\mathbf{a}(\underline{u}) \cdot \xi^*$ , so  $\phi(\underline{u}; \tau^*, \xi^*) = 0$ . To be on the unit circle we set  $\underline{\tau} = r \tau^*$  and  $\underline{\xi} = r \xi^*$  where  $r = ((\tau^*)^2 + |\xi^*|^2)^{-1/2}$ .  $\varphi(v) = \phi(v; \underline{\tau}, \underline{\xi})$  vanishes at  $v = \underline{u}$  and  $m_{\varphi}[\underline{u}] = d_{\mathbf{F}}$ . Furthermore, since  $\underline{u}$  is the point where the flux reaches its maximal degeneracy we also have  $m_{\varphi} = m_{\varphi}[\underline{u}] = d_{\mathbf{F}}$ . Now, by Lemma 5.1, more precisely the first inequality of (5.5), and the definition of  $\alpha$  (2.3), we have  $1/m_{\varphi} \geq \alpha$ . Thus the second step is proved.

$$\text{Finally } \frac{1}{d_{\mathbf{F}}} \leq \alpha_{\text{sup}} \leq \frac{1}{d_{\mathbf{F}}} \text{ and the proof is complete with Lemma 5.4. } \square$$

## 5.2. Comparisons with other nonlinear flux definitions.

There are more general definitions of nonlinear flux [15, 23]. Definitions 5.2, 5.3 below are qualitative. We use quantitative definitions 2.4, 5.1. Indeed, the precise smoothing is related to Definition 2.4 or Definition 5.1 and the parameter  $\alpha_{\text{sup}}$  or equivalently  $d_{\mathbf{F}}$ . Let us compare these definitions with Definition 5.1. It can be useful for other applications.

Notice also that definitions depend on the flux regularity, but, for our purpose, it is not an important point. For  $C^1$  flux, Lions, Perthame and Tadmor introduced Definition 2.4 and a more general definition of nonlinear flux, Definition 5.2. For  $C^\infty$  flux, we introduce Definition 5.1 related to stationary phase assumption on the flux. Indeed, by Theorem 5.1, Definition 5.1 is equivalent to Definition 2.4 for  $C^\infty$  flux. For  $C^2$  flux, Definition 5.3 related to the second derivative of the flux is more general than Definition 5.1, even in the context of  $C^\infty$  flux. Finally, for analytic flux, all these definitions are equivalent. For less smooth flux we refer to the works of E. Yu. Panov ([25, 26]).

Let us state Definitions 5.2 and 5.3 and prove our previous comparisons with Definition 5.1.

**Definition 5.2.** [General Nonlinear Flux [23]] *A flux  $\mathbf{F}$ , differentiable on  $[-M, M]$  is said to be nonlinear if the degeneracy set*

$$W(\tau, \xi) = \{|v| \leq M, \tau + \mathbf{F}'(v) \cdot \xi = 0\}$$

*has null Lebesgue measure for all  $(\tau, \xi)$  on the sphere.*

This definition is of a great importance since this condition implies the compactness of the semi-group  $\mathcal{S}_t$  associated with the conservation law (2.1).

**Proposition 5.1.** *Let  $\mathbf{F}$  be a smooth flux in  $C^\infty$ . Assume  $\mathbf{F}$  satisfies Definition 5.1. Then,  $\mathbf{F}$  is nonlinear for Definition 5.2 but the converse can be wrong.*

**Proof :** Lemma 5.4 and Theorem 5.1 show that nonlinearity of Definition 5.1 implies nonlinearity of Definition 2.4 and then of Definition 5.2. But we can give a direct proof from Lemma 2.5 and remark (2.3) p. 447 in [5], (see also [8] p. 84).

Notice that  $W(\tau, 0) = \emptyset$  since  $\tau = \pm 1$ . So we assume that  $\xi \neq 0$ . Set  $\phi(v) = \tau + \mathbf{F}'(v) \cdot \xi$ . Since  $\phi^{(k)}(v) = \mathbf{F}^{(k+1)}(v) \cdot \xi$ , for any  $v$ , there exists  $k > 0$  such that  $\phi^{(k)}(v) \neq 0$  by Definition 5.1. So the roots of  $\phi$  are isolated and the set  $W(\tau, \xi)$  is finite.

Conversely the counter-example  $\mathbf{F}'(u) = \exp(-1/u^2)(u, u^2, \dots, u^d)$  does not satisfies Definition 5.1 since  $d_{\mathbf{F}}[0] = +\infty$ .

But  $\mathbf{F}$  satisfies Definition 5.2. Indeed, with  $h(v) = \tau \exp(1/v^2) + \xi \cdot (v, v^2, \dots, v^d)$ , the set  $W(\tau, \xi) - \{0\}$  is the set of roots of  $h(\cdot)$ . If  $\tau = 0$ , we deal with the genuine nonlinear flux from Definition 5.1 and the degeneracy set  $W(\tau, \xi)$  is a null set. Indeed, it is finite. If  $\tau \neq 0$ ,  $h(\cdot)$  is analytic and non trivial on  $\mathbb{R}^*$ . Consequently  $W(\tau, \xi)$  is countable and also a null set which concludes the proof.  $\square$

Engquist and E in [15] gave another definition of strictly nonlinear flux generalizing Tartar [32].

**Definition 5.3.** [ **Strictly Nonlinear Flux** [15]]

*Let  $M$  be a positive constant, and  $\mathbf{F} : [-M, M] \rightarrow \mathbb{R}^d$  be a function twice differentiable on  $[-M, M]$ .*

*$F$  is said to be strictly nonlinear on  $[-M, M]$  if for any sub-interval  $I$  of  $[-M, M]$ , the functions  $F_1'', \dots, F_d''$  are linearly independent on  $I$ , i.e., for any constant vector  $\xi$ , if  $\xi \cdot \mathbf{F}''(u) = 0$  for all  $u \in I$  then  $\xi = 0$ .*

**Proposition 5.2.** *Let  $\mathbf{F}$  be a  $C^\infty([-M, M], \mathbb{R}^d)$  flux. Assume  $\mathbf{F}$  satisfying Definition 5.1, then  $\mathbf{F}$  satisfies Definition 5.3 but the converse is wrong.*

*Proof.* Assume  $\xi \cdot \mathbf{F}'' = 0$  on a open sub-interval  $I$ . Let  $u$  belong in  $I$ . Hence  $\xi \cdot \mathbf{F}^{(k)}(u) = 0$  for all  $k \geq 2$ . But  $\mathbf{F}$  satisfies Definition 5.1. It follows that  $\xi = 0$ .

