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Quiver varieties and the character ring of general linear groups over finite fields

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March 14, 2011

To Gus Lehrer and Jean Michel on the occasion of their 63th and 60th birthday

Abstract

Given a tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ we define a star-shaped quiver Γ together with a dimension vector \mathbf{v} . Assume that $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ is *generic*. Our first result is a formula which expresses the multiplicity of the trivial character in the tensor product $\mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_k$ as the trace of the action of some Weyl group on the intersection cohomology of some (non-affine) quiver varieties associated to (Γ, \mathbf{v}) . The existence of such a quiver variety is subject to some condition. Assuming that this condition is satisfied, we prove our second result: The multiplicity $\langle \mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_k, 1 \rangle$ is non-zero if and only if \mathbf{v} is a root of the Kac-Moody algebra associated with Γ . We conjecture that this remains true independently from the existence of the quiver variety. This is somehow similar to the connection between Horn's problem and the representation theory of $\mathrm{GL}_n(\mathbb{C})$ [20, Section 8].

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1 Introduction

1.1 Decomposing tensor products of irreducible characters

One problem is to study the decomposition

$$\mathcal{X}_1 \otimes \mathcal{X}_2 = \sum_{\mathcal{X}} \langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X} \rangle \mathcal{X}$$

of the tensor product $\mathcal{X}_1 \otimes \mathcal{X}_2$ of two irreducible complex characters of $\mathrm{GL}_n(\mathbb{F}_q)$ as a sum of irreducible characters. This reduces to the study of the multiplicities $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3, 1 \rangle$ of the trivial character 1 in the tensor products of three irreducible characters.

Although the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is known since 1955 by the work of Green [12], the computation of these multiplicities remains an open problem which does not seem to have been studied much in the literature.

When $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are unipotent characters, the multiplicities $\langle \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3, 1 \rangle$ were computed in [16] using CHEVIE for $n \leq 8$ and appeared to be polynomials in q with positive coefficients.

Let $\chi : \mathrm{GL}_n(\mathbb{F}_q) \rightarrow \mathbb{C}$ be the character of the conjugation action of $\mathrm{GL}_n(\mathbb{F}_q)$ on the group algebra $\mathbb{C}[\mathfrak{gl}_n(\mathbb{F}_q)]$. Fix a non-negative integer g and put $\Lambda := \chi^{\otimes g}$ ($\Lambda = 1$ when $g = 0$). In this paper we study the multiplicities $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ where $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ is a *generic* tuple of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$, see §6.7.

We now explain how to construct a comet-shaped quiver Γ together with a dimension vector \mathbf{v} of Γ from a tuple of irreducible characters.

We first show how to define a type A quiver together with a dimension vector from a single irreducible character \mathcal{X} .

Consider the lexicographic ordering \geq on the set \mathcal{P} of partitions and define a total ordering denoted again by \geq on the set $\mathbb{Z}_{>0} \times (\mathcal{P} - \{0\})$ as follows. If $\mu \neq \lambda$ then $(d, \mu) \geq (d', \lambda)$ if $\mu \geq \lambda$, and $(d, \lambda) \geq (d', \lambda)$ if $d \geq d'$. Denote by \mathbf{T}_n the set of non-increasing sequences $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r)$ such that $\sum_i d_i |\omega^i| = n$.

In §6.7, we associate to the irreducible character \mathcal{X} an element $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r) \in \mathbf{T}_n$ called the type of \mathcal{X} . The d_i 's are called the degrees of \mathcal{X} . If the degrees d_i 's are all equal to 1 we say that \mathcal{X} is *split*.

Consider the non-increasing sequence of partitions

$$\tilde{\omega} := \underbrace{\omega^1 \cdots \omega^1}_{d_1} \underbrace{\omega^2 \cdots \omega^2}_{d_2} \cdots \underbrace{\omega^r \cdots \omega^r}_{d_r}.$$

From this sequence we draw the Young diagrams of these partitions $\omega_1, \dots, \omega_r$ from the left to the right with diagram of ω^i repeated d_i times. Let l be the total number of columns and let n_i be the length of the i -th column. We obtain a strictly decreasing sequence $\mathbf{u} := (v_0 = n > v_1 > v_2 > \cdots > v_{l-1})$ by putting $v_1 := n - n_1, v_i := v_{i-1} - n_i$. We obtain then a type A_l -quiver with dimension vector \mathbf{u} . For instance if $\mathcal{X} = 1$, then $\omega = (1, (1, 1, \dots, 1))$ and so $A_l = A_1$ and $\mathbf{u} = n$. If \mathcal{X} is the Steinberg character then $\omega = (1, (n))$ and so $A_l = A_n$ and $\mathbf{u} = (n, n-1, n-2, \dots, 1)$. If \mathcal{X} is of type $(1, 1)(1, 1) \cdots (1, 1)$, then we still have $A_l = A_n$ and $\mathbf{u} = (n, n-1, n-2, \dots, 1)$.

Fix now a tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters. As just explained we obtain k type A quivers equipped with dimension vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$. Gluing together the first vertices of these k quivers and adding g loops we get a comet-shaped quiver Γ (see picture in §5.2) together with a dimension vector \mathbf{v} with coordinate n at the central vertex and whose other coordinates are given by the other coordinates of the \mathbf{u}_i 's.

Let $\Phi(\Gamma)$ be the root system associated with Γ (see [17]).

Assume now that the tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ is generic.

In §6.9.5 we give a formula for the multiplicity $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ in terms of Hall-Littlewood symmetric functions. This formula shows in particular that this multiplicity depends only on the types of $\mathcal{X}_1, \dots, \mathcal{X}_k$.

Conjecture 1.1.1. *We have:*

- (a) *The multiplicity $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ is a polynomial in q with integer coefficients. If moreover $\mathcal{X}_1, \dots, \mathcal{X}_k$ are split, then the coefficients are positive.*
- (b) *The coefficient of the highest power of q in $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ equals 1.*
- (c) *$\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle \neq 0$ if and only if $\mathbf{v} \in \Phi(\Gamma)$. If $g = 0$ then \mathbf{v} is a real root in $\Phi(\Gamma)$ if and only if $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle = 1$.*

We will prove (see Corollary 7.4.8) that if $g \geq 1$, then \mathbf{v} is always an imaginary root and so the second part of the assertion of (c) is relevant only when $g = 0$.

If $\mathcal{X}_1, \dots, \mathcal{X}_k$ are split *semisimple* irreducible characters, that is the type of each \mathcal{X}_i is of the form $(1, (1^{m_i})) \cdots (1, (1^{m_r}))$, then by [14] the multiplicity $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ coincides with $A_{\Gamma, \mathbf{v}}(q)$ which counts the number of absolutely indecomposable representations (up to isomorphism) of Γ of dimension \mathbf{v} over \mathbb{F}_q .

It is well-known [17] that $A_{\Gamma, \mathbf{v}}(q)$ is a polynomial in q with integer coefficients and that it is non-zero if and only if \mathbf{v} is a root (with $A_{\Gamma, \mathbf{v}}(q) = 1$ if and only if \mathbf{v} is a real root). It is also conjectured by Kac that $A_{\Gamma, \mathbf{v}}(q)$ has positive coefficients. This later conjecture is proved in [6] when \mathbf{v} is *indivisible* (i.e., the gcd of the coordinates of \mathbf{v} equals 1).

Hence, in the split semisimple case the assertion (c) of the conjecture is true and the assertion (a) is equivalent to Kac's conjecture for comet shaped quivers and so holds for indivisible \mathbf{v} .

Example 1.1.2. We review in details the statements (a), (b) and (c) of the conjecture in the case of characters of unipotent type (i.e., characters of type $(1, \lambda)$ with $\lambda \in \mathcal{P}$) when $g = 0$ and $n = k = 3$.

For a partition λ , we denote by R_λ the associated unipotent character of GL_3 . Recall that according to our parameterization (see beginning of this section), the trivial character corresponds to the partition $(1, 1, 1)$ and the Steinberg character to the partition (3) .

For a linear character $\alpha : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ we put $R_\lambda^\alpha := (\alpha \circ \det) \cdot R_\lambda$. This is again an irreducible character of type $(1, \lambda)$.

We say that $(R_\lambda^\alpha, R_\mu^\beta, R_\nu^\gamma)$ is *generic* if the subgroup $\langle \alpha\beta\gamma \rangle$ of $\mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$ is of size 3.

Assume now that $(R_\lambda^\alpha, R_\mu^\beta, R_\nu^\gamma)$ is generic. As mentioned earlier, the multiplicity $\langle R_\lambda^\alpha \otimes R_\mu^\beta \otimes R_\nu^\gamma, 1 \rangle$ depends only on λ, μ, ν and not on α, β, γ .

Put

$$R_{\lambda, \mu, \nu} := R_\lambda^\alpha \otimes R_\mu^\beta \otimes R_\nu^\gamma.$$

We can easily verify that the only non zero multiplicities (with unipotent type characters) are

$$\langle R_{(3), (3), (3)}, 1 \rangle = q, \tag{1.1.1}$$

$$\langle R_{(3), (3), (2, 1)}, 1 \rangle = 1. \tag{1.1.2}$$

In the first case the underlying graph of Γ is \tilde{E}_6 and \mathbf{v} is the indivisible positive imaginary root. In the second case the underlying graph of Γ is the Dynkin diagram E_6 and \mathbf{v} is the positive real root $\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ in the notation of [1, PLANCHE V]. Finally we can verify that there is no other pair (Γ, \mathbf{v}) arising from (λ, μ, ν) with $\mathbf{v} \in \Phi(\Gamma)$.

1.2 Quiver varieties

We now introduce the quiver varieties which provide a geometrical interpretation of $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ for a large class of generic tuples $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ which we call *admissible*.

Let P be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$, L a Levi factor of P and let $\Sigma = \sigma + C$ where C is a nilpotent orbit of the Lie algebra \mathfrak{l} of L and where σ is an element of the center z_1 of \mathfrak{l} . Put

$$\mathbb{X}_{L, P, \Sigma} := \{(X, gP) \in \mathfrak{gl}_n \times (\mathrm{GL}_n/P) \mid g^{-1}Xg \in \overline{\Sigma} + \mathfrak{u}_P\}$$

where \mathfrak{u}_P is the Lie algebra of the unipotent radical of P . We then denote by $\mathbb{X}_{L,P,\Sigma}^o$ the open subset of pairs (X, gP) which verify $g^{-1}Xg \in \Sigma + \mathfrak{u}_P$.

It is known (cf. §4.2 for more details) that the image of the projection $\mathbb{X}_{L,P,\Sigma} \rightarrow \mathfrak{gl}_n$ on the first coordinate is the Zariski closure of an adjoint orbit.

We assume without loss of generality that L is of the form $\prod_j \mathrm{GL}_{n_j}$ and that P is the unique parabolic subgroup of GL_n containing the upper triangular matrices and having L as a Levi factor (such a choice is only for convenience).

Consider triples $\{(L_i, P_i, \Sigma_i)\}_{i=1,\dots,k}$, with $\Sigma_i = \sigma_i + C_i$, as above and put $\mathbf{L} := L_1 \times \dots \times L_k$, $\mathbf{P} := P_1 \times \dots \times P_k$, $\Sigma := \Sigma_1 \times \dots \times \Sigma_k$ and $\mathbf{C} := C_1 \times \dots \times C_k$.

Let $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ be the tuple of adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$ such that the image of $\mathbb{X}_{L_i, P_i, \Sigma_i} \rightarrow \mathfrak{gl}_n$ is $\overline{\mathcal{O}_i}$.

We say that the pair (\mathbf{L}, Σ) is *generic* if the tuple $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ is *generic*. The existence of generic tuples of adjoint orbits with prescribed multiplicities of eigenvalues is subject to some restriction (cf. §5.1 for more details).

We assume now that (\mathbf{L}, Σ) is generic.

Fix a non-negative integer g , put $\mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma} = (\mathfrak{gl}_n)^{2g} \times \mathbb{X}_{\mathbf{L}, \mathbf{P}, \Sigma}$, $\mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}^o = (\mathfrak{gl}_n)^{2g} \times \mathbb{X}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ and define

$$\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} := \left\{ (A_1, B_1, \dots, A_g, B_g, (X_1, \dots, X_k, g_1 P_1, \dots, g_k P_k)) \in \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma} \mid \sum_j [A_j, B_j] + \sum_i X_i = 0 \right\}.$$

Put $\mathcal{O} := (\mathfrak{gl}_n)^{2g} \times \overline{\mathcal{O}_1} \times \dots \times \overline{\mathcal{O}_k}$, $\mathcal{O}^o := (\mathfrak{gl}_n)^{2g} \times \mathcal{O}_1 \times \dots \times \mathcal{O}_k$ and define

$$\mathcal{V}_{\mathcal{O}} := \left\{ (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathcal{O} \mid \sum_j [A_j, B_j] + \sum_i X_i = 0 \right\}.$$

Let $\rho : \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow \mathcal{V}_{\mathcal{O}}$ be the projection on the first $2g + k$ coordinates.

The group GL_n acts on $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ (resp. on $\mathcal{V}_{\mathcal{O}}$) diagonally by conjugating the first $2g + k$ coordinates and by left multiplication of the last k -coordinates (resp. diagonally by conjugating the $2g + k$ coordinates). Since the tuple $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ is generic, this action induces a set-theoretically free action of PGL_n on both $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ and $\mathcal{V}_{\mathcal{O}}$. The PGL_n -orbits of these two spaces are then all closed. Consider the affine GIT quotient

$$\mathcal{Q}_{\mathcal{O}} := \mathcal{V}_{\mathcal{O}} // \mathrm{PGL}_n = \mathrm{Spec}(\mathbb{C}[\mathcal{V}_{\mathcal{O}}]^{\mathrm{PGL}_n}).$$

The variety $\mathcal{Q}_{\mathcal{O}}$ can be identified with the orbit space $\mathcal{V}_{\mathcal{O}} / \mathrm{PGL}_n$. We will see that we can identify

$$\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma} := \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} / \mathrm{PGL}_n$$

with some GIT quotient $X //_{\chi} G$. In Nakajima's notation, the varieties $X // G$ and $X //_{\chi} G$ are quiver varieties $\mathfrak{M}_{\xi}(\mathbf{v})$ and $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ associated to some comet-shaped quiver Γ together with some dimension vector \mathbf{v} of Γ (cf. §5.3). The pair (Γ, \mathbf{v}) is obtained from $(\mathbf{L}, \mathbf{C}, g)$ as follows: Associate to each (L_i, C_i) a non-increasing sequence of partitions $\tilde{\omega}_i$ as in §4.2 and define (Γ, \mathbf{v}) from $(\tilde{\omega}_1, \dots, \tilde{\omega}_k)$ as explained for instance in §1.1.

The variety $\mathcal{Q}_{\mathcal{O}}$ is isomorphic to the image $\pi(\mathfrak{M}_{\xi, \theta}(\mathbf{v}))$ of $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}) \rightarrow \mathfrak{M}_{\xi}(\mathbf{v})$. By [3] it can also be identified with an affine quiver variety $\mathfrak{M}_{\xi}(\mathbf{u})$ so that the image $\mathcal{Q}_{\mathcal{O}}^o$ of

$$\mathcal{V}_{\mathcal{O}}^o := \mathcal{V}_{\mathcal{O}} \cap \mathcal{O}^o$$

in $\mathcal{Q}_{\mathcal{O}}$ corresponds to the subset $\mathfrak{M}_{\xi}^s(\mathbf{u}) \subset \mathfrak{M}_{\xi}(\mathbf{u})$ of simple representations. The image $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ of

$$\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^o := \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \cap \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$$

in $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ corresponds to the subset $\mathfrak{M}_{\xi, \theta}^s(\mathbf{v}) \subset \mathfrak{M}_{\xi, \theta}(\mathbf{v})$ of θ -stable representations.

The quiver varieties $\mathcal{Q}_{\mathcal{O}}$ and $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ will be the main focus of this paper.

Theorem 1.2.1. *Assume that $\mathcal{V}_{\mathbf{O}} \neq \emptyset$. Then:*

- (1) *The varieties $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}^{\circ}$ and $\mathbf{Q}_{\mathbf{O}}^{\circ}$ are both nonempty irreducible nonsingular dense open subsets of $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ and $\mathbf{Q}_{\mathbf{O}}$ respectively.*
- (2) *The map $\rho/\mathrm{PGL}_n : \mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow \mathbf{Q}_{\mathbf{O}}$ is a partial resolution, namely, it is proper and birational.*
- (3) *The restriction (pull back) of the intersection cohomology complex $\mathcal{IC}_{\mathbf{O}_{\mathbf{L}, \mathbf{P}, \Sigma}}^{\bullet}$ to $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is $\mathcal{IC}_{\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}}^{\bullet}$.*

Note that if $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \neq \emptyset$ if and only if $\mathcal{V}_{\mathbf{O}} \neq \emptyset$.

Recall that $\Phi(\Gamma)$ denotes the set of roots attached to the quiver Γ and denote by $W(\Gamma)$ the Weyl group, see [17]. Using Crawley-Boevey's results (cf. §2.2.1 and §5.3 for more details) we have a criterion for the non-emptiness of $\mathfrak{M}_{\xi}(\mathbf{u})$ in terms of roots of Γ . This criterion reads as follows: the variety $\mathbf{Q}_{\mathbf{O}} \simeq \mathfrak{M}_{\xi}(\mathbf{u})$ (and so $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$) is not empty if and only if $\mathbf{u} \in \Phi(\Gamma)$ which condition is equivalent to $\mathbf{v} \in \Phi(\Gamma)$ as \mathbf{u} is in the $W(\Gamma)$ -orbit of \mathbf{v} . Moreover, \mathbf{v} is a real root if and only if $\mathbf{Q}_{\mathbf{O}}$ (and so $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$) is reduced to a point.

For a pair (L, Σ) as above, we put

$$W(L, \Sigma) := \{n \in N_{\mathrm{GL}_n}(L) \mid n\Sigma n^{-1} = \Sigma\}/L.$$

Then it is known that the group $W(L, \Sigma)$ acts on the complex $p_*(\mathcal{IC}_{\mathbb{X}_{L, P, \Sigma}}^{\bullet})$ where $p : \mathbb{X}_{L, P, \Sigma} \rightarrow \mathfrak{gl}_n$ is the projection on the first coordinate, and $\mathcal{IC}_{\mathbb{X}_{L, P, \Sigma}}^{\bullet}$ is the simple perverse sheaf with coefficient in the constant local system \mathbb{C} .

From this, we find an action of

$$W(\mathbf{L}, \Sigma) := W(L_1, \Sigma_1) \times \cdots \times W(L_k, \Sigma_k)$$

on the complex $(\rho/\mathrm{PGL}_n)_*(\mathcal{IC}_{\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}}^{\bullet})$ and so on the hypercohomology $\mathbb{H}_c^i(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathcal{IC}_{\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}}^{\bullet})$ which we take as a definition for the compactly supported intersection cohomology $IH_c^i(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C})$.

From the theory of quiver varieties, we have $IH_c^i(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C}) = 0$ for odd i . Let us then consider the polynomials

$$P_c^{\mathbf{w}}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, q) := \sum_i \mathrm{Tr}(\mathbf{w} \mid IH_c^{2i}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C})) q^i,$$

with $\mathbf{w} \in W(\mathbf{L}, \Sigma)$.

As mentioned above, to each pair (L, C) with $L = \prod_{i=1}^r \mathrm{GL}_{n_i} \subset \mathrm{GL}_n$ and C a nilpotent orbit of $\bigoplus_{i=1}^r \mathfrak{gl}_{n_i}$ corresponds a unique sequence of partitions

$$\tilde{\omega} = \underbrace{\omega^1 \cdots \omega^1}_{a_1} \underbrace{\omega^2 \cdots \omega^2}_{a_2} \cdots \underbrace{\omega^l \cdots \omega^l}_{a_l}$$

with $\omega^1 \geq \omega^2 \geq \cdots \geq \omega^l$ and $\omega^j \neq \omega^s$ if $j \neq s$.

The group $W(L, C)$ is then isomorphic to $\prod_{j=1}^l S_{a_j}$ where S_d denotes the symmetric group in d letters.

The decomposition of the coordinates of an element $w \in W(L, C) \simeq \prod_{j=1}^l S_{a_j}$ as a product of disjoint cycles provides a partition $(d_j^1, d_j^2, \dots, d_j^{r_j})$ of a_j for each j , and so defines a unique type

$$\omega = (d_1^1, \omega^1) \cdots (d_1^{r_1}, \omega^1) (d_2^1, \omega^2) \cdots (d_2^{r_2}, \omega^2) \cdots (d_l^1, \omega^l) \cdots (d_l^{r_l}, \omega^l) \in \mathbf{T}_n.$$

We thus have a surjective map from the set of triples (L, C, w) with $w \in W(L, C)$ to the set \mathbf{T}_n .

Note that $W(\mathbf{L}, \Sigma) \subset W(\mathbf{L}, \mathbf{C})$.

Let $\mathbf{w} \in W(\mathbf{L}, \Sigma)$. The datum $(\mathbf{L}, \mathbf{C}, \mathbf{w})$ defines thus a multi-type $\omega = (\omega_1, \dots, \omega_k) \in (\mathbf{T}_n)^k$. We call *admissible* the multi-types arising in this way from generic pairs (\mathbf{L}, Σ) .

Let (X_1, \dots, X_k) be a generic tuple of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of type ω (generic tuples of irreducible characters of a given type always exist assuming that the characteristic of \mathbb{F}_q and q are large enough).

Theorem 1.2.2. *We have:*

$$P_c^{\mathbf{w}}(\mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma}, q) = q^{\frac{1}{2}\dim \mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma}} \langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle.$$

If $\mathbf{w} = 1$ and if the adjoint orbits $\mathcal{O}_1, \dots, \mathcal{O}_k$ are semisimple in which case $\mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma} \simeq \mathcal{Q}_{\mathbf{O}}$, the theorem is proved in [13].

One of the consequence of Theorem 1.2.2 is an explicit formula for $P_c^{\mathbf{w}}(\mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma}, q)$ in terms of Hall-Littlewood symmetric functions (cf. §6.9).

Note that if for each $i = 1, \dots, k$, we have $C_{\mathrm{GL}_n}(\sigma_i) = L_i$, then the projection $\mathbb{X}_{L_i, P_i, \Sigma_i} \rightarrow \overline{\mathcal{O}}_i$ is an isomorphism and so is the map $\rho/\mathrm{PGL}_n : \mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma} \rightarrow \mathcal{Q}_{\mathbf{O}}$. Hence our main results will give in particular explicit formulas for the Poincaré polynomial $P_c(\mathcal{Q}_{\mathbf{O}}, q)$ where we write P_c instead of $P_c^{\mathbf{w}}$ when $\mathbf{w} = 1$.

Let $\mathcal{A}_{(\mathbf{L},\mathbf{C})}$ be the set of $\sigma = (\sigma_1, \dots, \sigma_k) \in z_{l_1} \times \cdots \times z_{l_k}$ such that the pair $(\mathbf{L}, \sigma + \mathbf{C})$ is generic.

It follows from Theorem 1.2.2 that $P_c(\mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma}, q)$ depends only on (\mathbf{L}, \mathbf{C}) and not on $\sigma \in \mathcal{A}_{(\mathbf{L},\mathbf{C})}$.

We say that a generic tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters is *admissible* if it is of admissible type.

From Theorem 1.2.2 and the discussion after Theorem 1.2.1, we prove that Conjecture 1.1.1 is true for admissible tuples $(\mathcal{X}_1, \dots, \mathcal{X}_k)$, namely:

Theorem 1.2.3. *Let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be an admissible generic tuple of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$.*

(a) *The multiplicity $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ is a polynomial in q with integer coefficients. If moreover $\mathbf{w} = 1$, then it has positive coefficients.*

(b) *The coefficient of the highest power of q in $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ equals 1.*

(c) *We have $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle \neq 0$ if and only if $\mathbf{v} \in \Phi(\Gamma)$. If $g = 0$ then \mathbf{v} is a real root if and only if $\langle \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle = 1$.*

Now let us see some examples of generic tuples $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters which are not admissible. This is equivalent of giving examples of triples $(\mathbf{L}, \mathbf{C}, \mathbf{w})$ for which there is no $\sigma \in \mathcal{A}_{(\mathbf{L},\mathbf{C})}$ such that $\mathbf{w} \in W(\mathbf{L}, \sigma + \mathbf{C})$.

The condition for the existence of such a σ is subject to some restrictions which can be worked out explicitly using §5.1. Let us see the explicit situations (i), (ii) and (iii) below.

(i) Assume that \mathbf{L} is a maximal torus (in which case \mathbf{C} is the trivial nilpotent orbit) and that the coordinates of \mathbf{w} are all n -cycles. Then \mathbf{w} belongs to a subgroup $W(\mathbf{L}, \sigma + \mathbf{C})$ of $W(\mathbf{L}, \mathbf{C}) = W(\mathbf{L})$ if and only if the coordinates of $\sigma = (\sigma_1, \dots, \sigma_k)$ are all scalar matrices. But such a σ does not belong to $\mathcal{A}_{(\mathbf{L},\mathbf{C})}$.

(ii) When the dimension vector \mathbf{v} of the comet-shaped quiver Γ is divisible (i.e., the gcd of its coordinates is greater than 1), then $\mathcal{A}_{(\mathbf{L},\mathbf{C})} = \emptyset$.

(iii) If $\mathbf{L} = (\mathrm{GL}_n)^k$, then we also have $\mathcal{A}_{(\mathbf{L},\mathbf{C})} = \emptyset$.

When $\mathbf{C} = \{0\}$, then $\mathcal{A}_{(\mathbf{L},\mathbf{C})} \neq \emptyset$ if and only if \mathbf{v} is indivisible. This implies that a generic tuple of split semisimple irreducible characters is admissible if and only if \mathbf{v} is indivisible.

1.3 Character varieties: A conjecture

Now we propose a conjectural geometrical interpretation of $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle$ for any generic tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$.

Let P be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$, L a Levi factor of P and let $\Sigma = \sigma C$ where C is a unipotent conjugacy class of L and where σ is an element of the center Z_L of L . Put

$$\mathbb{Y}_{L,P,\Sigma} := \left\{ (x, gP) \in \mathrm{GL}_n \times (\mathrm{GL}_n/P) \mid g^{-1}xg \in \overline{\Sigma}.U_P \right\}$$

where U_P is the unipotent radical of P . The variety $\mathbb{Y}_{L,P,\Sigma}$ is the multiplicative analogue of $\mathbb{X}_{L,P,\Sigma}$.

We choose a tuple $(\mathfrak{D}_1, \dots, \mathfrak{D}_k)$ of conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ and for each $i = 1, \dots, k$ we let $\tilde{\mathfrak{D}}_i$ be the conjugacy class of the semisimple part of an element in \mathfrak{D}_i . We say that the tuple $(\mathfrak{D}_1, \dots, \mathfrak{D}_k)$ is

generic if $\prod_{i=1}^k \det(\mathfrak{D}_i) = 1$ and if V is a subspace of \mathbb{C}^n which is stable by some $x_i \in \tilde{\mathfrak{D}}_i$ (for each i) such that

$$\prod_{i=1}^k \det(x_i|_V) = 1$$

then either $V = 0$ or $V = \mathbb{C}^n$. Unlike the additive case, generic tuples of conjugacy classes always exist (the multiplicities of the eigenvalues being prescribed). For instance, while we can not form generic tuples of adjoint orbits of nilpotent type, we can always form generic tuples of conjugacy classes of unipotent type as follows. Let ζ be a primitive n -th root of unity, and $\mathfrak{D}_1 = \zeta C_1, \mathfrak{D}_2 = C_2, \dots, \mathfrak{D}_k = C_k$ where C_1, \dots, C_k are unipotent conjugacy classes, then $(\mathfrak{D}_1, \dots, \mathfrak{D}_k)$ is generic.

For each $i = 1, \dots, k$, let (L_i, P_i, Σ_i) be such that the image of the projection $\mathbb{Y}_{L_i, P_i, \Sigma_i} \rightarrow \mathfrak{gl}_n$ is $\overline{\mathfrak{D}}_i$. As in §1.2, we define $\mathbf{L}, \mathbf{P}, \Sigma, \mathbf{C}$ and we say that (\mathbf{L}, Σ) is *generic* if the tuple $(\mathfrak{D}_1, \dots, \mathfrak{D}_k)$ is generic which we now assume. We define the multiplicative analogue of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ as

$$\mathbb{U}_{\mathbf{L}, \mathbf{P}, \Sigma} := \left\{ (a_1, b_1, \dots, a_g, b_g, (x_1, \dots, x_k, g_1 P_1, \dots, g_k P_k)) \in (\mathrm{GL}_n)^{2g} \times \mathbb{Y}_{\mathbf{L}, \mathbf{P}, \Sigma} \mid (a_1, b_1) \cdots (a_g, b_g) x_1 \cdots x_k = 1 \right\}$$

where (a, b) denotes the commutator $aba^{-1}b^{-1}$. As in the quiver case, the genericity condition ensure that the group PGL_n acts freely on $\mathbb{U}_{\mathbf{L}, \mathbf{P}, \Sigma}$. Then consider the quotient $\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma} = \mathbb{U}_{\mathbf{L}, \mathbf{P}, \Sigma} / \mathrm{PGL}_n$. The projection $\mathbb{U}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow (\mathrm{GL}_n)^{2g+k}$ on the $2g+k$ first coordinates induces a morphism from $\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}$ onto the affine GIT quotient

$$\mathcal{M}_{\mathfrak{D}} := \left\{ (a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_k) \in (\mathrm{GL}_n)^{2g} \times \overline{\mathfrak{D}}_1 \times \cdots \times \overline{\mathfrak{D}}_k \mid \prod (a_i, b_i) \prod x_j = 1 \right\} / \mathrm{PGL}_n.$$

Remark 1.3.1. If S_g is a compact Riemann surface of genus g with punctures $p = \{p_1, \dots, p_k\} \subset S_g$, then $\mathcal{M}_{\mathfrak{D}}$ can be identified (hence the name of character varieties) with the affine GIT quotient

$$\left\{ \rho \in \mathrm{Hom}(\pi_1(S_g \setminus p), \mathrm{GL}_n) \mid \rho(\gamma_i) \in \overline{\mathfrak{D}}_i \right\} / \mathrm{PGL}_n,$$

where γ_i is the class of a simple loop around p_i with orientation compatible with that of S_g .

