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To cite this version:

Arnaud Debussche, Julien Vovelle. Diffusion limit for a stochastic kinetic problem. Communications on Pure and Applied Mathematics, 2012, 11 (6), pp.2305-2326. 10.3934/cpaa.2012.11.2305 . hal-00576610v3

HAL Id: hal-00576610

https://hal.science/hal-00576610v3

Submitted on 20 Nov 2018

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Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

DIFFUSION LIMIT FOR A STOCHASTIC KINETIC PROBLEM

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Dedicated to Michel Pierre in the occasion of his 60th birthday

Abstract. We study the limit of a kinetic evolution equation involving a small parameter and perturbed by a smooth random term which also involves the small parameter. Generalizing the classical method of perturbed test functions, we show the convergence to the solution of a stochastic diffusion equation.

1. **Introduction.** Our aim in this work is to develop new tools to study the limit of kinetic equations to fluid models in the presence of randomness. Without noise, this is a thoroughly studied field in the literature. Indeed, kinetic models with small parameters appear in various situations and it is important to understand the limiting equations which are in general much easier to simulate numerically.

In this article, we consider the following model problem

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon \text{ in } \mathbb{R}^+_t \times \mathbb{T}^d_x \times V_v, \tag{1}$$

with initial condition

$$f^\varepsilon(0) = f^\varepsilon_0 \text{ in } \mathbb{T}^d_x \times V_v, \tag{2}$$

where L is a *linear* operator (see (3) below) and m^ε a random process depending on $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$ (see Section 2.2). We will study the behavior in the limit $\varepsilon \to 0$ of its solution f^ε.

In the deterministic case $m^\varepsilon = 0$, such a problem occurs in various physical situations: we refer to [5] and references therein. The unknown $f^\varepsilon(t,x,v)$ is interpreted as a distribution function of particles, having position x and degrees of freedom v at time t. The variable v belongs to a measure space (V,μ) where μ is a probability measure. The actual velocity is $a(v)$, where $a \in L^\infty(V;\mathbb{R}^d)$. The operator L expresses the particle interactions. Here, we consider the most basic interaction operator, given by

$$Lf = \int_V f d\mu - f, \quad f \in L^1(V,\mu). \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35B25, 35Q35; Secondary: 60F05, 60H15, 82C40, 82D30.

Key words and phrases. Diffusion limit, kinetic equations, stochastic partial differential equations, perturbed test functions.

Note that L is dissipative since

$$-\int_V Lf \cdot f d\mu = \|Lf\|^2_{L^2(V,\mu)}, \quad f \in L^2(V,\mu). \tag{4}$$

In the absence of randomness, the density $\rho^\varepsilon = \int_V f^\varepsilon d\mu$ converges to the solution of the linear parabolic equation (see section 2.3 for a precise statement):

$$\partial_t \rho - \operatorname{div}(K\nabla\rho) = 0 \text{ in } \mathbb{R}^+_t \times \mathbb{T}^d,$$

where

$$K := \int_V a(v) \otimes a(v) d\mu(v) \tag{5}$$

is assumed to be positive definite. We thus have a diffusion limit in the partial differential equation (PDE) sense.

When a random term with the scaling considered here is added to a differential equation, it is classical that, at the limit $\varepsilon \to 0$, a stochastic differential equation with time white noise is obtained. This is also called a diffusion limit in the probabilistic language, since the solution of such a stochastic differential equation is generally called a diffusion. Such convergence has been proved initially by Khasminskii [11, 12] and then, using the martingale approach and perturbed test functions, in the classical article [17] (see also [8], [10], [14]).

The goal of the present article is twofold. First, we generalize the perturbed test function method to the context of a PDE and develop some tools for that. We believe that they will be of interest for future articles dealing with more complex PDEs. Second, we simultaneously take the diffusion limit in the PDE and in the probabilistic sense. This is certainly relevant in a situation where a noise with a correlation in time of the same order as a typical length of the deterministic mechanism is taken. Our main result states that under some assumptions on the random term m, in particular that it satisfies some mixing properties, the density $\rho^\varepsilon = \int_V f^\varepsilon d\mu$ converges to the solution of the stochastic partial differential equation

$$d\rho = \operatorname{div}(K\nabla\rho)dt + \rho \circ Q^{1/2} dW(t), \text{ in } \mathbb{R}^+_t \times \mathbb{T}^d,$$

where K is as above, W is a Wiener process in $L^2(\mathbb{T}^d)$ and the covariance operator Q can be written in terms of m. As is usual in the context of diffusion limit, the stochastic equation involves a Stratonovitch product.

As already mentioned, we use the concept of solution in the martingale sense. This means that the distribution of the process satisfies an equation written in terms of the generator (see section 3.2 for instance). This generator acts on test functions and the perturbed test function method is a clever way to choose the test functions such that one can identify the generator of the limiting equation. Instead of expanding the solution of the random PDE f^ε as is done in a Hilbert development in the PDE theory, we work on the test functions acting on the distributions of the solutions.

In section 2, we set some notations, describe precisely the random driving term, recall the deterministic result and finally state our main result. Section 3 studies the kinetic equation for ε fixed. In section 4, we build the correctors involved in the perturbed test function method and identify the limit generator. Finally, in section 5, we prove our result. We first show a uniform bound on the L^2 norm of the solutions, prove tightness of the distributions of the solutions and pass to the limit in the martingale formulation.

We are not aware of any result on probabilistic diffusion limit using perturbed test functions in the context of PDE, but the recent work [4] (in a context of nonlinear Schrödinger equations) and [16] (where the underlying PDE is parabolic and the limit $[\varepsilon \to 0]$ associated to homogenization effects). A diffusion limit is obtained for the nonlinear Schrödinger equation in [15], [3], [7] but there the driving noise is one dimensional and the solution of the PDE depends continuously on the noise so that in this case an easier argument can be used. Eventually, note that a method of perturbed test function has also been introduced in the context of viscosity solutions by Evans in [9]. Actually, in the case $m \equiv 0$, *i.e.* for the deterministic version of (1), the method of [9] allows to obtain the diffusive (in the PDE sense) limit $[\varepsilon \to 0]$ of (1) when the velocity set V is finite.

2. **Preliminary and main result.**

2.1. **Notations.** We work with PDEs on the torus $\mathbb{T}^d$, this means that the space variable $x \in [0,1]^d$ and periodic boundary conditions are considered. The variable v belongs to a measure space (V, μ) where μ is a probability measure. We shall write for simplicity $L^2_{x,v}$ instead of $L^2(\mathbb{T}^d \times V, dx \otimes d\mu)$, its scalar product being denoted by $(\cdot,\cdot)$. We use the same notation for the scalar product of $L^2(\mathbb{T}^d)$; note that this is consistent since $\mu(V) = 1$. Similarly, we denote by $\|u\|_{L^2}$ the norm $(u,u)^{1/2}$, whether $u \in L^2_{x,v}$ or $L^2(\mathbb{T}^d)$. We use the Sobolev spaces on the torus $H^\gamma(\mathbb{T}^d)$. For $\gamma \in \mathbb{N}$, they consist of periodic functions which are in $L^2(\mathbb{T}^d)$ as well as their derivatives up to order γ. For general $\gamma \geq 0$, they are easily defined by Fourier series for instance. For $\gamma < 0$, $H^\gamma(\mathbb{T}^d)$ is the dual of $H^{-\gamma}(\mathbb{T}^d)$. Classically, for $\gamma_1 > \gamma_2$, the injection of $H^{\gamma_1}(\mathbb{T}^d)$ in $H^{\gamma_2}(\mathbb{T}^d)$ is compact. We use also $L^\infty(\mathbb{T}^d)$ and $W^{1,\infty}(\mathbb{T}^d)$, the subspace of $L^\infty(\mathbb{T}^d)$ of functions with derivatives in $L^\infty(\mathbb{T}^d)$. Finally, $L^2(V; H^1(\mathbb{T}^d))$ is the space of functions f of v and x such that all derivatives with respect to x are in $L^2(\mathbb{T}^d)$ and the square of the norm

$$\|f\|^2_{L^2(H^1)} := \int_V \|f\|^2_{L^2} + \sum_{i=1}^d \|\partial_i f\|^2_{L^2} d\mu$$

is finite.

2.2. **The driving random term.** The random term m^ε has the scaling

$$m^\varepsilon(t,x) = m\left(\frac{t}{\varepsilon^2}, x\right),$$

where m is a stationary process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. Note that m^ε is adapted to the filtration $(\mathcal{F}^\varepsilon_t)_{t\in\mathbb{R}}$, with $\mathcal{F}^\varepsilon_t := \mathcal{F}_{\varepsilon^{-2}t}$, $t \in \mathbb{R}$.

Our basic assumption is that, considered as a random process with values in a space of spatially dependent functions, m is a stationary homogeneous Markov process taking values in a subset E of $W^{1,\infty}(\mathbb{T}^d)$. We assume that m is stochastically continuous. Note that m is supposed not to depend on the variable v. The law ν of $m(t)$ is supposed to be centered:

$$\mathbb{E}m(t) = \int_E n d\nu(n) = 0. \tag{6}$$

In fact, we also assume that m is uniformly bounded in $W^{1,\infty}(\mathbb{T}^d)$ so that E is included in a ball of $W^{1,\infty}(\mathbb{T}^d)$. We denote by $(P_t)_{t\geq 0}$ a transition semigroup on E associated to m and by M its infinitesimal generator.