Conversely take a flux  $\mathbf{F}$  such that  $\mathbf{F}''(u) = \exp(-1/u^2)(1, u, \dots, u^{d-1})$ . Obviously  $\mathbf{F}$  satisfies Definition 5.3. But  $\mathbf{F}$  does not satisfies Definition 5.1 since  $d_{\mathbf{F}}[0] = +\infty$ .  $\square$

By the same way, if  $\mathbf{F}$  satisfies Definition 5.2 then  $\mathbf{F}$  satisfies Definition 5.3.

For analytic flux, the situation is simpler.

**Proposition 5.3** (Analytic nonlinear flux). *Assume that the flux is an analytic function. Then, all previous Definitions 2.4, 5.1, 5.2, 5.3 are equivalent.*

*Proof.* Again we use Definition 5.1. There are two cases.

- (1) If  $\mathbf{F}$  is nonlinear for Definition 5.1. By Theorem 5.1, Propositions 5.1 and 5.2,  $\mathbf{F}$  is nonlinear for other definitions.
- (2) If  $\mathbf{F}$  is not nonlinear for Definition 5.1. By Theorem 5.1,  $\mathbf{F}$  does not satisfy Definition 2.4.

Let  $u$  be fixed. There exists an hyperplane  $H$  such that all derivatives  $\mathbf{F}^{(k)}(u) \in H$  for all  $k \geq 2$ , i.e. there exists  $\xi \neq 0$  such that  $\xi \cdot \mathbf{F}^{(k)}(u) = 0$  for all  $k \geq 2$ . Using the power series expansion of  $\mathbf{F}''$  near  $u$  we see that  $\mathbf{F}''$  stays in  $H$  near  $u$ . And by the unique analytic extension of  $\mathbf{F}''$ ,  $\mathbf{F}''$  stays always in  $H$ , i.e.  $\xi \cdot \mathbf{F}'' = 0$  everywhere. Thus  $\mathbf{F}$  does not satisfies Definition 5.3.

Integrating the relation  $\xi \cdot \mathbf{F}'' = 0$  we have  $\tau + \xi \cdot \mathbf{F}' = 0$  for some constant  $\tau$ . Dividing the relation by  $\sqrt{\tau^2 + |\xi|^2}$  we can assume that  $\tau^2 + |\xi|^2 = 1$ . Hence  $\mathbf{F}$  does not satisfies Definition 5.2.

We incidentally check that Definition 5.2 implies Definition 5.3.

□

## 6. SOBOLEV ESTIMATES

In this section, uniform and optimal Sobolev exponents of the family of highly oscillating solutions from Theorem 3.1 are investigated.

### **Theorem 6.1. [Sobolev exponent for highly oscillating solutions]**

*Let  $u_\varepsilon$  be the  $C^1([0, T_0] \times \mathbb{R}^d)$  oscillating solutions given in Theorem 3.1.*

*For all  $1 \leq p < +\infty$ , the family  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is uniformly bounded in*

$$C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})) \cap W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R}) \quad \text{with } s = \frac{1}{\gamma}.$$

*Furthermore, if  $U_0$  is a non constant function, then for all  $s > 1/\gamma$  the sequence  $(u_\varepsilon)_{0 < \varepsilon < 1}$  is unbounded in  $C^0([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}))$  and in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d, \mathbb{R})$ .*

Theorem 6.1 means that the Sobolev exponent  $s = \frac{1}{\gamma}$  is **optimal**. It is easily seen that the sequence  $(u_\varepsilon)_{0 < \varepsilon}$  is uniformly bounded in  $W_{loc}^{1/\gamma,p}$  by interpolation (see remark 6.1 below). The difficult part of the theorem is the optimality. That is to say the sequence is unbounded for larger  $s$ . For this purpose we need to get lower bound of Sobolev norms. Unfortunately, interpolation theory only gives upper bounds. Thus we use the intrinsic norm. It is rather elementary but quite long to achieve such lower bounds. This section is essentially devoted to compute these lower bounds to highlight the conjecture about the maximal smoothing effect in the next section.

Indeed, it is proved below that  $u_\varepsilon$  has order of  $\varepsilon^{1-s\gamma}$  in  $W_{loc}^{s,p}$  for any  $s \in [0, 1[$ . The case  $p = 1$  is the most important, since  $L^1$  norm plays an important role for conservation laws. The Sobolev estimates of the initial data are propagated by the semi-group  $\mathcal{S}_t$ , (see [23] for  $p = 1$  and also [28] for  $TV(|u_\varepsilon - \underline{u}|^s)$ ). A key point is there is no improvement of the Sobolev exponent of the family of initial data.

The basic idea of the proof is that the sequence of exact solutions  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  and the sequence of approximate oscillating solution given by  $\underline{u} + \varepsilon U \left( t, \frac{\phi(t, \mathbf{x})}{\varepsilon^\gamma} \right)$  have similar bounds in Sobolev spaces.

We use the  $W^{s,p}$  intrinsic semi-norm instead the interpolation theory as we explained before. More precisely, following semi-norms parametrized by  $Q = Q_d(\mathbf{x}_0, A) = \mathbf{x}_0 + ] - A, A[^d$ , where  $A > 0$ ,  $\mathbf{x}_0 \in \mathbb{R}^d$ , are used to estimate fractional derivatives in  $W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R})$  ([1]).