Unlike quiver varieties, the mixed Hodge structure on $IH_c^k(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C})$ is not pure (see for instance [13] in the case where the conjugacy classes \mathfrak{D}_i are semisimple).

We let W_\bullet be the weight filtration on $IH_c^k(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C})$ and put

$$H^{i,k}(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}) := W_i IH_c^k(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C}) / W_{i-1} IH_c^k(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C}).$$

The action of $W(\mathbf{L}, \Sigma)$ preserves the weight filtration and so, for $\mathbf{w} \in W(\mathbf{L}, \Sigma)$, we may consider the mixed Poincaré polynomial

$$H_c^{\mathbf{w}}(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}; q, t) := \sum_{i,k} \mathrm{Tr}(\mathbf{w} \mid H^{i,k}(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma})) q^i t^k$$

and its *pure part*

$$PH_c^{\mathbf{w}}(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma}, t) := \sum_i \mathrm{Tr}(\mathbf{w} \mid H^{i,2i}(\mathbb{M}_{\mathbf{L}, \mathbf{P}, \Sigma})) t^i.$$

Recall that $\Sigma = \sigma \mathbf{C}$ with \mathbf{C} a unipotent conjugacy class of \mathbf{L} and $\sigma \in Z_{\mathbf{L}}$.

Let $\mathbf{w} \in W(\mathbf{L}, \Sigma)$. As above Theorem 1.2.2, we can define a type $\omega \in \mathbf{T}_n$ from $(\mathbf{L}, \mathbf{C}, \mathbf{w})$. Let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be a generic tuple of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of type ω .

Conjecture 1.3.2. *We have*

$$PH_c^w(\mathbb{M}_{\mathbf{L},\mathbf{P},\Sigma}, q) = q^{\frac{1}{2}\dim \mathbb{M}_{\mathbf{L},\mathbf{P},\Sigma}} \langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle. \quad (1.3.1)$$

If $\mathbf{w} = 1$ and if the conjugacy classes \mathfrak{D}_i are semisimple, in which case $\mathbb{M}_{\mathbf{L},\mathbf{P},\Sigma} \simeq \mathcal{M}_{\mathfrak{D}}$, then this conjecture is already in [13].

Now put $\xi := (\zeta \cdot 1, 1, \dots, 1) \in (Z_{GL_n})^k$ where ζ is a primitive n -th root of unity. Then for any triple $(\mathbf{L}, \mathbf{C}, \mathbf{w})$ with $\mathbf{w} \in W(\mathbf{L}, \mathbf{C})$ the pair $(\mathbf{L}, \xi\mathbf{C})$ is always generic and $\mathbf{w} \in W(\mathbf{L}, \xi\mathbf{C}) = W(\mathbf{L}, \mathbf{C})$. Hence Conjecture 1.3.2 implies that for any generic tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters there exists a triple $(\mathbf{L}, \mathbf{C}, \mathbf{w})$ with $\mathbf{w} \in W(\mathbf{L}, \mathbf{C})$ such that if we put $\Sigma := \xi\mathbf{C}$, then Formula (1.3.1) holds.

Using the results of [5] we can prove that Conjecture 1.3.2 implies Conjecture 1.1.1.

Put $\mathbf{C}' := \mathbf{C} - 1$ and assume that there exists $\sigma' \in \mathcal{A}_{(\mathbf{L}, \mathbf{C}')} such that $C_{GL_n}(\sigma) = C_{GL_n}(\sigma')$. Then Conjecture 1.3.2 together with Theorem 1.2.2 implies the following conjecture.$

Conjecture 1.3.3. *We have*

$$PH_c^w(\mathbb{M}_{\mathbf{L},\mathbf{P},\Sigma}, q) = P_c^w(\mathbb{Q}_{\mathbf{L},\mathbf{P},\Sigma'}, q).$$

In the case where the adjoint orbits $\mathcal{O}_1, \dots, \mathcal{O}_k$ and the conjugacy classes $\mathfrak{D}, \dots, \mathfrak{D}_k$ are semisimple and $\mathbf{w} = 1$, then this conjecture is due to T. Hausel. If $g = 0$, he actually conjectured that the identity between the two polynomials is realized by the Hilbert monodromy map $\mathcal{Q}_{\mathcal{O}} \rightarrow \mathcal{M}_{\mathfrak{D}}$.

In [13] we gave a conjectural formula for the mixed Poincaré polynomial of $\mathcal{M}_{\mathfrak{D}}$ in terms of Macdonald polynomials when $\mathfrak{D}_1, \dots, \mathfrak{D}_k$ are semisimple. We will discuss the generalization of this conjecture for the twisted mixed Poincaré polynomial $H_c^w(\mathbb{M}_{\mathbf{L},\mathbf{P},\Sigma}; q, t)$ in a forthcoming paper.

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2 Preliminaries

In this section, \mathbb{K} is an algebraically closed field of arbitrary characteristic.

2.1 Preliminaries on geometric invariant theory

Let X be an affine algebraic variety over \mathbb{K} and let G be a reductive group over the same field acting (morphically) on X (we say that X is a G -variety). Given a character $\chi : G \rightarrow \mathbb{K}^\times$, i.e., a morphism of algebraic groups, Mumford [35] defined a quotient $X//_\chi G$ as follows. The group G acts on the line bundle $X \times \mathbf{A}^1$ by $g \cdot (x, z) = (g \cdot x, \chi(g)^{-1}z)$ and consider the subalgebra $\mathbb{K}[X \times \mathbf{A}^1]^G$ of the regular functions on $X \times \mathbf{A}^1$ which are G -invariant. A polynomial $f = \sum_{i=0}^r f_i \cdot z^i \in \mathbb{K}[X][z] \simeq \mathbb{K}[X \times \mathbf{A}^1]$ is G -invariant if and only if for each i , the function f_i is a χ^i -semi-invariant, that is

$$f_i(g \cdot x) = \chi(g)^i f_i(x)$$

for all $x \in X$. Let $\mathbb{K}[X]^{G, \chi^n} \subset \mathbb{K}[X]$ be the subspace of χ^n -semi-invariant functions. Then we have a decomposition

$$\mathbb{K}[X \times \mathbf{A}^1]^G = \bigoplus_{n \geq 0} \mathbb{K}[X]^{G, \chi^n}.$$

Then $X//_\chi G$ is defined as the projective spectrum of the graded algebra $\mathbb{K}[X \times \mathbf{A}^1]^G$ (the geometric points of this scheme corresponds to the maximal homogeneous ideals which do not contain the irrelevant ideal

$\bigoplus_{n>0} \mathbb{K}[X]^{G \cdot \chi^n}$). The first projection $X \times \mathbf{A}^1 \rightarrow X$ induces an embedding of algebras $\mathbb{K}[X]^G \subset \mathbb{K}[X \times \mathbf{A}^1]^G$ and therefore a projective morphism

$$\pi_X : X//_{\chi}G \rightarrow X//G := \text{Spec}(\mathbb{K}[X]^G). \quad (2.1.1)$$

Note that if $\chi = 1$, then π_X is an isomorphism.

For $J \subset \mathbb{K}[X]$, put $X_J := \{x \in X \mid f(x) \neq 0 \text{ for all } f \in J\}$. We say that a point $x \in X$ is χ -semistable if $x \in X_f$ for some $f \in \mathbb{K}[X]^{G \cdot \chi^n}$ with $n > 0$. If moreover the G -action map $G \times X_f \rightarrow X_f$ is closed and if the stabiliser of x in G is finite, we say that x is χ -stable. We denote by X^s and X^{ss} the open subsets of X respectively of stable and semistable points of X . If $\chi = 1$ then $X^{ss} = X$.

Let $q : X^{ss} \rightarrow X//_{\chi}G$ be the morphism that maps a point $x \in X^{ss}$ to the maximal homogeneous ideal (of $\mathbb{K}[X \times \mathbf{A}^1]^G$) of functions that vanish at x . If $\chi = 1$, then this is the map $X \rightarrow X//G$ induced by the inclusion $\mathbb{K}[X]^G \subset \mathbb{K}[X]$.

We recall the following well-known properties of the GIT quotient q .

Proposition 2.1.1. (1) *The quotient q is a categorical quotient (in the category of algebraic varieties).*

(2) *If $x, y \in X^{ss}$, we have $q(x) = q(y)$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$.*

(3) *If U is an q -saturated (i.e. $q^{-1}q(U) = U$) G -stable open subset of X^{ss} , then $q(U)$ is an open subset of $X//_{\chi}G$ and the restriction $U \rightarrow q(U)$ is a categorical quotient.*

(4) *If F is a closed G -stable subset of X , then $F^{ss} = F \cap X^{ss}$ and $q(F^{ss})$ is closed in $X//_{\chi}G$. If moreover $\mathbb{K} = \mathbb{C}$, then the restriction map $F^{ss} \rightarrow q(F^{ss})$ is the quotient map $F^{ss} \rightarrow F//_{\chi}G$.*

Since the Zariski closure of a G -orbit contains always a closed orbit, the assertion (2) shows that $X//_{\chi}G$ parameterizes the closed orbits of X^{ss} . If we identify $X//_{\chi}G$ with the set of closed orbits of X^{ss} , the map q sends an orbit \mathcal{O} of X^{ss} to the unique closed orbit contained in $\overline{\mathcal{O}}$.

2.2 Generalities on quiver varieties

We introduce the so-called quiver varieties $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ and $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w})$ over \mathbb{K} which were considered by many authors including Kronheimer, Lusztig, Nakajima and Crawley-Boevey. The second one, due to Nakajima and called *framed* quiver varieties, can be realized as the first one by an observation due to Crawley-Boevey [2, introduction]. For our application we found more convenient to introduce them separately. Here we recall the basic results we need.

From now we will only consider quotients by (finite) direct products of GL_n 's. If $G = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_r}$ is such a group and if $\chi : G \rightarrow \mathbb{K}^{\times}$, $(g_i) \mapsto \prod_i (\det g_i)^{-\theta_i}$ is the character given by $\theta \in \mathbb{Z}^{(1, \dots, r)}$, then we will use the notation $X//_{\theta}G$ instead of $X//_{\chi}G$.

2.2.1 Quiver varieties

Let Γ be a quiver and let I denote the set of its vertices. We assume that I is finite. A *dimension vector* for Γ is a collection of non-negative integers $\mathbf{v} = \{v_i\}_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ and a representation of Γ of dimension \mathbf{v} over \mathbb{K} is a collection of \mathbb{K} -linear maps $\varphi_{i,j} : \mathbb{K}^{v_i} \rightarrow \mathbb{K}^{v_j}$, for each arrow $i \rightarrow j$ of Γ , that we identify with matrices (using the canonical basis of \mathbb{K}^r). We define a morphism between two representations (possibly of different dimension) in the obvious way. A *subrepresentation* of φ is a representation φ' together with an injective morphism $\varphi' \rightarrow \varphi$. Let Ω be a set indexing the edges of Γ . For $\gamma \in \Omega$, let $h(\gamma), t(\gamma) \in I$ denote respectively the head and the tail of γ . The algebraic group $\text{GL}_{\mathbf{v}} := \prod_{i \in I} \text{GL}_{v_i}(\mathbb{K})$ acts on the space

$$\mathbf{M}(\Gamma, \mathbf{v}) := \bigoplus_{\gamma \in \Omega} \text{Mat}_{v_{h(\gamma)}, v_{t(\gamma)}}(\mathbb{K})$$

of representations of dimension \mathbf{v} in the obvious way, i.e., for $g = (g_i)_{i \in I} \in \text{GL}_{\mathbf{v}}$ and $B = (x_{\gamma})_{\gamma \in \Omega}$, we have $g \cdot B := (g_{v_{h(\gamma)}} x_{\gamma} g_{v_{t(\gamma)}}^{-1})$. As the diagonal center $Z = \{(\lambda \cdot \text{Id}_{v_i})_{i \in I} \mid \lambda \in \mathbb{K}^{\times}\} \subset \text{GL}_{\mathbf{v}}$ acts trivially, the action of

$\mathrm{GL}_{\mathbf{v}}$ induces an action of

$$\mathbf{G}_{\mathbf{v}} := \mathrm{GL}_{\mathbf{v}}/Z.$$

Clearly two elements of $\mathbf{M}(\Gamma, \mathbf{v})$ are isomorphic if and only if they are $\mathbf{G}_{\mathbf{v}}$ -conjugate.

We define a bilinear form on \mathbb{K}^I by $\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i$. Let $\boldsymbol{\theta} \in \mathbb{Z}^I$ be such that $\boldsymbol{\theta} \cdot \mathbf{v} = 0$. This defines a character $\chi : \mathbf{G}_{\mathbf{v}} \rightarrow \mathbb{K}^\times$ given by $(g_i)_i \mapsto \prod_i \det(g_i)^{-\theta_i}$.

Theorem 2.2.1. [19] *A point $V \in \mathbf{M}(\bar{\Gamma}, \mathbf{v})$ is χ -semistable if and only if*

$$\boldsymbol{\theta} \cdot \dim W \leq 0$$

for every subrepresentation W of V . It is χ -stable if and only if it is semistable and the inequality is strict unless $W = 0$ or $W = V$.

We will use the terminology “ $\boldsymbol{\theta}$ -semistable” instead of “ χ -semistable”. We denote respectively by $\mathbf{M}_{\boldsymbol{\theta}}^{\mathrm{ss}}(\Gamma, \mathbf{v})$ and $\mathbf{M}_{\boldsymbol{\theta}}^{\mathrm{s}}(\Gamma, \mathbf{v})$ the $\boldsymbol{\theta}$ -semistable and $\boldsymbol{\theta}$ -stable representations.

Let $\bar{\Gamma}$ be the *double quiver* of Γ i.e. $\bar{\Gamma}$ has the same vertices as Γ but the edges are given by $\bar{\Omega} := \{\gamma, \gamma^* \mid \gamma \in \Omega\}$ where $h(\gamma^*) = t(\gamma)$ and $t(\gamma^*) = h(\gamma)$. Then via the trace pairing we may identify $\mathbf{M}(\bar{\Gamma}, \mathbf{v})$ with the cotangent bundle $\mathbf{T}^*\mathbf{M}(\Gamma, \mathbf{v})$. Put $\mathfrak{gl}_{\mathbf{v}} = \mathrm{Lie}(\mathrm{GL}_{\mathbf{v}}) = \bigoplus_i \mathfrak{gl}_{v_i}(\mathbb{K})$ and $\mathfrak{g}_{\mathbf{v}} := \mathrm{Lie}(\mathbf{G}_{\mathbf{v}})$. Define the *moment map*

$$\begin{aligned} \mu_{\mathbf{v}} : \mathbf{M}(\bar{\Gamma}, \mathbf{v}) &\rightarrow \mathbf{M}(\mathbf{v})^0 \\ (x_{\gamma})_{\gamma \in \bar{\Omega}} &\mapsto \sum_{\gamma \in \Omega} [x_{\gamma}, x_{\gamma^*}], \end{aligned} \tag{2.2.1}$$

where

$$\mathbf{M}(\mathbf{v})^0 := \left\{ (f_i)_{i \in I} \in \mathfrak{gl}_{\mathbf{v}} \mid \sum_{i \in I} \mathrm{Tr}(f_i) = 0 \right\}.$$

Note that we can identify $\mathbf{M}(\mathbf{v})^0$ with $(\mathfrak{g}_{\mathbf{v}})^*$ via the trace pairing. The moment map $\mu_{\mathbf{v}}$ is $\mathbf{G}_{\mathbf{v}}$ -equivariant.

Let $\boldsymbol{\xi} = (\xi_i)_i \in \mathbb{K}^I$ be such that $\boldsymbol{\xi} \cdot \mathbf{v} = 0$. Then

$$(\xi_i \cdot \mathrm{Id})_i \in \mathfrak{gl}_{\mathbf{v}}$$

is in fact in $\mathbf{M}(\mathbf{v})^0$. By abuse of notation we denote by $\boldsymbol{\xi}$ the element $(\xi_i \cdot \mathrm{Id}_{v_i})_i \in \mathbf{M}(\mathbf{v})^0$. The affine variety $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})$ is $\mathbf{G}_{\mathbf{v}}$ -stable.

Define

$$\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}(\mathbf{v}) := \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi}) //_{\boldsymbol{\theta}} \mathbf{G}_{\mathbf{v}}.$$

We define $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}^{\mathrm{s}}(\mathbf{v})$ as the image of $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}}$ in $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}(\mathbf{v})$. By Proposition 2.1.1, it is an open subset of $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}(\mathbf{v})$.

Since stabilizers in $\mathbf{G}_{\mathbf{v}}$ of quiver representations are connected, the action of $\mathbf{G}_{\mathbf{v}}$ on the space $\mathbf{M}_{\boldsymbol{\theta}}^{\mathrm{s}}(\bar{\Gamma}, \mathbf{v})$ is set-theoretically free and so the restriction $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}} \rightarrow \mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}^{\mathrm{s}}(\mathbf{v})$ of φ is the set theoretical quotient $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}} \rightarrow \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}} / \mathbf{G}_{\mathbf{v}}$. By [41, Lemma 6.5], the map $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}} \rightarrow \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})^{\mathrm{s}} / \mathbf{G}_{\mathbf{v}}$ is actually a principal $\mathbf{G}_{\mathbf{v}}$ -bundle in the étale topology.

Let $\mathbf{C} = (c_{ij})_{i,j}$ be the Cartan matrix of the quiver Γ , namely

$$c_{ij} = \begin{cases} 2 - 2(\text{the number of edges joining } i \text{ to itself}) & \text{if } i = j \\ -(\text{the number of edges joining } i \text{ to } j) & \text{otherwise.} \end{cases}$$

We say that a variety X is of *pure dimension* d if its irreducible components are all of same dimension d . We have the following well-known theorem.

Theorem 2.2.2. *Let $\boldsymbol{\theta} \in \mathbb{Z}^I$ be such that $\boldsymbol{\theta} \cdot \mathbf{v} = 0$. If $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}^{\mathrm{s}}(\mathbf{v}) \neq \emptyset$, then it is of pure dimension $2 - {}^t \mathbf{v} \mathbf{C} \mathbf{v}$. If $\mathfrak{M}_{\boldsymbol{\xi}}^{\mathrm{s}}(\mathbf{v})$ is not empty, then $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}^{\mathrm{s}}(\mathbf{v})$ is also not empty and $\mathfrak{M}_{\boldsymbol{\xi}, \boldsymbol{\theta}}(\mathbf{v})$ is irreducible.*

Proof. First a simple representation is necessarily θ -stable, hence $\mathfrak{M}_\xi^s(\mathbf{v}) \neq \emptyset$ implies $\mathfrak{M}_{\xi,\theta}^s(\mathbf{v}) \neq \emptyset$. By [2, Theorem 1.2] the existence of simple representations in $\mu_{\mathbf{v}}^{-1}(\xi)$ implies the irreducibility of $\mu_{\mathbf{v}}^{-1}(\xi)$ and so the irreducibility of $\mathfrak{M}_{\xi,\theta}^s(\mathbf{v})$ and $\mathfrak{M}_{\xi,\theta}(\mathbf{v})$. Note that a point $\alpha \in \mu_{\mathbf{v}}^{-1}(\xi)$ is nonsingular if $\mu_{\mathbf{v}}$ is smooth at α , that is if the stabilizer of α in $G_{\mathbf{v}}$ is trivial. From this we deduce that the space $\mu_{\mathbf{v}}^{-1}(\xi)^s$ of θ -stable representations is a nonsingular space of dimension $\dim \mathbf{M}(\bar{\Gamma}, \mathbf{v}) - \dim G_{\mathbf{v}}$, and so that $\mathfrak{M}_{\xi,\theta}^s(\mathbf{v})$ is nonsingular of dimension

$$2 - {}^t \mathbf{v} \mathbf{C} \mathbf{v} = \dim \mathbf{M}(\bar{\Gamma}, \mathbf{v}) - 2 \dim G_{\mathbf{v}}.$$

□

If $\theta = 0$, we put $\mathfrak{M}_\xi(\mathbf{v}) := \mathfrak{M}_{\xi,0}(\mathbf{v})$. It is the affine GIT quotient $\mu_{\mathbf{v}}^{-1}(\xi) // G_{\mathbf{v}} = \text{Spec}(\mathbb{K}[\mu_{\mathbf{v}}^{-1}(\xi)]^{G_{\mathbf{v}}})$. The set $\mathfrak{M}_\xi(\mathbf{v})$ parameterizes the set of conjugacy classes of the semisimple representations of $\mu_{\mathbf{v}}^{-1}(\xi)$. Under this parameterization, the open subset $\mathfrak{M}_\xi^s(\mathbf{v})$ of 0-stable points coincides with the set of conjugacy classes of simple representations.

The natural projective morphism $\pi : \mathfrak{M}_{\xi,\theta}(\mathbf{v}) \rightarrow \mathfrak{M}_\xi(\mathbf{v})$ of §2.1 takes a representation to its semi-simplification.

We put an order on \mathbb{Z}^I as follows: we say that $\mathbf{w} \leq \mathbf{v}$ if we have $w_i \leq v_i$ for each $i \in I$. We denote by $\mathcal{E}(\mathbf{v})$ the set of \mathbf{w} such that $0 < \mathbf{w} < \mathbf{v}$, $\xi \cdot \mathbf{w} = 0$ and $\mu_{\mathbf{w}}^{-1}(\xi) \neq \emptyset$.

For $\mathbf{w} \in \mathbb{Z}_{\geq 0}^I$, we denote by $H_{\mathbf{w}}$ the hyperplane $\{\alpha \in \mathbb{Q}^I \mid \alpha \cdot \mathbf{w} = 0\}$ of \mathbb{Q}^I . Put $H_{\mathbf{vw}} := H_{\mathbf{v}} \cap H_{\mathbf{w}}$ and

$$D_{\mathbf{v}} := H_{\mathbf{v}} - \bigcup_{\mathbf{w} \in \mathcal{E}(\mathbf{v})} H_{\mathbf{vw}}.$$

We say that \mathbf{v} is *indivisible* if the gcd of $\{v_i\}_{i \in I}$ is 1. Note that $D_{\mathbf{v}}$ is not empty if and only if \mathbf{v} is *indivisible*.

When \mathbf{v} is indivisible, the spaces $H_{\mathbf{vw}}$ are hyperplanes of $H_{\mathbf{v}}$ and so defines a system of *faces* [1, Chapter 1, §1].

Definition 2.2.3. We say that θ is *generic* with respect to \mathbf{v} if $\theta \in D_{\mathbf{v}}$.

If θ is generic then θ -semistability coincides with θ -stability, and so

$$\mathfrak{M}_{\xi,\theta}^s(\mathbf{v}) = \mathfrak{M}_{\xi,\theta}(\mathbf{v}).$$

The variety $\mathfrak{M}_{\xi,\theta}(\mathbf{v})$ is thus nonsingular for generic θ .

We have [36][39, §2.5]:

Proposition 2.2.4. *Assume that θ is generic and that $\mathfrak{M}_{\xi,\theta}^s(\mathbf{v}) \neq \emptyset$. Then the map $\pi : \mathfrak{M}_{\xi,\theta}(\mathbf{v}) \rightarrow \mathfrak{M}_\xi(\mathbf{v})$ is a resolution of singularities.*

Definition 2.2.5. Let X be an algebraic variety over \mathbb{K} . We say that $X = \coprod_{\alpha \in I} X_{\alpha}$ is a *stratification* of X if the set $\{\alpha \in I \mid X_{\alpha} \neq \emptyset\}$ is finite, for each $\alpha \in I$ such that $X_{\alpha} \neq \emptyset$, the subset X_{α} is a locally closed nonsingular subvariety of X , and for each $\alpha, \beta \in I$, if C_{α} is a connected component of X_{α} such that $C_{\alpha} \cap \overline{X_{\beta}} \neq \emptyset$, then $C_{\alpha} \subset \overline{X_{\beta}}$.

We now define a stratification of the variety $\mathfrak{M}_\xi(\mathbf{v})$ as in [37, page 526]. Let H be a closed subgroup of $G_{\mathbf{v}}$ and let $\mathbf{M}(\bar{\Gamma}, \mathbf{v})_{(H)}$ be the set of all points in $\mathbf{M}(\bar{\Gamma}, \mathbf{v})$ whose stabilizer in $G_{\mathbf{v}}$ is conjugate to H . A $G_{\mathbf{v}}$ -orbit in $\mathbf{M}(\bar{\Gamma}, \mathbf{v})$ is said to be of type (H) if it lives in $\mathbf{M}(\bar{\Gamma}, \mathbf{v})_{(H)}$. Let $(\mathbf{M}(\bar{\Gamma}, \mathbf{v}) // G_{\mathbf{v}})_{(H)}$ denotes the set of closed $G_{\mathbf{v}}$ -orbits of type (H) . It is proved in [10, Theorem 5.1] that $\mathbf{M}(\bar{\Gamma}, \mathbf{v}) // G_{\mathbf{v}} = \coprod_{(H)} (\mathbf{M}(\bar{\Gamma}, \mathbf{v}) // G_{\mathbf{v}})_{(H)}$, where the union is over all conjugacy classes of subgroups of $G_{\mathbf{v}}$, is a stratification. This was first proved in [26, III, 2][22] when the characteristic of \mathbb{K} is zero.

Then we denote by $\mathfrak{M}_\xi(\mathbf{v})_{(H)} \subset \mathfrak{M}_\xi(\mathbf{v})$ the subset of closed $G_{\mathbf{v}}$ -orbits of type (H) .

We have the following theorem [25] (in characteristic zero see [39, Section 3]):

Theorem 2.2.6. *The decomposition*

$$\mathfrak{M}_\xi(\mathbf{v}) = \bigsqcup_{(H)} \mathfrak{M}_\xi(\mathbf{v})_{(H)}, \quad (2.2.2)$$

is a stratification. If H is a subgroup of $G_{\mathbf{v}}$, then the restriction $\pi^{-1}(\mathfrak{M}_\xi(\mathbf{v})_{(H)}) \rightarrow \mathfrak{M}_\xi(\mathbf{v})_{(H)}$ of the morphism $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}) \rightarrow \mathfrak{M}_\xi(\mathbf{v})$ is locally trivial in the étale topology.

We now give a criterion for the non-emptiness of $\mathfrak{M}_\xi^s(\mathbf{v})$. This criterion is due to Crawley-Boevey [2, Theorem 1.2]. For $i \in I$ let $\mathbf{e}_i \in \mathbb{Z}^I$ be the vector with 1 at the vertex i and zero elsewhere and let $\Phi(\Gamma) \subset \mathbb{Z}^I$ be the root system associated to Γ defined as in [17]. We denote by $\Phi^+(\Gamma)$ the set of positive roots. Let (\cdot, \cdot) be the symmetric bilinear form on the root lattice \mathbb{Z}^I given by $(\mathbf{e}_i, \mathbf{e}_j) = c_{ij}$. Note that vertices of Γ may support loops.

For $\alpha \in \mathbb{Z}^I$, we put $p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha)$. If α is a real root we have $p(\alpha) = 0$ and if α is an imaginary root then $p(\alpha) > 0$.

Theorem 2.2.7. (i) *The space $\mathfrak{M}_\xi(\mathbf{v})$ is non-empty if and only if $\mathbf{v} = \beta_1 + \beta_2 + \dots$ with $\beta_i \in \Phi^+(\Gamma)$ and $\beta_i \cdot \xi = 0$ for all i .*

(ii) *The space $\mathfrak{M}_\xi^s(\mathbf{v})$ is non-empty if and only if $\mathbf{v} \in \Phi^+(\Gamma)$ and $p(\mathbf{v}) > p(\beta_1) + p(\beta_2) + \dots$ for any nontrivial decomposition of \mathbf{v} as a sum $\mathbf{v} = \beta_1 + \beta_2 + \dots$ with $\beta_i \in \Phi^+(\Gamma)$ and $\beta_i \cdot \xi = 0$ for all i .*

2.2.2 Nakajima's framed quiver varieties

The construction of the so-called framed quiver varieties follows the above one's except that we have an additional graded vector space W .

Let Γ and \mathbf{v} be as in §2.2.1. Let $\mathbf{w} \in \mathbb{Z}_{\geq 0}^I$ be an other dimension vector. Then we put $L_{\mathbf{v}, \mathbf{w}} = \bigoplus_{i \in I} \text{Mat}_{w_i, v_i}(\mathbb{K}) \simeq \bigoplus_{i \in I} \text{Hom}(\mathbb{K}^{v_i}, \mathbb{K}^{w_i})$, $L_{\mathbf{w}, \mathbf{v}} = \bigoplus_{i \in I} \text{Mat}_{v_i, w_i}(\mathbb{K})$, and

$$\mathbf{M}(\overline{\Gamma}, \mathbf{v}, \mathbf{w}) := \mathbf{M}(\overline{\Gamma}, \mathbf{v}) \oplus L_{\mathbf{v}, \mathbf{w}} \oplus L_{\mathbf{w}, \mathbf{v}}.$$

An element of $\mathbf{M}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ is then denoted by (B, a, b) with $B \in \mathbf{M}(\overline{\Gamma}, \mathbf{v})$, $a \in L_{\mathbf{v}, \mathbf{w}}$ and $b \in L_{\mathbf{w}, \mathbf{v}}$. The group $\text{GL}_{\mathbf{v}}$ acts on $\mathbf{M}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ by

$$g \cdot (B, a, b) = (g \cdot B, a \cdot g^{-1}, g \cdot b) \quad (2.2.3)$$

where $g \cdot B$ is the action defined in §2.2.1.