As is usual in the context of diffusion limit, we use the notion of solution of the martingale problem and need mixing properties on m. We assume that there is a subset D_M of $C_b(E)$, the space of bounded continuous functions on E, such that, for every $\psi \in D_M$, $M\psi$ is well defined and

$$\psi(m(t)) - \int_0^t M\psi(m(s))ds$$

is a continuous and integrable martingale. Moreover, we suppose that m is ergodic and satisfies some mixing properties in the sense that there exists a subspace $\mathscr{P}_M$ of $C_b(E)$ such that for any $\theta \in \mathscr{P}_M$ the Poisson equation

$$M\varphi = \theta - \int_E \theta(n)d\nu(n) \tag{7}$$

has a unique solution $\varphi \in D_M$ satisfying $\displaystyle\int_E \varphi d\nu = 0$. When θ satisfies

$$\int_E \theta d\nu = 0, \tag{8}$$

we denote by $M^{-1}\theta \in D_M$ this solution and assume that it is given by:

$$M^{-1}\theta(n) = -\int_0^\infty P_t\theta(n)dt.$$

In particular, we suppose that the above integral is well defined. It implies that

$$\lim_{t\to+\infty} P_t\theta(n) = 0, \quad \forall n \in E. \tag{9}$$

We need that $\mathscr{P}_M$ contains sufficiently many functions. In particular, we assume that for each $x \in \mathbb{T}^d$, the evaluation function ψ_x defined by $\psi_x(n) = n(x),\ n \in E$, is in $\mathscr{P}_M$. Also, we assume that, for any $f, g \in L^2_{x,v}$, the function $\psi_{f,g}\ :\ n \mapsto (f, ng)$ is in $\mathscr{P}_M$ and we define $M^{-1}I$ from E into $W^{1,\infty}(\mathbb{T}^d)$ by

$$(f, M^{-1}I(n)g) := M^{-1}\psi_{f,g}(n), \quad \forall f, g \in L^2_{x,v}. \tag{10}$$

We need that $M^{-1}I$ takes values in a ball of $W^{1,\infty}(\mathbb{T}^d)$ and take C_* large enough so that

$$\|n\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_*, \quad \|M^{-1}I(n)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_*, \tag{11}$$

for all $n \in E$. It is natural to require the following compatibility assumption, which would follow from continuity properties of M^{-1}:

$$M^{-1}\psi_x(n) = M^{-1}I(n)(x), \quad \forall n \in E, x \in \mathbb{T}^d. \tag{12}$$

Note that by (6), $\psi_{f,g}$ and ψ_x satisfy the centering condition (8). Note also that, by (10) and (12), we have, taking $g = 1$,

$$\int_{\mathbb{T}^d} f(x)M^{-1}\psi_x(n)dx = M^{-1}\psi_{f,1}(n). \tag{13}$$

Eventually, we will also assume that for any $f, g \in L^2_{x,v}$ and for $x \in \mathbb{T}^d$, the functions

$$\Psi_{f,g}\ :\ n \mapsto (f, nM^{-1}I(n)g), \quad M^{-1}\psi_{f,1}, \quad M^{-1}\psi_x$$

are in $\mathscr{P}_M$.

To describe the limit equation, we remark that since $m(0)$ has law ν,

$$-\int_E \psi_y(n)M^{-1}\psi_x(n)d\nu(n) = -\mathbb{E}\left(\psi_y(m(0))M^{-1}\psi_x(m(0))\right) \tag{14}$$

$$= \mathbb{E}\left(\psi_y(m(0))\int_0^\infty P_t\psi_x(m(0))dt\right) \tag{15}$$

$$= \mathbb{E}\left(\psi_y(m(0))\int_0^\infty \psi_x(m(t))dt\right) \tag{16}$$

$$= \mathbb{E}\left(m(0)(y)\int_0^\infty m(t)(x)dt\right), \tag{17}$$

where we have used the Markov property in the identity (15)-(16). We define $k \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ by the formula

$$k(x,y) = \mathbb{E}\int_{\mathbb{R}} m(0)(y)m(t)(x)dt, \quad x,y \in \mathbb{T}^d.$$

Let $F \in L^\infty(\mathbb{T}^d)$ be the trace

$$F(x) = k(x,x) = \mathbb{E}\int_{\mathbb{R}} m(0)(x)m(t)(x)dt, \quad x \in \mathbb{T}^d.$$

Note that, m being stationary,

$$\begin{aligned} k(x,y) &= \mathbb{E}\left(\int_0^\infty m(0)(y)m(t)(x)dt\right) + \mathbb{E}\left(\int_{-\infty}^0 m(0)(y)m(t)(x)dt\right) \\ &= \mathbb{E}\left(\int_0^\infty m(0)(y)m(t)(x)dt\right) + \mathbb{E}\left(\int_{-\infty}^0 m(-t)(y)m(0)(x)dt\right) \\ &= \mathbb{E}\left(m(0)(y)\int_0^\infty m(t)(x)dt\right) + \mathbb{E}\left(m(0)(x)\int_0^\infty m(t)(y)dt\right), \end{aligned} \tag{18}$$

so that k is symmetric. Let Q be the linear operator on $L^2(\mathbb{T}^d)$ associated to the kernel k:

$$Qf(x) = \int_{\mathbb{T}^d} k(x,y)f(y)dy.$$

Lemma 2.1. *The operator Q is self-adjoint, compact and non-negative: $(Qf,f) \geq 0$ for all $f \in L^2(\mathbb{T}^d)$.*

Proof. Q is self-adjoint and compact since k is symmetric and bounded. To prove that $(Qf,f) \geq 0$, we will need the following fact: if $\psi \in \mathscr{P}_M$ satisfies (8), then

$$\lim_{T\to+\infty} \mathbb{E}|P_T\psi(m(0))| = 0. \tag{19}$$

Indeed

$$\mathbb{E}|P_T\psi(m(0))| = \int_E |P_T\psi(n)|d\nu(n) \text{ and } |P_T\psi(n)| \leq \|\psi\|_{C_b(E)},$$

whence (19) by the mixing property (9) and by the dominated convergence Theorem. In particular, if $\psi \in \mathscr{P}_M$ satisfies (8) and if, furthermore, $M^{-1}\psi \in \mathscr{P}_M$, then

$$P_t\psi = P_tMM^{-1}\psi = \frac{d}{dt}P_tM^{-1}\psi,$$

hence

$$\mathbb{E}\left|\int_T^\infty P_t\psi(m(0))\right| = \mathbb{E}|P_TM^{-1}\psi(m(0))| \to 0 \tag{20}$$

when $T \to +\infty$. For simplicity, let us denote by ψ_f the function $\psi_{f,1}$. By (13), (14), (18) and (20), we have

$$\begin{aligned}(Qf, f) &= -2\mathbb{E}\left[\psi_f(m(0))M^{-1}\psi_f(m(0))\right] \\ &= 2\mathbb{E}\left[\psi_f(m(0))\int_0^T P_t\psi_f(m(0))dt\right] + o(1), \end{aligned} \tag{21}$$

when $T \to +\infty$. On the other hand, for $T > 0$, we compute

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left|\int_0^T \psi_f(m(t))dt\right|^2 &= \frac{1}{T}\int_0^T\int_0^T \mathbb{E}[\psi_f(m(t)))\psi_f(m(\tau))]dtd\tau && (22)\\ &= \frac{2}{T}\int_0^T\int_0^t \mathbb{E}[\psi_f(m(t))\psi_f(m(\tau))]dtd\tau && (23)\\ &= \frac{2}{T}\int_0^T\int_0^t \mathbb{E}[\psi_f(m(t-\tau))\psi_f(m(0))]dtd\tau && (24)\\ &= \frac{2}{T}\int_0^T\int_0^t \mathbb{E}[\psi_f(m(\tau))\psi_f(m(0))]dtd\tau \\ &= \frac{2}{T}\int_0^T (T-\tau)\mathbb{E}[\psi_f(m(\tau))\psi_f(m(0))]d\tau \\ &= 2\int_0^T \mathbb{E}[\psi_f(m(\tau))\psi_f(m(0))] + r_T \\ &= 2\mathbb{E}\left[\psi_f(m(0))\int_0^T P_t\psi_f(m(0))\right] + r_T, && (25)\end{aligned}$$

where we have use the homogeneity of $m(t)$ in (23)-(24). The remainder r_T satisfies

$$r_T = -2\mathbb{E}\left(\psi_f(m(0))\frac{1}{T}\int_0^T \tau P_\tau\psi_f(m(0))d\tau\right).$$

Since $\psi_f \in \mathscr{P}_M$, $P_\tau\psi_f = \frac{d}{d\tau}P_\tau M^{-1}\psi_f$: this gives

$$r_T = -2\mathbb{E}\left(\psi_f(m(0))\left[P_T M^{-1}\psi_f(m(0)) - \frac{1}{T}\int_0^T P_\tau\psi_f(m(0))d\tau\right]\right).$$

By (19), we obtain $r_T = o(1)$. By (21), (Qf, f) is the limit of the left-hand side of (22), which is non-negative, hence $(Qf, f) \geq 0$. □

As a result of Lemma 2.1, we can define the square root $Q^{1/2}$. Note that $Q^{1/2}$ is Hilbert-Schmidt on $L^2(\mathbb{T}^d)$ and that, denoting by $\|Q^{1/2}\|_{\mathscr{L}_2}$ its Hilbert-Schmidt norm, we have

$$\|Q^{1/2}\|^2_{\mathscr{L}_2} = \text{Tr } Q = \int_{\mathbb{T}^d} k(x,x)dx.$$

We will not analyze here in detail which kind of processes satisfies our assumptions. The requirement (11) that m and $M^{-1}m$ are a.s. bounded in $W^\infty(\mathbb{T}^d)$ are quite strong. An example of process we may consider is

$$m(t) = \sum_{j\in\mathbb{N}} m_j(t)\eta_j$$

with $\eta_j \in W^{1,\infty}(\mathbb{T}^d)$,

$$\sum_{j\in\mathbb{N}} \|\eta_j\|_{W^{1,\infty}(\mathbb{T}^d)} < \infty,$$

where the processes $(m_j)_{j\in\mathbb{N}}$ are independent real valued centered stationary, satisfying the bound

$$|m_j(t)| \leq C, \ a.s., \ t \in \mathbb{R},$$

for a given $C > 0$. We are then reduced to analysis on a product space. The invariant measure of m is then easily constructed from the invariant measures of the m_j's. Also, the Poisson equation can be solved provided each Poisson equation associated to m_j can be solved. This can easily be seen by working first on functions ψ depending only on a finite number of j.

The precise description of the sets D_M and $\mathscr{P}_M$ depends on the specific processes $m_j, \ j \in \mathbb{N}$. For instance, if m_j are Poisson processes taking values in finite sets S_j, then D_M and $\mathscr{P}_M$ can be taken as the set of bounded functions on $\prod_{i\in\mathbb{N}} S_j$. More general Poisson processes could be considered (see [10]).