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))}^p = \int_{Q_d(\mathbf{x}_0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|V(\mathbf{x}) - V(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+sp}} d\mathbf{x}d\mathbf{y}.$$

The following classical Definitions are used in this section.

**Definition 6.1.** [ Estimates in  $W_{loc}^{s,p}(\mathbb{R}^d)$  ]

(i)  $u$  is said to be bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0,$$

$$\|u\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} = \|u\|_{L^p(Q_d(\mathbf{x}_0, A))} + |u|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C.$$

(ii)  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is said to be bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  if

$$\forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 0, \forall \varepsilon \in ]0, 1], \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C.$$

(iii) Let  $\beta \geq 0$ ,  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  has order of  $\varepsilon^{-\beta}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ , denoted by

$$u_\varepsilon \simeq \varepsilon^{-\beta},$$

$$\text{if } \forall \mathbf{x}_0 \in \mathbb{R}^d, \exists A > 0, \exists C \geq 1, \exists \varepsilon_0 \in ]0, 1], \forall \varepsilon \in ]0, \varepsilon_0],$$

$$C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q_d(\mathbf{x}_0, A))} \leq C \varepsilon^{-\beta}.$$

As usual if  $u$  is bounded in  $W_{loc}^{s,p}(\mathbb{R}^d)$  then for any cube  $Q$ ,  $u$  belongs to  $W^{s,p}(Q)$ . By the same way  $u_\varepsilon \simeq \varepsilon^{-\beta}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$  if for any cube  $Q$  there exists a constant  $C \geq 1$  and  $\varepsilon_0 \in ]0, 1]$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ ,  $C^{-1} \varepsilon^{-\beta} \leq \|u_\varepsilon\|_{W^{s,p}(Q)} \leq C \varepsilon^{-\beta}$ .

Since solutions of (2.1) are bounded in  $L^\infty$ , the key point is to focus on fractional derivatives. With  $|\mathbf{x}| = |x_1| + \dots + |x_d|$  and semi-norms

$$|V|_{\dot{W}^{\tilde{s},p}(Q_d(\mathbf{x}_0, A))}^p = \int_{Q_d(0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|V(\mathbf{x} + \mathbf{h}) - V(\mathbf{x})|^p}{|\mathbf{h}|^{d+sp}} d\mathbf{x}d\mathbf{h},$$

are also used. Notice that

$$|V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A/2))} \leq |V|_{\dot{W}^{\tilde{s},p}(Q_d(\mathbf{x}_0, A))} \leq |V|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, 2A))}.$$

Furthermore,  $|V|_{\dot{W}^{\tilde{s},p}(Q_1(\mathbf{x}_0, A))} = |V|_{\dot{W}^{s,p}(Q_1(\mathbf{x}_0, A))}$  when  $V$  is periodic with period  $A$  (or  $A/2$ ). Thus, these semi-norms can be useful to estimate bounds in  $W_{loc}^{s,1}$ .

The simplest example of high frequency oscillating functions with optimal estimates in Sobolev spaces is investigated in the following lemma. The remainder of the section is devoted to get the same estimates for the the family of highly oscillating solutions from Theorem 3.1.

**Lemma 6.1. [Highly oscillating periodic function on  $\mathbb{R}$ ]**

Let  $v$  belong to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ , and for all  $0 < \varepsilon \leq 1$ ,

$$V_\varepsilon(\theta) = v(\varepsilon^{-\gamma}\theta).$$

If  $v(\cdot)$  is a non constant periodic function then

$$V_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}).$$

Furthermore, if  $V_\varepsilon(\theta) = v_\varepsilon(\varepsilon^{-\gamma}\theta)$ ,  $v_\varepsilon$  is one periodic, and  $v_\varepsilon \rightarrow v$  in  $C^1$  then  $V_\varepsilon \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}(\mathbb{R})$ .

Notice that the magnitude of  $V_\varepsilon$  in  $W_{loc}^{s,p}$  is independent of  $p$ .

Notice also that if  $v_\varepsilon \rightarrow v$  in  $W_{loc}^{s,p}$  then  $v_\varepsilon(\varepsilon^{-\gamma}\theta) \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}(\mathbb{R})$ .

**Proof :** In the sequel one sets  $x_0 = 0$  in Definition 6.1 since computations are invariant under translation.

First the  $L_{loc}^1$  norm is easily bounded in [5]. Let  $A > 1/2$ ,  $X = \varepsilon^{-\gamma}x$ ,  $B_\varepsilon = \varepsilon^{-\gamma}A$ ,  $N_\varepsilon$  the integer such that  $N_\varepsilon \leq 2B_\varepsilon < N_\varepsilon + 1$  so  $2A - 1 \leq 2A - \varepsilon^\gamma \leq \varepsilon^\gamma N_\varepsilon \leq 2A$ .

$$\begin{aligned} \|V_\varepsilon\|_{L^p([-A,A])}^p &= \int_{-A}^A |V_\varepsilon(x)|^p dx = \varepsilon^{-\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \\ &= \varepsilon^{-\gamma} \left( \sum_{k=1}^{N_\varepsilon} \int_{-B_\varepsilon+k-1}^{-B_\varepsilon+k} |v(X)|^p dX + \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX \right) \\ &= \varepsilon^{-\gamma} N_\varepsilon \int_0^1 |v(X)|^p dX + \varepsilon^{-\gamma} \int_{-B_\varepsilon+N_\varepsilon}^{B_\varepsilon} |v(X)|^p dX. \end{aligned}$$

Finally one has

$$(6.1) \quad \|V_\varepsilon\|_{L^p([-A,A])} \leq (2A + 1)^{1/p} \|v\|_{L^p([0,1])},$$

$$(6.2) \quad \|V_\varepsilon\|_{L^p([-A,A])} \geq (2A - 1)^{1/p} \|v\|_{L^p([0,1])}$$

$$\|V_\varepsilon\|_{L^p([-A,A])} \sim (2A)^{1/p} \|v\|_{L^p([0,1])} \quad \text{when } \varepsilon \rightarrow 0.$$

$|V_\varepsilon|_{\dot{W}^{s,p}([-A,A])}$  is computed with the same notations and  $H = \varepsilon^{-\gamma}h$ ,

$$|V_\varepsilon|_{\dot{W}^{s,p}([-A,A])}^p = \varepsilon^{(1-sp)\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \int_{-B_\varepsilon}^{B_\varepsilon} \frac{|v(X+H) - v(X)|^p}{|H|^{1+sp}} dX dH.$$

Let  $Var(\cdot)$  be the one periodic function bounded in  $L^\infty$  by  $2^p \|v\|_{L^p([0,1])}^p$ ,

$$Var(H) = \int_0^1 |v(X+H) - v(X)|^p dX.$$

Notice that  $Var \equiv 0$  if and only if  $v$  is constant a.e.