Consider the moment map

$$\mu_{\mathbf{v}, \mathbf{w}} : \mathbf{M}(\overline{\Gamma}, \mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{gl}_{\mathbf{v}} \simeq (\mathfrak{gl}_{\mathbf{v}})^*$$

that maps (B, a, b) to $\mu_{\mathbf{v}}(B) + ba$. For $\xi \in \mathbb{Z}^I$ we denote by $\mathfrak{M}_\xi(\mathbf{v}, \mathbf{w})$ the affine framed quiver variety $\mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi) // \text{GL}_{\mathbf{v}}$ as in [37]. Note that unlike in §2.2.1, we do not assume that $\xi \cdot \mathbf{v} = 0$.

Definition 2.2.8. Let $\theta \in \mathbb{Z}^I$. A point $(B, a, b) \in \mathbf{M}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ is θ -semistable if the two following conditions are satisfied:

- (i) For any B -invariant subspace S of V such that S_i is contained in $\text{Ker}(a_i)$ for all $i \in I$, the inequality $\theta \cdot \dim S \leq 0$ holds.
- (ii) For any B -invariant subspace T of V such that T_i contains $\text{Im}(b_i)$ for all $i \in I$, the inequality $\theta \cdot \dim T \leq \theta \cdot \mathbf{v}$ holds.

The point (B, a, b) is called θ -stable if strict inequalities hold in (i), (ii) unless $S = 0$, $T = V$ respectively.

We denote respectively by $\mathbf{M}_\theta^{ss}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ and $\mathbf{M}_\theta^s(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ the set of θ -semistable and θ -stable points. Then $\mathbf{M}_\theta^s(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ is an open subset of $\mathbf{M}_\theta^{ss}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$ on which the group $\text{GL}_{\mathbf{v}}$ acts set-theoretically freely.

Remark 2.2.9. (i) If $\theta_i \geq 0$ for all $i \in I$, then the condition (ii) of Definition 2.2.8 is always satisfied and so a representation is θ -semistable if and only if the condition (i) is satisfied.

(ii) Let $\theta, \theta' \in \mathbb{Z}_{\geq 0}^I$ and let $J_\theta := \{i \in I \mid \theta_i = 0\}$ and $J_{\theta'} := \{i \in I \mid \theta'_i = 0\}$. If $J_\theta \subset J_{\theta'}$, then $\mathbf{M}_\theta^{ss}(\overline{\Gamma}, \mathbf{v}, \mathbf{w}) \subset \mathbf{M}_{\theta'}^{ss}(\overline{\Gamma}, \mathbf{v}, \mathbf{w})$.

Let $\chi : \mathrm{GL}_{\mathbf{v}} \rightarrow \mathbb{K}^{\times}$, $(g_i) \mapsto \prod_i \det(g_i)^{-\theta_i}$ be the character associated to θ . Then a representation in $\mathbf{M}(\bar{\Gamma}, \mathbf{v}, \mathbf{w})$ is χ -semistable if and only if it is θ -semistable. The framed quiver variety $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w})$ is defined as

$$\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) := \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi) //_{\theta} \mathrm{GL}_{\mathbf{v}}.$$

Define also $\mathfrak{M}_{\xi, \theta}^s(\mathbf{v}, \mathbf{w})$ as the image of $\mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)^s$ in $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w})$. If not empty, the variety $\mathfrak{M}_{\xi, \theta}^s(\mathbf{v}, \mathbf{w})$ is a nonsingular open subset of $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w})$.

Note that $\mathfrak{M}_{\xi, 0}(\mathbf{v}, \mathbf{w})$ is the affine framed quiver variety $\mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w})$ as all points of $\mathbf{M}(\bar{\Gamma}, \mathbf{v}, \mathbf{w})$ are 0-semistable. We thus have a natural projective morphism $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w})$.

It was observed by Crawley-Boevey [2, Introduction] that any framed quiver variety can be in fact realized as an ‘‘unframed’’ quiver variety of §2.2.1. This is done as follows.

From Γ and W we construct a new quiver Γ^* by adding to Γ a new vertex ∞ and for each vertex i of Γ , we add w_i arrows starting at ∞ toward i . Put $I^* = I \cup \{\infty\}$. We then define $(\mathbf{v}^*, \theta^*) \in \mathbb{Z}_{\geq 0}^{I^*} \times \mathbb{Z}^{I^*}$ as follows. We put

- (i) $v_i^* = v_i$ if $i \in I$ and $v_{\infty}^* = 1$,
- (ii) $\theta_i^* = \theta_i$ if $i \in I$ and $\theta_{\infty}^* = -\theta \cdot \mathbf{v}$.

We have a natural group embedding $\mathrm{GL}_{\mathbf{v}} \hookrightarrow \mathrm{GL}_{\mathbf{v}^*}$ that sends an element $g = (g_i)_{i \in I}$ to the element $g^* = (g_i^*)_{i \in I^*}$ with $g_i^* := g_i$ if $i \in I$ and $g_{\infty}^* := 1$. This induces an isomorphism $\mathrm{GL}_{\mathbf{v}} \simeq \mathbf{G}_{\mathbf{v}^*} = \mathrm{GL}_{\mathbf{v}^*} / \mathbb{K}^{\times}$. We have a $\mathrm{GL}_{\mathbf{v}}$ -equivariant isomorphism $\mathbf{M}(\bar{\Gamma}^*, \mathbf{v}^*) \rightarrow \mathbf{M}(\bar{\Gamma}, \mathbf{v}, \mathbf{w})$. Under this isomorphism, the θ -semistability (resp. stability) of Definition 2.2.8 coincides with the θ^* -semistability (resp. stability) in §2.2.1.

In the context of framed quiver, we say that θ is *generic* if θ^* is generic with respect to \mathbf{v}^* in the sense of Definition 2.2.3. In this case we have

$$\mathbf{M}_{\theta}^{ss}(\bar{\Gamma}, \mathbf{v}, \mathbf{w}) = \mathbf{M}_{\theta^*}^s(\bar{\Gamma}^*, \mathbf{v}^*, \mathbf{w})$$

We have [36]:

Proposition 2.2.10. *Assume that θ is generic and that $\mathfrak{M}_{\xi}^s(\mathbf{v}, \mathbf{w}) \neq \emptyset$. Then $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) = \mathfrak{M}_{\xi, \theta}^s(\mathbf{v}, \mathbf{w})$ and the map $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w})$ is a resolution of singularities.*

Remark 2.2.11. If $\theta_i > 0$ for all i , then θ^* is always generic with respect to \mathbf{v}^* .

3 Intersection cohomology

3.1 Generalities and notation

Let X be an algebraic variety over the algebraically closed field \mathbb{K} . Let ℓ be a prime which does not divide the characteristic of \mathbb{K} . The letter κ denotes the field $\bar{\mathbb{Q}}_{\ell}$.

We denote by $\mathcal{D}_c^b(X)$ the bounded ‘‘derived category’’ of κ -(constructible) sheaves on X . For $K \in \mathcal{D}_c^b(X)$ we denote by $\mathcal{H}^i K$ the i -th cohomology sheaf of K . If m is an integer, then we denote by $K[m]$ the m -th shift of K ; we have $\mathcal{H}^i K[m] = \mathcal{H}^{i+m} K$. For a morphism $f : X \rightarrow Y$, we have the usual functors $f_*, f^! : \mathcal{D}_c^b(X) \rightarrow \mathcal{D}_c^b(Y)$ and $f^*, f^! : \mathcal{D}_c^b(Y) \rightarrow \mathcal{D}_c^b(X)$. If $i : Y \hookrightarrow X$ is a closed immersion, the restriction $i^* K$ of $K \in \mathcal{D}_c^b(X)$ is denoted by $K|_Y$. We denote by $D_X : \mathcal{D}_c^b(X) \rightarrow \mathcal{D}_c^b(X)$ the Verdier dual operator.

Let now Y be an irreducible open smooth subset of X such that $\bar{Y} = X$. Then for a local system ξ on Y , we let $\mathcal{IC}_{X, \xi}^{\bullet} \in \mathcal{D}_c^b(X)$ be the intersection cohomology complex defined by Goresky-McPherson and Deligne. The perverse sheaf $K = \underline{\mathcal{IC}}_{X, \xi}^{\bullet} := \mathcal{IC}_{X, \xi}^{\bullet}[\dim X]$ is characterized by the following properties:

- $\mathcal{H}^i K = 0$ if $i < -\dim X$,
- $\mathcal{H}^{-\dim X} K|_Y = \xi$,
- $\dim(\mathrm{Supp}(\mathcal{H}^i K)) < -i$ if $i > -\dim X$,
- $\dim(\mathrm{Supp}(\mathcal{H}^i D_X K)) < -i$ if $i > -\dim X$.

If U is another open smooth subset of X and if ζ is any local system on U such that $\zeta|_{U \cap Y} = \xi|_{U \cap Y}$, then $\mathcal{IC}_{X,\xi}^\bullet = \mathcal{IC}_{X,\zeta}^\bullet$. This is why we omit the open set Y from the notation $\mathcal{IC}_{X,\xi}^\bullet$. We will simply denote by \mathcal{IC}_X^\bullet the complex $\mathcal{IC}_{X,\mathbb{Q}_\ell}^\bullet$.

Remark 3.1.1. Note that if U is a locally closed subvariety of X such that $\overline{U} \subsetneq X$ then $\mathcal{H}^{-\dim U} K|_U = 0$.

We define the compactly supported i -th intersection cohomology groups $IH_c^i(X, \xi)$ with coefficient in the local system ξ as the compactly supported i -th ℓ -adic hypercohomology group $\mathbb{H}_c^i(X, \mathcal{IC}_{X,\xi}^\bullet)$. If f is the unique morphism $X \rightarrow \{pt\}$, then $IH_c^i(X, \xi) = \mathcal{H}^i(f_! \mathcal{IC}_{X,\xi}^\bullet)$.

If X is nonsingular, then \mathcal{IC}_X^\bullet is the constant sheaf κ concentrated in degree 0 and so $IH_c^i(X, \kappa) = H_c^i(X, \kappa)$.

Definition 3.1.2. A proper surjective morphism $f : Z \rightarrow X$ is *semi-small* if and only if one of the following equivalent conditions is satisfied:

- (i) $\dim \{x \in X \mid \dim f^{-1}(x) \geq i\} \leq \dim X - 2i$ for all $i \in \mathbb{Z}_{\geq 0}$.
- (ii) There exists a filtration $X := F_0 \supset F_1 \supset \dots \supset F_r = \emptyset$ of X by closed subsets such that, for all $i \in \{0, \dots, r-1\}$ and $x \in F_i - F_{i+1}$, we have $2 \dim f^{-1}(x) \leq \dim X - \dim F_i$.

We will need the following decomposition theorem of Beilinson, Bernstein, Deligne and Gabber.

Theorem 3.1.3. *Suppose that $\varphi : X \rightarrow X'$ is a proper map with X irreducible. Then*

$$\varphi_*(\mathcal{IC}_X^\bullet) \simeq \bigoplus_{Z,\xi,r} V_{Z,\xi,r} \otimes \mathcal{IC}_{Z,\xi}^\bullet[r]$$

where ξ is an irreducible local system on some open subset of a closed irreducible subvariety Z of X' and where by abuse of notation we still denote by $\mathcal{IC}_{Z,\xi}^\bullet$ its extension by zero on $X' - Z$. If moreover $\varphi_*(\underline{\mathcal{IC}}_X^\bullet)$ is a perverse sheaf, then

$$\varphi_*(\underline{\mathcal{IC}}_X^\bullet) \simeq \bigoplus_{Z,\xi} V_{Z,\xi} \otimes \underline{\mathcal{IC}}_{Z,\xi}^\bullet \quad (3.1.1)$$

Remark 3.1.4. Let Y be a closed irreducible subvariety of X' and let U be a non-empty nonsingular open subset of Y . Note that

$$\mathcal{H}^{-\dim Y} \left(\bigoplus_{Z,\xi} V_{Z,\xi} \otimes \underline{\mathcal{IC}}_{Z,\xi}^\bullet \right) \Big|_U \simeq \bigoplus_{\xi} V_{Y,\xi} \otimes \xi$$

where the direct sum on the right hand side is over the irreducible local systems on Y .

Corollary 3.1.5. *If $\varphi : X \rightarrow X'$ is a semi-small resolution of singularities and $\underline{\kappa}$ the constant sheaf on X concentrated in degree $-\dim X$ then (3.1.1) becomes*

$$\varphi_*(\underline{\kappa}) \simeq \underline{\mathcal{IC}}_{X'}^\bullet \oplus \left(\bigoplus_{Z,\xi} V_{Z,\xi} \otimes \underline{\mathcal{IC}}_{Z,\xi}^\bullet \right) \quad (3.1.2)$$

with $Z \subsetneq X'$. In particular

$$H_c^i(X, \kappa) \simeq IH_c^i(X', \kappa) \oplus \left(\bigoplus_{Z,\xi} V_{Z,\xi} \otimes IH_c^{i+d_Z-d_X}(Z, \xi) \right). \quad (3.1.3)$$

where d_Z the dimension of Z .

The isomorphism (3.1.3) is obtained from (3.1.2) by applying the functor $f_!$ with $f : X' \rightarrow \{pt\}$.

Corollary 3.1.6. *Assume that $\varphi : X \rightarrow X'$ is a semi-small resolution. If $X' = \bigcup_{\alpha \in I} X'_\alpha$ where I is a finite set and where the X'_α are locally closed irreducible subvarieties of X' such that the restriction of $\mathcal{H}^i(\varphi_*(\kappa))$ to X'_α is a locally constant sheaf for all i and all $\alpha \in I$, then*

$$\varphi_*(\kappa) \simeq \underline{IC}_{X'}^\bullet \oplus \left(\bigoplus_{\alpha, \xi_\alpha} V_{\alpha, \xi_\alpha} \otimes \underline{IC}_{\overline{X'_\alpha}, \xi_\alpha}^\bullet \right)$$

where the direct sum is over the α such that $\overline{X'_\alpha} \subseteq Y$.

Proof. Let Z be an irreducible closed subvariety of X' such that $\underline{IC}_{Z, \xi}^\bullet$ is a direct summand of $\varphi_*(\kappa)$. We have $Z = \bigcup_{\alpha} (X'_\alpha \cap Z)$. Since Z is irreducible, there exists an α such that $X'_\alpha \cap Z$ is dense in Z . By Remark 3.1.4, we have $\mathcal{H}^{-\dim Z} \varphi_*(\kappa)|_{X'_\alpha \cap Z} \neq 0$. Since $\mathcal{H}^{-\dim Z} \varphi_*(\kappa)|_{X'_\alpha}$ is locally constant and non-zero, we have $X'_\alpha \subset \text{Supp}(\mathcal{H}^{-\dim Z} \varphi_*(\kappa))$. Hence

$$\dim X'_\alpha \leq \dim \left(\text{Supp}(\mathcal{H}^{-\dim Z} \varphi_*(\kappa)) \right) \leq \dim Z.$$

The right inequality holds because $\varphi_*(\kappa)$ is a perverse sheaf. Since $\dim(X'_\alpha \cap Z) = \dim Z$, we deduce that the inclusion $X'_\alpha \cap Z \subset X'_\alpha$ is an equality, i.e., $X'_\alpha \subset Z$, and so that $\overline{X'_\alpha} = Z$. \square

Assume that \mathbb{K} is an algebraic closure of a finite field \mathbb{F}_q and that X is an irreducible algebraic variety defined over \mathbb{F}_q , i.e., there is a scheme X_o over \mathbb{F}_q such that X is the set of $\overline{\mathbb{F}_q}$ -points of $X_o \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. We denote by $F : X \rightarrow X$ the corresponding Frobenius endomorphism. Let $K \in \mathcal{D}_c^b(X)$ and assume that there exists an isomorphism $\varphi : F^*(K) \simeq K$. The *characteristic function* $\mathbf{X}_{K, \varphi} : X^F \rightarrow \kappa$ of (K, φ) is defined by

$$\mathbf{X}_{K, \varphi}(x) = \sum_i (-1)^i \text{Trace}(\varphi_x^i, \mathcal{H}_x^i K).$$

If $r \in \mathbb{Z}$, we denote by $K(r)$ the r -th Tate twist of K . Then $\mathbf{X}_{K(r), \varphi(r)} = q^{-r} \mathbf{X}_{K, \varphi}$.

We have the following relation between the trace on the local intersection cohomology and the trace on the hypercohomology of K

$$\sum_{x \in X^F} \mathbf{X}_{K, \varphi}(x) = \sum_i (-1)^i \text{Trace}(F_\varphi^*, \mathbb{H}_c^i(X, K))$$

where F_φ is the Frobenius on the hypercohomology induced by φ . Assume that the locally closed irreducible subvariety Y of X is also defined over \mathbb{F}_q , i.e., it is F -stable. Let $F^*(\kappa) \simeq \kappa$ be the isomorphism (in the category of sheaves on Y) which induces the identity on the stalk at $x \in Y^F$. It induces a canonical isomorphism $\varphi : F^*(K) \simeq K$ with $K = \underline{IC}_X^\bullet$. We then put $\mathbf{X}_K(x) := \mathbf{X}_{K, \varphi}(x)$. Hence:

$$\sum_{x \in X^F} \mathbf{X}_{\underline{IC}_X^\bullet}(x) = \sum_i (-1)^i \text{Trace}(F^*, IH_c^i(X, \kappa)). \quad (3.1.4)$$

In particular, if X is nonsingular then we recover the usual Grothendieck trace formula as the left hand side of the above identity is $\#\{X(\mathbb{F}_q)\}$.

3.2 Restriction

Assume that X is irreducible. Let Z be an irreducible closed subvariety of X et let $i : Z \hookrightarrow X$ denotes the inclusion. We give a condition for $i^*(\underline{IC}_X^\bullet) = \underline{IC}_Z^\bullet$ to be true.

Proposition 3.2.1. *Assume that there is a decomposition $X = \bigcup_{\alpha \in I} X_\alpha$ of X where I is a finite set and where the X_α are locally closed irreducible subvarieties such that*

- (i) if $Z_\alpha := X_\alpha \cap Z$ is not empty, then it is equidimensional and $\text{codim}_X X_\alpha = \text{codim}_Z Z_\alpha$.

Assume moreover that there exists a Cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & X \\ \tilde{i} \uparrow & & \uparrow i \\ \tilde{Z} & \xrightarrow{g} & Z \end{array}$$

such that the conditions (ii) and (iii) below are satisfied.

(ii) f and g are semi-small resolutions of singularities.

(iii) The restriction of the sheaf $\mathcal{H}^i(f_*(\kappa))$ to X_α is a locally constant sheaf for all i .

Then $i^*(IC_X^\bullet) = IC_Z^\bullet$.

Proof. If Y is a variety, let d_Y denote its dimension. Let $\alpha_o \in I$ be such that X_{α_o} is the open stratum of X . To avoid any confusion we will use the notation $IC_Z^\bullet[d_Z]$ instead of \underline{IC}_Z^\bullet . By Corollary 3.1.6, we have

$$f_*(\kappa[d_X]) = IC_X^\bullet[d_X] \oplus \left(\bigoplus_{\alpha \neq \alpha_o, \xi_\alpha} V_{\alpha, \xi_\alpha} \otimes IC_{\tilde{X}_\alpha, \xi_\alpha}^\bullet [d_{X_\alpha}] \right). \quad (3.2.1)$$

By (iii) and $i^*f_*(\kappa) = g_*(\kappa)$ we see that the restriction of $\mathcal{H}^i(g_*(\kappa))$ to Z_α is locally constant. Hence by Corollary 3.1.6, we have

$$g_*(\kappa[d_Z]) = IC_Z^\bullet[d_Z] \oplus \left(\bigoplus_{\alpha \neq \alpha_o, \beta \in I_\alpha, \zeta_{\alpha, \beta}} W_{(\alpha, \beta), \zeta_{\alpha, \beta}} \otimes IC_{\tilde{Z}_{(\alpha, \beta), \zeta_{\alpha, \beta}}}^\bullet [d_{Z_\alpha}] \right) \quad (3.2.2)$$

where $\{Z_{(\alpha, \beta)}\}_{\beta \in I_\alpha}$ is the set of irreducible components of Z_α . Using again $i^*f_*(\kappa) = g_*(\kappa)$ we see from (3.2.1) and (3.2.2) that the complex $i^*(IC_X^\bullet)[d_Z]$ is a direct summand of the semisimple perverse sheaf $g_*(\kappa[d_Z])$. It is therefore a semisimple perverse subsheaf of $g_*(\kappa[d_Z])$. Since the open stratum Z_{α_o} of Z is contained in the open stratum of X , the restriction of $i^*(IC_X^\bullet)[d_Z]$ to Z_{α_o} is the constant sheaf $\kappa[d_Z]$. Hence $i^*(IC_X^\bullet)[d_Z]$ contains $IC_Z^\bullet[d_Z]$ as a direct summand, i.e.,

$$i^*(IC_X^\bullet)[d_Z] = IC_Z^\bullet[d_Z] \oplus \left(\bigoplus_{\alpha \neq \alpha_o, \beta \in I_\alpha, \zeta_{\alpha, \beta}} W'_{(\alpha, \beta), \zeta_{\alpha, \beta}} \otimes IC_{\tilde{Z}_{(\alpha, \beta), \zeta_{\alpha, \beta}}}^\bullet [d_{Z_\alpha}] \right)$$

for some subspaces $W'_{(\alpha, \beta), \zeta_{\alpha, \beta}} \subset W_{(\alpha, \beta), \zeta_{\alpha, \beta}}$. It remains to see that $W'_{(\alpha, \beta), \zeta_{\alpha, \beta}} = 0$ for all $\alpha \neq \alpha_o$.

Put $K := i^*(IC_X^\bullet)[d_Z]$. Then for $\alpha \neq \alpha_o$ we have

$$\begin{aligned} \mathcal{H}^{-d_{Z_\alpha}} K|_{Z_\alpha} &= \mathcal{H}^{d_Z - d_{Z_\alpha}} IC_X^\bullet|_{Z_\alpha} \\ &= \mathcal{H}^{d_X - d_{X_\alpha}} IC_X^\bullet|_{Z_\alpha} \\ &= \mathcal{H}^{-d_{X_\alpha}} IC_X^\bullet[d_X]|_{Z_\alpha} \\ &= 0. \end{aligned}$$

The last equality follows from Remark 3.1.1. Hence $W'_{(\alpha, \beta), \zeta_{\alpha, \beta}} = 0$ by Remark 3.1.4 and we proved the proposition. \square

3.3 E-polynomial

Recall that a mixed Hodge structure on a rational vector space H consist of a finite increasing filtration W_\bullet (the weight filtration) on H , and a finite decreasing filtration F^\bullet (the Hodge filtration) on the complexification $H_\mathbb{C}$, which induces a pure Hodge structure of weight k on the complexified graded pieces $\text{Gr}_k^W H_\mathbb{C} = (W_k H / W_{k-1} H)_\mathbb{C}$, i.e.,

$$\mathrm{Gr}_k^W H_{\mathbb{C}} = \bigoplus_{p+q=k} \left(\mathrm{Gr}_k^W H_{\mathbb{C}} \right)^{p,q}$$

with

$$\left(\mathrm{Gr}_k^W H_{\mathbb{C}} \right)^{p,q} = F^p \mathrm{Gr}_k^W H_{\mathbb{C}} \cap \overline{F^q \mathrm{Gr}_k^W H_{\mathbb{C}}}.$$

We call the integers $\left\{ h^{p,q} := \dim \left(\mathrm{Gr}_{p+q}^W H_{\mathbb{C}} \right)^{p,q} \right\}$ the *mixed Hodge numbers*.

Recall (see [43], [40, Chapter 14]) that for any complex algebraic variety X , the intersection cohomology group $IH_c^k(X, \mathbb{C})$ is endowed with a mixed Hodge structure. If X is nonsingular, it coincides with Deligne's mixed Hodge structure on $H_c(X, \mathbb{C})$ which is defined in [7].

We then denote by $\{ih_c^{p,q;k}(X)\}_{p,q}$ the mixed Hodge numbers of $IH_c^k(X, \mathbb{C})$ and we define the mixed Hodge polynomial of X as

$$IH_c(X; x, y, z) = \sum_{p,q,k} ih_c^{p,q;k}(X) x^p y^q z^k.$$

The compactly supported Poincaré polynomial of X is then $IH_c(X; 1, 1, t)$.

In this paper we will say that X is *pure* if the mixed Hodge structure on $IH_c^k(X, \mathbb{C})$ is pure for all k , i.e., if $ih_c^{p,q;k}(X) = 0$ when $p + q \neq k$.

The E -polynomial of X is defined as

$$E^ic(X; x, y) := IH_c(X; x, y, -1) = \sum_{p,q} \left(\sum_k (-1)^k ih_c^{p,q;k}(X) \right) x^p y^q.$$

Let R be a subring of \mathbb{C} which is finitely generated as a \mathbb{Z} -algebra and let \mathcal{X} be a separated R -scheme of finite type. According to [15, Appendix], we say that \mathcal{X} is *strongly polynomial count* if there exists a polynomial $P(T) \in \mathbb{C}[T]$ such that for any finite field \mathbb{F}_q and any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$, the \mathbb{F}_q -scheme \mathcal{X}^φ obtained from \mathcal{X} by base change is polynomial count with counting polynomial P , i.e., for every finite extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, we have

$$\#\{\mathcal{X}^\varphi(\mathbb{F}_{q^n})\} = P(q^n).$$

According to Katz terminology (cf. [Appendix][15]), we call a separated R -scheme \mathcal{X} which gives back X after extension of scalars from R to \mathbb{C} a *spreading out* of X .

The complex variety X is said to be *polynomial count* if there exists a spreading out of X which is strongly polynomial count.

Let us now denote by $\{h_c^{i,j;k}(X)\}_{i,j}$ the mixed Hodge numbers of $H_c^k(X, \mathbb{C})$ and put

$$E(X; x, y) := \sum_{i,j} \left(\sum_k (-1)^k h_c^{i,j;k}(X) \right) x^i y^j.$$

We recall the result of Katz in the appendix of [15].

Theorem 3.3.1. *Assume that X is polynomial count with counting polynomial $P \in \mathbb{C}[T]$. Then*

$$E(X; x, y) = P(xy).$$

Let $X = \coprod_{\alpha \in I} X_\alpha$ be a stratification and let X_{α_o} be the open stratum, i.e., $X = \overline{X_{\alpha_o}}$. Put $\alpha \leq \beta$ if $X_\alpha \subset \overline{X_\beta}$, and $r_\alpha := (\dim X - \dim X_\alpha)/2$.

We say that X satisfies the property (E) with respect to this stratification and the ring R if there exist a spreading out $\tilde{\mathcal{X}}$ of X , a stratification $\mathcal{X} = \coprod_{\alpha} \mathcal{X}_\alpha$, and a morphism $\nabla : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ of R -schemes such that:

(1) $\tilde{\mathcal{X}}$ and the closed strata \mathcal{X}_α are strongly polynomial count,

(2) for each α , the stratum X_α is a spreading out of X_α , the morphism $r : \tilde{X} \rightarrow X$ obtained from ∇ after extension of scalars from R to \mathbb{C} yields an isomorphism of mixed Hodge structures

$$H_c^i(\tilde{X}, \mathbb{Q}) \simeq IH_c^i(X, \mathbb{Q}) \oplus \left(\bigoplus_{\alpha \neq \alpha_o} W_\alpha \otimes (IH_c^{i+2r_\alpha}(\bar{X}_\alpha, \mathbb{Q}) \otimes \mathbb{Q}(r_\alpha)) \right), \quad (3.3.1)$$

(3) for any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$, the morphism $\nabla^\varphi : \tilde{X}^\varphi \rightarrow X^\varphi$ obtained from ∇ by base change yields an isomorphism

$$(\nabla^\varphi)_*(\underline{\kappa}) \simeq \underline{IC}_{X^\varphi}^\bullet \oplus \left(\bigoplus_{\alpha \neq \alpha_o} W_\alpha \otimes \underline{IC}_{\bar{X}_\alpha^\varphi}^\bullet(r_\alpha) \right) \quad (3.3.2)$$

of perverse sheaves.

Assume now that all complex varieties \bar{X}_α (in particular X) satisfy the property (E) with respect to the stratification $\bar{X}_\alpha = \coprod_{\beta \leq \alpha} X_\beta$ and the ring R_α . Since there is only a finite number of strata we may assume without loss of generalities that the rings R_α are all equal to the same ring R .

Theorem 3.3.2. *With the above assumption, there exists a polynomial $P(T) \in \mathbb{Z}[T]$ such that for any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$, we have*

$$\sum_{x \in \mathcal{X}^\varphi(\mathbb{F}_q)} \mathbf{X}_{IC_{X^\varphi(\mathbb{F}_q)}^\bullet}(x) = P(q) \quad (3.3.3)$$

and

$$E^{ic}(X; x, y) = P(xy).$$

Proof. If there is only one stratum, i.e., if X is nonsingular, then the theorem is true by Theorem 3.3.1. The theorem is now easy to prove by induction on $\alpha < \beta$. Assume that the theorem is true for all $\alpha < \alpha_o$. By Formula (3.3.1), we have

$$E(\tilde{X}; x, y) = E^{ic}(X; x, y) + \sum_{\alpha < \alpha_o} (\dim W_\alpha) E^{ic}(\bar{X}_\alpha; x^{-r_\alpha}, y^{-r_\alpha}).$$

By induction hypothesis and since \tilde{X} is polynomial count, this formula shows that $E^{ic}(X; x, y)$ depends only on the product xy , i.e., that there exists a unique polynomial P such that $E^{ic}(X; x, y) = P(xy)$, more precisely P is defined as $P = \tilde{P} - \sum_{\alpha < \alpha_o} (\dim W_\alpha) P_\alpha(x^{-r_\alpha} y^{-r_\alpha})$ where \tilde{P} is the counting polynomial of X and P_α (with $\alpha \neq \alpha_o$) is the polynomial which satisfies the theorem for $X = \bar{X}_\alpha$. It remains to see that P satisfies Formula (3.3.3).