Actually, the hypothesis (11) can be slightly relaxed. The boundedness assumption is used two times. First, in the proof of (30) and (31), but there it would be sufficient to know that m has finite exponential moments. It is used in a more essential way in Proposition 4. There, we need that the square of the norm of m and $M^{-1}m$ have some exponential moments. However, (under suitable assumptions on the variance of the processes for example), we may consider driving random terms given by Gaussian processes, or more generally diffusion processes.

2.3. **The deterministic equation.** There are also some structure hypotheses on the first and second moments of μ: we assume

$$\int_V a(v)d\mu(v) = 0, \tag{26}$$

and suppose that the following symmetric matrix is definite positive:

$$K := \int_V a(v) \otimes a(v)d\mu(v) > 0. \tag{27}$$

An example of (V, μ, a) satisfying the hypotheses above is given by $V = \S^{d-1}$ (the unit sphere of $\mathbb{R}^d$) with $\mu = d-1$-dimensional Hausdorff measure and $a(v) = v$.

In the deterministic case $m = 0$, the limit problem when $\varepsilon \to 0$ is a diffusion equation, as asserted in the following theorem.

Theorem 2.2 (Diffusion Limit in the deterministic case)**.** *Suppose $m \equiv 0$. Assume that (f_0^ε) is bounded in $L^2_{x,v}$ and that*

$$\rho_{0,\varepsilon} := \int_V f_0^\varepsilon d\mu \to \rho_0 \ \textit{in} \ H^{-1}(\mathbb{T}^d).$$

Assume (26)-(27). Then the density $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ converges in weak-$L^2_{t,x}$ to the solution ρ to the diffusion equation

$$\partial_t \rho - \operatorname{div}(K\nabla\rho) = 0 \ \textit{in} \ \mathbb{R}^+_t \times \mathbb{T}^d,$$

with initial condition: $\rho(0) = \rho_0$ in $\mathbb{T}^d$.

This result is a contained is [5] where a more general diffusive limit is analyzed. Note that, actually, strong convergence of (ρ^ε) can be proved by using compensated compactness, see [5] also.

2.4. **Main result.** In our context, the limit of the Problem (1)-(2) is a stochastic diffusion equation.

Theorem 2.3 (Diffusion Limit in the stochastic case)**.** *Assume that* (f_0^ε) *is bounded in* $L^2_{x,v}$ *and that*

$$\rho_{0,\varepsilon} := \int_V f_0^\varepsilon d\mu \to \rho_0 \text{ in } L^2(\mathbb{T}^d).$$

Assume (6)-(11)-(26)-(27). Then, for all $\eta > 0$*, the density* $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ *converges in law on* $C([0,T]; H^{-\eta})$ *to the solution* ρ *to the stochastic diffusion equation:*

$$d\rho = \operatorname{div}(K\nabla\rho)dt + \frac{1}{2}F\rho + \rho Q^{1/2}dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d, \tag{28}$$

with initial condition: $\rho(0) = \rho_0$ *in* $\mathbb{T}^d$*. In (28),* W *is a cylindrical Wiener process on* $L^2(\mathbb{T}^d)$*.*

It is not difficult to see that formally, (28) is the Itô form of the Stratonovitch equation

$$d\rho = \operatorname{div}(K\nabla\rho)dt + \rho \circ Q^{1/2}dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d. \tag{29}$$

Theorem 2.3 remains true in the slightly more general situation where the coefficient in factor of the noise in (1) is in the form $\frac{1}{\varepsilon}\sigma(f)m^\varepsilon$ with

$$\sigma(f) = \bar{\sigma}(\rho) + f, \quad \rho := \int_V f d\mu,$$

where $\bar{\sigma}$ is a smooth, sublinear function.

3. Resolution of the kinetic Cauchy Problem.

3.1. **Pathwise solutions.** Problem (1)-(2) is linear and solved for instance as follows. Let $A := a(v)\cdot\nabla_x$ denote the unbounded, skew-adjoint operator on $L^2_{x,v}$ with domain

$$D(A) := \{f \in L^2_{x,v}; a(v)\cdot\nabla_x f \in L^2_{x,v}\}.$$

Since A is closed and densely defined, by the Hille-Yosida Theorem [2], it defines a unitary group e^{tA} on $L^2_{x,v}$.

Theorem 3.1. *Assume (11). Then, for any* $f_0^\varepsilon \in L^2_{x,v}$ *and* $T > 0$*, there exists a unique solution* f^ε $\mathbb{P}$*-a.s. in* $C([0,T]; L^2_{x,v})$ *of (1)-(2) on* $[0,T]$*, in the sense that,*

$$f^\varepsilon(t) = e^{-\frac{t}{\varepsilon}A}f_0^\varepsilon + \int_0^t e^{-\frac{t-s}{\varepsilon}A}\left(\frac{1}{\varepsilon^2}Lf^\varepsilon(s) + f^\varepsilon(s)m^\varepsilon(s)\right)ds,$$

$\mathbb{P}$*-a.s., for all* $t \in [0,T]$*. Besides, if* $f_0^\varepsilon \in L^2(V; H^1(\mathbb{T}^d))$*, then,* $\mathbb{P}$*-a.s.* $f^\varepsilon \in C^1([0,T]; L^2_{x,v}) \cap C([0,T]; L^2(V; H^1(\mathbb{T}^d)))$*.*

The proof of this result is not difficult and left to the reader. The last statement is easily obtained since A commutes with derivatives with respect to x.

Energy estimates can be obtained. Indeed, for smooth integrable solutions f^ε to (1)-(2), we have the a priori estimate

$$\begin{aligned}\frac{d}{dt}\|f^\varepsilon(t)\|^2_{L^2} - \frac{2}{\varepsilon^2}(Lf^\varepsilon, f^\varepsilon) &= -\frac{2}{\varepsilon}(a(v)\cdot\nabla f^\varepsilon, f^\varepsilon) + \frac{2}{\varepsilon}(f^\varepsilon m^\varepsilon, f^\varepsilon)\\ &= \frac{2}{\varepsilon}(f^\varepsilon m^\varepsilon, f^\varepsilon).\end{aligned}$$

By (4) and (11), this gives the bound

$$\|f^\varepsilon(t)\|_{L^2}^2 + \frac{2}{\varepsilon^2}\int_0^t \|Lf^\varepsilon(s)\|_{L^2}^2 ds \le \|f_0^\varepsilon\|_{L^2}^2 + \frac{2C_*}{\varepsilon}\int_0^t \|f^\varepsilon(s)\|_{L^2_{x,v}}^2 ds,$$

hence, by Gronwall's Lemma, the following bound (depending on ε):

$$\|f^\varepsilon(t)\|_{L^2}^2 \le e^{\frac{2C_*}{\varepsilon}t}\|f_0^\varepsilon\|_{L^2}^2. \tag{30}$$

Similarly, we have

$$\|f^\varepsilon(t)\|_{L^2(H^1)}^2 \le e^{\frac{4C_*}{\varepsilon}t}\|f_0^\varepsilon\|_{L^2(H^1)}^2. \tag{31}$$

It is sufficient to assume $f_0^\varepsilon \in L^2(V;H^1(\mathbb{T}^d))$ (*resp.* $f_0^\varepsilon \in L^2(V;H^2(\mathbb{T}^d))$) to prove (30) (*resp.* (31)). By density, the inequality holds true for $f_0^\varepsilon \in L^2_{x,v}$ (*resp.* $f_0^\varepsilon \in L^2(V;H^1(\mathbb{T}^d))$). In particular, $\|f^\varepsilon(t)\|_{L^2}$ is uniformly bounded in $\omega \in \Omega$ if $f_0^\varepsilon \in L^2_{x,v}$ and $\|f^\varepsilon(t)\|_{L^2(H^1)}$ also if $f_0^\varepsilon \in L^2(V;H^1(\mathbb{T}^d))$.

3.2. **Generator.** The process f^ε is not Markov but the couple $(f^\varepsilon, m^\varepsilon)$ is. Its infinitesimal generator is given by :

$$\mathscr{L}^\varepsilon\varphi = \frac{1}{\varepsilon}\mathscr{L}_{A*}\varphi + \frac{1}{\varepsilon^2}\mathscr{L}_{L*}\varphi, \tag{32}$$

with

$$\begin{cases} \mathscr{L}_{A*}\varphi(f,n) = -(Af, D\varphi(f,n)) + (fn, D\varphi(f,n)), \\ \\ \mathscr{L}_{L*}\varphi(f,n) = (Lf, D\varphi(f,n)) + M\varphi(f,n). \end{cases}$$

These are differential operators with respect to the variables $f \in L^2_{x,v}$, $n \in E$. Here and in the following, D denotes differentiation with respect to f and we identify the differential with the gradient. For a C^2 function on $L^2_{x,v}$, we also use the second differential $D^2\varphi$ of a function φ, it is a bilinear form and we sometimes identify it with a bilinear operator on $L^2_{x,v}$, by the formula:

$$D^2\varphi(f)\cdot(h,k) = (D^2\varphi(f)h, k).$$

Let us define a set of test functions for the martingale problem associated to the generator $\mathscr{L}^\varepsilon$.

Definition 3.2. We say that Ψ is a good test function if

- $\Psi : L^2(V;H^1(\mathbb{T}^d)) \times E \to \mathbb{R}$, $(f,m) \mapsto \Psi(f,m)$ is differentiable with respect to f
- $(f,m) \mapsto D\Psi(f,m)$ is continuous from $L^2(V;H^1(\mathbb{T}^d)) \times E$ to $L^2_{x,v}$ and maps bounded sets onto bounded sets
- $(f,m) \mapsto M\Psi(f,m)$ is continuous from $L^2(V;H^1(\mathbb{T}^d)) \times E$ to $\mathbb{R}$ and maps bounded sets onto bounded sets of $\mathbb{R}$
- for any $f \in L^2(V;H^1(\mathbb{T}^d))$, $\Psi(f,\cdot) \in D_M$.

We have the following result.