Using one periodicity of  $v$  with respect to  $X$  yields as in (6.1)

$$\begin{aligned} |V_\varepsilon|_{\dot{W}^{s,p}([-A,A])}^p &= \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} \left( \varepsilon^\gamma \int_{-B_\varepsilon}^{B_\varepsilon} |v(X+H) - v(X)|^p dX \right) \frac{dH}{|H|^{1+sp}}, \\ &\leq \varepsilon^{-sp\gamma} \int_{-B_\varepsilon}^{B_\varepsilon} ((2A+1)Var(H)) \frac{dH}{|H|^{1+sp}} \leq \varepsilon^{-sp\gamma} (2A+1) D_\infty^p, \\ D_B^p = (D_B)^p &= \int_{-B}^{+B} Var(H) \frac{dH}{|H|^{1+sp}}. \end{aligned}$$

Notice that  $D_B$  is a true constant related to the fractional derivative of  $v$  since for  $B = 1/2$ ,  $D_{1/2} = |v|_{\dot{W}^{s,p}([-1/2,1/2])}$  and for  $B = \infty$  the integral converges.

The lower bound is obtained by the same way and finally one has

$$\begin{aligned} |V_\varepsilon|_{\dot{W}^{s,p}([-A,A])} &\leq \varepsilon^{-s\gamma} (2A+1)^{1/p} D_\infty, \\ |V_\varepsilon|_{\dot{W}^{s,p}([-A,A])} &\geq \varepsilon^{-s\gamma} (2A-1)^{1/p} D_1, \\ |V_\varepsilon|_{\dot{W}^{s,p}([-A,A])} &\sim \varepsilon^{-s\gamma} (2A)^{1/p} D_\infty. \end{aligned}$$

Notice also that  $D_B > 0$  for  $B > 1/2$ . Otherwise  $D_B = 0$  implies  $Var \equiv 0$  a.e. which implies  $v$  is a constant function on  $[x_0 - 2B, x_0 + 2B]$  and on  $\mathbb{R}$  by periodicity.

A key point in this paper is the lower bound to get sharp estimates. Since  $D_B$  is non decreasing with respect to  $B$ , the previous lower bound of  $V_\varepsilon$  in  $W^{s,p}$  implies the following lower bound

$$|V_\varepsilon|_{\dot{W}^{s,p}([-A,A])} \geq \varepsilon^{-s\gamma} (2A-1)^{1/p} |v|_{\dot{W}^{s,p}([-1/2,1/2])}.$$

With more work, similar estimates are still valid for  $|V_\varepsilon|_{\dot{W}^{s,1}([-A,A])}$ , see lemmas in [5] about triangular changes of variables for oscillatory integrals. But it is enough for our purpose.

Same computations when  $v$  replaced by  $v_\varepsilon$  are still valid, which complete the proof.  $\square$

The following lemma is useful to check that  $W^{s,1}$  semi-norms of  $V : \mathbb{R} \mapsto \mathbb{R}$  and  $W : \mathbb{R}^d \mapsto \mathbb{R}$  have the same order, where  $W(x_1, \dots, x_d) = V(x_1)$ .

**Lemma 6.2.** *Let  $d \geq 2$ ,  $s > 0$ ,  $A > 0$ ,  $h_1 > 0$ ,*

$$(6.3) \quad \mu_{d,s}(h_1) = \int_0^A \cdots \int_0^A \frac{h_1^{1+s}}{(h_1 + h_2 + \cdots + h_d)^{d+s}} dh_2 \cdots dh_d.$$

*Then, there exist two positive numbers  $c_{d,s}, C_{d,s}$  such that*

$$(6.4) \quad 0 < c_{d,s} \leq \mu_{d,s}(h_1) \leq C_{d,s} < +\infty, \quad \forall A > 0, \quad \forall h_1 \in ]0, A].$$

*Furthermore, the optimal constant  $C_{d,s}$  is  $\gamma_{d,s}$  where*

$$\gamma_{d,s} = \lim_{\substack{h_1 \rightarrow 0 \\ h_1 > 0}} \mu_{d,s}(h_1) = \frac{1}{(d-1+s) \cdots (1+s)}$$

The constants  $c_{d,s}$  and  $C_{d,s}$  are independent of  $A > 0$ . Notice that there is a singularity for  $\mu_{d,s}$  at  $h_1 = 0$  since  $\mu_{d,s}(0) = 0$  and  $\mu_{d,s} > 0$  on  $]0, A]$ .

**Proof :** It seems that  $\mu_{d,s}(h_1)$  is depending on  $A$ ,  $\mu_{d,s}(h_1) = \mu_{d,s}^A(h_1)$ . But by homogeneity the problem is reduced to the case  $A = 1$  with the change of variable  $h_i = t_i A$ ,  $0 < t_i < 1$ .

Now  $\mu_{d,s}(t_1) = \mu_{d,s}^1(t_1) = \mu_{d,s}^A(h_1)$  is computed explicitly .

Let  $\mu_{d,s}(t_1, B) = \int_0^1 \cdots \int_0^1 \frac{t_1^{1+s}}{(t_1 + t_2 + \cdots + t_d + B)^{d+s}} dt_2 \cdots dt_d$  for  $d > 1$ ,  $B \geq 0$ . Notice that  $\mu_{d,s}(t_1) = \mu_{d,s}(t_1, 0)$ .