By Formula (3.3.2), we have

$$\mathbf{X}_{(\nabla^\varphi)_*(\underline{\kappa})} = \mathbf{X}_{IC_{X^\varphi}^\bullet} + \sum_{\alpha < \alpha_o} (\dim W_\alpha) q^{-r_\alpha} \mathbf{X}_{IC_{\bar{X}_\alpha^\varphi}^\bullet}. \quad (3.3.4)$$

By Grothendieck trace formula we have

$$\sum_{x \in \mathcal{X}^\varphi(\mathbb{F}_q)} \mathbf{X}_{(\nabla^\varphi)_*(\underline{\kappa})}(x) = \#\{\tilde{X}^\varphi(\mathbb{F}_q)\} = \tilde{P}(q).$$

Now integrating Formula (3.3.4) over $\mathcal{X}^\varphi(\mathbb{F}_q)$ proves Formula (3.3.3). \square

Remark 3.3.3. Assume that X satisfies the assumptions of Theorem 3.3.2 and that X is pure. Since X is pure we have $E^{ic}(X; x, y) = \sum_{p,q} (-1)^{p+q} ih_c^{p,q;p+q}(X) x^p y^q$. By Theorem 3.3.2, the polynomial $E^{ic}(X; x, y)$ depends only on the product xy , hence $ih_c^{p,q;p+q}(X) = 0$ if $p \neq q$. The mixed Hodge numbers of X are thus all of the form $ih_c^{p,p;2p}(X)$ and so $E^{ic}(X; x, y) = P_c(X; xy)$ where $P_c(X; t) := \sum_i (\dim IH_c^{2i}(X, \mathbb{C})) t^i$.

4 Quiver varieties of type A

We review known results [21] [37] [38] [3] [5] [42] and give a slight generalization of some of them.

4.1 Partitions and types

We denote by \mathcal{P} the set of all partitions including the unique partition 0 of 0, by \mathcal{P}^* the set of non-zero partitions and by \mathcal{P}_n the set of partitions of n . Partitions λ are denoted by $\lambda = (\lambda_1, \lambda_2, \dots)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, or by $(1^{n_1}, 2^{n_2}, \dots)$ where n_i denotes the number of parts of λ_i equal to i . We put $|\lambda| := \sum_i \lambda_i$ for the size of λ . The length of λ is the maximum i with $\lambda_i > 0$ and we denote by λ' the dual partition of λ . For two partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ we define the partition $\lambda + \mu$ as $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$, and for $\lambda = (1^{n_1}, 2^{n_2}, \dots)$, $\mu = (1^{m_1}, 2^{m_2}, \dots)$, we define the union $\lambda \cup \mu$ as $(1^{n_1+m_1}, 2^{n_2+m_2}, \dots)$. For a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ and a positive integer d , we denote by $d \cdot \lambda$ the partition $(d\lambda_1, \dots, d\lambda_s)$. Recall that $(\lambda + \mu)' = \lambda' \cup \mu'$. We denote by $\lambda \leq \mu$ the lexicographic ordering on \mathcal{P} and by $\lambda \trianglelefteq \mu$ the dominance partial ordering on \mathcal{P} , namely $\lambda \trianglelefteq \mu$ if $\lambda_1 + \lambda_2 + \dots + \lambda_i \leq \mu_1 + \mu_2 + \dots + \mu_i$ for all i .

We denote by $\tilde{\mathbf{T}}$ the set of non-increasing sequences $\tilde{\omega} = \omega^1 \omega^2 \dots \omega^r$ with $\omega^i \in \mathcal{P}$ and let $\tilde{\mathbf{T}}_n$ be the subset of sequences $\tilde{\omega}$ such that $\sum_i |\omega^i| = n$. We will see in §4.2 that the set $\tilde{\mathbf{T}}_n$ parametrizes the types of the adjoint orbits in $\mathfrak{gl}_n(\mathbb{K})$.

We choose a total order \geq on the set $\{(d, \lambda) \mid d \in \mathbb{Z}_{\geq 0}^*, \lambda \in \mathcal{P}^*\}$ as follows. If $\mu \neq \lambda$, we say that $(d, \lambda) \geq (d', \mu)$ if $\lambda \geq \mu$, and we say that $(d, \lambda) \geq (d', \lambda)$ if $d \geq d'$. We denote by \mathbf{T} the set of all non-increasing sequences $\omega = (d_1, \lambda^1)(d_2, \lambda^2) \dots (d_r, \lambda^r)$ and by \mathbf{T}_n the subset of \mathbf{T} of these sequences which satisfy $|\omega| := \sum_i d_i |\lambda^i| = n$. The first coordinate of a pair (d, λ) is called the *degree*. We will see in §6.7 that \mathbf{T}_n parametrizes both the types of the adjoint orbits in $\mathfrak{gl}_n(\mathbb{F}_q)$ and the types of the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$.

Since the terminology ‘‘type’’ has two meanings in this paper, we use the letter $\{\omega, \tau, \dots\}$ to denote the elements of \mathbf{T} and the symbols $\{\tilde{\omega}, \tilde{\tau}, \dots\}$ for the elements of $\tilde{\mathbf{T}}$.

Given a type $\omega = (d_1, \omega^1) \dots (d_r, \omega^r) \in \mathbf{T}$, we assign the type

$$\tilde{\omega} = \overbrace{\omega^1 \dots \omega^1}^{d_1} \overbrace{\omega^2 \dots \omega^2}^{d_2} \dots \overbrace{\omega^r \dots \omega^r}^{d_r}$$

of $\tilde{\mathbf{T}}$. We thus have a surjective map $\mathfrak{S} : \mathbf{T} \rightarrow \tilde{\mathbf{T}}$, $\omega \mapsto \tilde{\omega}$.

Let

$$\tilde{\omega} = \overbrace{\omega^1 \dots \omega^1}^{a_1} \overbrace{\omega^2 \dots \omega^2}^{a_2} \dots \overbrace{\omega^r \dots \omega^r}^{a_r} \in \tilde{\mathbf{T}}$$

with $\omega^i \neq \omega^j$ if $i \neq j$ and put

$$W_{\tilde{\omega}} := \prod_{i=1}^r S_{a_i}.$$

Note that the elements in the fiber $\mathfrak{S}^{-1}(\tilde{\omega})$ are parametrized by $\mathcal{P}_{a_1} \times \dots \times \mathcal{P}_{a_r}$, and so by the conjugacy classes of $W_{\tilde{\omega}}$.

4.2 Zariski closure of adjoint orbits as quiver varieties

Let $A \in \mathfrak{gl}_n(\mathbb{K})$ we denote by σ , resp. ν , the semisimple part, resp. the nilpotent part, of A . We assume for simplicity that σ is diagonal matrix so that its centralizer L in GL_n is exactly a product of GL_{m_i} 's. We have $A = \sigma + \nu$ with $[\sigma, \nu] = 0$ where $[x, y] = xy - yx$. We put $C_{\mathfrak{gl}_n}(A) := \{X \in \mathfrak{gl}_n \mid [A, X] = 0\} = \mathrm{Lie}(L)$. Let C be the L -orbit of ν . Then the GL_n -conjugacy class of the pair (L, C) is called the *type* of \mathcal{O} . The types of the adjoint orbits of \mathfrak{gl}_n are parameterized by the set $\tilde{\mathbf{T}}_n$ as follows.

Let m_1, \dots, m_r be the multiplicities of the r distinct eigenvalues $\alpha_1, \dots, \alpha_r$ of A . We may assume that σ is the diagonal matrix

$$\left(\overbrace{\alpha_1, \dots, \alpha_1}^{m_1}, \overbrace{\alpha_2, \dots, \alpha_2}^{m_2}, \dots, \overbrace{\alpha_r, \dots, \alpha_r}^{m_r} \right).$$

The Jordan form of the element $v \in C_{\mathfrak{gl}_n}(\sigma) = \mathfrak{gl}_{m_1} \oplus \mathfrak{gl}_{m_2} \oplus \dots \oplus \mathfrak{gl}_{m_r}$ defines a unique partition ω^i of m_i for each $i \in \{1, 2, \dots, r\}$. Re-indexing if necessary we may assume that $\omega^1 \geq \omega^2 \geq \dots \geq \omega^r$ in which case we have $\tilde{\omega} = \omega^1 \cdots \omega^r \in \tilde{\mathbf{T}}_n$. Conversely, any element of $\tilde{\mathbf{T}}_n$ arises as the type of some adjoint orbit of \mathfrak{gl}_n . Types of semisimple orbits are of the form $(1^{n_1}) \cdots (1^{n_r})$ and types of nilpotent orbits are just partitions of n .

Lemma 4.2.1. *The dimension of \mathcal{O} is*

$$n^2 - \sum_{j=1}^r \langle \omega^j, \omega^j \rangle \quad (4.2.1)$$

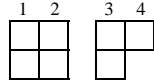
where for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we put $\langle \lambda, \lambda \rangle = 2n(\lambda) + |\lambda|$ with $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$.

We now explain how to construct a quiver Γ and pairs (ξ, \mathbf{v}) from \mathcal{O} such that $\mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) \simeq \overline{\mathcal{O}}$. We draw the Young diagrams respectively of $\omega^1, \dots, \omega^r$ from the left to the right and we label the columns from the left to the right. We define a pair $(\xi, \mathbf{v}) \in \mathbb{K}^l \times \mathbb{Z}^l$ as follows. Let n_i be the length of the i -th column with respect to this labeling. We define the dimension vector $\mathbf{v} = (v_1, \dots, v_{d-1})$ by $v_1 := n - n_1$ and $v_i := v_{i-1} - n_i$ for $i > 1$ and the parameter $\zeta = (\zeta_1, \dots, \zeta_d)$ as follows. If the i -th column belongs to the Young diagram of ω^j then we put $\zeta_i = \alpha_j$.

We then have

$$(A - \zeta_1 \text{Id}) \cdots (A - \zeta_d \text{Id}) = 0.$$

Example 4.2.2. Assume that \mathcal{O} is of type $(2, 2)(2, 1)$ with eigenvalues α_1 and α_2 respectively of multiplicity 4 and 3. The corresponding Young diagrams are



Then the vector dimension is $\mathbf{v} = (5, 3, 1)$ and $\zeta = (\alpha_1, \alpha_1, \alpha_2, \alpha_2)$.

We have

Lemma 4.2.3. *For $i > 0$, the integer v_i is the rank of the partial product*

$$(A - \zeta_1 \text{Id}) \cdots (A - \zeta_i \text{Id}).$$

Let Γ be the quiver

$$\bullet^1 \longrightarrow \bullet^2 \longrightarrow \cdots \longrightarrow \bullet^{d-1}$$

whose underlying graph is the Dynkin diagram of type A_{d-1} and put $I := \{1, \dots, d-1\}$. Put $\mathbf{w} := (n, 0, \dots, 0)$ and define $\xi = (\xi_1, \dots, \xi_{d-1})$ by $\xi_j := \zeta_{j+1} - \zeta_j$.

Proposition 4.2.4. *The map $q : \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi) \rightarrow \overline{\mathcal{O}}$ given by $(B, a, b) \mapsto ab + \zeta_1 \text{Id}$ is well-defined. It induces a bijective morphism $\tilde{q} : \mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) \rightarrow \overline{\mathcal{O}}$. If $\mathbb{K} = \mathbb{C}$, then q is a categorical quotient by $\text{GL}_{\mathbf{v}}$, i.e., the map $\mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) \rightarrow \overline{\mathcal{O}}$ is an isomorphism. The bijective morphism \tilde{q} restricts to $\mathfrak{M}_{\xi}^s(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{O}$.*

Proof. The first assertion follows from [5, Theorem 2.1]. The second assertion can be proved using the “First Fundamental Theorem of Invariant Theory” as in [21, §2]. The third assertion follows from the second one using the assertion (4) of Proposition 2.1.1, cf. [21, §2][4, Lemma 9.1]. For an arrow of $\bar{\Gamma}$ with tail i and head j , we denote by $B_{i,j}$ the corresponding coordinate of B . By [3, §3], we have $f(B, a, b) \in O$ if and only if the $B_{i+1,i}$ ’s and a are all injective and if the maps $B_{i,i+1}$ ’s and b are all surjective, i.e., (B, a, b) is a 0-stable representation. \square

Remark 4.2.5. From Proposition 4.2.4 it follows that if C is the $\mathrm{GL}_{\mathbf{v}}$ -orbit of any representation $(B, a, b) \in \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)$ then $a'b' = ab$ for any $(B', a', b') \in \bar{C}$.

We say that $(n_1, \dots, n_{d-1}) \in (\mathbb{Z}_{>0})^{d-1}$ is *monotone* if $n_1 > \dots > n_{d-1}$.

Remark 4.2.6. Let $\mathbf{v} = (v_1, \dots, v_{d-1})$ be a monotone sequence with $n > v_1$, and let $\xi = (\xi_1, \dots, \xi_{d-1})$. Then (ξ, \mathbf{v}) arises from an adjoint orbit as above if and only if the following condition is satisfied, see [5, §2].

(*) For any $j \in I$ with $\xi_j = 0$ we have $v_{j-1} - v_j \geq v_j - v_{j+1}$ with $v_0 := n$.

4.3 Partial resolutions of Zariski closure of adjoint orbits

Let $(L, P, \Sigma, \sigma, C)$ be as in §1.2. The aim of this section is to see the varieties $\mathbb{X}_{L,P,\Sigma}$ as quiver varieties of the form $\mathfrak{M}_{\xi,\theta}(\mathbf{v}, \mathbf{w})$ when $\mathbb{K} = \mathbb{C}$ (in positive characteristic we only have a bijective morphism $\mathfrak{M}_{\xi,\theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathbb{X}_{L,P,\Sigma}$).

Note that the varieties $\mathbb{X}_{L,P,\Sigma}$ are irreducible of dimension $\dim \mathrm{GL}_n - \dim L + \dim \Sigma$.

Taking a GL_n -conjugate of L if necessary, we may assume that $L = \mathrm{GL}_{s_{p+1}} \times \mathrm{GL}_{s_p} \times \dots \times \mathrm{GL}_{s_1}$. Since σ is in the center of \mathfrak{l} , we may write σ as the diagonal matrix

$$\left(\overbrace{\epsilon_{p+1}, \dots, \epsilon_{p+1}}^{s_{p+1}}, \overbrace{\epsilon_p, \dots, \epsilon_p}^{s_p}, \dots, \overbrace{\epsilon_1, \dots, \epsilon_1}^{s_1} \right).$$

The nilpotent orbit C of \mathfrak{l} decomposes as

$$C = C_{p+1} \times \dots \times C_1$$

with C_i a nilpotent orbit of \mathfrak{gl}_{s_i} . For $i = 1, \dots, p+1$, let μ_i be the partition of s_i which gives the size of the blocks of the Jordan form of C_i . Re-ordering if necessary, we may assume that $\mu^1 \geq \mu^2 \geq \dots \geq \mu^{p+1}$. Let $\alpha_1, \dots, \alpha_k$ be the distinct eigenvalues of σ with respective multiplicities m_1, \dots, m_k . For each $i = 1, \dots, k$, we define a partition λ_i of m_i as the sum of the partitions μ_r where r runs over the set $\{r \mid \epsilon_r = \alpha_i\}$. The partitions $\lambda_1, \dots, \lambda_k$ defines a unique nilpotent orbits of the Lie algebra \mathfrak{m} of $M := C_{\mathrm{GL}_n}(\sigma)$. Let v be an element in this orbit and let O be the unique adjoint orbit of \mathfrak{gl}_n that contains $\sigma + v$. The following proposition is well-known.

Proposition 4.3.1. *The image of the projection $p : \mathbb{X}_{L,P,\Sigma} \rightarrow \mathfrak{gl}_n$ is \bar{O} . Moreover it induces an isomorphism $p^{-1}(O) \simeq O$. If $M = L$, the map p is an isomorphism $\mathbb{X}_{L,P,\Sigma} \simeq \bar{O}$.*

We have $\dim O = \dim \mathbb{X}_{L,P,\Sigma}$ and so

$$\dim O = \dim G - \dim L + \dim \Sigma \tag{4.3.1}$$

We now denote by \mathcal{F} the variety of partial flags $\{0\} = E^{p+1} \subset E^p \subset \dots \subset E^1 \subset E^0 = \mathbb{K}^n$ with $\dim E^{r-1}/E^r = s_r$. For an element $X \in \mathfrak{gl}_n$ that leaves stable a partial flag

$$\left(\{0\} = E^{p+1} \subset E^p \subset \dots \subset E^1 \subset E^0 = \mathbb{K}^n \right) \in \mathcal{F}$$

we denote by X_r , $r = 1, \dots, p+1$, the induced endomorphism of $E^{r-1}/E^r \simeq \mathbb{K}^{s_r}$.

We denote by $\mathbb{Z}_{L,P,\Sigma}$ (resp. $\mathbb{Z}_{L,P,\Sigma}^o$) the subvariety of $\mathfrak{gl}_n \times \mathcal{F}$ of pairs (X, f) such that $X \cdot f = f$ and such that for all $r = 1, \dots, p+1$, we have $X_r \in \epsilon_r \text{Id} + \overline{C}_r$ (resp. $X_r \in \epsilon_r \text{Id} + C_r$).

Note that $\mathcal{F} \simeq \text{GL}_n/P$ and so the two varieties $\mathbb{Z}_{L,P,\Sigma}$ and $\mathbb{X}_{L,P,\Sigma}$ are isomorphic.

There exist a unique positive integer d , a monotone sequence of positive integers $\mathbf{v} = (v_1, \dots, v_{d-1}) \in (\mathbb{Z}_{>0})^{d-1}$, and p elements $i_1 < \dots < i_p$ in $\{1, \dots, d-1\}$ such that if we put $i_0 := 0$, $v_0 := n$, $i_{p+1} = d$, and $v_d := 0$, then for each $r = 1, \dots, p+1$, we have $v_{i_{r-1}} - v_{i_r} = s_r$, and $(v_{i_{r-1}} - v_{i_{r-1}+1}, \dots, v_{i_r-1} - v_{i_r})$ is the dual partition of μ_r .

This defines a type A_{d-1} quiver Γ as in §4.2. We keep the same \mathbf{w} as in §4.2 and we define $\zeta = (\zeta_1, \dots, \zeta_d)$ by $\zeta_j = \epsilon_{r+1}$ if $i_r < j \leq i_{r+1}$ with $r = 0, \dots, p$.

As in §4.2, this defines a unique parameter $\xi \in \mathbb{K}^I$ such that $\xi_i = \zeta_{i+1} - \zeta_i$. We now choose a stability parameter $\theta \in (\mathbb{Z}_{\geq 0})^I$ with the requirement that $\theta_j \neq 0$ exactly when $j \in \{i_1, \dots, i_p\}$.

Example 4.3.2. Assume that $L = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_3$, $C = C_{(1,1)} \times C_{(2)} \times C_{(2,1)}$ where C_μ denotes the nilpotent orbit corresponding to the partition μ , and that σ is the diagonal matrix $(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta)$ with $\alpha \neq \beta$. We have $\epsilon_1 = \beta$ and $\epsilon_2 = \epsilon_3 = \alpha$. Clearly σ is in the center of \mathfrak{l} and $M = \text{GL}_4 \times \text{GL}_3$. The underlying graph of Γ is A_4 and $\mathbf{w} = (7, 0, 0, 0)$, $\mathbf{v} = (5, 4, 2, 1)$, $\theta = (0, \theta_2, 0, \theta_4)$ with $\theta_2, \theta_4 > 0$, $\zeta = (\beta, \beta, \alpha, \alpha, \alpha)$, $\xi = (0, \alpha - \beta, 0, 0)$.

The aim of the section is to show that there is a bijective morphism $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathbb{Z}_{L,P,\Sigma}$ which is an isomorphism when $\mathbb{K} = \mathbb{C}$.

Given $(B, a, b) \in \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)$ and an arrow of $\overline{\Gamma}$ with tail i and head j , we denote by $B_{i,j}$ the corresponding coordinate of B .

For a parameter $x \in \mathbb{K}^I$, put $J_x = \{i \in I \mid x_i = 0\}$. We will need the following lemma:

Lemma 4.3.3. *Let $(B, a, b) \in \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)$. Then (B, a, b) is θ -semistable if and only if for all $i \in I - J_\theta$ the map $a \circ B_{2,1} \circ \dots \circ B_{i,i-1} : \mathbb{K}^{v_i} \rightarrow \mathbb{K}^n$ is injective.*

Proof. Put $V := \bigoplus_i \mathbb{K}^{v_i}$. We first construct for each $s \in I$ a B -invariant graded subspace $L^s = \bigoplus_i L_i^s$ of V . Put $L_1^s := \text{Ker}(a)$, for all $i \in \{2, \dots, s\}$ put $L_i^s := \text{Ker}(a \circ B_{2,1} \circ \dots \circ B_{i,i-1})$, and for $i > s$ put $L_i^s := B_{i-1,i} \circ B_{i-2,i-1} \circ \dots \circ B_{s+1,s+2} \circ B_{s,s+1}(L_s^s)$. Let us see that L^s is a B -invariant subspace of V . For $i < s$ we need to see that $B_{i,i+1}(L_i^s) \subset L_{i+1}^s$. We first prove it when $i = 1$. We have $ba - B_{2,1}B_{1,2} = \xi_1 \text{Id}$, hence $(a \circ B_{2,1})(B_{1,2}(\text{Ker}(a))) = a \circ (ba - \xi_1 \text{Id})(\text{Ker}(a)) = 0$ and so $B_{2,1}(L_1^s) \subset L_2^s$. Assume that this is true for all $j < i$. At the vertex i , we have the relation $B_{i-1,i}B_{i,i-1} - B_{i+1,i}B_{i,i+1} = \xi_2 \text{Id}$. For $x \in L_i^s$ we have

$$\begin{aligned} a \circ B_{2,1} \circ \dots \circ B_{i,i-1} \circ B_{i+1,i}(B_{i,i+1}(x)) &= a \circ B_{2,1} \circ \dots \circ B_{i,i-1} \circ (B_{i-1,i}B_{i,i-1} - \xi_2 \text{Id})(x) \\ &= a \circ B_{2,1} \circ \dots \circ B_{i,i-1} \circ (B_{i-1,i}B_{i,i-1}(x)). \end{aligned}$$

We need to see that the RHS is 0. By definition of L^s it is clear that $B_{i,i-1}(L_i^s) \subset L_{i-1}^s$ hence $B_{i,i-1}(x) \in L_{i-1}^s$. By induction hypothesis we then have $B_{i-1,i}(B_{i,i-1}(x)) \subset L_i^s$. By definition of L_i^s we thus have $a \circ B_{2,1} \circ \dots \circ B_{i,i-1}(B_{i-1,i} \circ B_{i,i-1}(x)) = 0$. To see that L^s is a B -invariant subspace of V it remains to see that for all $i \geq s$ we have $B_{i+1,i}(L_{i+1}^s) \subset L_i^s$ which again can be proved by induction using the relations at the vertices.

Assume that (B, a, b) is θ -semistable. Assume that $s \in I - J_\theta$. If the map $a_s := a \circ B_{2,1} \circ \dots \circ B_{s,s-1}$ is not injective then L^s is a non-trivial B -invariant subspace of V such that $\theta \cdot \dim L^s > 0$ (as $\theta_s \neq 0$) which contradicts the stability condition (i) of Definition 2.2.8. Hence the map a_s must be injective for all $s \in I - J_\theta$.

Let us prove the converse. Assume that V' is a B -invariant subspace of V such that $V'_1 \subset \text{ker}(a)$. Hence for all i and $x \in V'_i$ we have $B_{2,1} \circ \dots \circ B_{i,i-1}(x) \in \text{Ker}(a)$, i.e., $a \circ B_{2,1} \circ \dots \circ B_{i,i-1}(x) = 0$, and so $V'_i \subset \text{Ker}(a \circ B_{2,1} \circ \dots \circ B_{i,i-1})$. Hence for $i \in I - J_\theta$ we have $V'_i = 0$ by assumption. Therefore $\theta \cdot \dim V' = 0$ and so the condition (i) of Definition 2.2.8 is satisfied. \square

For $(B, a, b) \in \mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w})$, we denote by $f_{(B, a, b)}$ the partial flag $\{0\} = \mathcal{E}^{p+1} \subset \mathcal{E}^p \subset \dots \subset \mathcal{E}^1 \subset \mathcal{E}^0 = \mathbb{K}^n$ with $\mathcal{E}^r := \text{Im}(a \circ B_{2,1} \circ \dots \circ B_{i_r, i_r-1})$. By Lemma 4.3.3, we have $f_{(B, a, b)} \in \mathcal{F}$.

Proposition 4.3.4. *The map $\mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)^{ss} \rightarrow \mathbb{Z}_{L, P, \Sigma}$, $(B, a, b) \mapsto (ab + \zeta_1 \text{Id}, f_{(B, a, b)})$ is well-defined and induces a canonical bijective morphism $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathbb{Z}_{L, P, \Sigma}$ which restricts to $\mathfrak{M}_{\xi, \theta}^s(\mathbf{v}, \mathbf{w}) \rightarrow \mathbb{Z}_{L, P, \Sigma}^o$ and which makes the following diagram commutative*

$$\begin{array}{ccc} \mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) & \longrightarrow & \mathbb{Z}_{L, P, \Sigma} \\ \downarrow \pi & & \downarrow pr_1 \\ \mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) & \xrightarrow{\rho} & \mathfrak{gl}_n \end{array}$$

where ρ maps a semisimple representation (B, a, b) to $ab + \zeta_1 \text{Id}$. If $\mathbb{K} = \mathbb{C}$ this bijective map is an isomorphism $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \xrightarrow{\sim} \mathbb{Z}_{L, P, \Sigma}$.

If $\theta_i > 0$ for all i and if $\xi = 0$, then this is [36, Theorem 7.3], see also [42] for more details.

Proof. The fact that the diagram is commutative follows from a generalization of Remark 4.2.5 to any monotone \mathbf{v} (see [21, Proposition 3.4]). Let us see that the map

$$h : \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)^{ss} \rightarrow \mathbb{Z}_{L, P, \Sigma}, (B, a, b) \mapsto (ab + \zeta_1 \text{Id}, f_{(B, a, b)})$$

is well-defined. Let $(B, a, b) \in \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\xi)^{ss}$ and put $X := ab + \zeta_1 \text{Id}$ and $\mathcal{E}^r := \text{Im}(a \circ B_{2,1} \circ \dots \circ B_{i_r, i_r-1})$. The fact that X leaves stable the partial flag $f_{(B, a, b)}$ is straightforward from the preprojective relations

$$B_{i-1, i} B_{i, i-1} - B_{i+1, i} B_{i, i+1} = \xi_i \text{Id}$$

with $B_{0,1} := b$ and $B_{1,0} := a$.

To alleviate the notation, for all $i < j$ we denote by $f_{j,i}$ the map $B_{i+1, i} \circ \dots \circ B_{j, j-1} : \mathbb{K}^{v_j} \rightarrow \mathbb{K}^{v_i}$.

Fix $r \in \{1, \dots, p+1\}$ and define $H = \bigoplus_{i \in I \cup \{0\}} H_i$ by $H_i = \mathbb{K}^{v_i}$ if $i \geq i_r$ and by $H_i = \text{Im}(f_{i_r, i})$ if not. From the preprojective relations we see that (B, a, b) leaves H stable and so we can consider the restriction (B_H, a_H, b_H) of (B, a, b) to H and the quotient $(\bar{B}, \bar{a}, \bar{b})$ of (B, a, b) by (B_H, a_H, b_H) . Put $U_i := \mathbb{K}^{v_i} / H_i$. Then $U_i \simeq \mathbb{K}^{v_i - v_{i_r}}$ if $i < i_r$ and $U_i = \{0\}$ otherwise. From the preprojective relations we see that $X_r : \mathcal{E}^{r-1} / \mathcal{E}^r \rightarrow \mathcal{E}^{r-1} / \mathcal{E}^r$ coincides, with the map $Y_r : U_{i_{r-1}} \rightarrow U_{i_{r-1}}$ induced by $B_{i_{r-1}+1, i_{r-1}} B_{i_{r-1}, i_{r-1}+1} + \zeta_{i_{r-1}+1} \text{Id}$. In other words the diagram

$$\begin{array}{ccc} \mathcal{E}^{r-1} / \mathcal{E}^r & \xleftarrow{f_{i_{r-1}, 0} / H_{i_{r-1}}} & U_{i_{r-1}} \\ X_r \downarrow & & \downarrow Y_r \\ \mathcal{E}^{r-1} / \mathcal{E}^r & \xleftarrow{f_{i_{r-1}, 0} / H_{i_{r-1}}} & U_{i_{r-1}} \end{array}$$

is commutative.

We want to see that the map $Y_r \in \text{End}(U_{i_{r-1}}) \simeq \text{End}(\mathbb{K}^{s_r})$ leaves in $\zeta_{i_{r-1}+1} \text{Id} + \bar{C}_r$.

Consider the *subquiver* Γ'

$$\bullet_{i_{r-1}+1} \longrightarrow \dots \longrightarrow \bullet_{i_r-1}$$

of Γ . Put $d' := i_r$, $\mathbf{w}' := (v_{i_{r-1}} - v_{i_r}, 0, \dots, 0)$, $\mathbf{v}' := (v_{i_{r-1}+1} - v_{i_r}, v_{i_{r-1}+2} - v_{i_r}, \dots, v_{i_r-1} - v_{i_r})$, and $\zeta' = (\zeta_{i_{r-1}+1}, \zeta_{i_{r-1}+2}, \dots, \zeta_{i_r-1+d'})$. We have $\xi'_i = 0$ for all $i = i_{r-1}+1, \dots, i_r-1$, i.e., $\zeta_{i_{r-1}+1} = \zeta_{i_{r-1}+2} = \dots = \zeta_{i_r-1+d'}$. Consider the projection of $(\bar{B}, \bar{a}, \bar{b})$ on

$$\left(\bigoplus_{i \in \{i_{r-1}, \dots, i_r-2\}} \text{Hom}(U_i, U_{i+1}) \oplus \bigoplus_{i \in \{i_{r-1}+1, \dots, i_r-1\}} \text{Hom}(U_i, U_{i-1}) \right) \simeq \mathbf{M}(\bar{\Gamma}', \mathbf{v}', \mathbf{w}')$$

and denote by (B', a', b') the corresponding element in $\mathbf{M}(\overline{\Gamma}', \mathbf{v}', \mathbf{w}')$. Note that a' and b' come from $B_{i_{r-1}+1, i_{r-1}}$ and $B_{i_{r-1}, i_{r-1}+1}$ respectively. The map $Y_r : U_{i_{r-1}} \rightarrow U_{i_{r-1}}$ is thus $a'b' + \zeta_{i_{r-1}+1}\text{Id}$.