Proposition 1. *Let Ψ be a good test function. Let $f_0^\varepsilon \in L^2(V;H^1(\mathbb{T}^d))$ and let f^ε be the solution to Problem (1)-(2). Then*

$$M_\Psi^\varepsilon(t) := \Psi(f^\varepsilon(t), m^\varepsilon(t)) - \int_0^t \mathscr{L}^\varepsilon\Psi(f^\varepsilon(s), m^\varepsilon(s))ds$$

is a continuous and integrable $(\mathcal{F}_t^\varepsilon)$ martingale with quadratic variation

$$\langle M_\Psi^\varepsilon, M_\Psi^\varepsilon\rangle(t) = \int_0^t (\mathscr{L}^\varepsilon|\Psi|^2 - 2\Psi\mathscr{L}^\varepsilon\Psi)(f^\varepsilon(s), m^\varepsilon(s))ds. \tag{33}$$

Proof. Let $s,t\geq 0$ and let $s=t_1<\cdots<t_n=t$ be a subdivision of $[s,t]$ such that $\max_i|t_{i+1}-t_i|=\delta$. We have for any $\mathcal{F}_s^\varepsilon$ measurable and bounded g

$$\begin{aligned}&\mathbb{E}\Big(\Big(\Psi(f^\varepsilon(t),m^\varepsilon(t))-\Psi(f^\varepsilon(s),m^\varepsilon(s))\Big)g\Big)\\&=\mathbb{E}\Big(\Big(\int_s^t\mathscr{L}^\varepsilon\Psi(f^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\Big)g\Big)+A+B,\end{aligned}$$

With

$$\begin{aligned}A=\sum_{i=1}^{n-1}\mathbb{E}\Big(\Big(&\Psi(f^\varepsilon(t_{i+1}),m^\varepsilon(t_{i+1}))-\Psi(f^\varepsilon(t_i),m^\varepsilon(t_{i+1}))\\&-\int_{t_i}^{t_{i+1}}(-\frac{1}{\varepsilon}Af^\varepsilon(\sigma)+\frac{1}{\varepsilon^2}Lf^\varepsilon(\sigma)+\frac{1}{\varepsilon}f^\varepsilon(\sigma)m^\varepsilon(\sigma),D\Psi(f^\varepsilon(\sigma),m^\varepsilon(\sigma)))d\sigma\Big)g\Big)\end{aligned}$$

and

$$\begin{aligned}B\quad=\sum_{i=1}^{n-1}\mathbb{E}\Big(\Big(&\Psi(f^\varepsilon(t_i),m^\varepsilon(t_{i+1}))-\Psi(f^\varepsilon(t_i),m^\varepsilon(t_i))\\&-\int_{t_i}^{t_{i+1}}M\Psi(f^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\Big)g\Big).\end{aligned}$$

We write

$$A=\mathbb{E}\Big(\Big(\int_0^t a_\delta(s)ds\Big)g\Big),$$

with

$$a_\delta(s)=\sum_{i=1}^{n-1}\mathbf{1}_{[t_i,t_{i+1}]}(s)\big(D\Psi(f^\varepsilon(s),m^\varepsilon(t_{i+1}))-D\Psi(f^\varepsilon(s),m^\varepsilon(s))\big)\frac{df^\varepsilon}{dt}(s).$$

Since $f_0^\varepsilon\in L^2(V;H^1(\mathbb{T}^d))$, we deduce from (31) and the assumption on Ψ that a_δ is uniformly integrable with respect to (s,ω). Also f^ε is almost surely continuous and m^ε is stochastically continuous. It follows that $D\Psi(f^\varepsilon(s),m^\varepsilon(t_{i+1}))-D\Psi(f^\varepsilon(s),m^\varepsilon(s))$ converges to 0 in probability when δ goes to zero for any s. By uniform integrability, we deduce that A converges to 0. Similarly, we have

$$B\quad=\sum_{i=1}^{n-1}\mathbb{E}\Big(\Big(\int_{t_i}^{t_{i+1}}M\Psi(f^\varepsilon(t_i),m^\varepsilon(\sigma))-M\Psi(f^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\Big)g\Big),$$

and, by the same argument, B converges to zero when δ goes to zero. The result follows : M_Ψ^ε is a continuous martingale. Since Ψ is a good test function and $f_0^\varepsilon\in L^2(V;H^1(\mathbb{T}^d))$, it follows from (31) and the bound (11) that $t\mapsto\Psi(f^\varepsilon(t),m^\varepsilon(t))$ and $t\mapsto\mathscr{L}^\varepsilon\Psi(f^\varepsilon(t),m^\varepsilon(t))$ are a.s. bounded. The expression (33) for the quadratic variation can then either be computed by expanding

$$\begin{aligned}\mathbb{E}|M_\Psi^\varepsilon(t)|^2=\mathbb{E}\Bigg(\Bigg[&\sum_{i=1,\dots,n-1}\Psi(f^\varepsilon,m^\varepsilon)(t_{i+1})-\Psi(f^\varepsilon,m^\varepsilon)(t_i)\\&-\int_{t_i}^{t_{i+1}}\mathscr{L}^\varepsilon\Psi(f^\varepsilon(s),m^\varepsilon(s))ds\Bigg]^2\Bigg),\end{aligned}$$

where $0=t_1<\cdots<t_n=t$ is an arbitrary subdivision of $[0,t]$ with step $\delta\downarrow 0$, or, quite similarly, by proceeding as in Appendix 6.9.1 in [10]. □

4. **The limit generator.** To prove the convergence of (ρ^ε), we use the method of the perturbed test-function [17]. The method of [17] has two steps: first construct a corrector φ^ε to φ so that $\mathscr{L}^\varepsilon\varphi^\varepsilon$ is controlled, then, in a second step, use this with particular test-functions to show the tightness of (ρ^ε). In the first step, we are led to identify the limit generator acting on φ.

4.1. **Correctors.** In this section, we try to understand the limit equation at $\varepsilon \to 0$. To that purpose, we investigate the limit of the generator $\mathscr{L}^\varepsilon$ by the method of perturbed test-function.

We restrict our study to smooth test functions and introduce the following class of functions. Let $\varphi \in C^3(L^2_{x,v})$. We say that φ is regularizing and subquadratic if there exists a constant $C_\varphi \geq 0$ such that

$$\begin{cases} |\varphi(f)| \leq C_\varphi(1+\|f\|_{L^2})^2, \\ \|A^m D\varphi(f)\|_{L^2} \leq C_\varphi(1+\|f\|_{L^2}), \\ |D^2\varphi(f)\cdot(A^{m_1}h, A^{m_2}k)| \leq C_\varphi\|h\|_{L^2}\|k\|_{L^2}, \\ |D^3\varphi(f)\cdot(A^{m_1}h, A^{m_2}k, A^{m_3}l)| \leq C_\varphi\|h\|_{L^2}\|k\|_{L^2}\|l\|_{L^2}, \end{cases} \tag{34}$$

for all $f, h, k, l \in L^2_{x,v}$, for all $m, m_i \in \{0, \cdots, 3\}$, $i = \{1, 2, 3\}$. Note that regularizing and subquadratic functions define good test functions (depending on f only).

Given φ regularizing and subquadratic, we want to construct φ_1, φ_2 good test functions, such that

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = \mathscr{L}\varphi(f,n) + \mathcal{O}(\varepsilon), \quad \varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

The limit generator $\mathscr{L}$ is to be determined. By the decomposition (32), this is equivalent to the system of equations

$$\mathscr{L}_{L*}\varphi = 0, \tag{35a}$$

$$\mathscr{L}_{A*}\varphi + \mathscr{L}_{L*}\varphi_1 = 0, \tag{35b}$$

$$\mathscr{L}_{A*}\varphi_1 + \mathscr{L}_{L*}\varphi_2 = \mathscr{L}\varphi(f,n), \tag{35c}$$

$$\mathscr{L}_{A*}\varphi_2 = \mathcal{O}(1). \tag{35d}$$

4.1.1. *Order ε^{-2}.* Equation (35a) constrains φ to depends on $\rho = \bar{f} = \int_V f d\mu$ uniquely:

$$\varphi(f) = \varphi(\rho), \quad \rho := \int_V f d\mu, \tag{36}$$

and imposes that the limit generator $\mathscr{L}$ acts on $\varphi(\bar{f})$ uniquely, as expected in the diffusive limit, in which we obtain an equation on the unknown $\int_V f d\mu$. Indeed, since φ is independent on n, (35a) reads

$$(Lf, D\varphi(f)) = 0. \tag{37}$$

Let $(g(t,f))_{t\geq 0}$ denote the flow of L on $L^2(V,\mu)$:

$$\frac{d}{dt}g(t,f) = Lg(t,f), \quad g(0,f) = f. \tag{38}$$

An explicit expression for g is

$$g(t,f) = \rho + e^{-t}(f-\rho), \quad \rho = \int_V f(v)d\mu(v).$$

In particular, $g(t,f) \to \rho$ exponentially fast in $L^2(V,\mu)$ when $t \to +\infty$. By (38), equation (37) is equivalent to

$$\varphi(f) = \varphi(g(t,f)), \quad \forall t \in \mathbb{R},$$

i.e. (36) by letting $t \to +\infty$.