For  $d = 1$ , set  $\mu_{1,s}(t_1, B) = \frac{t_1^{1+s}}{(t_1 + B)^{1+s}}$ ,  $\mu_{1,s}(t_1) = \mu_{1,s}(t_1, 0) = 1$ . The identity

$$\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} = (j + s)^{-1} (B^{-(j+s)} - (B + 1)^{-(j+s)}),$$

yields  $(j + s)\mu_{1+j,s}(t_1, B) = \mu_{j,s}(t_1, B) - \mu_{j,s}(t_1, B + 1)$ , and proceeding by induction with the notation  $C_n^k = \frac{n!}{k!(n-k)!}$ ,

$$\mu_{d,s}(t_1, B) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \mu_1(t_1, B + k).$$

Hence, for  $B = 0$ ,

$$(6.5) \quad \mu_{d,s}(t_1) = \gamma_{d,s} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \frac{t_1^{1+s}}{(t_1 + k)^{1+s}},$$

which gives  $\mu_{d,s}(0+) = \gamma_{d,s} > 0$ . Now,  $\mu_{d,s}(\cdot)$  belongs in  $C^0(]0, 1], \mathbb{R}^+)$ ,  $\mu_{d,s}(\cdot)$  is positive on  $]0, 1]$  with a positive right limit at  $t_1 = 0$ , thus positive constants stated in the lemma exist,

$$0 < c_{d,s} = \inf_{]0,1]} \mu_{d,s} \leq \gamma_{d,s} \leq C_{d,s} = \sup_{]0,1]} \mu_{d,s} < +\infty.$$

For  $d = 2$  we can show that  $C_{2,s} = \gamma_{2,s} = 1/(1+s)$  and  $c_{2,s} = (1 - 2^{-(1+s)})\gamma_{2,s}$ . That follows from the explicit formula  $\mu_{2,s}(t_1) = \gamma_{2,s} \left(1 - \left(\frac{t_1}{t_1+1}\right)^s\right)$  from (6.5), and the fact that this function is decreasing on  $]0, 1]$ .

For  $d \geq 2$ , let us show that  $C_{d,s}$  is still  $\gamma_{d,s}$ . It suffices to show that  $\mu_{d,s}(t_1) \leq \gamma_{d,s}$ . Notice that  $\int_0^1 \frac{dt}{(t + B)^{(1+j+s)}} \leq \frac{1}{(j + s) B^{j+s}}$ . Now, we have the following



computations for  $0 < t_1 \leq 1$ :

$$\begin{aligned}
\mu_{d,s}(t_1) &= \int_0^1 \cdots \int_0^1 \frac{t_1^{1+s}}{(t_1 + t_2 + \cdots + t_d)^{d+s}} dt_2 \cdots dt_d, \\
&\leq \int_0^1 \cdots \int_0^1 \frac{1}{(0 + t_2 + \cdots + t_d)^{d+s}} dt_2 \cdots dt_d, \\
&= \int_0^1 \cdots \left( \int_0^1 \frac{1}{(t_d + [t_2 + \cdots + t_{d-1}])^{d+s}} dt_d \right) dt_2 \cdots dt_{d-1}, \\
&\leq \frac{1}{d-1+s} \int_0^1 \cdots \int_0^1 \frac{1}{(t_2 + \cdots + t_{d-1})^{d-1+s}} dt_2 \cdots dt_{d-1}, \\
&\leq \cdots \leq \gamma_{d,s}.
\end{aligned}$$

Then  $C_{d,s} = \gamma_{d,s}$  which completes the proof.  $\square$

Our example of oscillating solutions is related to the following key example. For instance  $V_\varepsilon$ , defined by  $u_\varepsilon = \varepsilon V_\varepsilon$  where  $u_\varepsilon$  is the solution of  $\partial_t(u_\varepsilon) + \partial_x |u_\varepsilon|^{1+\gamma} = 0$ ,  $u_\varepsilon(0, x) = 0 + \varepsilon U(0, \varepsilon^{-\gamma} x)$ , satisfies the assumption of the lemma on a bounded time interval ([4]).

**Lemma 6.3. [Example of highly periodic oscillations on  $[0, T] \times \mathbb{R}$ ]**

Let  $T, \gamma$  be positive. If  $U$  belongs to  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  and non constant, then  $V_\varepsilon(t, x) = U(t, \varepsilon^{-\gamma} x) \simeq \varepsilon^{-s\gamma}$  in  $C^0([0, T], W_{loc}^{s,p}(\mathbb{R})) \cap W_{loc}^{s,p}([0, T[ \times \mathbb{R})$ .

*Remark 6.1.* Notice that the upper bound is quite easy to get. It directly follows from the fact that  $W^{s,p}$  is an interpolated space of exponent  $\theta = s$  between  $L^p = W^{0,p}$  and  $W^{1,p}$ , [33]. But we also want a lower bound to obtain an optimal estimate. This is a very crucial point in our study. For this purpose we use the intrinsic semi-norm in the proofs. The computations are elementary but long.

The same remark is still valid for all the next lemmas in this section.

**Proof :** First the fractional derivative w.r.t.  $\mathbf{x}$  is estimated. Second the whole fractional derivative in  $(t, \mathbf{x})$  is obtained.

Bounds in  $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$ : There exists  $t_0 \in ]0, T[$  such that  $\theta \mapsto U(t_0, \theta)$  is non constant since  $U$  is non constant and continuous on  $[0, T] \times \mathbb{R}/\mathbb{Z}$ . For fixed  $t_0$  the sharp estimate is a consequence of Lemma 6.1. For another  $t$ , we get the same order  $\varepsilon^{-s\gamma}$  or  $\varepsilon^0 = 1$ . Finally, constants involved in this estimate depend continuously on  $t$  so the bound in  $L^\infty([0, T], W_{loc}^{s,p}(\mathbb{R}))$  is obtained. Since  $U \in C^1$ , this previous bound is automatically in  $C^0([0, T], W_{loc}^{s,p}(\mathbb{R}))$ .