The sequence $(w'_1 - v'_1, v'_1 - v'_2, v'_2 - v'_3, \dots, v'_{d_r-1})$ is the partition μ'_r . Now apply Proposition 4.2.4 to $(\Gamma', \mathbf{v}', \mathbf{w}', \xi')$. Then we see that $a'b'$ belongs to the Zariski closure of nilpotent orbit \overline{C}_r , proving thus that $Y_r \in \zeta_{i_{r-1}+1} + \overline{C}_r$. \square

By Proposition 4.3.4 and Proposition 4.3.1 we have

Corollary 4.3.5. *The image of the composition $\mathfrak{M}_{\xi, \theta}(\mathbf{v}, \mathbf{w}) \xrightarrow{\pi} \mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) \xrightarrow{\rho} \mathfrak{gl}_n$ is $\overline{\mathcal{O}}$. Moreover if $J_{\theta} = J_{\xi}$, then $\pi \circ \rho$ is a bijective morphism onto its image (if $\mathbb{K} = \mathbb{C}$, it is an isomorphism).*

Remark 4.3.6. Assume that $\mathbb{K} = \mathbb{C}$. The condition in Remark 4.2.6 to have $\mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w}) \simeq \overline{\mathcal{O}}$ may not be satisfied here. For instance in the example [42, Example 4.3] where $\mathbf{v} = (4, 1)$, $\mathbf{w} = (5, 0)$, $\zeta = (0, 0)$, $\theta = (1, 1)$, the adjoint orbit \mathcal{O} is the nilpotent orbit with partition $(3, 1, 1)$ while $\mathfrak{M}_{\xi}(\mathbf{v}, \mathbf{w})$ is isomorphic to the Zariski closure of the nilpotent orbit with partition $(3, 2)$.

Recall that $L = \text{GL}_{s_{p+1}} \times \dots \times \text{GL}_{s_1} \subset \text{GL}_n$ and assume for simplicity that P contains the upper triangular matrices. Recall also that μ_i is a partition of s_i . For each $i = 1, \dots, p+1$, the dual partition $\mu'_i = (\mu'_{i,1}, \dots, \mu'_{i,r_i})$ of μ_i defines a Levi subgroup $\hat{L}_i = \prod_j \text{GL}_{\mu'_{i,j}} \subset \text{GL}_{s_i}$. Let \hat{P}_i be a parabolic subgroup of GL_{s_i} having \hat{L}_i as a Levi subgroup and containing the upper triangular matrices. Then $\tilde{P} := \prod_i \hat{P}_i$ is a parabolic subgroup of L having $\hat{L} := \prod_{i=1}^{p+1} \hat{L}_i$ as a Levi factor. Put $\hat{P} := \tilde{P}.U_P$. It is the unique parabolic subgroup of GL_n having \hat{L} as a Levi factor and contained in P .

Consider the following maps

$$\mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}} \xrightarrow{\tilde{\pi}} \mathbb{X}_{L, P, \Sigma} \xrightarrow{f} \overline{\mathcal{O}} \quad (4.3.2)$$

where $\tilde{\pi}(X, g\hat{P}) = (X, gP)$.

Note that the variety $\mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}}$ is nonsingular and that $\tilde{\pi}$ is surjective.

The decomposition $\overline{C} = \coprod_{\alpha} C_{\alpha}$ as a disjoint union of L -orbits provides a stratification $\overline{\Sigma} = \coprod_{\alpha} \Sigma_{\alpha}$ with $\Sigma_{\alpha} = \sigma + C_{\alpha}$ and therefore a stratification of $\mathbb{X}_{L, P, \Sigma} = \coprod_{\alpha} \mathbb{X}_{L, P, \Sigma_{\alpha}}^{\circ}$ where

$$\mathbb{X}_{L, P, \Sigma_{\alpha}}^{\circ} := \{(X, gP) \in \mathfrak{g} \times (\text{GL}_n/P) \mid g^{-1}Xg \in \Sigma_{\alpha} + \mathfrak{u}_P\}$$

is the smooth locus of $\mathbb{X}_{L, P, \Sigma_{\alpha}}$.

The following proposition is a particular case of a result of Lusztig [29] (cf. [24, proof of Proposition 5.1.19] for more details).

Proposition 4.3.7. *For $x \in \overline{\mathcal{O}}$, put $f^{-1}(x)_{\alpha} := f^{-1}(x) \cap \mathbb{X}_{L, P, \Sigma_{\alpha}}^{\circ}$. Then*

$$\dim \left\{ x \in \overline{\mathcal{O}} \mid \dim f^{-1}(x)_{\alpha} \geq \frac{i}{2} - \frac{1}{2}(\dim \Sigma - \dim \Sigma_{\alpha}) \right\} \leq \dim \overline{\mathcal{O}} - i$$

for all $i \in \mathbb{Z}_{\geq 0}$.

In particular if we apply the proposition to $(\hat{L}, \hat{P}, \{\sigma\})$ instead of (L, P, Σ) we find that $f \circ \tilde{\pi}$ is semi-small.

Proposition 4.3.8. *The morphism $\tilde{\pi}$ is a semi-small with respect to $\mathbb{X}_{L, P, \Sigma} = \coprod_{\alpha} \mathbb{X}_{L, P, \Sigma_{\alpha}}^{\circ}$.*

Define $\mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}} := \{(X, g\hat{P}) \in \mathfrak{l} \times (L/\hat{P}) \mid g^{-1}Xg \in \sigma + \mathfrak{u}_{\hat{P}}\}$, $\mathbb{Y}_{L, P, \Sigma} := \{(X, g) \in \mathfrak{gl}_n \times \text{GL}_n \mid g^{-1}Xg \in \overline{\Sigma} + \mathfrak{u}_P\}$, and let \mathbb{Y} be the variety $\{(y, z, g) \in P \times \mathfrak{gl}_n \times \text{GL}_n \mid g^{-1}zg \in \sigma + \mathfrak{u}_{\hat{P}}\}$ modulo the action of \hat{P} given by $p \cdot (y, z, g) := (yp^{-1}, z, gp^{-1})$.

Consider the following Cartesian diagram which was considered by Lusztig [30, §4] to prove the transitivity of parabolic induction.

$$\begin{array}{ccccc}
\mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}} & \xleftarrow{a_1} & \mathbb{Y} & \xrightarrow{a_2} & \mathbb{X}_{\hat{L}, \hat{P}, \sigma} \\
\downarrow \rho & & \downarrow c & & \downarrow \tilde{\pi} \\
\bar{\Sigma} & \xleftarrow{b_1} & \mathbb{Y}_{L, P, \Sigma} & \xrightarrow{b_2} & \mathbb{X}_{L, P, \Sigma} \\
& & & & \downarrow f \\
& & & & \bar{\mathcal{O}}
\end{array} \tag{4.3.3}$$

where $a_1(y, z, g) = (\pi_{\mathfrak{p}}(y g^{-1} z g y^{-1}), \pi_P(y) \hat{P})$, $a_2(y, z, g) = (z, g \hat{P})$, $c(y, z, g) = (z, g y^{-1})$, $b_1(X, g) = \pi_{\mathfrak{q}}(g^{-1} X g)$, $b_2(X, g) = (X, g P)$. Here $\pi_{\mathfrak{p}} : \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P \rightarrow \mathfrak{l}$ and $\pi_P : L \ltimes U_P \rightarrow L$ are the canonical projections.

By Proposition 4.3.7 applied to $(\hat{L}, \hat{P}, \{\sigma\})$ instead of (L, P, Σ) we find that the projection (on the first coordinate) $\rho : \mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}} \rightarrow \bar{\Sigma}$ is semi-small with respect to the stratification $\bar{\Sigma} = \coprod_{\alpha} \Sigma_{\alpha}$. On the other hand

$$\text{codim}_{\bar{\Sigma}}(\Sigma_{\alpha}) = \text{codim}_{\mathbb{Y}_{L, P, \Sigma}} \mathbb{Y}_{L, P, \Sigma_{\alpha}} = \text{codim}_{\mathbb{X}_{L, P, \Sigma}} \mathbb{X}_{L, P, \Sigma_{\alpha}}.$$

Hence Proposition 4.3.8 follows from the fact that the two squares of the diagram (4.3.3) are Cartesian.

Proposition 4.3.9. *The restriction of the sheaves $\mathcal{H}^i(\tilde{\pi}_!(\kappa))$ to $\mathbb{X}_{L, P, \Sigma_{\alpha}}^{\circ}$ are locally constant for all i and α .*

Proof. By the above diagram 4.3.3, it is enough to prove that the restriction of $\mathcal{H}^i(\rho_!(\kappa))$ to Σ_{α} are locally constant for all i and α . The map ρ is semi-small and L -equivariant if we let L acts on $\mathbb{X}_{\hat{L}, \hat{P}, \{\sigma\}}$ by $v \cdot (X, m \hat{P}) = (v X v^{-1}, v m \hat{P})$. The complex $\rho_!(\kappa)$ is thus a semisimple L -equivariant perverse sheaf. Since $\bar{\Sigma}$ has only a finite number of L -orbits, the simple constituents of $\rho_!(\kappa)$ are of the form $\underline{IC}_{\Sigma_{\alpha}}^{\bullet}$. \square

5 Comet-shaped quiver varieties

5.1 Generic tuples of adjoint orbits

Let O_1, \dots, O_k be k -orbits of $\mathfrak{gl}_n(\mathbb{K})$ and let $\tilde{\omega}_i$ be the type of O_i , then $\tilde{\omega} := (\tilde{\omega}_1, \dots, \tilde{\omega}_k)$ is called the type of (O_1, \dots, O_k) .

Definition 5.1.1. A k -tuple (C_1, \dots, C_k) of semisimple adjoint orbits is said to be *generic* if $\sum_{i=1}^k \text{Tr } C_i = 0$ and the following holds. If $V \subset \mathbb{K}^n$ is a subspace stable by some $X_i \in C_i$ for each i such that

$$\sum_{i=1}^k \text{Tr}(X_i|_V) = 0,$$

then either $V = 0$ or $V = \mathbb{K}^n$.

Let C_i be the adjoint orbit of the semisimple part of an element of O_i . Then we say that (O_1, \dots, O_k) is *generic* if the tuple (C_1, \dots, C_k) of semisimple orbits is generic.

By [13, Lemma 2.2.2] we have:

Lemma 5.1.2. *For $i = 1, \dots, k$, put $\tilde{\omega}_i = \omega_i^1 \omega_i^2 \cdots \omega_i^{r_i}$ with $\omega_i^j \in \mathcal{P}^*$ such that $\sum_j |\omega_i^j| = n$. Put $D = \min_i \max_j |\omega_i^j|$ and let $d = \gcd\{|\omega_i^j|\}$. Assume that*

$$\text{char}(\mathbb{K}) \nmid D!$$

If $d > 1$, generic k -tuples of adjoint orbits of \mathfrak{gl}_n of type $(\tilde{\omega}_1, \dots, \tilde{\omega}_k)$ do not exist. If $d = 1$, they do.

Remark 5.1.3. The tuple (O_1, \dots, O_k) is generic if and only if it is generic in the sense of [3, §6].

5.2 Affine comet-shaped quiver varieties

Assume that (O_1, \dots, O_k) is a generic k -tuple of adjoint orbits of \mathfrak{gl}_n of type $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_k)$. Let $g \geq 0$ be an integer. We denote by $\mathcal{V}_{\tilde{\omega}}^o$ the locally closed subvariety of \mathfrak{gl}_n^{2g+k} of tuples $(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k)$ such that

$$[A_1, B_1] + \dots + [A_g, B_g] + X_1 + \dots + X_k = 0 \quad (5.2.1)$$

with X_i in O_i .

Let $\mathrm{PGL}_n(\mathbb{K})$ acts on $\mathcal{V}_{\tilde{\omega}}^o$ by simultaneously conjugating the $2g + k$ matrices.

Proposition 5.2.1. *The group PGL_n acts set-theoretically freely on $\mathcal{V}_{\tilde{\omega}}^o$ and for an element*

$$(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathcal{V}_{\tilde{\omega}}^o,$$

there is no non-zero proper subspace of \mathbb{K}^n which is stable by $A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k$.

Proof. The proof is similar to the case of semisimple adjoint orbits [13, Proposition 2.2.3]. \square

Let $\mathcal{V}_{\tilde{\omega}}$ be the affine closed subvariety of $\mathfrak{gl}_n(\mathbb{K})^{2g+k}$ of matrices $(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k)$ which satisfy Equation (5.2.1) with X_i in the Zariski closure \overline{O}_i of O_i . Define

$$\mathcal{Q}_{\tilde{\omega}} := \mathcal{V}_{\tilde{\omega}} // \mathrm{PGL}_n = \mathrm{Spec}(\mathbb{K}[\mathcal{V}_{\tilde{\omega}}]^{\mathrm{PGL}_n}).$$

We denote by $\mathcal{Q}_{\tilde{\omega}}^o$ the quotient $\mathcal{V}_{\tilde{\omega}}^o // \mathrm{PGL}_n$ (it is defined as the image of $\mathcal{V}_{\tilde{\omega}}^o$ by the quotient map $\mathcal{V}_{\tilde{\omega}} \rightarrow \mathcal{Q}_{\tilde{\omega}}$).

We have the following theorem (see [13, Theorem 2.2.4] in the case of semisimple orbits):

Theorem 5.2.2. *If not empty the varieties $\mathcal{V}_{\tilde{\omega}}^o$ and $\mathcal{Q}_{\tilde{\omega}}^o$ are both nonsingular respectively of pure dimension $(2g + k - 1)n^2 + 1 - \sum_{i,j} \langle \omega_i^j, \omega_i^j \rangle$ and*

$$d_{\tilde{\omega}} := (2g + k - 2)n^2 + 2 - \sum_{i,j} \langle \omega_i^j, \omega_i^j \rangle.$$

Proof. Similar to [13, Theorem 2.1.5]. \square

Let $\tilde{\tau}$ be a multi-type. We say that $\tilde{\tau} \preceq \tilde{\omega}$ if there exists a k -tuple (C_1, \dots, C_k) of adjoint orbits of \mathfrak{gl}_n of type $\tilde{\tau}$ such that

$$C_i \subset \overline{O}_i \quad (5.2.2)$$

for all $i = 1, \dots, k$. If such a tuple (C_1, \dots, C_k) exists, it is unique.

For $\tilde{\tau} \preceq \tilde{\omega}$ we denote by $\mathcal{V}_{\tilde{\tau}}^o$ the corresponding subvariety (possibly empty) of $\mathcal{V}_{\tilde{\omega}}$.

Note that if (C_1, \dots, C_k) satisfies (5.2.2) then it is necessarily generic (as the genericity condition is a condition on eigenvalues only) and so the subvarieties $\mathcal{V}_{\tilde{\tau}}^o$ are all nonsingular by Theorem 5.2.2.

Obviously we have

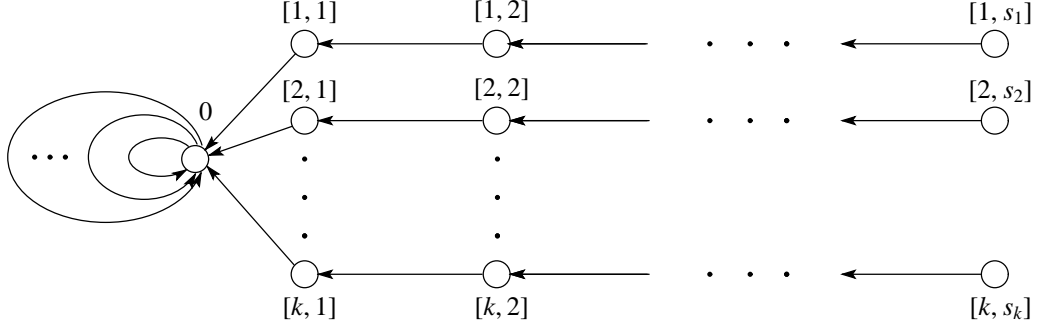
$$\mathcal{V}_{\tilde{\omega}} = \bigsqcup_{\tilde{\tau} \preceq \tilde{\omega}} \mathcal{V}_{\tilde{\tau}}^o.$$

We deduce that:

Proposition 5.2.3. *The group PGL_n acts freely on $\mathcal{V}_{\tilde{\omega}}$ and for an element $(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathcal{V}_{\tilde{\omega}}$, there is no non-zero proper subspace of \mathbb{K}^n which is stable by $A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k$. The quotient $\pi_{\tilde{\omega}} : \mathcal{V}_{\tilde{\omega}} \rightarrow \mathcal{Q}_{\tilde{\omega}}$ is a principal PGL_n -bundle in the étale topology.*

Now we follow [3] to identify $\mathcal{Q}_{\tilde{\omega}}$ with $\mathfrak{M}_{\xi}(\mathbf{v})$ for some Γ , ξ and \mathbf{v} .

Now let Γ be the following quiver¹ with g loops at the central vertex 0 :



Now we define a dimension vector \mathbf{v} of Γ as follows. We put $v_0 = n$ and for each i , we define the sequence $\mathbf{v}_i := v_{[i,1]} > v_{[i,2]} > \dots > v_{[i,s_i]}$ as the dimension vector of \mathcal{O}_i , see §4.

We also define $\xi \in \mathbb{K}^I$ as follows. For each i , let $\zeta_i = (\zeta_{i,1}, \dots, \zeta_{i,s_i+1})$ and $\xi_i = (\xi_{[i,1]}, \dots, \xi_{[i,s_i]})$ be the two sequences defined from \mathcal{O}_i , see §4. We also put $\xi_0 = -\sum_{i=1}^k \zeta_{i,1}$. This defines an element $\xi = \{\xi_0\} \cup \{\xi_{[i,j]}\}_{i,j} \in \mathbb{K}^I$ such that $\xi \cdot \mathbf{v} = 0$. For a representation φ of Γ , denote by $\varphi_{[i,1]}$ the linear map associated to the arrow whose tail is $[i, 1]$, by $\varphi_1, \dots, \varphi_g$ the matrices associated to the loops in Ω and by $\varphi_1^*, \dots, \varphi_g^*$ the ones associated to the loops in $\bar{\Omega} - \Omega$. We have the following consequence of Proposition 4.2.4 (see also [13, §2.2] for the semisimple case).

Proposition 5.2.4. *The map $\mu_{\mathbf{v}}^{-1}(\xi) \rightarrow (\mathfrak{gl}_n)^{2g+k}$ defined by $\varphi \mapsto (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k)$, with*

$$A_i = \varphi_i, B_i = \varphi_i^*, X_i = \varphi_{[i,1]} \varphi_{[i,1]}^* + \zeta_{i,1} \text{Id}, \quad (5.2.3)$$

induces a bijective morphism

$$\mathfrak{M}_{\xi}(\mathbf{v}) \longrightarrow \mathcal{Q}_{\tilde{\omega}}$$

which maps $\mathfrak{M}_{\xi}^s(\mathbf{v})$ to $\mathcal{Q}_{\tilde{\omega}}^o$. If $\mathbb{K} = \mathbb{C}$, this bijective map is an isomorphism.

Remark 5.2.5. Note that the dimension vector \mathbf{v} depends only on the type $\tilde{\omega}$. The condition in Lemma 5.1.2 for the existence of generic tuple $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ of semisimple adjoint orbits reads as follows. There exists a generic tuple $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ of a given type $\tilde{\omega}$ if and only if \mathbf{v} is *indivisible*, i.e., \mathbf{v} can not be written as $r \cdot \mathbf{u}$ for some dimension vector \mathbf{u} and some integer $r > 1$.

5.3 General comet-shaped quiver varieties

Let $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ be the generic tuple of adjoint orbits of type $\tilde{\omega}$, and for each $i = 1, \dots, k$, let (L_i, P_i, Σ_i) be a triple as in §4.3 such that the image of $\mathbb{X}_{L_i, P_i, \Sigma_i} \rightarrow \mathfrak{gl}_n$ is $\overline{\mathcal{O}_i}$. As in the introduction we put $\mathbf{P} = P_1 \times \dots \times P_k$, $\mathbf{L} = L_1 \times \dots \times L_k$ and $\mathbf{\Sigma} = \Sigma_1 \times \dots \times \Sigma_k$. Put $\mathbb{O}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}} = (\mathfrak{gl}_n)^{2g} \times \mathbb{X}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}}$, $\mathbb{O}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}}^o = (\mathfrak{gl}_n)^{2g} \times \mathbb{X}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}}^o$ and

$$\mathbb{V}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}} := \left\{ (A_1, B_1, \dots, A_g, B_g, (X_1, \dots, X_k, g_1 P_1, \dots, g_k P_k)) \in \mathbb{O}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}} \mid \sum_j [A_j, B_j] + \sum_i X_i = 0 \right\}.$$

Because of the genericity condition, the diagonal action of the group PGL_n is free. Put

$$\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}} := \mathbb{V}_{\mathbf{L}, \mathbf{P}, \mathbf{\Sigma}} / \text{PGL}_n.$$

¹The picture is from [45].

When $\mathbb{K} = \mathbb{C}$, we show in this section that $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ can be identified with a quiver variety $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ for appropriate choices of ξ, θ, \mathbf{v} .

By §4.3, for each $i = 1, \dots, k$, we can define a type A quiver Γ_i together with parameters $\xi_i, \theta_i, \mathbf{v}_i$ such that there is a canonical bijective morphism $\mathbb{X}_{L_i, P_i, \Sigma_i} \rightarrow \mathfrak{M}_{\xi_i, \theta_i}(\mathbf{v}_i, \mathbf{w})$ which is an isomorphism when $\mathbb{K} = \mathbb{C}$.

We now define a comet shaped quiver Γ as in §5.2 in such that each leg with vertices $[1, 1], \dots, [1, s_i]$ is exactly the quiver Γ_i . I.e., if we delete the central vertex $\{0\}$ from Γ , we recover the k type A quivers $\Gamma_1, \dots, \Gamma_k$. We denote by I the set of vertices of Γ , and we define a dimension vector \mathbf{v} as follows. We put $v_0 = n$ and for each $i = 1, \dots, k$, we define $(v_{[i, 1]}, \dots, v_{[i, s_i]}) := \mathbf{v}_i$. Multiplying the vectors θ_i by a strictly positive integer if necessary, there is a dimension vector $\theta \in \mathbb{Z}^I$ of Γ such that its projection on Γ_i is θ_i for each i and such that $\theta \cdot \mathbf{v} = 0$. There is a unique $\xi \in \mathbb{K}^I$ whose projection on Γ_i is ξ_i for all i and $\xi \cdot \mathbf{v} = 0$. Note that θ_0 must be negative.

Note that the quiver Γ is completely determined by the triple $(\mathbf{L}, \mathbf{C}, g)$. The dimension vector \mathbf{v} , as well as the position of the non-zero coordinates of θ , is determined by (\mathbf{L}, \mathbf{C}) , and the complex parameter ξ depends on the choice of the σ_i 's.

Remark 5.3.1. The quiver Γ and the parameter ξ are the same as the ones obtained from (O_1, \dots, O_k) as in §5.2. However in general the dimension vector obtained from (O_1, \dots, O_k) in §5.2 differs from \mathbf{v} . They agree if and only if \mathbf{v} satisfies the condition (*) in Remark 4.2.6. For instance in the situation of Remark 4.3.6, this condition (*) is not satisfied.

Let Γ^\dagger be the quiver obtained from Γ by deleting the central vertex (i.e., it is the union of $\Gamma_1, \dots, \Gamma_k$). We denote by $I^\dagger = \{[i, j]\}_{i, j}$ the set of vertices of Γ^\dagger . For a parameter $x \in \mathbb{K}^{I^\dagger}$, we denote by x^\dagger its projection on \mathbb{K}^{I^\dagger} .

We put

$$\mathbf{Z}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w}) := (\mathfrak{gl}_n)^{2g} \times \mathbf{M}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w}).$$

We let $\mathrm{GL}_{\mathbf{v}^\dagger}$ acts on $\mathbf{Z}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w})$ by the trivial action on $(\mathfrak{gl}_n)^{2g}$ and by the usual action on the second coordinate. Put $\mathbf{Z}_{\theta^\dagger}^{ss}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w}) := (\mathfrak{gl}_n)^{2g} \times \mathbf{M}_{\theta^\dagger}^{ss}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w})$ and define

$$\mathfrak{Z}_{\xi^\dagger, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) := \left((\mathfrak{gl}_n)^{2g} \times \mu_{\mathbf{v}^\dagger, \mathbf{w}}^{-1}(\xi^\dagger) \right) //_{\theta^\dagger} \mathrm{GL}_{\mathbf{v}^\dagger} \simeq (\mathfrak{gl}_n)^{2g} \times \mathfrak{M}_{\xi^\dagger, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}).$$

There is a canonical bijective map $\mathfrak{Z}_{\xi^\dagger, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) \rightarrow \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}$.

We identify in the obvious way $\mathbf{M}(\bar{\Gamma}, \mathbf{v})$ with $\mathbf{Z}(\bar{\Gamma}^\dagger, \mathbf{v}^\dagger, \mathbf{w})$ so that we can regard $\mu_{\mathbf{v}}^{-1}(\xi)$ as a $\mathrm{GL}_{\mathbf{v}^\dagger}$ -stable closed subvariety of $(\mathfrak{gl}_n)^{2g} \times \mu_{\mathbf{v}^\dagger, \mathbf{w}}^{-1}(\xi^\dagger)$. From the definition of semistability (see §2.2.1 and §2.2.2) it is clear that a θ -semistable point of $\mu_{\mathbf{v}}^{-1}(\xi)$ is θ^\dagger -semistable. Since the tuple (O_1, \dots, O_k) is generic we actually have:

Proposition 5.3.2. *An element of $\mu_{\mathbf{v}}^{-1}(\xi)$ is θ -semistable (resp. θ -stable) if and only if it is θ^\dagger -semistable (resp. θ^\dagger -stable).*

Proof. Assume that $\varphi \in \mu_{\mathbf{v}}^{-1}(\xi)$ is θ^\dagger -semistable. Let ψ be a subrepresentation of φ . It is an element in $\mu_{\mathbf{v}'}^{-1}(\xi)$ for some $\mathbf{v}' \leq \mathbf{v}$. We need to verify that $\theta \cdot \mathbf{v}' \leq 0$. If $v'_0 = v_0$, then we must have $\theta \cdot \mathbf{v}' \leq \theta \cdot \mathbf{v} = 0$ since $\theta^\dagger \in \mathbb{Z}_{\geq 0}^{I^\dagger}$. If $v'_0 = 0$, then the subspaces $V'_{[i, 1]}$ are contained in $\mathrm{Ker}(a_i)$ for all $i = 1, \dots, k$ and so $\theta \cdot \mathbf{v}' = \theta^\dagger \cdot (\mathbf{v}')^\dagger \leq 0$ since φ is θ^\dagger -semistable. For each $i \in \{1, \dots, k\}$, let X_i^φ be given by Formula (5.2.3). The preprojective relation at the central vertex reads $\sum_{i=1}^k X_i^\varphi = 0$. Because ψ is a subrepresentation of φ we also have

$$\sum_{i=1}^k X_i^\psi = 0. \quad (5.3.1)$$

Since the tuple (O_1, \dots, O_k) is generic, the equation (5.3.1) holds if and only if $v'_0 = 0$ or $v'_0 = n$. \square

Remark 5.3.3. Assume that the θ_i 's, $i = 1, \dots, k$, have strictly positive coordinates. Then $\mu_{\mathbf{v}}^{-1}(\xi)^{ss} = \mu_{\mathbf{v}}^{-1}(\xi)^s$. This identity also happens when θ is generic. We want to notice that in this situation we can

actually choose our θ_i (taking larger values of the coordinates if necessary) such that θ is generic. Indeed the set $\mu_v^{-1}(\xi)^{ss}$ depends only on the position of the non-zero coordinates of the θ_i 's and not on their values (cf. Remark 2.2.9 (ii)).

By the above proposition we can use the same notation $\mu_v^{-1}(\xi)^{ss}$ to denote the open subset of $\mu_v^{-1}(\xi)$ of θ -semistable and that of θ^\dagger -semistable points. Define

$$\mathfrak{M}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) := \mu_v^{-1}(\xi) //_{\theta^\dagger} \mathrm{GL}_{\mathbf{v}^\dagger}.$$

Since the quotient map $\mu_v^{-1}(\xi)^{ss} \rightarrow \mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is constant on $\mathrm{GL}_{\mathbf{v}^\dagger}$ -orbits, we have a canonical map $q : \mathfrak{M}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) \rightarrow \mathfrak{M}_{\xi, \theta}(\mathbf{v})$ which is a categorical quotient with respect to GL_n (the action of GL_n induces a free action of PGL_n on $\mathfrak{M}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w})$).

By the assertion (4) of Proposition 2.1.1, we have a canonical morphism $\alpha : \mathfrak{M}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) \rightarrow \mathfrak{Z}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w})$ which is an immersion when $\mathbb{K} = \mathbb{C}$.

Put $O_{\tilde{\omega}} := (\mathfrak{gl}_n)^{2g} \times \overline{O}_1 \times \cdots \times \overline{O}_k$. We have the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{Z}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) & \xrightarrow{f_1} & \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma} & \xrightarrow{\pi_1} & O_{\tilde{\omega}} \\ \uparrow \alpha & & \uparrow i & & \uparrow i \\ \mathfrak{M}_{\xi, \theta^\dagger}(\mathbf{v}^\dagger, \mathbf{w}) & \xrightarrow{f_2} & \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} & \xrightarrow{\pi_2} & \mathcal{V}_{\tilde{\omega}} \\ \downarrow q & & & & \downarrow q \\ \mathfrak{M}_{\xi, \theta}(\mathbf{v}) & \xrightarrow{\rho} & & & Q_{\tilde{\omega}} \end{array} \quad (5.3.2)$$

where i, i are the canonical inclusions, and where ρ is the factorization morphism (as $q \circ \pi_2 \circ f_2$ is constant on GL_n -orbits).