4.1.2. *Order* ε^{-1}. Let us now solve the second equation (35b). To that purpose, we need to invert $\mathscr{L}_{L*}$. Let us work formally in a first step to derive a solution. Assume that $m(t,n)$ is a Markov process with generator M, let g be defined by (38) and consider the Markov process $(g(t,f), m(t,n))$. Its generator is precisely $\mathscr{L}_{L*}$. Denote by $(Q_t)_{t\geq 0}$ its transition semigroup. Since both g and m satisfy mixing properties, the couple (g,m) also. In particular, we have

$$Q_t\psi(f,n) \to \langle\psi\rangle(\bar{f}) := \int_E \psi(\bar{f},n)d\nu(n), \tag{39}$$

and it is expected that, under the necessary condition $\langle\mathscr{L}_{A*}\varphi\rangle = 0$, a solution to (35b) is given by

$$\varphi_1 = \int_0^\infty Q_t\mathscr{L}_{A*}\varphi dt.$$

Let us now compute $\mathscr{L}_{A*}\varphi$. By (36), we have for $h \in L^2_{x,v}$, $(h, D\varphi(f)) = (\bar{h}, D\varphi(\rho))$, where as above the upper bar denotes the average with respect to v and $\rho := \bar{f}$. Hence

$$\mathscr{L}_{A*}\varphi(f,n) = -(\overline{Af}, D\varphi(\rho)) + (\rho n, D\varphi(\rho)).$$

Since the first moments of $a(v)$ and $m(t)$ vanish, we have

$$\overline{A\rho} = 0 \quad \text{and} \quad \int_E (\rho n, D\varphi(\rho))d\nu(n) = 0,$$

and the cancellation condition $\langle\mathscr{L}_{A*}\varphi\rangle = 0$ is satisfied. We then write

$$\begin{aligned}\varphi_1(f,n) &= \int_0^\infty Q_t\mathscr{L}_{A*}\varphi(f,n)dt \\ &= \int_0^\infty \mathbb{E}\left(\mathscr{L}_{A*}\varphi(g(t,f), m(t,n))\right)dt.\end{aligned}$$

Note that g is deterministic and $\bar{g} = \rho$, so that

$$\begin{aligned}\varphi_1(f,n) &= \int_0^\infty -(\overline{Ag(t,f)}, D\varphi(\rho)) + \mathbb{E}\left((\rho m(t,n), D\varphi(\rho))\right)dt \\ &= -\int_0^\infty (\overline{Ag(t,f)}, D\varphi(\rho))dt - (\rho M^{-1}I(n), D\varphi(\rho)).\end{aligned}$$

Furthermore, regarding the term $\overline{Ag(t,f)}$, we have

$$\frac{d}{dt}Ag(t,f) = A\frac{d}{dt}g(t,f) = ALg(t,f) = A\bar{g}(t,f) - Ag(t,f).$$

Since $\overline{A\bar{f}} = 0$, we obtain $\dfrac{d}{dt}\overline{Ag(t,f)} = -\overline{Ag(t,f)}$, *i.e.*

$$\overline{Ag(t,f)} = e^{-t}\overline{Af}.$$

It follows that

$$\varphi_1(f,n) = -(\overline{Af}, D\varphi(\rho)) - (\rho M^{-1}I(n), D\varphi(\rho)).$$

By (36), this is also equivalent to

$$\varphi_1(f,n) = -(Af, D\varphi(f)) - (fM^{-1}I(n), D\varphi(f)). \tag{40}$$

This computation is formal but it is now easy to define φ_1 by (40) and to check that it satisfies (35b). It is also clear that φ_1 is a good test function.

Proposition 2 (First corrector)**.** *Let $\varphi \in C^3(L^2_{x,v})$ be regularizing and subquadratic according to (34). Assume that φ satisfy (36). Then (35b) has a solution $\varphi_1 \in C^1(L^2_{x,v} \times E)$ given by*

$$\varphi_1(f,n) = -(Af, D\varphi(f)) - (fM^{-1}I(n), D\varphi(f)), \tag{41}$$

for all $f \in L^2_{x,v}$, $n \in E$. Moreover φ_1 is a good test function.

4.1.3. *Order ε^0.* Let us now analyze Equation (35c). Setting $\rho = \bar{f}$, it gives

$$\mathscr{L}\varphi(\rho) = \langle \mathscr{L}_{A*}\varphi_1\rangle(\rho) = \int_E \mathscr{L}_{A*}\varphi_1(\rho,n)d\nu(n). \tag{42}$$

We have

$$\mathscr{L}_{A*}\psi(f,n) = (-Af + fn, D\psi(f,n))$$

and

$$\begin{aligned}\varphi_1(f,n) = (-Af - fM^{-1}I(n), D\varphi(f)) &= -(\overline{Af}, D\varphi(\rho)) - (\rho M^{-1}I(n), D\varphi(\rho)) \\ &=: \varphi_1^\sharp(f,n) + \varphi_1^*(f,n).\end{aligned}$$

By (42), the limit generator is therefore the sum of two terms:

$$\mathscr{L}\varphi(\rho) = \mathscr{L}_\sharp\varphi(\rho) + \mathscr{L}_*\varphi(\rho).$$

The first term $\mathscr{L}_\sharp\varphi(\rho)$ corresponds to the deterministic part of the equation. We compute, for $h \in L^2_{x,v}$,

$$(h, D\varphi_1^\sharp(f,n)) = -(\overline{Ah}, D\varphi(\rho)) - D^2\varphi(\rho)\cdot(\overline{Af}, \bar{h}).$$

In particular, evaluating at $f = \rho$ we have

$$(h, D\varphi_1^\sharp(\rho,n)) = -(\overline{Ah}, D\varphi(\rho))$$

since $\overline{A\rho} = 0$. Taking then $h = -A\rho + \rho n$ and using once again the cancellation property $\overline{A\rho} = 0$, we obtain

$$\mathscr{L}_\sharp\varphi(\rho) = \int_E (\overline{A^2\rho}, D\varphi(\rho))d\nu(n),$$

i.e.

$$\mathscr{L}_\sharp\varphi(\rho) = (\overline{A^2\rho}, D\varphi(\rho)). \tag{43}$$

The second part $\mathscr{L}_*$ corresponds to the random part of the equation: since $\overline{A\rho} = 0$,

$$\mathscr{L}_*\varphi(\rho) = \int_E (\rho n, D\varphi_1^*(\rho,n))d\nu(n), \quad \varphi_1^*(f,n) = -(\rho M^{-1}I(n), D\varphi(\rho)). \tag{44}$$

Now that $\mathscr{L}\varphi = \langle \mathscr{L}_{A*}\varphi_1\rangle$ has been identified, we go on with the resolution of (35c). At least formally at a first stage, we can set

$$\varphi_2(f,n) = -\int_0^\infty Q_t\{\langle \mathscr{L}_{A*}\varphi_1\rangle - \mathscr{L}_{A*}\varphi_1\}(f,n)dt.$$

To the decomposition $\varphi_1 = \varphi_1^\sharp + \varphi_1^*$ then corresponds a similar decomposition

$$\varphi_2 = \varphi_2^\sharp + \varphi_2^*$$

for φ_2. Since $\varphi_1^*(n)$ is linear with respect to n, the term

$$\mathscr{L}_{A*}\varphi_1^*(f,n) := (-Af + fn, D\varphi_1^*(f,n))$$

can be decomposed into two parts: one that is linear with respect to n, the second that is quadratic in n. The first (linear) part does not contribute to φ_2^* since $m(t)$ is centered: $\mathbb{E}m(t) = 0$. Let us thus compute the remaining part

$$q(f,n) := (fn, D\varphi_1^*(f,n)).$$

Since $\varphi_1^*(f,n)$ depends on ρ only, we have $q(f,n) = (\rho n, D\varphi_1^*(\rho,n))$. Since, along the flow of L, the density $\overline{g(t,f)} = \rho$ is constant, we obtain

$$\varphi_2^*(f,n) = -\int_0^\infty P_t \left\{ \int_E q(\rho,n)d\nu(n) - q(\rho,\cdot) \right\}(n)dt.$$

In particular, from the expression (44) for φ_1^* and the fact that φ is subquadratic and regularizing, it follows that $\varphi_2^* \in C(L^2_{x,v} \times E)$ is a good test function and satisfies

$$|\varphi_2^*(f,n)| \le C\left(1 + \|f\|^2_{L^2}\right), \tag{45}$$

$$|\mathscr{L}_{A*}\varphi_2^*(f,n)| \le C\left(1 + \|f\|^2_{L^2}\right), \tag{46}$$

for all $f \in L^2_{x,v}$, $n \in E$, where C is a constant depending on the constant C_* in (11) and on the constant C_φ. Similarly, $\mathscr{L}_{A*}\varphi_1^\sharp(f,n)$ is the sum of one term independent on n and one term linear with respect to n. This latter does not contribute to $\varphi_2^\sharp$ by the centering condition $\mathbb{E}m(t) = 0$. We explicitly compute the first term:

$$(-Af, D\varphi_1^\sharp(f,n)) = (\overline{A^2 f}, D\varphi(\rho)) + D^2\varphi(\rho)\cdot(\overline{Af}, \overline{Af}).$$

We have already proved (*cf* Section 4.1.2) that, along the flow $g(t,f)$ of L, we have $\overline{Ag(t,f)} = e^{-t}\overline{Af}$. Similarly, we have

$$\overline{A^2 g(t,f)} - \overline{A^2\rho} = e^{-t}(\overline{A^2 f} - \overline{A^2\rho}).$$

By integrating the exponential e^{-t} with respect to t, it follows that

$$\varphi_2^\sharp(f,n) = (\overline{A^2 f} - \overline{A^2\rho}, D\varphi(\rho)) + \frac{1}{2}D^2\varphi(\rho)(\overline{Af}, \overline{Af}).$$

In particular, $\varphi_2^\sharp \in C(L^2_{x,v} \times E)$ is a good test function and satisfies (45)-(46). By introducing $\mathcal{A}\rho = \operatorname{div}(K\nabla\rho)$, where K is given by (5), to identify $\mathscr{L}_\sharp$ in (43), and by developing the expression (44) for $\mathscr{L}_*$, we obtain the following result.

Proposition 3 (Second corrector)**.** *Let $\varphi \in C^3(L^2_{x,v})$ be regularizing and subquadratic according to (34). Assume (36) and (6), (11), (26), (27). Let $\mathcal{A}$ denote the unbounded operator defined by*

$$\mathcal{A}\rho = \operatorname{div}(K\nabla\rho), \quad D(\mathcal{A}) = H^2(\mathbb{T}^d) \subset L^2(\mathbb{T}^d).$$

Then (35c) is satisfied for $\mathscr{L}$ defined by: $\forall \psi \in C^2(L^2(\mathbb{T}^d))$,

$$\begin{aligned}\mathscr{L}\psi(\rho) &= (\mathcal{A}\rho, D\psi(\rho)) \\ &\quad - \int_E \left\{ (\rho n M^{-1} I(n), D\varphi(\rho)) + D^2\varphi(\rho)\cdot(\rho M^{-1} I(n), \rho n) \right\} d\nu(n),\end{aligned} \tag{47}$$

and a corrector $\varphi_2 \in C(L^2_{x,v} \times E)$ *which is a good test function and satisfies*

$$|\varphi_2(f,n)| \leq C\left(1 + \|f\|^2_{L^2}\right),$$
$$|\mathscr{L}_{A*}\varphi_2(f,n)| \leq C\left(1 + \|f\|^2_{L^2}\right),$$

for all $f \in L^2_{x,v}$, $n \in E$, *where* C *is a constant depending on the constant* C_* *in (11) and on the constant* C_φ.