Bounds in  $W_{loc}^{s,p}([0, T[ \times \mathbb{R})$ : The only problem is to estimate for  $x_0 \in \mathbb{R}$ ,  $t_0 \in ]0, T[$  and  $\min(t_0, T - t_0) > A > 0$ , the quadruple integral

$$\begin{aligned}
IA &= |V_\varepsilon|_{\widehat{W_{loc}^{s,p}([t_0-A, t_0+A] \times [x_0-A, x_0+A])}}^p \\
&= \int_{t_0-A}^{t_0+A} \int_{x_0-A}^{x_0+A} \int_{-A}^A \int_{-A}^A \frac{|U(t+\tau, \varepsilon^{-\gamma}(x+\xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt.
\end{aligned}$$

*Upper bound of IA:*

Let  $Num$  be the numerator of the previous fraction,  $Q$  be  $U(t, \varepsilon^{-\gamma}(x + \xi)) - U(t, \varepsilon^{-\gamma}x)$ ,  $R$  be  $U(t + \tau, \varepsilon^{-\gamma}(x + \xi)) - U(t, \varepsilon^{-\gamma}(x + \xi))$ . Then,  $Num = |Q + R|^p \leq 2^{p-1}(|Q|^p + |R|^p)$ .

Previous inequality implies  $IA \leq 2^{p-1}(IQ + IR)$  with obvious notations.

$$\begin{aligned} IQ &= \int \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x + \xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &= \int \int \int \frac{|U(t, \varepsilon^{-\gamma}(x + \xi)) - U(t, \varepsilon^{-\gamma}x)|^p}{|\xi|^{1+sp}} \mu_{2,sp}(\xi) d\xi dx dt, \end{aligned}$$

with  $\mu_{2,sp}(\cdot)$  is defined in Lemma 6.2. Lemmas 6.1, 6.2 yield  $IQ \simeq \varepsilon^{-s\gamma}$ .  $IR$  is easily bounded since

$$\begin{aligned} IR &= \int \int \int \int \frac{|U(t + \tau, \varepsilon^{-\gamma}(x + \xi)) - U(t, \varepsilon^{-\gamma}(x + \xi))|^p}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt, \\ &\leq \int \int \int \int \frac{\|\partial_t U\|_{L^\infty}^p |\tau|^p}{(|\tau| + |\xi|)^{2+sp}} d\tau d\xi dx dt \\ &\leq 8A^2 \|\partial_t U\|_{L^\infty}^p \int_0^A |\tau|^{p(1-s)-1} \mu_{2,sp}(\tau) d\tau, \end{aligned}$$

which is finite, so  $IA \leq IQ + IR = \mathcal{O}(\varepsilon^{-sp\gamma})$ .

*Lower bound of IA:*

We again use notations  $Q$ ,  $R$ ,  $Num$ . By a convex inequality, the numerator satisfies:  $Num = |Q + R|^p \geq |Q|^p - p|Q|^{p-1}|R| = |Q|^p - \mathcal{O}(|\tau||Q|^{p-1})$  since  $R = \mathcal{O}(\tau)$ . Then  $IA \geq IQ - \mathcal{O}(IS)$ , where  $IQ$  has order of  $\varepsilon^{-sp\gamma}$ . The term  $IS$  has a lower order as we can find after the following computations as in the proof of Lemma 6.1. Notice first that for all positive numbers  $A$ ,  $b$ ,  $\int_0^A \frac{\tau}{(\tau + b)^{2+\beta}} d\tau \leq \frac{C}{2b^\beta}$  where  $\beta > 0$  and  $C = 2 \int_0^{+\infty} \frac{\tau}{(\tau+1)^{2+\beta}} d\tau < +\infty$ . Now integrating on  $\tau$  yields

$$IS = \int \int \int \int \frac{|\tau||Q|^{p-1}}{(|\tau| + |\xi|)^{2+sp}} d\xi d\tau dx dt \leq C \int \int \int \frac{|Q|^{p-1}}{|\xi|^{sp}} d\xi dx dt.$$

We set  $\eta = \varepsilon^\gamma$ ,  $X = x/\eta$ ,  $\Xi = \xi/\eta$ . Then, the previous inequality becomes

$$IS \leq C \eta^{2-sp} \int_0^T \int_{-A/\eta}^{A/\eta} \int_{-A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi dX dt.$$

We now focus on the integral with respect to  $\Xi$  and remark that  $Q = O(1)$  and also  $Q = O(\Xi)$  since  $U$  is  $C^1$ .

$$\begin{aligned} \int_{-A/\eta}^{A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi &= \int_{|\Xi| < 1} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{|Q|^{p-1}}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} \frac{O(|\Xi|^{p-1})}{|\Xi|^{sp}} d\Xi + \int_{1 < |\Xi| < A/\eta} \frac{O(1)}{|\Xi|^{sp}} d\Xi \\ &\leq \int_{|\Xi| < 1} O(|\Xi|^{p(1-s)-1}) d\Xi + O(g(\eta)) = O(1) + O(g(\eta)), \end{aligned}$$

where  $g(\eta) = \eta^{sp-1}$  if  $sp \neq 1$ , else  $g(\eta) = \ln(\eta)$ .

To bound  $IS$ , we notice that the integral  $\eta \int_{-A/\eta}^{A/\eta} dX$  is bounded by periodicity and we can take the supremum with respect  $t$  on  $[0, T]$ . So  $IS = O(1)$  if  $sp \neq 1$  else  $IS = O(\ln(\eta))$  which is enough to have a lower order than  $IQ$ .

In conclusion, the bounds of  $IA$  yield  $V_\varepsilon \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R})$ .  $\square$

Now, we estimate the Sobolev norm for the multidimensional case with one phase.