Remark 5.3.4. When $\mathbb{K} = \mathbb{C}$, the maps f_1, f_2 are isomorphisms and the squares $(\alpha, \pi_1 \circ f_1, \pi_2 \circ f_2, i)$ and $(q, \pi_2 \circ f_2, \rho, q)$ are Cartesian.

When $\mathbb{K} = \mathbb{C}$ we identify the orbit space $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ with the quiver variety $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$.

Recall that $\Sigma_i = \sigma_i + C_i$. Put $\mathbf{C} = C_1 \times \cdots \times C_k$. Then the decomposition of $\overline{\mathbf{C}} = \coprod_{\alpha} \mathbf{C}_{\alpha}$ as a union of \mathbf{L} -orbits provides a stratification $\overline{\Sigma} = \coprod_{\alpha} \Sigma_{\alpha}$. We thus have a decomposition

$$\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} = \coprod_{\alpha} \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}^{\circ}$$

where $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}^{\circ} := \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}} \cap \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}^{\circ}$. By Proposition 4.3.4, the subset $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^{\circ} \subset \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ corresponds to the stable points, i.e., $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^{\circ} \simeq \mathfrak{M}_{\xi, \theta^\dagger}^s(\mathbf{v}^\dagger, \mathbf{w}) = \mu_v^{-1}(\xi)^s / \mathrm{GL}_{\mathbf{v}^\dagger}$. The image of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}^{\circ}$ by the projective morphism

$$\rho : \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \longrightarrow \mathcal{V}_{\tilde{\omega}}$$

is of the form $\overline{\mathcal{V}}_{\tilde{\tau}_{\alpha}}$ for some $\tilde{\tau}_{\alpha} \preceq \tilde{\omega}$.

Theorem 5.3.5. *If $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}$ is not empty, then the piece $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}^{\circ}$ is also not empty and is an irreducible nonsingular dense open subset of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma_{\alpha}}$ of dimension*

$$(2g + k - 1)n^2 + 1 - \dim \mathbf{L} + \dim \Sigma_{\alpha}.$$

We need an intermediate result.

Let \mathbf{u} be the dimension vector of Γ obtained from the tuple (O_1, \dots, O_k) as in §5.2. We now explain the relation between \mathbf{u} and \mathbf{v} . For each $i \in I^\dagger$, we let $s_i : \mathbb{Z}^I \rightarrow \mathbb{Z}^I$ be the reflection defined by

$$s_i(x) = x - (x, \mathbf{e}_i) \mathbf{e}_i,$$

where $(,)$ is the form defined by $(\mathbf{e}_i, \mathbf{e}_j) = c_{ij}$, cf. §2.2.1.

We look for the vertices $[i, j]$ of Γ^\dagger which satisfy $\xi_{[i,j]} = 0$ and $v_{[i,j-1]} - v_{[i,j]} < v_{[i,j]} - v_{[i,j+1]}$ with the convention that $v_{[i,0]} = n$ and $v_{[i,s_i+1]} = 0$. If there are not such vertices, then $\mathbf{u} = \mathbf{v}$. If such a vertex $[i, j]$ does exist, then apply $s_{[i,j]}$ to \mathbf{v} and we get a dimension vector $\mathbf{v}' \in (\mathbb{Z}_{\geq 0})^I$ which satisfies $v'_{[i,j-1]} - v'_{[i,j]} \geq v'_{[i,j]} - v'_{[i,j+1]}$. Reiterating the process as many times as needed, we end up with a dimension vector \mathbf{u} which satisfies $u_{[i,j-1]} - u_{[i,j]} \geq u_{[i,j]} - u_{[i,j+1]}$ whenever $\xi_{[i,j]} = 0$. We denote by w the composition $s_{i_1} \circ \dots \circ s_{i_r}$ of the various reflections we used to get $w(\mathbf{v}) = \mathbf{u}$. From Proposition 5.2.4 there is a bijective morphism $\mathfrak{M}_\xi(\mathbf{u}) \rightarrow \mathcal{Q}_{\tilde{\omega}}$.

Proposition 5.3.6. *The following three assertions are equivalent.*

(i) *The set $\mathcal{V}_{\tilde{\omega}}$ is not empty.*

(ii) *The set $\mathcal{V}_{\tilde{\omega}}^o$ is not empty.*

(iii) $\mathbf{u} \in \Phi^+(\Gamma)$.

(iv) $\mathbf{v} \in \Phi^+(\Gamma)$.

If $g = 0$ then \mathbf{v} is a real root if and only if $\mathcal{V}_{\tilde{\omega}}^o$ consists of a single PGL_n -orbit.

Recall that $\mathcal{V}_{\tilde{\omega}}$ (resp. $\mathcal{V}_{\tilde{\omega}}^o$) is not empty if and only if $\mu_{\mathbf{u}}^{-1}(\xi)$ is not empty, (resp. $\mu_{\mathbf{u}}^{-1}(\xi)$ contains a simple representation).

Proof. The last assertion is proved in [3]. We already proved the equivalence between (iii) and (iv). The equivalence between (ii) and (iii) is proved in [3, §6]. We prove the equivalence between (i) and (iii). By Theorem 2.2.7 only one implication needs to be proved. As in [5] we say that a dimension vector β of Γ is *strict* if for any leg i of Γ we have $n \geq \beta_{[i,1]} \geq \dots \geq \beta_{[i,s_i]}$. Assume that $\mathcal{V}_{\tilde{\omega}}$ is not empty. We first prove that \mathbf{u} is a sum $\beta_1 + \beta_2 + \dots$ of strict positive roots with $\xi \cdot \beta_1 = \xi \cdot \beta_2 = \dots = 0$. By [5, Proof of Theorem 2.1] we see that we can choose a representation $B \in \mu_{\mathbf{u}}^{-1}(\xi)$ such that the coordinates B_h with $h \in \Omega$ are all injective (recall that the arrows of Γ are oriented towards the central vertex). Let $\pi : \mu_{\mathbf{u}}^{-1}(\xi) \rightarrow \mathrm{Rep}(\Gamma)$ be the projection $A \mapsto \{A_h\}_{h \in \Omega}$. Write $\pi(B)$ as a direct sum $I_1 \oplus I_2 \oplus \dots \oplus I_r$ of indecomposables and let β_m be the dimension vector of I_m . Then $\mathbf{u} = \beta_1 + \dots + \beta_r$ and by Kac's theorem [17] the dimension vectors β_m are positive roots. Since the linear maps B_h are all injective, the maps $(I_m)_h$ are also injective and so β_m is a strict dimension vector of Γ . Finally by [2, Theorem 3.3], we have $\xi \cdot \beta_m = 0$ for all m . Now, using the fact that ξ arises from a generic tuple of adjoint orbits, we proceed exactly as in [3, §6] to see that $r = 1$. \square

Proof of Theorem 5.3.5. We prove it for $\Sigma = \Sigma_\alpha$ as the proof will be the same for any Σ_α . Assume that $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}$ is not empty. Then $\mathcal{V}_{\tilde{\omega}}$ is not empty and so by Proposition 5.3.6 the set $\mathcal{V}_{\tilde{\omega}}^o$ is also not empty. Since the inverse image of $\mathcal{V}_{\tilde{\omega}}^o$ by the map $\rho : \mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma} \rightarrow \mathcal{V}_{\tilde{\omega}}$ is contained in $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}^o$, the open subset $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}^o$ is not empty.

Consider $\mathbb{Y}_{\mathbf{L},\mathbf{P},\Sigma}^o := \{(X, g) \in \mathfrak{gl}_n \times \mathrm{GL}_n \mid g^{-1}Xg \in \Sigma + \mathfrak{u}_\rho\}$. Then the canonical map $\mathbb{Y}_{\mathbf{L},\mathbf{P},\Sigma}^o \rightarrow \mathbb{X}_{\mathbf{L},\mathbf{P},\Sigma}^o$, $(X, g) \mapsto (X, gP)$ is a locally trivial principal P -bundle (for the Zariski topology). Note that $\mathbb{Y}_{\mathbf{L},\mathbf{P},\Sigma}^o \simeq G \times (\Sigma + \mathfrak{u}_\rho)$. Now consider the set $\mathbb{L}_{\mathbf{L},\mathbf{P},\Sigma}^o$ of $(2g + k)$ -tuples $(A_1, B_1, \dots, A_g, B_g, (g_1, \sigma_1), \dots, (g_k, \sigma_k))$ in $(\mathfrak{gl}_n)^{2g} \times (G \times (\Sigma_1 + \mathfrak{u}_\rho)) \times \dots \times (G \times (\Sigma_k + \mathfrak{u}_\rho))$ such that

$$\sum_j [A_j, B_j] + \sum_i g_i \sigma_i g_i^{-1} = 0.$$

The natural map $\mathbb{L}_{\mathbf{L},\mathbf{P},\Sigma}^o \rightarrow \mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}^o$ is then a locally trivial principal \mathbf{P} -bundle. Hence we are reduced to prove that $\mathbb{L}_{\mathbf{L},\mathbf{P},\Sigma}^o$ is nonsingular. A point $\mathbf{x} \in \mathbb{L}_{\mathbf{L},\mathbf{P},\Sigma}^o$ is nonsingular when the differential $d_{\mathbf{x}}\mu$ of the map

$$\mu : (\mathfrak{gl}_n)^{2g} \times (G \times (\Sigma_1 + \mathfrak{u}_\rho)) \times \dots \times (G \times (\Sigma_k + \mathfrak{u}_\rho)) \longrightarrow \mathrm{SL}_n$$

given by $(A_1, B_1, \dots, A_g, B_g, (g_1, \sigma_1), \dots, (g_k, \sigma_k)) \mapsto \sum_j [A_j, B_j] + \sum_i g_i \sigma_i g_i^{-1}$ is surjective.

Let σ_k be the coordinate of \mathbf{x} in $\Sigma_k + \mathfrak{u}_\rho$. Consider the restriction λ of μ to the closed subset $(\mathfrak{gl}_n)^{2g} \times (G \times \{\sigma_1\}) \times \dots \times (G \times \{\sigma_k\})$. It is enough to prove that the differential $d_{\mathbf{x}}\lambda$ is surjective. But this what

we prove to see that the variety $\mathcal{V}_{\tilde{\tau}}$ is nonsingular ($\tilde{\tau}$ being the type of the adjoint orbits of $\sigma_1, \dots, \sigma_\ell$), see Theorem 5.2.2. The variety $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ is thus nonsingular and its irreducible components are all of same dimension. To compute the dimension of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ we may use what we just said or use the fact that there is a bijective morphism $\mathfrak{M}_{\xi, \theta^i}^s(\mathbf{v}^\dagger, \mathbf{w}) \rightarrow \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ and use Theorem 2.2.2 (a straightforward calculation shows that ${}^t\mathbf{v}\mathbf{C}\mathbf{v} = 2n^2 - \dim O_{\tilde{\omega}}$).

Let us see now that $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is irreducible. The canonical map $\mathbb{V}_{\hat{\mathbf{L}}, \hat{\mathbf{P}}, \{\sigma\}} \rightarrow \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ deduced from the diagram (4.3.2) being surjective it is enough to show that $\mathbb{V}_{\hat{\mathbf{L}}, \hat{\mathbf{P}}, \{\sigma\}}$ is irreducible. We are thus reduced to prove the irreducibility of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ when Σ is reduced to a point $\{\sigma\}$ which we now assume. Hence $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \{\sigma\}} = \mathbb{V}_{\mathbf{L}, \mathbf{P}, \{\sigma\}}^o$ and the parameter θ satisfies $\theta_i > 0$ for all $i \in I^\dagger$. By Remark 5.3.3, we may assume that θ is generic with respect to \mathbf{v} . We now need to prove the irreducibility of $\mathfrak{M}_{\xi, \theta^i}(\mathbf{v}, \mathbf{w})$. Since $\mathfrak{M}_{\xi, \theta^i}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is a principal PGL_n -bundle, we are reduced to prove that $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is irreducible.

Assume first that $\mathbb{K} = \mathbb{C}$. It is known (see [13, Proof of Theorem 2.2.6]) that $H_c^i(\mathfrak{M}_{\xi, \theta}(\mathbf{v}), \mathbb{C}) \simeq H_c^i(\mathfrak{M}_{\theta, \theta}(\mathbf{v}), \mathbb{C})$. Recall that the dimension of $H_c^{2e}(X, \mathbb{C})$ where e is the dimension of X equals the number of irreducible components of X of dimension e . The varieties $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ and $\mathfrak{M}_{\theta, \theta}(\mathbf{v})$ are both of pure dimension by Theorem 2.2.2. Hence we are reduced to see that $\mathfrak{M}_{\theta, \theta}(\mathbf{v})$ is irreducible. The representations in $\mu_{\mathbf{v}}^{-1}(\theta)$ are all simple because θ is generic, hence $\mathfrak{M}_{\theta}(\mathbf{v})$ is irreducible and nonsingular. The canonical map $\mathfrak{M}_{\theta, \theta}(\mathbf{v}) \rightarrow \mathfrak{M}_{\theta}(\mathbf{v})$ being a resolution of singularities is thus an isomorphism and so $\mathfrak{M}_{\theta, \theta}(\mathbf{v})$ is irreducible.

Assume that $\mathbb{K} = \overline{\mathbb{F}}_q$ and that $\mathfrak{M}_{\xi, \theta^i}(\mathbf{v}, \mathbf{w})$ is defined over \mathbb{F}_q with q large enough (the characteristic being fixed). Then by Nakajima's appendix [6] and [13, Theorem 2.2.7] we have $\#\{\mathfrak{M}_{\xi, \theta}(\mathbf{v})(\mathbb{F}_q)\} = \#\{\mathfrak{M}_{\theta, \theta}(\mathbf{v})(\mathbb{F}_q)\}$. As the canonical map $\mathfrak{M}_{\theta, \theta}(\mathbf{v}) \rightarrow \mathfrak{M}_{\theta}(\mathbf{v})$ is an isomorphism we actually have

$$\#\{\mathfrak{M}_{\xi, \theta}(\mathbf{v})(\mathbb{F}_q)\} = \#\{\mathfrak{M}_{\theta}(\mathbf{v})(\mathbb{F}_q)\}. \quad (5.3.3)$$

Note that the dimension of the compactly supported ℓ -adic cohomology group $H_c^{2e}(X, \overline{\mathbb{Q}}_\ell)$ with ℓ invertible in \mathbb{K} and $e = \dim X$ also equals the number m of irreducible components of X of dimension e . Moreover if X is defined over \mathbb{F}_q , then the Frobenius F^* acts on $H_c^{2e}(X, \overline{\mathbb{Q}}_\ell)$ as multiplication by q^e . Therefore, the coefficient of q^e in $\#\{X(\mathbb{F}_q)\}$ equals m . From the identity (5.3.3) we deduce that $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is irreducible if and only if $\mathfrak{M}_{\theta}(\mathbf{v})$ is irreducible. But as above the variety $\mathfrak{M}_{\theta}(\mathbf{v})$ is irreducible as θ is generic. \square

Until the end of this § we assume that $\mathbb{K} = \mathbb{C}$.

Let $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}) \rightarrow \mathfrak{M}_{\xi}(\mathbf{v})$ be the canonical map. By §4.3, the image of π is isomorphic to $Q_{\tilde{\omega}}$.

Under the identification $Q_{\tilde{\omega}} \simeq \pi(\mathfrak{M}_{\xi, \theta}(\mathbf{v})) \subset \mathfrak{M}_{\xi}(\mathbf{v})$ the stratification

$$Q_{\tilde{\omega}} = \coprod_{\tilde{\tau} \leq \tilde{\omega}} Q_{\tilde{\tau}}^o$$

is a refinement of (2.2.2). Hence by Theorem 2.2.6 we have:

Theorem 5.3.7. *The restriction $\pi^{-1}(Q_{\tilde{\tau}}^o) \rightarrow Q_{\tilde{\tau}}^o$ of $\pi : \mathfrak{M}_{\xi, \theta}(\mathbf{v}) \rightarrow Q_{\tilde{\omega}}$ is locally trivial in the étale topology.*

5.4 A restriction property

We keep the notation of §5.3 and we assume that $\mathcal{V}_{\tilde{\omega}}$ is not empty. Note that $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}^o$ is then also not empty by Theorem 5.3.5.

The aim of this section is to prove the following theorem.

Theorem 5.4.1. *Let i be the natural inclusion $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \hookrightarrow \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}$. Then*

$$i^*(IC_{\mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}}^\bullet) = IC_{\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}}^\bullet.$$

By §5.3, we have a stratification

$$\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma} = \coprod_{\alpha} \mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ}$$

with $\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ} := (\mathfrak{gl}_n)^{2g} \times \mathbb{X}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ}$. It satisfies the conditions (i) of Proposition 3.2.1.

We consider the semi-small resolution $\tilde{\pi} : \mathbb{O}_{\hat{\mathbf{L}},\hat{\mathbf{P}},\{\sigma\}} \rightarrow \mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma}$, see the end of §4.3.

Proposition 5.4.2. *The morphism $\tilde{\rho} : \mathbb{V}_{\hat{\mathbf{L}},\hat{\mathbf{P}},\{\sigma\}} \rightarrow \mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}$ is a semi-small resolution. Moreover the diagram*

$$\begin{array}{ccc} \mathbb{O}_{\hat{\mathbf{L}},\hat{\mathbf{P}},\{\sigma\}} & \xrightarrow{\tilde{\pi}} & \mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma} \\ \uparrow & & \uparrow \\ \mathbb{V}_{\hat{\mathbf{L}},\hat{\mathbf{P}},\{\sigma\}} & \xrightarrow{\tilde{\rho}} & \mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma} \end{array}$$

is Cartesian (the vertical arrows being the canonical inclusions) and the restriction of the sheaf $\mathcal{H}^i(\tilde{\pi}_*(\kappa))$ to each piece $\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ}$ is a locally constant sheaf.

Proof. The diagram is Cartesian by definition of the varieties $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}$. The variety $\mathbb{V}_{\hat{\mathbf{L}},\hat{\mathbf{P}},\{\sigma\}}$ is also nonsingular by Theorem 5.3.5. Hence $\tilde{\rho}$ is a resolution of singularities.

By Proposition 4.3.8 the map $\tilde{\pi}$ is semi-small with respect to $\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma} = \coprod_{\alpha} \mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ}$. By Theorem 5.3.5 we see that the codimension of $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}^{\circ}$ in $\mathbb{V}_{\mathbf{L},\mathbf{P},\Sigma}$ equals the codimension of $\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma_{\alpha}}^{\circ}$ in $\mathbb{O}_{\mathbf{L},\mathbf{P},\Sigma}$, hence $\tilde{\rho}$ is also semi-small. The last assertion of the proposition follows from Proposition 4.3.9. \square

Theorem 5.4.1 is now a consequence of Proposition 5.4.2 and Proposition 3.2.1.

We have the following particular case of Theorem 5.4.1.

Proposition 5.4.3. *Let i denotes the inclusion $\mathcal{V}_{\hat{\omega}} \hookrightarrow \mathcal{O}_{\hat{\omega}}$. Then $i^*(IC_{\mathcal{O}_{\hat{\omega}}}^{\bullet}) \simeq IC_{\mathcal{V}_{\hat{\omega}}}^{\bullet}$.*

6 Characters and Fourier transforms

Here \mathbb{K} is an algebraic closure of a finite field \mathbb{F}_q . In this section we put $G := \mathrm{GL}_n(\mathbb{K})$ and $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$. We denote by F the standard Frobenius endomorphism $\mathfrak{g} \rightarrow \mathfrak{g}$ that maps a matrix $(a_{ij})_{i,j}$ to $(a_{ij}^q)_{i,j}$ so that $G^F = \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathfrak{g}^F = \mathfrak{gl}_n(\mathbb{F}_q)$.

6.1 Preliminaries on finite groups

Let κ be an algebraically closed field of characteristic 0. Let $z \mapsto \bar{z}$ be an involution of κ that maps roots of unity to their inverses. For a finite set E , we define \langle , \rangle_E on the space of all functions $E \rightarrow \kappa$ by

$$\langle f, g \rangle_E = \frac{1}{|E|} \sum_{x \in E} f(x) \overline{g(x)}.$$

Now let H be a subgroup of a finite group K and let \tilde{H} be a subgroup of $N_K(H)$ containing H . Let $\rho^1 : \tilde{H} \rightarrow \mathrm{GL}(V^1)$ and $\rho^2 : \tilde{H} \rightarrow \mathrm{GL}(V^2)$ be two representations of \tilde{H} in the finite dimensional κ -vector spaces V^1, V^2 . We denote by χ^1 and χ^2 their associated characters. The group \tilde{H} acts on the space $\mathrm{Hom}(V^1, V^2)$ as follows. For $f \in \mathrm{Hom}(V^1, V^2)$, we define $r \cdot f : V^1 \rightarrow V^2$ by $(r \cdot f)(v) = r \cdot f(r^{-1} \cdot v)$. Moreover we see that the subspace $\mathrm{Hom}_H(V^1, V^2)$ of fixed points of $\mathrm{Hom}(V^1, V^2)$ by H is clearly \tilde{H} -stable (it is therefore a $\kappa[\tilde{H}/H]$ -module).

Proposition 6.1.1. *For any $r \in \tilde{H}$, we have*

$$\mathrm{Tr}(r | \mathrm{Hom}(V^1, V^2)) = \chi^1(r) \chi^2(r^{-1}).$$

Proof. It follows from a straightforward calculation. Namely, we identify $\text{Hom}(V^1, V^2)$ with matrices with respect to some fixed bases and we calculate the trace of the left hand side in the basis $\{E_{ij}\}_{i,j}$ where $E_{ij} = (\delta_{p,i}\delta_{q,j})_{p,q}$ (here $\delta_{r,s} = 1$ if $r = s$ and $\delta_{r,s} = 0$ otherwise). \square

For $s \in \tilde{H}$, we denote by χ_s^i the restriction of χ^i to the coset $Hs := \{hs \mid h \in H\}$.

Proposition 6.1.2. *Let $s \in \tilde{H}$. We have*

$$\text{Tr}(s \mid \text{Hom}_H(V^1, V^2)) = \langle \chi_s^1, \chi_s^2 \rangle_{Hs}.$$

Proof. Put $E := \text{Hom}(V^1, V^2)$ and $E_H := \text{Hom}_H(V^1, V^2)$ and denote $p : E \rightarrow E_H$ the map $p(x) = \frac{1}{|H|} \sum_{h \in H} h \cdot x$. Then $E' := \text{Ker } p$ is an \tilde{H} -stable subspace of E and $E = E_H \oplus E'$. Since

$$\left(\frac{1}{|H|} \sum_{h \in H} hs \right) \Big|_{E'} = 0$$

we deduce that

$$\text{Tr}(s \mid E_H) = \frac{1}{|H|} \sum_{h \in H} \text{Tr}(hs \mid E).$$

By Proposition 6.1.1, the right hand side of this equation is $\langle \chi_s^1, \chi_s^2 \rangle_{Hs}$. \square

We now let φ and ψ be the characters of \tilde{H} and K associated respectively two representations $\tilde{H} \rightarrow \text{GL}(V)$ and $K \rightarrow \text{GL}(W)$. The group \tilde{H} acts on the K -module $\text{Ind}_H^K(V) := \kappa[K] \otimes_{\kappa[H]} V$ by $(x \otimes v) \cdot t = xt \otimes t^{-1} \cdot v$. Its restriction to H being trivial, it factorizes through an action of \tilde{H}/H on $\text{Ind}_H^K(V)$. Under the natural isomorphism (Frobenius reciprocity)

$$\text{Hom}_H(V, W) \simeq \text{Hom}_K(\text{Ind}_H^K(V), W) \quad (6.1.1)$$

the action of \tilde{H}/H on $\text{Hom}_H(V, W)$ described earlier corresponds to the action of \tilde{H}/H on the κ -vector space $\text{Hom}_K(\text{Ind}_H^K(V), W)$ given by $(t \cdot f)(x \otimes v) = f(t^{-1} \cdot (x \otimes v))$. For a subset E of K and a function $f : E \rightarrow \kappa$, we define $\text{Ind}_E^K(f) : K \rightarrow \kappa$ by

$$\text{Ind}_E^K(f)(k) = \frac{1}{|E|} \sum_{\{g \in K \mid g^{-1}kg \in E\}} f(g^{-1}kg).$$

Then we have the following generalization of Frobenius reciprocity for functions:

Lemma 6.1.3. *Let $h : K \rightarrow \kappa$ be a function. Then*

$$\langle \text{Ind}_E^K(f), h \rangle_K = \langle f, \text{Res}_E^K(h) \rangle_E.$$

Proof. It follows from a straightforward calculation. \square

By Proposition 6.1.2, (6.1.1) and the above lemma, we have the following proposition:

Proposition 6.1.4. *Let $v \in \tilde{H}/H$ and let $\dot{v} \in \tilde{H}$ be a representative of v . Then*

$$\text{Tr}(v \mid \text{Hom}_K(\text{Ind}_H^K(V), W)) = \langle \text{Ind}_{H\dot{v}}^K(\varphi_v), \psi \rangle_K$$

where φ_v denotes the restriction of φ to $H\dot{v}$.

6.2 Littlewood-Richardson coefficients

For a positive integer m , we denote by S_m the symmetric group in m letters.

Notation 6.2.1. For a subgroup H of a group K , we denote by $W_K(H)$ the quotient $N_K(H)/H$.

Fix a sequence $\tau_o = (a_1, m_1)(a_2, m_2) \cdots (a_s, m_s)$ with $a_i, m_i \in \mathbb{Z}_{>0}$ such that $\sum_i a_i m_i = n$ and $m_i \neq m_j$ if $i \neq j$. Put

$$S := (S_{m_1})^{a_1} \times \cdots \times (S_{m_r})^{a_s} \subset S_n$$

where $(S_m)^d$ stands for $S_m \times \cdots \times S_m$ (d times). Then the action of an element of $N_{S_n}(S)$ on each component $(S_{m_i})^{a_i}$ writes uniquely in the form $v_i w_i$ where $v_i : (S_{m_i})^{a_i} \rightarrow (S_{m_i})^{a_i}$ is a permutation of the coordinates (i.e., an element in S_{a_i}) and where $w_i \in (S_{m_i})^{a_i}$ acts on $(S_{m_i})^{a_i}$ by $z \mapsto w_i z w_i^{-1}$. In other words

$$W_{S_n}(S) \simeq \prod_{i=1}^s S_{a_i}.$$

The group $N_{S_n}(S)$ acts on the category of $\kappa[S]$ -modules in the natural way, i.e., if $\rho : S \rightarrow \text{GL}(V)$ and $n \in N_{S_n}(S)$, we denote by $n^*(\rho)$ the representation $\rho \circ n^{-1} : S \rightarrow \text{GL}(V)$.

For a representation $\rho : S \rightarrow \text{GL}(V)$, we denote by $W_{S_n}(S, \rho)$ the quotient $N_{S_n}(S, \rho)/S$ where

$$N_{S_n}(S, \rho) = \{n \in N_{S_n}(S) \mid n^*(\rho) \simeq \rho\}.$$

The group $W_{S_n}(S, \rho)$ acts (on the right) on the S_n -module $\text{Ind}_{S_n}^{S_n}(V)$ and so acts on

$$\text{Hom}_{S_n}(\text{Ind}_{S_n}^{S_n}(V), W)$$

for any $\kappa[S_n]$ -module W , see §6.1.

We now explain how to compute

$$\text{Tr} \left(v \mid \text{Hom}_{S_n}(\text{Ind}_{S_n}^{S_n}(V), W) \right)$$

for $v \in W_{S_n}(S, \rho)$.

Fix $v \in W_{S_n}(S) \simeq \prod_i S_{a_i}$ and decomposes each coordinate of v in S_{a_i} as a product of disjoint cycles. Then decompose S as

$$(S_{n_1})^{d_1} \times \cdots \times (S_{n_r})^{d_r}$$

such that the action of v on S is given by circular permutation on each component $(S_{n_i})^{d_i}$, i.e.,

$$v \cdot (g_1, \dots, g_{d_i}) = (g_2, g_3, \dots, g_{d_i}, g_1).$$

For each $i = 1, \dots, r$, we denote by v_i the automorphism of $(S_{n_i})^{d_i}$ given by $v_i \cdot (g_1, \dots, g_{d_i}) = (g_2, g_3, \dots, g_{d_i}, g_1)$, so that $v = (v_1, \dots, v_r)$.

Let V_i be an S_{n_i} -module with character χ_i . For each i , the group $(S_{n_i})^{d_i} \rtimes \langle v_i \rangle$ acts on $T^{d_i}(V_i) = V_i \otimes \cdots \otimes V_i$ as $(w, s) \cdot (x_1 \otimes \cdots \otimes x_{d_i}) = (w_1 \cdot x_{s(1)} \otimes \cdots \otimes w_{d_i} \cdot x_{s(d_i)})$. This defines an action of $\tilde{S} := S \rtimes \langle v \rangle$ on $\bigotimes_{i=1}^r T^{d_i}(V_i)$ and we denote by $\tilde{\chi} : \tilde{S} \rightarrow \kappa$ the corresponding character.