4.2. **Limit equation.** We will show here that $\mathscr{L}_*$ is the generator of the semi-group associated to a diffusion process on $L^2(\mathbb{T}^d)$. Then (44) is a form of $\mathscr{L}_*$ corresponding to the Stratonovitch formulation of the corresponding stochastic differential equation. Actually, we use the expanded form of (47) (which corresponds to a stochastic differential equation in Itô form) to identify more precisely the limit generator $\mathscr{L}$. The notations for F, k, Q are those introduced in Section 2.2. We have first:

$$\begin{aligned}
-\int_E (\rho n M^{-1} I(n), D\varphi(\rho)) d\nu(n) &= \mathbb{E}\int_0^\infty (\rho m(0) m(t), D\varphi(\rho)) dt \\
&= \frac{1}{2}\mathbb{E}\int_{\mathbb{R}} (\rho m(0) m(t), D\varphi(\rho)) dt \\
&= \frac{1}{2}(\rho F, D\varphi(\rho)),
\end{aligned}$$

where

$$F(x) := \mathbb{E}\int_{\mathbb{R}} m(0)(x) m(t)(x) dt = k(x,x).$$

To recognize the part containing $D^2\varphi$, we identify $D^2\varphi$ with its Hessian and first assume that it is associated to a kernel Φ. Then, we write:

$$\begin{aligned}
&-\int_E D^2\varphi(\rho)\cdot(\rho M^{-1} n, \rho n) d\nu(n) \\
&= \mathbb{E}\int_0^\infty D^2\varphi(\rho)\cdot(\rho m(t), \rho m(0)) dt \\
&= \frac{1}{2}\mathbb{E}\int_{\mathbb{R}} D^2\varphi(\rho)\cdot(\rho m(t), \rho m(0)) dt \\
&= \frac{1}{2}\mathbb{E}\int_{\mathbb{R}} (D^2\varphi(\rho)(\rho m(t)), \rho m(0)) dt \\
&= \frac{1}{2}\mathbb{E}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \Phi(x,y)\rho(x) m(t)(x)\rho(y) m(0)(y) dx dy dt \\
&= \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \Phi(x,y) k(x,y)\rho(x)\rho(y) dx dy.
\end{aligned}$$

Denote by q the kernel of $Q^{1/2}$, then

$$k(x,y) = \int_{\mathbb{T}^d} q(x,z) q(y,z) dz,$$

which gives

$$\int_E D^2\varphi(\rho)\cdot(\rho M^{-1}n,\rho n)d\nu(n)$$
$$= \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\Phi(x,y)q(x,z)q(y,z)\rho(x)\rho(y)dxdydz$$
$$= \frac{1}{2}\mathrm{Trace}[(\rho Q^{1/2})D^2\varphi(\rho)(\rho Q^{1/2})^*]. \quad (48)$$

By approximation, this formula holds for all C^2 function φ. We conclude that $\mathscr{L}$ is the generator associated to the stochastic PDE

$$d\rho = \mathrm{div}(K\nabla\rho)dt + \frac{1}{2}F\rho dt + \rho Q^{1/2}dW(t), \quad (49)$$

where W is a cylindrical Wiener process.

4.3. **Summary.** By Proposition 1, Proposition 2 and Proposition 3, we deduce:

Corollary 1. *Let $\varphi \in C^3(L^2_{x,v})$ be a regularizing and subquadratic function satisfying (36). There exist two good test functions φ_1, φ_2 such that, defining $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$,*

$$|\varphi_1(f,n)| \le C\left(1+\|f\|^2_{L^2}\right),$$
$$|\varphi_2(f,n)| \le C\left(1+\|f\|^2_{L^2}\right), \quad (50)$$
$$|\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) - \mathscr{L}\varphi(f,n)| \le C\varepsilon\left(1+\|f\|^2_{L^2}\right),$$

for all $f \in L^2_{x,v}$, $n \in E$, where C is a constant depending on the constant C_* in (11) and C_φ. Moreover

$$M^\varepsilon(t) := \varphi^\varepsilon(f^\varepsilon(t),m^\varepsilon(t)) - \int_0^t \mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s),m^\varepsilon(s))ds, \quad t\ge 0,$$

is a continuous integrable martingale for the filtration $(\mathcal{F}^\varepsilon_t)$ generated by m^ε with quadratic variation

$$\langle M^\varepsilon, M^\varepsilon\rangle(t) = \int_0^t \left\{(M|\varphi_1|^2 - 2\varphi_1 M\varphi_1)(f^\varepsilon(s),m^\varepsilon(s)) + r_\varepsilon(t)\right\}ds, \quad (51)$$

where

$$|r_\varepsilon(t)| \le C\varepsilon\int_0^t (1+\|f^\varepsilon(t)\|^4_{L^2})ds, \quad (52)$$

for a constant C depending on C_* and C_φ. Finally, for $0\le s_1\le\cdots\le s_n\le s\le t$ and $\psi\in C_b((L^2_{x,v})^n)$,

$$\left|\mathbb{E}\left(\left(\varphi(\rho^\varepsilon(t)) - \varphi(\rho^\varepsilon(s)) - \int_s^t \mathscr{L}\varphi(\rho^\varepsilon(\sigma))d\sigma\right)\psi(\rho^\varepsilon(s_1),\dots,\rho^\varepsilon(s_n))\right)\right|$$
$$\le C\varepsilon\left(1+\sup_{s\in[0,T]}\mathbb{E}\|f^\varepsilon(t)\|^2_{L^2}\right), \quad (53)$$

with another constant C depending on the constant C_* in (11), on C_φ and on the supremum of ψ.

Proof. Everything has already been proved except for (51) and the last statement (53). For this latter, it suffices to write:

$$\varphi(\rho^\varepsilon(t)) - \varphi(\rho^\varepsilon(s)) - \int_s^t \mathscr{L}\varphi(\rho^\varepsilon(\sigma))d\sigma$$

$$= M^\varepsilon(t) - M^\varepsilon(s) - \varepsilon\varphi_1(\rho^\varepsilon(t)) - \varepsilon^2\varphi_2(\rho^\varepsilon(t)) + \varepsilon\varphi_1(\rho^\varepsilon(s)) + \varepsilon^2\varphi_2(\rho^\varepsilon(s))$$

$$- \int_s^t \Big(\mathscr{L}\varphi(\rho^\varepsilon(\sigma)) - \mathscr{L}^\varepsilon\varphi^\varepsilon(\rho^\varepsilon(\sigma))\Big)d\sigma.$$

Then, we multiply by $\psi(\rho^\varepsilon(s_1), \dots, \rho^\varepsilon(s_n))$, take the expectation and use the bounds (50) to conclude. Furthermore,

$$\mathcal{M}|\varphi|^2 - 2\varphi\mathcal{M}\varphi = 0 \tag{54}$$

if $\varphi \mapsto \mathcal{M}\varphi$ is a linear first order operator in φ. Applying (54) to

$$\mathcal{M}\varphi(f,n) = \frac{1}{\varepsilon}\mathscr{L}_{A*}\varphi(f,n) + \frac{1}{\varepsilon^2}(Lf, D\varphi(f,n))$$

gives

$$\mathscr{L}^\varepsilon|\varphi^\varepsilon|^2 - 2\varphi^\varepsilon\mathscr{L}^\varepsilon\varphi^\varepsilon = M|\varphi_1|^2 - 2\varphi_1 M\varphi_1 + r_\varepsilon.$$

By (50), the remainder r_ε satisfies (52). □

5. **Diffusive limit.** Our aim now is to prove the convergence in law of $\rho^\varepsilon = \int_V f^\varepsilon d\mu$ to ρ, solution to (29), or equivalently Equation (49). To that purpose, we use again the perturbed test function method to get a bound on the solutions in $L^2_{x,v}$ then we prove that ρ^ε is tight in $C([0,T]; H^{-\eta})$, $\eta > 0$. We follow (and adapt to our context) the method in [10], paragraph 6.3.5. In particular, we use Kolmogorov criterion to get tightness in section 5.2; an alternative method would be to use Aldous' criterion for tightness (*e.g.* Theorem 4.5 in [13]).

5.1. Bound in $L^2_{x,v}$.

Proposition 4 (Uniform $L^2_{x,v}$ bound)**.** *Assume (11). Then, for all $T > 0$, $p \geq 1$, we have*

$$\mathbb{E} \sup_{t\in[0,T]} \|f^\varepsilon(t)\|^p_{L^2} \leq C$$

where the constant $C \geq 0$ depends on T, on p, on $\|a\|_{L^\infty(V)}$, on the constant C_* in (11) and $\sup_{\varepsilon>0}\|f^\varepsilon_0\|_{L^2}$ only.

Proof. Fix $p \geq 2$. Let us write $a(\varepsilon,t) \lesssim b(\varepsilon,t)$ if there exists a constant C depending on T, on p, on $\|a\|_{L^\infty(V)}$ and on the constant C_* in (11) only such that $a(\varepsilon,t) \leq Cb(\varepsilon,t)$ for all $t \in [0,T]$. Set $\varphi(f) := \frac{1}{2}\|f\|^2_{L^2}$. We want to apply Corollary 1 to φ. This requires some care since φ is actually a function of f and not of ρ. Thus, we first seek for one corrector $\varphi_1 \in C^2(L^2_{x,v} \times E)$ such that, for the modified test-function

$$\varphi^\varepsilon := \varphi + \varepsilon\varphi_1,$$

the term $\mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon, m^\varepsilon)$ can be accurately controlled: for $f \in L^2(V; H^1(\mathbb{T}^d))$, $n \in E$, we compute

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n) = \varepsilon^{-2}\mathscr{L}_{L*}\varphi(f) + \varepsilon^{-1}(\mathscr{L}_{A*}\varphi + \mathscr{L}_{L*}\varphi_1)(f,n) + \mathscr{L}_{A*}\varphi_1(f,n). \tag{55}$$