**Lemma 6.4.** [Example of highly periodic oscillations on  $\mathbb{R}^d$ ]

Let  $v$  belong to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ ,  $\psi(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + b$  where  $\mathbf{v} \in \mathbb{R}^d$ ,  $\mathbf{v} \neq 0$ ,  $b \in \mathbb{R}$  and  $0 < \varepsilon < 1$ ,

$$W_\varepsilon(\mathbf{x}) = v(\varepsilon^{-\gamma} \psi(\mathbf{x})).$$

If  $v$  is a non constant periodic function and  $\nabla \psi \neq 0$ , then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}(\mathbb{R}^d, \mathbb{R}).$$

Furthermore, when functions  $v_\varepsilon$  are one periodic functions for all  $\varepsilon \in ]0, 1]$ , which converge towards  $v$  in  $C^1$  and  $W_\varepsilon(\mathbf{x}) = v_\varepsilon(\varepsilon^{-\gamma} \psi(\mathbf{x}))$ , the same conclusion holds.

**Proof :** We first choose a new variable  $X = (X_1, \dots, X_d)$  such that  $X_1 = \psi(\mathbf{x})$ . This is possible since  $\nabla_{\mathbf{x}} \psi \equiv \mathbf{v} \neq 0$ . Moreover,  $\psi$  is an affine function so we choose an affine change of variables,  $X = M\mathbf{x} + B$  where  $M$  is a  $d \times d$  non-degenerate matrix and  $B \in \mathbb{R}^d$ . With the new variable  $X$ , the expounded proof has three steps.

Step 1: When  $W(\mathbf{x}) = U(M\mathbf{x} + b)$ ,  $W$  and  $U$  are the same order in  $W_{loc}^{s,p}$  since  $\det M \neq 0$ . More precisely, fix following positive constants  $m_0 = |\det M| > 0$ ,  $m_1 = \|M\| = \sup\{|M\mathbf{x}|, |\mathbf{x}| = 1\} > 0$ ,  $m_{-1} = \|M^{-1}\| > 0$ ,  $0 < r < R$  such that  $Q_d(X_0, r) \subset MQ_d(\mathbf{x}_0, 1) \subset Q_d(X_0, R)$  where  $X_0 = M\mathbf{x}_0 + B$ . Performing the change of variables  $X = M\mathbf{x} + B$ ,  $Y = M\mathbf{y} + B$  yields for any  $\mathbf{x}_0 \in \mathbb{R}^d$

and any  $A > 0$

$$\begin{aligned} m_0^{-1} \|U\|_{L^p(Q_d(X_0, rA))} &\leq \|W\|_{L^p(Q_d(\mathbf{x}_0, A))} \leq m_0^{-1} \|U\|_{L^p(Q_d(X_0, RA))}, \\ \frac{m_0^{-2}}{m_{-1}^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, rA))} &\leq |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} \leq \frac{m_0^{-2}}{m_1^{-(d+sp)}} |U|_{\dot{W}^{s,p}(Q_d(X_0, RA))}. \end{aligned}$$

Step 2: Assume  $\psi(\mathbf{x}) = x_1$ , i.e.  $W(\mathbf{x}) = W(x_1, \dots, x_d) = w(x_1)$ ,  $x_0 = \psi(\mathbf{x}_0)$ . Then,  $W$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$  and  $w$  in  $W_{loc}^{s,p}(\mathbb{R})$  have the same order. More precisely, elementary computations yield

$$\begin{aligned} \|W\|_{L^1(Q_d(\mathbf{x}_0, A))} &= (2A)^{d-1} \|w\|_{L^1(Q_1(x_0, A))}, \\ |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &\leq (2A)^{d-1} C_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))} \\ &\geq (2A)^{d-1} c_{d,sp} |U|_{\dot{W}^{s,p}(Q_1(x_0, A))}. \end{aligned}$$

The two last inequalities and constants come from Lemma 6.2 since

$$\begin{aligned} |W|_{\dot{W}^{s,p}(Q_d(\mathbf{x}_0, A))} &= \int_{Q_d(0, A)} \int_{Q_d(\mathbf{x}_0, A)} \frac{|w(x_1 + h_1) - w(x_1)|}{|\mathbf{h}|^{d+sp}} d\mathbf{x} d\mathbf{h} \\ &= (2A)^{d-1} \int_{-A}^A \int_{x_0-A}^{x_0+A} \frac{|w(x_1 + h_1) - w(x_1)|}{|h_1|^{1+sp}} \mu_{d,sp}(h_1) dx_1 dh_1. \end{aligned}$$

Step 3: By step 1,  $W_\varepsilon(\mathbf{x}) = V_\varepsilon(\varepsilon^{-\gamma}\psi(\mathbf{x})) \simeq V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ , by step 2,  $\mathbf{x} \mapsto V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$  and  $x_1 \mapsto V_\varepsilon(\varepsilon^{-\gamma}\mathbf{x}_1)$  have the same order in  $W_{loc}^{s,p}(\mathbb{R}^d)$  and  $W_{loc}^{s,p}(\mathbb{R})$ . Finally we have by Lemma 6.1  $W_\varepsilon \simeq \varepsilon^{-s\gamma}$  in  $W_{loc}^{s,p}(\mathbb{R}^d)$ .  $\square$

It is the last step to estimate the Sobolev norm for the multidimensional case before proving Theorem 6.1.

**Lemma 6.5.** [Example of highly periodic oscillations on  $[0, T] \times \mathbb{R}^d$ ] Let  $U$  belong to  $W_{loc}^{s,p}(\mathbb{R}, \mathbb{R})$ ,  $\gamma > 0$ ,  $\varphi(t, \mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + bt$  where  $\mathbf{v} \in \mathbb{R}^d$ ,  $b \in \mathbb{R}$  and  $0 < \varepsilon < 1$ ,

$$W_\varepsilon(t, \mathbf{x}) = U(t, \varepsilon^{-\gamma}\varphi(t, \mathbf{x})).$$

If  $U$  is a non constant function in  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  and  $\mathbf{v} \neq 0_{\mathbb{R}^d}$ , then

$$W_\varepsilon \simeq \varepsilon^{-s\gamma} \quad \text{in } W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R}).$$

Furthermore, when  $U_\varepsilon$  belongs to  $C^1([0, T] \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$  for all  $\varepsilon \in ]0, 1]$  converging towards  $U$  in  $C^1$  and  $W_\varepsilon(t, \mathbf{x}) = U_\varepsilon(t, \varepsilon^{-\gamma}\varphi(t, \mathbf{x}))$ , the same conclusion holds.