Lemma 6.2.2. *Let $w_i = (w_{i,1}, w_{i,2}, \dots, w_{i,d_i}) \in (S_{n_i})^{d_i}$ and put $w = (w_1, \dots, w_r) \in S$. We have*

$$\tilde{\chi}(w, v) = \prod_{i=1}^r \chi_i(w_{i,1} w_{i,2} \cdots w_{i,d_i}).$$

Assume now that $v \in W_{S_n}(S, \rho)$, i.e., that our space V afforded by the representation ρ is of the form $\bigotimes_i T^{d_i}(V_i)$, or more generally a direct sum of modules of this form. By Proposition 6.1.4 we have:

Proposition 6.2.3. *For any $\kappa[S_n]$ -module W with character ψ , we have*

$$\mathrm{Tr} \left(\nu \mid \mathrm{Hom}_{S_n} \left(\mathrm{Ind}_S^{S_n}(V), W \right) \right) = \left\langle \mathrm{Ind}_{S^v}^{S_n}(\tilde{\chi}), \psi \right\rangle_{S_n}.$$

We now show that this trace is also a Littlewood-Richardson coefficient (or more precisely a twisted version of it). We will use this result later on.

Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be an infinite set of variables and let $\Lambda(\mathbf{x})$ be the corresponding ring of symmetric functions. For a partition λ , let $s_\lambda(\mathbf{x})$ be the associated Schur symmetric function. For a type $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r) \in \mathbf{T}$, we define $\{c_\omega^\mu\}_{\mu \in \mathcal{P}}$ by

$$s_\omega(\mathbf{x}) := s_{\omega^1}(\mathbf{x}^{d_1}) s_{\omega^2}(\mathbf{x}^{d_2}) \cdots s_{\omega^r}(\mathbf{x}^{d_r}) = \sum_{\mu} c_\omega^\mu s_\mu(\mathbf{x})$$

where $\mathbf{x}^d := \{x_1^d, x_2^d, \dots\}$. We call the coefficients c_ω^μ the *twisted Littlewood-Richardson coefficients*. If $d_1 = d_2 = \cdots = d_r = 1$, these are the usual Littlewood-Richardson coefficients.

For $\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P}$, put

$$z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$$

It is also the cardinality of the centralizer in $S_{|\lambda|}$ of an element of type λ (i.e. whose decomposition as a product of disjoint cycles is given by λ). We denote by χ^λ the irreducible character corresponding to λ (see for instance [34, I, §7]) and by χ_μ^λ its value at an element of type μ .

Proposition 6.2.4. *We have*

$$c_\omega^\mu = \sum_{\rho} \chi_\rho^\mu \sum_{\alpha} \left(\prod_{i=1}^r z_{\alpha^i}^{-1} \chi_{\alpha^i}^{\omega^i} \right)$$

where the second sum runs over the $\alpha = (\alpha^1, \dots, \alpha^r) \in \mathcal{P}_{|\omega^1|} \times \cdots \times \mathcal{P}_{|\omega^r|}$ such that $\cup_i d_i \cdot \alpha^i = \rho$.

Proof. We have $s_\lambda(\mathbf{x}^d) = \sum_{\rho} z_\rho^{-1} \chi_\rho^\lambda p_\rho(\mathbf{x}^d)$ where p_ρ is the power symmetric function (see [34]). On the other hand, $p_{\rho^1}(\mathbf{x}^{d_1}) \cdots p_{\rho^r}(\mathbf{x}^{d_r}) = p_{\cup_i d_i \cdot \rho^i}(\mathbf{x})$. Hence

$$s_\omega(\mathbf{x}) = \sum_{\rho} \left(\sum_{\alpha} \prod_i z_{\alpha^i}^{-1} \chi_{\alpha^i}^{\omega^i} \right) p_\rho(\mathbf{x})$$

where the second sum runs over the $\alpha = (\alpha^1, \dots, \alpha^r) \in \mathcal{P}_{|\omega^1|} \times \cdots \times \mathcal{P}_{|\omega^r|}$ such that $\cup_i d_i \cdot \alpha^i = \rho$. We now decompose p_ρ in the basis $\{s_\lambda\}_\lambda$ and we get the result. \square

For $\lambda \in \mathcal{P}$, we denote by V_λ an irreducible $\kappa[S_{|\lambda|}]$ -module with corresponding character χ^λ .

Proposition 6.2.5. *Assume that ω^i is a partition of n_i and put $V_\omega = \bigotimes_{i=1}^r T^{d_i} V_{\omega^i}$. Then*

$$\mathrm{Tr} \left(\nu \mid \mathrm{Hom}_{S_n} \left(\mathrm{Ind}_S^{S_n}(V_\omega), V_\mu \right) \right) = c_\omega^\mu.$$

Proof. This is a consequence of Proposition 6.2.3 and Proposition 6.2.4. \square

6.3 Rational Levi subgroups and Weyl groups

By a *Levi subgroup* of G , we shall mean a Levi subgroup of a parabolic subgroup of G , i.e., a subgroup of G which is GL_n -conjugate to some subgroup of the form $\prod_{i=1}^r \mathrm{GL}_{n_i}$ with $\sum_i n_i = n$. A maximal torus of G is a Levi subgroup which is isomorphic to $(\mathbb{K}^\times)^n$. Let L be an F -stable Levi subgroup of G . An F -stable subtorus of S of L of rank r is said to be *split* if there is an isomorphism $S \simeq (\mathbb{K}^\times)^r$ which is defined over \mathbb{F}_q , i.e., $S^F \simeq (\mathbb{F}_q^\times)^r$. The \mathbb{F}_q -rank of L is defined as the maximal value of the ranks of the split subtori of L . Since the maximal torus of diagonal matrices is split, any F -stable Levi subgroup that contains diagonal matrices is of \mathbb{F}_q -rank n .

If T is an F -stable maximal torus of L of same \mathbb{F}_q -rank as L , in which case we say that T is an L -split maximal torus of L . In this case we denote by W_L , instead of $W_L(T)$ (see Notation 6.2.1), the Weyl group of L with respect to T .

If f is a group automorphism of K , we say that two elements k and h of K are *f-conjugate* if there exists $g \in K$ such that $k = ghf(g)^{-1}$.

The identification of the symmetric group S_n with the monomial matrices in GL_n with entries in $\{0, 1\}$ gives an isomorphism $S_n \simeq W_G$. Fix a sequence of integers $\mathbf{m} = (m_1, \dots, m_r)$ such that $\sum_i m_i = n$ and consider the Levi subgroup $L_o = \mathrm{GL}_{\mathbf{m}} := \prod_{i=1}^r \mathrm{GL}_{m_i}$. Then $W_{L_o} = S_{\mathbf{m}} := \prod_{i=1}^r S_{m_i}$. The G^F -conjugacy classes of the F -stable Levi subgroups of G that are G -conjugate to L_o are parametrized by the conjugacy classes of $W_G(L_o) = W_{S_n}(S_{\mathbf{m}})$ [9, Proposition 4.3]. For $v \in N_{S_n}(S_{\mathbf{m}})$, we denote by L_v a representative of the G^F -conjugacy class (of F -stable Levi subgroups) which corresponds to the conjugacy class of v in $W_{S_n}(S_{\mathbf{m}})$. Then $(L_v, F) \simeq (L_o, vF)$, i.e., the action of the Frobenius F on L_v corresponds to the action of vF on L_o given by $vF(g) := vF(g)v^{-1}$ for any $g \in L_o$. Since F acts trivially on $W_G \simeq S_n$, we have $(W_{L_v}, F) \simeq (S_{\mathbf{m}}, v)$. By §6.2, there exists a decomposition

$$S_{\mathbf{m}} = (S_{n_1})^{d_1} \times \cdots \times (S_{n_r})^{d_r}$$

for some sequence $(d_1, n_1)(d_2, n_2) \cdots (d_r, n_r)$ and a specific choice of an element σ in the coset $vS_{\mathbf{m}}$ which acts on each component $(S_{n_i})^{d_i}$ by circular permutation of the coordinates. Taking the G^F -conjugate L_{σ} of L_v if necessary we may assume that $v = \sigma$. We also have

$$L_o = \prod_{i=1}^r (\mathrm{GL}_{m_i})^{d_i}, \text{ and } (L_v)^F \simeq (L_o)^{vF} \simeq \prod_{i=1}^r \mathrm{GL}_{m_i}(\mathbb{F}_{q^{d_i}}).$$

Now let L be any F -stable Levi subgroup of G . Consider the semi-direct product $W_L \rtimes \langle F \rangle$ where $\langle F \rangle$ is the cyclic group generated by the Frobenius automorphism on W_L . If ψ is a character of $W_L \rtimes \langle F \rangle$, then for all $a \in W_L$, we have $\psi(F(a)) = \psi(a)$ since $(F(a), 1) \in W_L \rtimes \langle F \rangle$ is the conjugate of $(a, 1)$ by $(1, F)$. Hence the restriction of ψ to W_L is an F -stable character of W_L . Conversely, given an F -stable character χ of W_L , we now define an extension $\tilde{\chi}$ of χ to $W_L \rtimes \langle F \rangle$ as follows. We have $L = L_v$ for some \mathbf{m} and $v \in N_{S_n}(S_{\mathbf{m}})$ by the above discussion so that we may identify $W_L \rtimes \langle F \rangle$ with $S_{\mathbf{m}} \rtimes \langle v \rangle$. For an v -stable character χ of $S_{\mathbf{m}}$ we define the extension $\tilde{\chi}$ of $S_{\mathbf{m}} \rtimes \langle v \rangle$ as in §6.2.

The L^F -conjugacy classes of the F -stable maximal tori of L are parametrized by the F -conjugacy classes of W_L [9, Proposition 4.3]. If $w \in W_L$, we denote by T_w an F -stable maximal torus of L which is in the L^F -conjugacy class associated to the F -conjugacy class of w . We put $\mathfrak{t}_w := \mathrm{Lie}(T_w)$.

6.4 Springer correspondence for relative Weyl groups

Let P be a parabolic subgroup of G and L a Levi factor of P . Let \mathfrak{l} be the Lie algebra of L and let z_1 denotes its center. Recall that the classical Springer correspondence gives a bijection

$$\mathbb{C} : \mathrm{Irr} W_L \rightarrow \{\text{nilpotent orbits of } \mathfrak{l}\}$$

which maps the trivial character to the regular nilpotent orbit. Moreover if L is F -stable then \mathfrak{C} restricts to a bijection between the F -stable irreducible characters of W_L and the F -stable nilpotent orbits of \mathfrak{l} . Recall that if $L = G$ and $\lambda \in \mathcal{P}_n$, then the size of the Jordan blocks of the nilpotent orbit $\mathfrak{C}(\chi^\lambda)$ are given by the partition λ .

Let $\epsilon \in \text{Irr } W_L$ be the sign character. For $\chi \in \text{Irr } W_L$ put $\chi' := \chi \otimes \epsilon$. Then let $\mathfrak{C}_\epsilon : \text{Irr } W_L \xrightarrow{\sim} \{\text{nilpotent orbits of } \mathfrak{l}\}$ be the map which sends χ to $\mathfrak{C}(\chi')$. The bijection \mathfrak{C}_ϵ was actually the first correspondence to be discovered [44].

Let C be a nilpotent orbit of \mathfrak{l} and put $\Sigma = \sigma + C$ with $\sigma \in z_1$. Consider the relative Weyl group

$$W_G(L, \Sigma) := \{n \in N_G(L) \mid n\Sigma n^{-1} = \Sigma\}/L.$$

Recall that Σ is of the form $\sigma + C$ with C a nilpotent orbit of \mathfrak{l} and $\sigma \in z_1$. Put $M := C_G(\sigma)$, then $W_G(L, \Sigma) = W_M(L, C)$. Let O be the orbit of \mathfrak{gl}_n whose Zariski closure is the image of the projection $f : \mathbb{X}_{L,P,\Sigma} \rightarrow \mathfrak{g}$ on the first coordinate.

Let \mathfrak{g}_σ be the set of elements $x \in \mathfrak{g}$ whose semisimple part is G -conjugate to σ . Note that the image of f is contained in \mathfrak{g}_σ . The set \mathfrak{g}_σ has a finite number of G -orbits which are indexed by the irreducible characters of W_M by \mathfrak{C} . If χ is an irreducible character of W_M we denote by O_χ the corresponding adjoint orbit in \mathfrak{g}_σ . For $\chi \in \text{Irr } W_M$, put

$$A_\chi = \text{Hom}_{W_M} \left(\text{Ind}_{W_L}^{W_M} (V_C), V_\chi \right)$$

where $\rho : W_L \rightarrow \text{GL}(V_C)$ is an irreducible representation of W_L (unique up to isomorphism) corresponding to the nilpotent orbit C under \mathfrak{C} .

In the notation of §6.2, the group $W_M(L, C)$ is isomorphic to $W_{W_M}(W_L, \rho)$ and so the spaces A_χ are $W_M(L, C)$ -modules. We have the following theorem [31, 2.5].

Theorem 6.4.1. *We have*

$$f_* \left(\underline{\text{IC}}_{\mathbb{X}_{L,P,\Sigma}}^\bullet \right) \simeq \bigoplus_{\chi \in \text{Irr } W_M} A_\chi \otimes \underline{\text{IC}}_{\overline{O}_\chi}^\bullet$$

where $A_\chi = 0$ if O_χ is not included in \overline{O} .

In this correspondence we have $O = O_{\text{Id}}$ and A_{Id} is the trivial character of $W_M(L, C)$. If O is regular nilpotent, $L = T$ and if $\Sigma = \{0\}$, then this is the classical Springer correspondence.

To alleviate the notation put $K := f_* \left(\underline{\text{IC}}_{\mathbb{X}_{L,P,\Sigma}}^\bullet \right)$ and $K_\chi := A_\chi \otimes \underline{\text{IC}}_{\overline{O}_\chi}^\bullet$. Assume now that (M, Q, L, P, Σ) is F -stable and let $F : \mathbb{X}_{L,P,\Sigma} \rightarrow \mathbb{X}_{L,P,\Sigma}$ be the Frobenius given by $F(X, gP) = (F(x), F(g)P)$. Then the morphism f commutes with the Frobenius endomorphisms. Let $\varphi : F^*(\kappa) \simeq \kappa$ be the isomorphism (in the category of sheaves on $\mathbb{X}_{L,P,\Sigma}^q$) which induces the identity on stalks at \mathbb{F}_q -points. It induces a canonical isomorphism $F^* \left(\underline{\text{IC}}_{\mathbb{X}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{\text{IC}}_{\mathbb{X}_{L,P,\Sigma}}^\bullet$ which in turns induces a canonical isomorphism $\tilde{\varphi} : F^*(K) \simeq K$. Note that the orbits O_χ are F -stable and F acts trivially on W_M . Hence $F^*(K_\chi) \simeq K_\chi$ and so $\tilde{\varphi}$ induces an isomorphism $\tilde{\varphi}_\chi : F^*(K_\chi) \simeq K_\chi$ for each χ . Now we define an isomorphism $\phi_\chi : F^* \left(\underline{\text{IC}}_{\overline{O}_\chi}^\bullet \right) \simeq \underline{\text{IC}}_{\overline{O}_\chi}^\bullet$ with the requirement that its tensor product with the identity on A_χ gives $\tilde{\varphi}_\chi$.

We then have

$$\mathbf{X}_{\underline{\text{IC}}_{\overline{O}_\chi}^\bullet, \phi_\chi} = q^{\frac{1}{2}(\dim O - \dim O_\chi)} \mathbf{X}_{\underline{\text{IC}}_{\overline{O}_\chi}^\bullet}.$$

Since the A_χ are $W_M(L, C)$ -modules, each $v \in W_M(L, C)$ induces an isomorphism $K_\chi \simeq K_\chi$ and so an isomorphism $\theta_v : K \simeq K$ such that

$$\mathbf{X}_{K, \theta_v \circ \tilde{\varphi}} = \sum_{\chi} \text{Tr} \left(v | A_\chi \right) q^{\frac{1}{2}(\dim O - \dim O_\chi)} \mathbf{X}_{\underline{\text{IC}}_{\overline{O}_\chi}^\bullet}.$$

6.5 Characters of finite general linear groups

The character table of G^F was first computed by Green [12]. Here we recall how to construct it from the point of view of Deligne-Lusztig theory [33].

Recall that for any F -stable Levi subgroup L of G we have the Lusztig induction $R_L^G : C(L^F) \rightarrow C(G^F)$ where $C(L^F)$ denotes the $\overline{\mathbb{Q}}_\ell$ -vector space of class functions $L^F \rightarrow \overline{\mathbb{Q}}_\ell$ (see for instance [8][28]).

Let L be an F -stable Levi subgroup of G and let φ be an F -stable irreducible character of W_L . The function $\chi_\varphi^L : L^F \rightarrow \overline{\mathbb{Q}}_\ell$ defined by

$$\chi_\varphi^L = |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^L(\text{Id}_{T_w}) \quad (6.5.1)$$

is an irreducible character of L^F (here $\tilde{\varphi}$ is the extension of φ defined in §6.3). The characters χ_φ^L are called the *unipotent characters* of L^F .

For $g \in G^F$ and $\theta \in \text{Irr}(L^F)$, let ${}^s\theta \in \text{Irr}(gL^Fg^{-1})$ be defined by ${}^s\theta(glg^{-1}) = \theta(l)$. We say that a linear character $\theta : L^F \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is *regular* if for $n \in N_{G^F}(L)$, we have $n\theta = \theta$ only if $n \in L^F$. We denote by $\text{Irr}_{\text{reg}}(L^F)$ the set of regular linear characters of L^F . Put $\epsilon_L = (-1)^{\mathbb{F}_q - \text{rank}(L)}$. Then for $\theta^L \in \text{Irr}_{\text{reg}}(L^F)$, the virtual character

$$\mathcal{X} := \epsilon_G \epsilon_L R_L^G(\theta^L \cdot \chi_\varphi^L) = \epsilon_G \epsilon_L |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^G(\theta^{T_w}) \quad (6.5.2)$$

where $\theta^{T_w} := \theta^L|_{T_w}$, is an irreducible true character of G^F and any irreducible character of G^F is obtained in this way [33]. An irreducible character of G^F is thus completely determined by the G^F -conjugacy class of a datum (L, θ^L, φ) with L an F -stable Levi subgroup of G , $\theta^L \in \text{Irr}_{\text{reg}}(L^F)$ and $\varphi \in \text{Irr}(W_L)^F$. Characters associated to triples of the form $(L, \theta^L, 1)$ are called *semisimple*.

The characters $\epsilon_G \epsilon_{T_w} R_{T_w}^G(\theta)$ are called *Deligne-Lusztig characters*.

6.6 Fourier transforms

We fix once for all a non-trivial additive character $\Psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and we denote by $\mu : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ the trace map $(a, b) \mapsto \text{Trace}(ab)$. It is a non-degenerate G -invariant symmetric bilinear form defined over \mathbb{F}_q . Let $\text{Fun}(\mathfrak{g}^F)$ be the $\overline{\mathbb{Q}}_\ell$ -vector space of all functions $\mathfrak{g}^F \rightarrow \overline{\mathbb{Q}}_\ell$. We define the Fourier transform $\mathcal{F}^{\mathfrak{g}} : \text{Fun}(\mathfrak{g}^F) \rightarrow \text{Fun}(\mathfrak{g}^F)$ with respect to (Ψ, μ) by

$$\mathcal{F}^{\mathfrak{g}}(f)(x) = \sum_{y \in \mathfrak{g}^F} \Psi(\mu(x, y)) f(y).$$

We have also the Deligne-Fourier transform $\mathcal{F}^{\mathfrak{g}} : \mathcal{D}_c^b(\mathfrak{g}) \rightarrow \mathcal{D}_c^b(\mathfrak{g})$. It is defined as follows.

We denote by \mathbb{A}^1 the affine line over \mathbb{K} . Let $h : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the Artin-Schreier covering defined by $h(t) = t^q - t$. Then, since h is a Galois covering of \mathbb{A}^1 with Galois group \mathbb{F}_q , the sheaf $h_*(\overline{\mathbb{Q}}_\ell)$ is a local system on \mathbb{A}^1 on which \mathbb{F}_q acts. We denote by \mathcal{L}_Ψ the subsheaf of $h_*(\overline{\mathbb{Q}}_\ell)$ on which \mathbb{F}_q acts as Ψ^{-1} . There exists an isomorphism $\varphi_\Psi : F^*(\mathcal{L}_\Psi) \rightarrow \mathcal{L}_\Psi$ such that for any integer $i \geq 1$, we have $\mathbf{X}_{\mathcal{L}_\Psi, \varphi_\Psi^{(i)}} = \Psi \circ \text{Trace}_{\mathbb{F}_q/\mathbb{F}_q} : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell$ (see [18, 3.5.4]). Then for a complex $K \in \mathcal{D}_c^b(\mathfrak{g})$ we define

$$\mathcal{F}^{\mathfrak{g}}(K) := (p_1)_!((p_2)^*(K) \otimes \mu^*(\mathcal{L}_\Psi))[\dim \mathfrak{g}]$$

where $p_1, p_2 : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ are the two projections. If $\varphi : F^*(K) \rightarrow K$ is an isomorphism, then it induces a natural isomorphism $\mathcal{F}(\varphi) : F^*(\mathcal{F}^{\mathfrak{g}}(K)) \rightarrow \mathcal{F}^{\mathfrak{g}}(K)$. Moreover,

$$\mathbf{X}_{\mathcal{F}^{\mathfrak{g}}(K), \mathcal{F}(\varphi)} = (-1)^{\dim \mathfrak{g}} \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{K, \varphi}).$$

We will need to compute the characteristic functions of the perverse sheaves $\mathcal{F}^{\mathfrak{g}}(\underline{IC}_O^\bullet)$, where O an F -stable adjoint orbit of \mathfrak{g} . It is known that these perverse sheaves are closely related to the character sheaves

on G [32] and that the characteristic functions of character sheaves on G give the irreducible characters of G^F [27]. We thus expect to have a tight connection between the characteristic functions of the sheaves $\mathcal{F}^{\mathfrak{g}}(\underline{IC}_{\overline{\mathcal{O}}})$ on \mathfrak{g} and the irreducible characters of G^F .

More precisely, let $x \in \mathcal{O}^F$, and let σ and n be the semisimple part and the nilpotent part of x . Put $L = C_G(x_s)$, and let φ be the F -stable irreducible character of W_L that corresponds to the nilpotent orbit \mathcal{O}_n^L of $\mathfrak{l} = \text{Lie}(L)$ via the Springer correspondence \mathfrak{C}_ϵ .

Theorem 6.6.1. *We have*

$$\mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\mathcal{O}}}}) = \epsilon_G \epsilon_L q^{\frac{1}{2} \dim \mathcal{O}} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) \mathcal{R}_{t_w}^{\mathfrak{g}}(\theta_\sigma^w) \quad (6.6.1)$$

where $\mathcal{R}_{t_w}^{\mathfrak{g}}$ is the Deligne-Lusztig induction [24] and $\theta_{x_s}^w : \mathfrak{l}_{x_s}^F \rightarrow \overline{\mathbb{Q}}_\ell$ is the character $z \mapsto \Psi(\mu(\sigma, z))$.

Remark 6.6.2. Note that Formula (6.5.2) is similar to Formula (6.6.1). It shows that $\mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\mathcal{O}}}})$ arises from the G^F -conjugacy class of a triple (L, η^1, φ) with $\eta^1 : \mathfrak{l}^F \rightarrow \overline{\mathbb{Q}}_\ell^\times, z \mapsto \Psi(\mu(\sigma, z))$ exactly as in the group setting.

Proof of Theorem 6.6.1. Let $x \in \mathcal{O}^F$ and let σ, n be respectively the semisimple part and the nilpotent part of x . Let $L = C_G(\sigma)$ and let \mathcal{O}^L be the L -orbit of x in $\mathfrak{l} := \text{Lie}(L)$. Then $\overline{\mathcal{O}}^L$ decomposes as $\sigma + \overline{\mathcal{O}}_n^L$ where \mathcal{O}_n^L denotes the L -orbit of n in \mathfrak{l} . Let $1_\sigma : \mathfrak{l}^F \rightarrow \overline{\mathbb{Q}}_\ell$ be the characteristic function of σ , i.e., it is the function that takes the value 1 on σ and zero elsewhere. Then

$$\mathbf{X}_{IC_{\overline{\mathcal{O}}^L}} = 1_\sigma * \mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}}$$

where $*$ is the usual convolution product on functions, i.e., $(f * h)(x) = \sum_y f(y)h(x-y)$. By [24, Proposition 7.1.8], we have

$$\mathbf{X}_{IC_{\overline{\mathcal{O}}}} = \mathcal{R}_1^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\mathcal{O}}^L}}) \quad (6.6.2)$$

Hence from the commutation formula of Fourier transforms with Deligne-Lusztig induction [24, Corollary 6.2.17] we have

$$\begin{aligned} \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\mathcal{O}}}}) &= \epsilon_G \epsilon_L q^{\frac{1}{2}(\dim G - \dim L)} \mathcal{R}_1^{\mathfrak{g}} \circ \mathcal{F}^1(\mathbf{X}_{IC_{\overline{\mathcal{O}}^L}}) \\ &= \epsilon_G \epsilon_L q^{\frac{1}{2}(\dim G - \dim L)} \mathcal{R}_1^{\mathfrak{g}} \left(\mathcal{F}^1(1_\sigma) \cdot \mathcal{F}^1(\mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}}) \right) \end{aligned}$$

We also have:

$$\mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}} = q^{-\delta} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}'(wF) \mathcal{R}_{t_w}^1(1_0) \quad (6.6.3)$$

where $\delta = \frac{1}{2}(\dim C_L(n) - \dim T)$ and 1_0 denotes the characteristic function of 0. Note that $\mathcal{R}_{t_w}^1(1_0)$ is a Green function in the sense of Deligne-Lusztig [8]. Let us prove the formula (6.6.3).

Let $\omega : L_{\text{uni}} \rightarrow \mathfrak{l}_{\text{nil}}$ be the L -equivariant isomorphism $u \mapsto u - 1$ from the variety of unipotent elements of L onto the variety of nilpotent elements of \mathfrak{l} . Put $C^L = \omega^{-1}(\mathcal{O}_n^L)$. By [27], we have $\mathcal{X}_{\varphi'}^L|_{L_{\text{uni}}} = q^\delta \mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}}$ where $\mathcal{X}_{\varphi'}^L$ is the unipotent character of L^F associated to φ' . Hence Formula (6.6.3) is obtained from Formula (6.5.1) via the isomorphism ω .

We now deduce from Formula (6.6.3) and the commutation formula [24, Corollary 6.2.17] that

$$\mathcal{F}^1(\mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}}) = q^{-\delta} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}'(wF) \epsilon_L \epsilon_{T_w} q^{\frac{1}{2}(\dim L - \dim T_w)} \mathcal{R}_{t_w}^1(\text{Id}_{t_w})$$

Since σ is central in \mathfrak{l} , we deduce that

$$\mathcal{F}^1(1_\sigma) \cdot \mathcal{F}^1(\mathbf{X}_{IC_{\overline{\mathcal{O}}_n^L}}) = q^{-\delta} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}'(wF) \epsilon_L \epsilon_{T_w} q^{\frac{1}{2}(\dim L - \dim T_w)} \mathcal{R}_{t_w}^1(\theta_\sigma^w).$$

From the transitivity property of Deligne-Lusztig induction [23] and the fact that $C_G(x) = C_L(n)$ we deduce that:

$$\mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\mathcal{O}}}) = \epsilon_G \epsilon_L q^{\frac{1}{2} \dim \mathcal{O}} |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}'(wF) \epsilon_L \epsilon_{T_w} \mathcal{R}_{T_w}^{\mathfrak{g}}(\theta_{\sigma}^v).$$

The map $W_L \rightarrow \{1, -1\}$, $w \mapsto \epsilon_L \epsilon_{T_w}$ is the sign character ϵ of W_L . \square

Lemma 6.6.3. *The functions $\mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\mathcal{O}}})$ are G^F -invariant (i.e. constant on adjoint orbits) characters of the finite abelian group $(\mathfrak{g}^F, +)$.*

Proof. The functions $\mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\mathcal{O}}})$ are clearly G^F -invariant. For an adjoint orbit \mathcal{O} of \mathfrak{g}^F , let $1_{\mathcal{O}}$ denotes the characteristic function of \mathcal{O} , i.e., $1_{\mathcal{O}}(x) = 1$ if $x \in \mathcal{O}$ and $1_{\mathcal{O}}(x) = 0$ otherwise. Then $\mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}})$ is a sum of linear characters of \mathfrak{g}^F and therefore is character of \mathfrak{g}^F . We thus need to see that if we write $\mathbf{X}_{IC_{\mathcal{O}}} = \sum_C n_C 1_C$ as a sum over the adjoint orbits of \mathfrak{g}^F , then $n_C \in \mathbb{Z}_{\geq 0}$. Let us use the notation introduced in the proof of Theorem 6.6.1. Write

$$\mathbf{X}_{IC_{\mathcal{O}_L}} = 1_{\sigma} * \mathbf{X}_{IC_{\mathcal{O}_h}^*} = 1_{\sigma} * \left(\sum_{C'} n_{C'} 1_{C'} \right) = \sum_{C'} n_{C'} 1_{\sigma+C'}$$

where the sum runs over the nilpotent L^F -orbits of \mathfrak{l}^F (note that $\sigma+C'$ is an L^F -orbit of \mathfrak{l}^F since σ is central). By [24, Proposition 3.2.24], for a nilpotent adjoint orbit of \mathfrak{l}^F , the function $\mathcal{R}_{\mathfrak{l}^F}(1_{\sigma+C'})$ is the characteristic function of the G^F -orbit of an element in $\sigma+C'$. By Formula (6.6.2) we are reduced to see that $n_{C'} \in \mathbb{Z}_{\geq 0}$. We have $L^F \simeq \prod_i \mathrm{GL}_{n_i}(\mathbb{F}_{q^{d_i}})$ for some $n_i, d_i \in \mathbb{Z}_{\geq 0}$, and so $\mathbf{X}_{IC_{\mathcal{O}_h}^*}$ is a product of functions of the form $\mathbf{X}_{IC_{\mathcal{O}_i}^*}$ on $\mathfrak{gl}_{n_i}(\mathbb{F}_{q^{d_i}})$ where \mathcal{O}_i is a nilpotent orbit of $\mathfrak{gl}_{n_i}(\overline{\mathbb{F}}_q)$. By [27], the values of the functions $\mathbf{X}_{IC_{\mathcal{O}_i}^*}$ are non-negative integers. \square

6.7 Generic characters and generic orbits

Let (L, θ^L, φ) be a triple as in §6.5 with L an F -stable Levi subgroup, $\theta^L \in \mathrm{Irr}_{\mathrm{reg}}(L^F)$ and $\varphi \in \mathrm{Irr}(W_L)^F$ and let χ be the associated irreducible character of G^F . Then we say that the G^F -conjugacy class of the pair (L, φ) is the *type* of χ . Similarly we define the *type* of an adjoint orbit \mathcal{O}^F of \mathfrak{g}^F as follows. Let $x \in \mathcal{O}^F$ and let $M = C_G(x_s)$ and let C^M be the M -orbit of $x_n \in \mathfrak{m}$. Then the G^F -conjugacy class of the pair (M, C^M) is called the *type* of \mathcal{O}^F .