Since $M\varphi(f,n)=0$ (φ being independent on n), and since $D\varphi(f,n)=f$, the first term in (55) is

$$\varepsilon^{-2}\mathscr{L}_{L*}\varphi(f)=-\frac{1}{\varepsilon^2}\|Lf\|_{L^2}^2, \tag{56}$$

which has a favorable sign. Since A is skew-symmetric, $\mathscr{L}_{A*}\varphi(f,n)=(fn,f)$. This term is difficult to control and we choose φ_1 to compensate it. We set

$$\varphi_1(f,n)=-(fM^{-1}I(n),f),$$

so that $M\varphi_1=-(fn,f)$ and the second term in (55) is

$$\varepsilon^{-1}(\mathscr{L}_{A*}\varphi+\mathscr{L}_{L*}\varphi_1)=\frac{1}{\varepsilon}(Lf,D\varphi_1(f,n))=-\frac{2}{\varepsilon}(Lf,fM^{-1}I(n)).$$

By (11), we obtain

$$\varepsilon^{-1}(\mathscr{L}_{A*}\varphi+\mathscr{L}_{L*}\varphi_1)(f^\varepsilon(t),m^\varepsilon(t))\leq\frac{1}{4\varepsilon^2}\|Lf^\varepsilon(t)\|_{L^2}^2+4C_*^2\|f^\varepsilon(t)\|_{L^2}^2. \tag{57}$$

The remainder $\mathscr{L}_{A*}\varphi_1$ satisfies the following bounds

$$\begin{aligned}\mathscr{L}_{A*}\varphi_1(f,n)&=-(Af,fM^{-1}I(n))+(fn,fM^{-1}I(n))\\&=\frac{1}{2}(f^2,AM^{-1}I(n))+(fn,fM^{-1}I(n))\\&\leq\left(\frac{1}{2}\|AM^{-1}I(n)\|_{L^\infty_{x,v}}+\|M^{-1}I(n)\|_{L^\infty_{x,v}}\|n\|_{L^\infty_{x,v}}\right)\|f\|_{L^2}^2\\&\leq\left(\frac{1}{2}\|a\|_{L^\infty_v}\|M^{-1}I(n)\|_{W^{1,\infty}_x}+\|M^{-1}I(n)\|_{L^\infty_{x,v}}\|n\|_{L^\infty_{x,v}}\right)\|f\|_{L^2}^2.\end{aligned}$$

By (11), (56), (57), we obtain

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(t),m^\varepsilon(t))\lesssim\|f^\varepsilon(t)\|_{L^2}^2. \tag{58}$$

Set

$$M^\varepsilon(t):=\varphi^\varepsilon(f^\varepsilon(t),m^\varepsilon(t))-\varphi^\varepsilon(f_0^\varepsilon,m^\varepsilon(0))-\int_0^t\mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s),m^\varepsilon(s))ds.$$

By definition of φ, φ^ε and M^ε, we have

$$\begin{aligned}\frac{1}{2}\|f^\varepsilon(t)\|_{L^2_{x,v}}^2=\frac{1}{2}\|f_0^\varepsilon\|_{L^2_{x,v}}^2-\varepsilon(\varphi_1(f^\varepsilon(t),m^\varepsilon(t)))-\varphi_1(f_0^\varepsilon,m^\varepsilon(0)))\\+\int_0^t\mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s),m^\varepsilon(s))ds+M^\varepsilon(t).\end{aligned}$$

By (58) and the estimate

$$|\varphi_1(f^\varepsilon(t),m^\varepsilon(t))|\lesssim\|f^\varepsilon(t)\|_{L^2_{x,v}}^2, \tag{59}$$

we deduce the bound

$$\|f^\varepsilon(t)\|_{L^2_{x,v}}^2\lesssim\|f_0^\varepsilon\|_{L^2_{x,v}}^2+\varepsilon\|f^\varepsilon(t)\|_{L^2_{x,v}}^2+\int_0^t\|f^\varepsilon(s)\|_{L^2_{x,v}}^2ds+\sup_{t\in[0,T]}|M^\varepsilon(t)|.$$

For ε small enough, it follows that

$$\|f^\varepsilon(t)\|_{L^2_{x,v}}^2\lesssim\|f_0^\varepsilon\|_{L^2_{x,v}}^2+\int_0^t\|f^\varepsilon(s)\|_{L^2_{x,v}}^2ds+\sup_{t\in[0,T]}|M^\varepsilon(t)|,$$

and, by Gronwall Lemma,

$$\|f^\varepsilon(t)\|_{L^2_{x,v}}^2\lesssim\|f_0^\varepsilon\|_{L^2_{x,v}}^2+\sup_{t\in[0,T]}|M^\varepsilon(t)|. \tag{60}$$

On the other hand, similarly to (51), we have

$$\langle M^\varepsilon, M^\varepsilon\rangle(t) = \int_0^t (M|\varphi_1|^2 - 2\varphi_1 M\varphi_1)(f^\varepsilon(s), m^\varepsilon(s))ds.$$

(Note that there is no remaining terms here since $\varphi_2 \equiv 0$, *cf.* the proof of (51) in Corollary 1.) In particular, by (59) and the similar estimate for $M\varphi_1$, we have

$$|\langle M^\varepsilon, M^\varepsilon\rangle(t)| \lesssim \int_0^t \|f^\varepsilon(s)\|^4_{L^2_{x,v}} ds.$$

Since M^ε is a martingale with $\mathbb{E}M^\varepsilon(t) = 0$, Burkholder-Davis-Gundy inequality gives

$$\mathbb{E}[\sup_{t\in[0,T]} |M^\varepsilon(t)|^p|] \leq C_p \mathbb{E}|\langle M^\varepsilon, M^\varepsilon\rangle(T)|^{p/2} \lesssim \int_0^T \mathbb{E}\|f^\varepsilon(s)\|^{2p}_{L^2_{x,v}} ds. \tag{61}$$

By (60), $\mathbb{E}\|f^\varepsilon(t)\|^{2p}_{L^2_{x,v}} \lesssim \mathbb{E}\|f^\varepsilon_0\|^{2p}_{L^2_{x,v}} + \mathbb{E}[\sup_{t\in[0,T]} |M^\varepsilon(t)|^p]$. Hence, by (61),

$$\mathbb{E}\|f^\varepsilon(T)\|^{2p}_{L^2_{x,v}} \lesssim \mathbb{E}\|f^\varepsilon_0\|^{2p}_{L^2_{x,v}} + \int_0^T \mathbb{E}\|f^\varepsilon(s)\|^{2p}_{L^2_{x,v}} ds.$$

By Gronwall Lemma, we obtain $\mathbb{E}\|f^\varepsilon(T)\|^{2p}_{L^2_{x,v}} \lesssim \mathbb{E}\|f^\varepsilon_0\|^{2p}_{L^2_{x,v}}$. This actually holds true for any $t \in [0,T]$. Thus, using (61) and then (60) gives finally $\mathbb{E}[\sup_{t\in[0,T]} \|f^\varepsilon(t)\|^{2p}_{L^2_{x,v}}] \lesssim \mathbb{E}\|f^\varepsilon_0\|^{2p}_{L^2_{x,v}}$. □

5.2. **Tightness.**

Proposition 5 (Tightness)**.** *Let $T > 0$, $\eta > 0$. Assume (6), (11), (26), (27) and assume that (f^ε_0) is bounded in L^2. Then (ρ^ε) is tight in $C([0,T]; H^{-\eta}(\mathbb{T}^d))$.*

Proof. Let $\varphi(\rho) = \rho$, or, more precisely (since the perturbed test-function method has been developed for real-valued, regularizing functions), define the test-function φ_j as follows. Let $\{p_j; j \geq 1\}$ be a complete orthonormal system in $L^2(\mathbb{T}^d)$, let $\gamma > \max\{3, d\}$ and let

$$J = (\mathrm{Id} - \Delta_x)^{-1/2},$$

where Id is the identity on $L^2(\mathbb{T}^d)$. Note that $\|\rho\|_{H^{-\gamma}(\mathbb{T}^d)} = \|J^\gamma\rho\|_{L^2}$ and that J^γ is Hilbert-Schmidt on $L^2(\mathbb{T}^d)$ since $\gamma > d$. We set $\varphi_j(\rho) = (J^\gamma\rho, p_j)$. It is clear that φ_j is subquadratic (it is linear) and regularizing as in (34) (the operator $\nabla^3 J^\gamma$ is of order ≤ 0). Let φ^ε_j be the correction of φ_j given by Corollary 1. Let

$$M^\varepsilon_j(t) = \varphi^\varepsilon_j(f^\varepsilon(t), m^\varepsilon(t)) - \varphi^\varepsilon_j(f^\varepsilon_0, m^\varepsilon(0)) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon_j(f^\varepsilon(s), m^\varepsilon(s))ds$$

and

$$\theta^\varepsilon_j(t) = \varphi_j(\rho^0) + \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon_j(f^\varepsilon(s), m^\varepsilon(s))ds + M^\varepsilon_j(t).$$

Then

$$\varphi_j(\rho^\varepsilon(t)) - \theta^\varepsilon_j(t) = [\varphi_j(\rho^\varepsilon(t)) - \varphi^\varepsilon_j(f^\varepsilon(t), m^\varepsilon(t))] - [[\varphi_j(\rho^\varepsilon_0) - \varphi^\varepsilon_j(f^\varepsilon_0, m^\varepsilon(0))]].$$

By the estimates (50) on the correctors of φ_j and the L^2-bounds of Proposition 4, we deduce that

$$\mathbb{E}[\sup_{t\in[0,T]} |\varphi_j(\rho^\varepsilon(t)) - \theta^\varepsilon_j(t)|] \lesssim \varepsilon, \tag{62}$$

where we write $a(j,\varepsilon) \lesssim b(j,\varepsilon)$ if there exists a constant C depending on $\|a\|_{L^\infty(V)}$, on T, on the constant C_* in (11) and on $\sup_{\varepsilon>0}\|f_0^\varepsilon\|_{L^2}$, but not on ε and j such that $a(j,\varepsilon) \le Cb(j,\varepsilon)$. Note also that, by (50), $\mathbb{E}\sup_{t\in[0,T]}|\theta_j^\varepsilon(t)| \lesssim 1$, hence

$$\theta^\varepsilon(t) := J^{-\gamma}\sum_{j\ge 1}\theta_j^\varepsilon(t)J^\gamma p_j$$

is a.s. well defined for all $t \in [0,T]$ in $H^{-\gamma}(\mathbb{T}^d)$ since the sum is convergent in $L^2(\mathbb{T}^d)$. By (62), we obtain

$$\mathbb{E}[\sup_{t\in[0,T]}\|\rho^\varepsilon(t)-\theta^\varepsilon(t)\|_{H^{-\gamma}(\mathbb{T}^d)}] \lesssim \varepsilon. \tag{63}$$