**Proof :** We proceed the proof as in the previous proofs. First with a linear change of variable  $(t, \mathbf{x}) \mapsto (t, \mathbf{y})$  with  $\mathbf{y}_1 = \varphi(t, \mathbf{x})$ .  $W_\varepsilon$  has the same estimates  $V_\varepsilon = U(t, \varepsilon^{-\gamma}y_1)$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$ . Notice that the change of variable depends on  $t$  varying in the compact set  $[0, T]$ . So we have uniform estimates of positive constants  $m_0, m_1, m_{-1}$  used in the proof of Lemma 6.4. Now, the estimates of  $V_\varepsilon$  in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$  and in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}, \mathbb{R})$  have

the same order since

$$\begin{aligned}
& \int_{-A}^A \cdots \int_{-A}^A \frac{dh_0 dh_1 \cdots dh_d}{(|h_0| + |h_1| + \cdots + |h_d|)^{1+d+sp}} \\
= & \int_{-A}^A \cdots \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \frac{(|h_0| + |h_1|)^{1+(sp+1)} dh_2 \cdots dh_d}{(|h_0| + |h_1| + \cdots + |h_d|)^{d+(sp+1)}} \\
= & \int_{-A}^A \int_{-A}^A \frac{dh_0 dh_1}{(|h_0| + |h_1|)^{2+sp}} \mu_{2,(sp+1)}(|h_0| + |h_1|)
\end{aligned}$$

where  $h_0$  plays the role of time. From the bounds of  $\mu_{2,(sp+1)}(|h_0| + |h_1|)$  on  $]0, 2A]$  in Lemma 6.2, we can complete the proof by Lemma 6.3

With a smooth extension of  $U$  on  $[-\delta, T + \delta] \times \mathbb{R}/\mathbb{Z}$ , for a small positive  $\delta$ , we obtain estimates in  $W_{loc}^{s,p}([0, T] \times \mathbb{R}^d, \mathbb{R})$ .  $\square$

We are now able to prove the Theorem by using Lemma 6.4 and the method of characteristics.

**Proof of Theorem 6.1:** Bounds  $L^\infty([0, T_0], W_{loc}^{s,p}(\mathbb{R}^d))$ : Such bounds give bounds in  $C^0([0, T_0], W_{loc}^{s,p})$  since  $u_\varepsilon$  is in  $C^1$ .

For  $t = 0$ , it is only an application of Lemma 6.4. The profile  $U(t, \cdot)$  is non constant for each  $t$ , else  $U_0$  must be constant by the method of characteristics. And the estimates are uniform.

Bounds in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$  The semi-norms  $|\cdot|_{\tilde{W}^{s,p}(Q_{d+1}(\mathbf{y}_0, A))}$ , where  $\mathbf{y}_0 = (t_0, \mathbf{x}_0)$ , needs some precautions to use on  $[0, T_0] \times \mathbb{R}^d$ .  $\mathbf{y}_0$  must be such that  $0 < t_0 < T_0$  and  $A < \min(t_0, T_0 - t_0)$ . Furthermore, only  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$  smoothness can be estimate. Indeed,  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  is bounded in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ . To prove this, let us use the following trick. By the methods of characteristics the family of solutions  $(u_\varepsilon)_{0 < \varepsilon \leq 1}$  exists on a maximal time interval  $] - \delta, T_1[$ , with  $0 < \delta < T_0 < T_1$ . Notice that solutions exist for negative time since the initial data is smooth. Now estimates in  $W_{loc}^{s,p}([ - \delta, T_1] \times \mathbb{R}^d)$  can be obtained which is sufficient to get smoothness in  $W_{loc}^{s,p}([0, T_0] \times \mathbb{R}^d)$ . Now using Lemma 6.4 we complete the proof.  $\square$

## 7. SUPER CRITICAL GEOMETRIC OPTICS AND MAXIMAL SMOOTHING EFFECT

In this short and final section we prove the Theorem 3.2 which gives a bound for the maximal uniform smoothing effect. This theorem requires almost all the previous results proven in this paper. Indeed it is a consequence of

- smooth critical geometric expansions in Theorem 3.1 under a "stationary phase assumption" with respect to the flux, namely condition (3.5) in Section 3,

- links between "stationary phase assumption" (3.5) and Lions-Perthame-Tadmor definition 2.4 on nonlinear flux, by Definition 5.1 and Theorem 5.1 in Subsection 5.1,
- Sobolev estimates on our family of highly frequency waves in Theorem 6.1, Section 6 .

**Proof of Theorem 3.2:**

The proof is a consequence of three previous theorems.

By Theorem 5.1, there exists  $\underline{u} \in [-M, M]$  such that  $\alpha = \frac{1}{d_{\mathbf{F}}[\underline{u}]}$ . Let  $U_0$  be a non constant smooth periodic function such that:  $-M \leq \underline{u} + U_0(\theta) \leq M$  for all  $\theta$ . Let  $\mathbf{v} \in \mathbb{R}^d$  such that  $\mathbf{a}^k(\underline{u}) \cdot \mathbf{v} = 0$  and  $\mathbf{v} \neq 0$  for  $k = 1, \dots, d_{\mathbf{F}}[\underline{u}] - 1$ . Such  $\mathbf{v}$  exists by Definition of  $d_{\mathbf{F}}[\underline{u}]$ .

Now, let  $(u_\varepsilon)$  be the family of smooth solutions given by Theorem 3.1.

Theorem 6.1 is the desired conclusion.  $\square$

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