Let $\eta > 0$. Let

$$w(\rho,\delta) := \sup_{|t-s|<\delta}\|\rho(t)-\rho(s)\|_{H^{-\eta}(\mathbb{T}^d)}$$

denote the modulus of continuity of a function $\rho \in C([0,T];H^{-\eta}(\mathbb{T}^d))$. Since the injection $L^2(\mathbb{T}^d) \subset H^{-\eta}(\mathbb{T}^d)$ is compact, the set

$$K_R = \left\{\rho \in C([0,T];H^{-\eta}(\mathbb{T}^d));\ \sup_{t\in[0,T]}\|\rho(t)\|_{L^2} \le R; w(\rho,\delta) \le \varepsilon(\delta)\right\},$$

where $R >$ and $\varepsilon(\delta) \to 0$ when $\delta \to 0$, is compact in $C([0,T];H^{-\eta}(\mathbb{T}^d))$ (Ascoli's Theorem). By Prokhorov's Theorem, the tightness of (ρ^ε) will follow if we prove that, for all $\alpha > 0$, there exists $R > 0$, such that

$$\mathbb{P}(\sup_{t\in[0,T]}\|\rho^\varepsilon(t)\|_{L^2} > R) < \alpha, \tag{64}$$

and

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}(w(\rho^\varepsilon,\delta) > \alpha) = 0. \tag{65}$$

The estimate (64) follows from the L^2-bound of Proposition 4 by the estimate

$$\mathbb{P}(\sup_{t\in[0,T]}\|\rho^\varepsilon(t)\|_{L^2} > R) \le \frac{1}{R}\mathbb{E}[\sup_{t\in[0,T]}\|\rho^\varepsilon(t)\|_{L^2}].$$

Similarly, we will deduce (65) from a bound on $\mathbb{E}w(\rho^\varepsilon,\delta)$ for $\delta > 0$. Actually, by the L^2-bound of Proposition 4 and by interpolation, we have

$$\mathbb{E}\sup_{|t-s|<\delta}\|\rho(t)-\rho(s)\|_{H^{-\eta^\flat}(\mathbb{T}^d)} \le \mathbb{E}\sup_{|t-s|<\delta}\|\rho(t)-\rho(s)\|^\sigma_{H^{-\eta^\sharp}(\mathbb{T}^d)}$$

for a certain $\sigma > 0$ if $\eta^\sharp > \eta^\flat > 0$. Therefore it is indeed sufficient to work with $\eta = \gamma$. Besides, by (63), it is sufficient to obtain an estimate on the increments of θ^ε. By definition

$$\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s) = \int_s^t \mathscr{L}^\varepsilon\varphi_j^\varepsilon(f^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma + M_j^\varepsilon(t) - M_j^\varepsilon(s),$$

for $0 \le s \le t \le T$. By the L^2-bound of Proposition 4 and by (50), we have

$$\mathbb{E}\left|\int_s^t \mathscr{L}^\varepsilon\varphi_j^\varepsilon(f^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\right|^4 \lesssim |t-s|^4.$$

By Burkholder-Davis-Gundy inequality,

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim \mathbb{E}|\langle M_j^\varepsilon, M_j^\varepsilon\rangle(t) - \langle M_j^\varepsilon, M_j^\varepsilon\rangle(s)|^2,$$

where $\langle M_j^\varepsilon, M_j^\varepsilon\rangle$ is the quadratic variation of M_j^ε. By (51) and the L^2-bound of Proposition 4, we obtain

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim |t-s|^2.$$

Finally, we have $\mathbb{E}|\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s)|^4 \lesssim |t-s|^2$, and thus

$$\mathbb{E}\|\theta^\varepsilon(t) - \theta^\varepsilon(s)\|^4_{H^{-\gamma}(\mathbb{T}^d)} \lesssim |t-s|^2.$$

It follows (by the Kolmogorov's criterion) that, for $\alpha < 1/2$,

$$\mathbb{E}\|\theta^\varepsilon\|^4_{W^{\alpha,4}(0,T;H^{-\gamma}(\mathbb{T}^d))} \lesssim 1.$$

By the embedding

$$W^{\alpha,4}(0,T;H^{-\gamma}(\mathbb{T}^d)) \subset C^{0,\mu}([0,T];H^{-\gamma}(\mathbb{T}^d)), \quad \mu < \alpha - \frac{1}{4},$$

we obtain $\mathbb{E}w(\theta^\varepsilon,\delta) \lesssim \delta^\mu$ for a certain positive μ. This concludes the proof of the proposition. □

5.3. **Convergence.** We conclude here the proof of Theorem 2.3. Fix $\eta > 0$. By Proposition 5, there is a subsequence still denoted by (ρ^ε) and a probability measure P on $C([0,T];H^{-\eta}(\mathbb{T}^d))$ such that

$$P^\varepsilon \to P \quad \text{weakly on} \quad C([0,T];H^{-\eta}(\mathbb{T}^d))$$

where P^ε is the law of ρ^ε. We then show that P is a solution of the martingale problem, with a set of test functions specified below, associated to the limit equation (28).

By Skohorod representation Theorem [1], and since $C([0,T];H^{-\eta}(\mathbb{T}^d))$ is separable, there exists a new probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and some random variables

$$\widetilde{\rho}^\varepsilon, \widetilde{\rho}\colon \widetilde{\Omega} \to C([0,T];H^{-\eta}(\mathbb{T}^d)),$$

with respective law P^ε and P such that $\widetilde{\rho}^\varepsilon \to \widetilde{\rho}$ in $C([0,T];H^{-\eta}(\mathbb{T}^d))$, $\widetilde{\mathbb{P}}$ a.s.

Let $\varphi \in C^3(L^2(\mathbb{T}^d))$ be regularizing and subquadratic according to (34). By Corollary 1 and the L^2-bound of Proposition 4, we have for $0 \le s_1 \le \cdots \le s_n \le s \le t$ and $\psi \in C_b((L^2_{x,v})^n)$,

$$\left|\mathbb{E}\left[\left(\varphi(\rho^\varepsilon(t)) - \varphi(\rho^\varepsilon(s)) - \int_s^t \mathscr{L}\varphi(\rho^\varepsilon(\sigma))d\sigma\right)\psi(\rho^\varepsilon(s_1),\dots,\rho^\varepsilon(s_n))\right]\right| \le C\varepsilon \quad (66)$$

with a constant C depending on the constant C_* in (11), on C_φ, on $\sup_{\varepsilon>0}\|f_0^\varepsilon\|_{L^2}$ and on the supremum of ψ. Since ρ^ε and $\widetilde{\rho}^\varepsilon$ have the same law, this is still true if ρ^ε is replaced by $\widetilde{\rho}^\varepsilon$. Assume furthermore that φ is bounded and continuous from $H^{-\eta}(\mathbb{T}^d)$ into $\mathbb{R}$, then it is easy to take the limit $\varepsilon \to 0$ in (66) and to obtain

$$\widetilde{\mathbb{E}}\left\{\left[\varphi(\widetilde{\rho}(t)) - \varphi(\widetilde{\rho}(s)) - \int_s^t \mathscr{L}\varphi(\widetilde{\rho}(\sigma))d\sigma\right]\psi(\widetilde{\rho}(s_1),\dots,\widetilde{\rho}(s_n))\right\} = 0. \quad (67)$$

The additional hypothesis on φ can be relaxed. Indeed, thanks to Proposition 4, we can approximate every subquadratic and regularizing functions by functions in $C_b(H^{-\eta}(\mathbb{T}^d))$ which are subquadratic and regularizing with a uniform constant in (34) and which converge pointwise.

We have thus proved that P solves the martingale problem associated to $\mathscr{L}$ with subquadratic and regularizing test functions. In particular, for all such φ:

$$M_\varphi(t) = \varphi(\rho(t)) - \int_0^t \mathscr{L}\varphi(\rho(s))ds, \quad t \geq 0, \tag{68}$$

is a martingale with respect to the filtration $\mathcal{F}_s$ generated by $(\rho(s))$. The quadratic variation of M_φ is (*cf* (33))

$$\langle M_\varphi, M_\varphi\rangle(t) = \mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi.$$

Let $D\varphi(\rho) \otimes D\varphi(\rho)$ denote the bilinear form

$$(h,k) \mapsto (h, D\varphi(\rho))(k, D\varphi(\rho))$$

on $L^2(\mathbb{T}^d)$. By (48), we have

$$\begin{aligned}\mathscr{L}\varphi^2(\rho) - 2\varphi(\rho(s))\mathscr{L}\varphi(\rho(s)) &= \frac{1}{2}\mathrm{Trace}[\rho Q^{1/2})D\varphi(\rho) \otimes D\varphi(\rho)\rho Q^{1/2})^*] \\ &= \left\|\rho Q^{1/2} D\varphi(\rho)\right\|_{L^2}^2.\end{aligned}$$

We deduce that

$$M(t) = \rho(t) - \rho(0) - \int_0^t \mathcal{A}\rho(s) + \frac{1}{2}F\rho(s)ds,\ t \geq 0,$$

is a martingale with quadratic variation $\int_0^t \rho(s)Q^{1/2}\left(\rho(s)Q^{1/2}\right)^* ds$. Thanks to martingale representation results (see for instance [6]), up to a change of probability space, there exist a cylindrical Wiener process W such that

$$\rho(t) - \rho(0) - \int_0^t \mathcal{A}\rho(s) + \frac{1}{2}F\rho(s)ds = \int_0^t \rho(s)Q^{1/2}dW(s), \quad t \geq 0.$$

It is well known that this equation has a unique solution with paths in the space $C([0,T]; H^{-\eta}(\mathbb{R}^d))$. This can be shown for instance by energy estimates using Itô formula after a suitable regularization argument. Moreover pathwise uniqueness implies uniqueness in law and we deduce that P is the law of this solution and is uniquely determined. Finally, by uniqueness of the limit, the whole sequence (P^ε) converges to P weakly in the space of probability measures on $C([0,T]; H^{-\eta}(\mathbb{R}^d))$.

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Received xxxx 20xx; revised xxxx 20xx.

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