From the pair (L, φ) we define $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r) \in \mathbf{T}_n$ as follows. There exist positive integers d_i, n_i such that $L \simeq \prod_{i=1}^r \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}$ and $L^F \simeq \prod_{i=1}^r \mathrm{GL}_{n_i}(\mathbb{F}_{q^{d_i}})$. The F -stable irreducible characters of W_L correspond then to $\mathrm{Irr}(S_{n_1}) \times \cdots \times \mathrm{Irr}(S_{n_r})$ and the later set is in bijection with $\mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_r}$ via Springer correspondence \mathfrak{C}_{ϵ} that sends the trivial character of S_m to the partition (1^m) . If $q > n$, the set of types of irreducible characters of G^F is thus parametrized by \mathbf{T}_n . Under this parameterisation, semisimple irreducible characters correspond to types of the form $(d_1, (1^{n_1})) \cdots (d_r, (1^{n_r}))$ and unipotent characters to types of the form $(1, \lambda)$.

From the pair (M, C^M) we define $\tau = (d_1, \tau^1)(d_2, \tau^2) \cdots (d_r, \tau^r) \in \mathbf{T}_n$ as follows. There exist positive integers d_i, n_i such that $M \simeq \prod_{i=1}^r \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}$ and $M^F \simeq \prod_{i=1}^r \mathrm{GL}_{n_i}(\mathbb{F}_{q^{d_i}})$. The Jordan form of C^M defines partitions τ^1, \dots, τ^r of n_1, \dots, n_r respectively. If $q \geq n$, the set of types of adjoint orbits of \mathfrak{g}^F is thus parametrized by \mathbf{T}_n .

Remark 6.7.1. Note that if \mathcal{O}^F is an orbit of \mathfrak{g}^F of type $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r)$, then in the sense of §4 the G -orbit \mathcal{O} is of type

$$\tilde{\omega} := \underbrace{\omega^1 \cdots \omega^1}_{d_1} \underbrace{\omega^2 \cdots \omega^2}_{d_2} \cdots \underbrace{\omega^r \cdots \omega^r}_{d_r}.$$

In particular, the two notions coincide if the eigenvalues of \mathcal{O} are in \mathbb{F}_q .

Definition 6.7.2. Let O_1^F, \dots, O_k^F be k adjoint orbits of g^F . We say that the tuple (O_1^F, \dots, O_k^F) is *generic* if (O_1, \dots, O_k) is generic in the sense of Definition 5.1.1.

Assume that L is an F -stable Levi subgroup of G and let z_l denotes the center of its Lie algebra \mathfrak{l} . We say that a linear additive character of z_l^F is *generic* if its restriction to z_g^F is trivial and its restriction to z_m^F is non-trivial for any proper F -stable Levi subgroup M of G which contains L .

Lemma 6.7.3. Let (O_1^F, \dots, O_k^F) be a generic tuple of adjoint orbits of g^F . Let (L_i, η_i, φ_i) be a datum defining the character $\mathcal{F}^{\eta_i}(\mathbf{X}_{\mathcal{I}O_i^F})$, see Remark 6.6.2. Then $\prod_{i=1}^k ({}^{g_i}\eta_i)|_{z_m}$ is a generic character of z_m^F for any F -stable Levi subgroup M of G which satisfies the following condition: For all $i \in \{1, \dots, k\}$, there exists $g_i \in G^F$ such that its center Z_M is contained in $g_i L_i g_i^{-1}$.

Proof. We may write $\eta_i = \mathcal{F}^{\eta_i}(1_{\sigma_i})$ where $\sigma_i \in z_l$ is the semisimple part of an element of O_i^F . Note that $g_i \sigma_i g_i^{-1}$ is in the center of $g_i L_i g_i^{-1}$ and so it commutes with the elements of $z_m \subset g_i L_i g_i^{-1}$, i.e., $g_i \sigma_i g_i^{-1} \in C_g(z_m) = \mathfrak{m}$. Let $z \in z_m^F$. Then

$$\prod_{i=1}^k ({}^{g_i}\eta_i)(z) = \prod_{i=1}^k \mathcal{F}^{\eta_i}(1_{\sigma_i})(g_i^{-1} z g_i) = \prod_{i=1}^k \Psi(\mu(\sigma_i, g_i^{-1} z g_i)) = \prod_{i=1}^k \Psi(\mu(g_i \sigma_i g_i^{-1}, z)) = \Psi\left(\mu\left(\sum_i g_i \sigma_i g_i^{-1}, z\right)\right).$$

If $z = \lambda \cdot \text{Id} \in z_g$, then $\mu(\sum_i g_i \sigma_i g_i^{-1}, z) = \lambda \text{Tr}(\sum_i g_i \sigma_i g_i^{-1}) = 0$ by the first genericity condition (see Definition 5.1.1). Let L be an F -stable Levi subgroup such that $M \subsetneq L \subsetneq G$, i.e., such that $z_g \subsetneq z_l \subsetneq z_m$ and assume that $\prod_{i=1}^k ({}^{g_i}\eta_i)|_{z_l}$ is trivial. There is a decomposition $\mathbb{K}^n = V_1 \oplus V_2 \oplus \dots \oplus V_r$ such that $\mathfrak{l} \simeq \bigoplus_i \mathfrak{gl}(V_i)$. Then any element $z \in z_l$ is a of the form $(\lambda_1 \cdot \text{Id}, \dots, \lambda_r \cdot \text{Id})$ for some $\lambda_1, \dots, \lambda_r \in \mathbb{K}$. Since $g_i \sigma_i g_i^{-1} \in \mathfrak{m} \subset \mathfrak{l}$ for all i , we may write $\sum_i g_i \sigma_i g_i^{-1} = (x_1, \dots, x_r) \in \mathfrak{gl}(V_1) \oplus \dots \oplus \mathfrak{gl}(V_r)$. Since $\prod_{i=1}^k ({}^{g_i}\eta_i)|_{z_l}$ is trivial we have $\sum_{i=1}^r \lambda_i \text{Tr}(x_i) = 0$ for all $\lambda_1, \dots, \lambda_r \in \mathbb{K}$. Hence $\text{Tr}(x_i) = 0$ for all $i = 1, \dots, r$. This contradicts the second genericity assumption. \square

Assume that L is an F -stable Levi subgroup of G and let Z_L denotes its center. Let Γ be a linear character of Z_L^F . It is said to be *generic* if its restriction to Z_G^F is trivial and its restriction to Z_M^F is non-trivial for any F -stable proper Levi subgroup M of G such that $L \subset M$. We have the following definitions [13]:

Definition 6.7.4. Let $\mathcal{X}_1, \dots, \mathcal{X}_k$ be k -irreducible characters of G^F . For each i , let $(L_i, \theta_i, \varphi_i)$ be a datum defining \mathcal{X}_i . We say that the tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ is *generic* if $\prod_{i=1}^k ({}^{g_i}\theta_i)|_{Z_M}$ is a generic character of Z_M^F for any F -stable Levi subgroup M of G which satisfies the following condition: For all $i \in \{1, \dots, k\}$, there exists $g_i \in G^F$ such that $Z_M \subset g_i L_i g_i^{-1}$.

Example 6.7.5. Let μ^1, \dots, μ^k be k partitions of n and denote by $R_{\mu^1}, \dots, R_{\mu^k}$ the corresponding unipotent characters of G^F (see beginning of this section). Consider k linear characters $\alpha_1, \dots, \alpha_k$ of \mathbb{F}_q^\times . For each i , put $\mathcal{X}_i := (\alpha_i \circ \det) \cdot R_{\mu^i}$. Then \mathcal{X}_i is an irreducible character of G^F of same type as R_{μ^i} . Then according to Definition 6.7.4, the tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ is generic if and only if the size of the subgroup of $\text{Irr } \mathbb{F}_q^\times$ generated by $\rho := \alpha_1 \cdots \alpha_k$ equals n .

Assume that $\text{char}(\mathbb{F}_q)$ and q are large enough. Then for any $\omega \in (\mathbf{T}_n)^k$, we can always find a generic tuple $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ of irreducible characters of G^F of type ω . The proof of this is similar to the proof of the existence of generic tuples of conjugacy classes of GL_n of a given type, see [13].

Definition 6.7.6. We say that an adjoint orbit of g^F (or an irreducible character of G^F) is *split* if the degrees of its type are all equal to 1.

6.8 Multiplicities in tensor products

Let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be a generic tuple of irreducible characters of G^F of type ω and assume that there exists a generic tuple (O_1^F, \dots, O_k^F) of adjoint orbits of g^F of same type $\omega = (\omega_1, \dots, \omega_k) \in (\mathbf{T}_n)^k$ (cf, §5.1). For

$i = 1, 2, \dots, k$, write $\omega_i = (d_1, \omega_i^1) \cdots (d_r, \omega_i^r)$. By Lemma 4.2.1 we have $\dim O_i = n^2 - \sum_j d_j \langle \omega_i^j, \omega_i^j \rangle$. As in Theorem 5.2.2, we put $d_\omega = (2g - 2)n^2 + 2 + \sum_i \dim O_i$.

Let $\Theta : \mathfrak{g}^F \rightarrow \overline{\mathbb{Q}}_\ell$ be given by $x \mapsto q^{gn^2 + g \dim C_G(x)}$, and let $\Lambda : G^F \rightarrow \overline{\mathbb{Q}}_\ell$ be given by $x \mapsto q^{g \dim C_G(x)}$. If $g = 1$, note that Λ is the character of the representation of G^F in the group algebra $\overline{\mathbb{Q}}_\ell[\mathfrak{g}^F]$ where G^F acts on \mathfrak{g}^F by conjugation.

Theorem 6.8.1. *We have*

$$\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle_{G^F} = \frac{q^{-d_\omega/2}(q-1)}{|G^F|} \left\langle \Theta \otimes \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\sigma}_1}}) \otimes \cdots \otimes \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\sigma}_k}}), 1 \right\rangle_{\mathfrak{g}^F}.$$

Proof. For each $i = 1, \dots, k$, let $(L_i, \theta_i, \varphi_i)$ be a datum defining \mathcal{X}_i . Then

$$\begin{aligned} |G^F| \langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle_{G^F} &= \sum_{x \in G^F} q^{g \dim C_G(x)} \prod_{i=1}^k \left(\epsilon_{G \in L_i} |W_{L_i}|^{-1} \sum_{w \in W_{L_i}} \tilde{\varphi}_i(wF) R_{T_w}^G(\theta_i)(x) \right) \\ &= \prod_{i=1}^k \left(\epsilon_{G \in L_i} |W_{L_i}|^{-1} \right) \sum_{x \in G^F} q^{g \dim C_G(x)} \sum_{(w_1, \dots, w_k) \in W_{L_1} \times \cdots \times W_{L_k}} \prod_{i=1}^k \tilde{\varphi}_i(w_i F) R_{T_{w_i}}^G(\theta_i)(x) \\ &= \sum_{(w_1, \dots, w_k) \in W_{L_1} \times \cdots \times W_{L_k}} \left(\prod_{i=1}^k \epsilon_{G \in L_i} |W_{L_i}|^{-1} \tilde{\varphi}_i(w_i F) \right) \sum_{x \in G^F} q^{g \dim C_G(x)} \prod_{i=1}^k R_{T_{w_i}}^G(\theta_i)(x). \end{aligned}$$

The type of O_i is the G^F -conjugacy class of $(L_i, O_i^{L_i})$ where $O_i^{L_i}$ is an F -stable nilpotent orbit of \mathfrak{l}_i that corresponds to φ_i via Springer's correspondence. Using Theorem 6.6.1 we may proceed as above to get

$$\begin{aligned} &\left\langle \Theta \otimes \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\sigma}_1}}) \otimes \cdots \otimes \mathcal{F}^{\mathfrak{g}}(\mathbf{X}_{IC_{\overline{\sigma}_k}}), 1 \right\rangle_{\mathfrak{g}^F} \\ &= |G^F|^{-1} \sum_{(w_1, \dots, w_k) \in W_{L_1} \times \cdots \times W_{L_k}} \left(\prod_{i=1}^k \epsilon_{G \in L_i} q^{\frac{1}{2} \dim O_i} |W_{L_i}|^{-1} \tilde{\varphi}_i(w_i F) \right) \sum_{x \in \mathfrak{g}^F} q^{gn^2 + g \dim C_G(x)} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(x) \\ &= q^{gn^2 - n^2 + \frac{1}{2} \sum_i \dim O_i} \sum_{(w_1, \dots, w_k) \in W_{L_1} \times \cdots \times W_{L_k}} \left(\prod_{i=1}^k \epsilon_{G \in L_i} |W_{L_i}|^{-1} \tilde{\varphi}_i(w_i F) \right) \sum_{x \in \mathfrak{g}^F} q^{g \dim C_G(x)} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(x). \end{aligned}$$

Since $d_\omega/2 = gn^2 - n^2 + 1 + \frac{1}{2} \sum_i \dim O_i$, we need to see that:

$$(q-1) \sum_{x \in \mathfrak{g}^F} q^{g \dim C_G(x)} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(x) = q \sum_{x \in G^F} q^{g \dim C_G(x)} \prod_{i=1}^k R_{T_{w_i}}^G(\theta_i)(x).$$

Since the functions $R_{T_{w_i}}^G(\theta_i)$ and $R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)$ are constant respectively on conjugacy classes and adjoint orbits, we need to verify that for a given type $\omega \in \mathbf{T}_n$:

$$(q-1) \sum_{x \sim \omega} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(x) = q \sum_{x \sim \omega} \prod_{i=1}^k R_{T_{w_i}}^G(\theta_i)(x). \quad (6.8.1)$$

where $x \sim \omega$ means that the G -conjugacy class of x is of type ω . Let (M, C) with M an F -stable Levi subgroup and C an F -stable nilpotent orbit of \mathfrak{m} such that the G^F -conjugacy class of (M, C) corresponds to ω as in §6.7. Recall that $x \in \mathfrak{g}^F$ is of type (M, C) if there exists y in the G^F -orbit of x such that $M = C_G(y_s)$ and $y_n \in C^F$. Similarly, an element $x \in G^F$ is of type (M, C) if there exists y in the G^F -orbit of x such that $M = C_G(y_s)$ and $y_u - 1 \in C^F$ where y_s , resp. y_u , is the semisimple part, resp. the unipotent part, of y . Put

$$(Z_M)_{\text{reg}} = \{x \in M^F \mid C_G(y) = M\},$$

$$(z_m)_{\text{reg}} = \{x \in m^F \mid C_G(x) = M\}.$$

Then the proof of Formula (6.8.1) reduces to the proof of the following identity:

$$(q-1) \sum_{z \in (z_m)_{\text{reg}}} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(z+v) = q \sum_{z \in (Z_M)_{\text{reg}}} \prod_{i=1}^k R_{T_{w_i}}^G(\theta_i)(zu)$$

where v is a fixed element in C^F and $u = v + 1$. By [8][24] we have:

$$R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(z+v) = |M^F|^{-1} \sum_{\{h \in G^F \mid z \in hT_{w_i}h^{-1}\}} Q_{hT_{w_i}h^{-1}}^M(u) \eta_i(h^{-1}zh),$$

$$R_{T_{w_i}}^G(\theta_i)(zu) = |M^F|^{-1} \sum_{\{h \in G^F \mid z \in hT_{w_i}h^{-1}\}} Q_{hT_{w_i}h^{-1}}^M(u) \theta_i(h^{-1}zh),$$

where Q_T^M denotes the Green function of Deligne-Lusztig [8]. Since $C_G(z) = M$, we have $\{h \in G^F \mid z \in hT_{w_i}h^{-1}\} = \{h \in G^F \mid hT_{w_i}h^{-1} \subset M\}$. We thus have:

$$\sum_{z \in (z_m)_{\text{reg}}} \prod_{i=1}^k R_{T_{w_i}}^{\mathfrak{g}}(\eta_i)(z+v) = \sum_{h_1, \dots, h_k} \left(\prod_{i=1}^k |M^F|^{-1} Q_{h_i T_{w_i} h_i^{-1}}^M(u) \right) \sum_{z \in (z_m)_{\text{reg}}} \prod_{i=1}^k \eta_i(h_i^{-1}zh_i)$$

where the first sum runs over the set $\prod_{i=1}^k \{h \in G^F \mid hT_{w_i}h^{-1} \subset M\}$. Similarly we have

$$\sum_{z \in (Z_M)_{\text{reg}}} \prod_{i=1}^k R_{T_{w_i}}^G(\theta_i)(zu) = \sum_{h_1, \dots, h_k} \left(\prod_{i=1}^k |M^F|^{-1} Q_{h_i T_{w_i} h_i^{-1}}^M(u) \right) \sum_{z \in (Z_M)_{\text{reg}}} \prod_{i=1}^k \theta_i(h_i^{-1}zh_i).$$

The inclusion $h_i T_{w_i} h_i^{-1} \subset M$ implies that $Z_M \subset h_i T_{w_i} h_i^{-1} \subset h_i L_i h_i^{-1}$. By Lemma 6.7.3, the character $(\prod_{i=1}^k \eta_i)_{|z_m}$ is a generic character of z_m and so by the Lie algebra version of [13, Proposition 4.2.1], see proof of [13, Lemma 6.2.3] we have

$$\sum_{z \in (z_m)_{\text{reg}}} \prod_{i=1}^k \eta_i(h_i^{-1}zh_i) = qK_{\omega}^o$$

for some constant K_{ω}^o which depends only on the G^F -conjugacy class of M . Similarly, by [13, Proposition 4.2.1] we have

$$\sum_{z \in (Z_M)_{\text{reg}}} \prod_{i=1}^k \theta_i(h_i^{-1}zh_i) = (q-1)K_{\omega}^o.$$

□

6.9 Multiplicities and symmetric functions

6.9.1 Definitions

Consider k separate sets $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ of infinitely many variables and denote by $\Lambda := \mathbb{Q}(q) \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}_1) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}_k)$ the ring of functions separately symmetric in each set $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ with coefficients in $\mathbb{Q}(q)$ where q is an indeterminate. On $\Lambda(\mathbf{x}_i)$ consider the Hall pairing $\langle \cdot, \cdot \rangle_i$ that makes the set $\{m_{\lambda}(\mathbf{x}_i)\}_{\lambda \in \mathcal{P}}$ of monomial symmetric functions and the set $\{h_{\lambda}(\mathbf{x}_i)\}_{\lambda \in \mathcal{P}}$ of complete symmetric functions dual bases. On Λ , put $\langle \cdot, \cdot \rangle = \prod_i \langle \cdot, \cdot \rangle_i$. For an infinite set of variable \mathbf{x} , the transformed Hall-Littlewood symmetric function $\tilde{H}_{\lambda}(\mathbf{x}, q) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(q)$ is defined as

$$\tilde{H}_{\lambda}(\mathbf{x}, q) := \sum_{\lambda} \tilde{K}_{\nu, \lambda}(q) s_{\nu}(\mathbf{x})$$

where $\tilde{K}_{\nu, \lambda}(q) = q^{n(\lambda)} K_{\nu, \lambda}(q^{-1})$ is the transformed Kostka polynomial [34, III (7.11)].

For a partition λ , put

$$\mathcal{H}_\lambda(q) := \frac{q^{g(\lambda, \lambda)}}{a_\lambda(q)}$$

where $a_\lambda(q)$ denotes the cardinality of the centralizer of a unipotent element of $\mathrm{GL}_n(\mathbb{F}_q)$ with Jordan form of type λ [34, IV, (2.7)]. Define the k -points Cauchy function

$$\Omega(q) := \sum_{\lambda \in \mathcal{P}} \left(\prod_{i=1}^n \tilde{H}_\lambda(\mathbf{x}_i, q) \right) \mathcal{H}_\lambda(q).$$

Given a family of symmetric functions $u_\lambda(\mathbf{x}, q) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(q)$ indexed by partitions, we extend its definition to a type $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r) \in \mathbf{T}_n$ by $u_\omega(\mathbf{x}, q) := \prod_{i=1}^r u_{\omega^i}(\mathbf{x}^{d_i}, q^{d_i})$ where $\mathbf{x}^d := \{x_1^d, x_2^d, \dots\}$. For a multitype $\omega = (\omega_1, \dots, \omega_k) \in (\mathbf{T}_n)^k$, put $u_\omega := u_{\omega_1}(\mathbf{x}_1, q) \cdots u_{\omega_k}(\mathbf{x}_k, q) \in \Lambda$.

Recall that λ' denotes the dual partition of λ . For a type $\omega = (d_1, \lambda_1) \cdots (d_r, \lambda_r)$, we denote by ω' the type $(d_1, \lambda'_1) \cdots (d_r, \lambda'_r)$.

Let $\omega = (\omega_1, \dots, \omega_k) \in (\mathbf{T}_n)^k$ with $\omega_i = (d_i^1, \omega_i^1) \cdots (d_i^{r_i}, \omega_i^{r_i})$ and define

$$\mathbb{H}_\omega(q) := (-1)^{r(\omega)} (q-1) \langle s_{\omega'}, \mathrm{Log}(\Omega(q)) \rangle \quad (6.9.1)$$

where $r(\omega) := kn + \sum_{i,j} |\omega_i^j|$, and where Log is as in [15] [13, 2.3].

Note that if the degrees d_i^j are all equal to 1, then $r(\omega) = 2kn$.

We rewrite Formula (6.9.1) in some special cases:

6.9.2 The split semisimple case

We say that $\omega \in \mathbf{T}_n$ is a *semisimple type* if it is the type of a semisimple adjoint orbit of \mathfrak{g}_n^F (or equivalently the type of a semisimple character of G^F). It is then of the form $(d_1, (1^{n_1})) \cdots (d_r, (1^{n_r}))$. If moreover ω is split, i.e., $d_i = 1$ for all i , then $\lambda = (n_1, \dots, n_r)$ is a partition of n and any partition of n is obtained in this way from a unique split semisimple type of \mathbf{T}_n . Note that for a split semisimple type ω with corresponding partition λ , we have $s_{\omega'}(\mathbf{x}) = h_\lambda(\mathbf{x})$.

For a multipartition $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathcal{P}_n)^k$ with corresponding split semisimple multitype $\omega \in (\mathbf{T}_n)^k$ we put $\mathbb{H}_\lambda^{ss}(q) := \mathbb{H}_\omega(q)$. Then Formula (6.9.1) reads

$$\mathbb{H}_\lambda^{ss}(q) = (q-1) \langle h_\lambda, \mathrm{Log}(\Omega(q)) \rangle.$$

Since $\{h_\lambda\}$ and $\{m_\lambda\}$ are dual bases with respect to the Hall pairing, we may recover $\Omega(q)$ from $\mathbb{H}_\lambda^{ss}(q)$ by the formula

$$\Omega(q) = \mathrm{Exp} \left(\sum_{\lambda \in (\mathcal{P}_n)^k} \frac{\mathbb{H}_\lambda^{ss}(q)}{q-1} m_\lambda \right) \quad (6.9.2)$$

where Exp is the plethystic exponential which is the inverse of Log (see [15] [13] for more details).

6.9.3 The nilpotent case

We say that a type $\omega \in \mathbf{T}_n$ is nilpotent if it is the type of a nilpotent adjoint orbit of \mathfrak{g}^F (or the type of a unipotent character of G^F) in which case it is of the form $\omega = (1, \lambda)$ for some partition λ of n , and $s_\omega(\mathbf{x}) = s_\lambda(\mathbf{x})$.

For a multipartition $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathcal{P}_n)^k$, we put $\mathbb{H}_\lambda^n(q) := \mathbb{H}_\omega(q)$, where $\omega = ((1, \lambda_1), \dots, (1, \lambda_k))$.

Since the base $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ is auto-dual, we recover $\Omega(q)$ from the $\mathbb{H}_\lambda^n(q)$ by the formula

$$\Omega(q) = \mathrm{Exp} \left(\sum_{\lambda \in (\mathcal{P}_n)^k} \frac{\mathbb{H}_\lambda^n(q)}{q-1} s_\lambda \right). \quad (6.9.3)$$

6.9.4 The regular semisimple case

We say that a type $\omega \in \mathbf{T}_n$ is semisimple regular if it is the type of a semisimple regular adjoint orbit of G^F (or the type of an irreducible Deligne-Lusztig character, see §6.5). Then it is of the form $\omega = (d_1, 1) \cdots (d_r, 1)$ and so $\lambda = (d_1, \dots, d_r)$ is a partition of n . In this case, the fonction $s_\omega(\mathbf{x})$ is the power symmetric function $p_\lambda(\mathbf{x})$.

For a multipartition λ with corresponding regular semisimple multitype ω , we use the notation $\mathbb{H}_\lambda^{r,ss}(q)$ and $r(\lambda)$ instead of $\mathbb{H}_\omega(q)$ and $r(\omega)$.

Recall that for any two partitions λ, μ , we have $\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x}) \rangle = z_\lambda \delta_{\lambda\mu}$.

Then we recover $\Omega(q)$ from $\mathbb{H}_\lambda^{r,ss}(q)$ by the formula

$$\Omega(q) = \text{Exp} \left(\sum_{\lambda \in (\mathcal{P})^k} \frac{(-1)^{r(\lambda)} \mathbb{H}_\lambda^{r,ss}(q)}{(q-1)z_\lambda} p_\lambda \right). \quad (6.9.4)$$

6.9.5 Multiplicities

Let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be a generic tuple of irreducible characters of G^F of type $\omega = (\omega_1, \dots, \omega_k) \in (\mathbf{T}_n)^k$ and put

$$R_\omega := \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k.$$

Theorem 6.9.1. *We have*

$$\langle \Lambda \otimes R_\omega, 1 \rangle_{G^F} = \mathbb{H}_\omega(q).$$

If the irreducible characters $\mathcal{X}_1, \dots, \mathcal{X}_k$ are all split semisimple with corresponding multipartition $\mu \in (\mathcal{P}_n)^k$, then $\mathbb{H}_\omega(q) = (q-1) \langle h_\mu, \text{Log}(\Omega(q)) \rangle$ by §6.9.2. Hence in the split semisimple case, this theorem is exactly [13, Theorem 7.1.1].

Since the main ingredient [13, §4.3] in the proof of [13, Theorem 7.1.1] is available for any type $\omega \in \mathbf{T}_n$, we may follow line by line the proof of [13, Theorem 7.1.1] for arbitrary types (not necessarily split semisimple) to obtain the formula of Theorem 6.9.1.

Remark 6.9.2. The theorem shows that the multiplicities of generic irreducible characters depend only on the types and not on the choices of irreducible characters of a given type.

7 Poincaré polynomials of quiver varieties and multiplicities

Unless specified \mathbb{K} is an arbitrary algebraically closed field.

7.1 Decomposition theorem and Weyl group action

We keep the notation and assumptions of §5.3.

Let $\pi : \mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow \mathcal{V}_{\tilde{\omega}}$ and $f : \mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow \mathcal{O}_{\tilde{\omega}}$ be the canonical projective maps.

By §6.4, we have

$$f_* \left(\underline{IC}_{\mathbb{O}_{\mathbf{L}, \mathbf{P}, \Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathcal{O}_{\tilde{\omega}}}^\bullet \oplus \left(\bigoplus_{\tilde{\tau} \triangleleft \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \underline{IC}_{\mathcal{O}_{\tilde{\tau}}}^\bullet \right)$$

for some explicit representations $\{A_{\tilde{\tau}}\}_{\tilde{\tau}}$ of the relative Weyl group $W_{\mathbf{M}}(\mathbf{L}, \mathbf{C}) = W_{M_1}(L_1, C_1) \times \cdots \times W_{M_k}(L_k, C_k)$.

With this notation we have

$$A_{\tilde{\tau}} = \text{Hom}_{W_{\mathbf{M}}} \left(\text{Ind}_{W_{\tilde{\mathbf{L}}}}^{W_{\mathbf{M}}}(V_{\mathbf{C}}), V_{\chi^{\tilde{\tau}}} \right)$$

where $V_C = \bigotimes_i V_{C_i}$ and $\chi^{\tilde{\tau}}$ is the irreducible character of W_M associated to $\tilde{\tau}$ via the correspondence $\lambda \mapsto \chi^\lambda$.

Theorem 7.1.1. *We have*

$$\pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathcal{V}_{\tilde{\omega}}}^\bullet \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \underline{IC}_{\mathcal{V}_{\tilde{\tau}}}^\bullet \right). \quad (7.1.1)$$

If $\mathbb{K} = \mathbb{C}$, let $\rho : \mathbb{Q}_{L,P,\Sigma} \rightarrow \mathcal{Q}_{\tilde{\omega}}$, then

$$\rho_* \left(\underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathcal{Q}_{\tilde{\omega}}}^\bullet \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \underline{IC}_{\mathcal{Q}_{\tilde{\tau}}}^\bullet \right). \quad (7.1.2)$$

By the above theorem we thus have an action of $W_M(\mathbf{L}, \mathbf{C})$ on the complex $\rho_* \left(\underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet \right)$. This action induces an action on the hypercohomology $\mathbb{H}_c^i(\mathbb{Q}_{L,P,\Sigma}, \underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet) = IH_c^i(\mathbb{Q}_{L,P,\Sigma}, \kappa)$. For $v \in W_M(\mathbf{L}, \mathbf{C})$, we denote by $\theta_v : \rho_* \left(\underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet \right) \simeq \rho_* \left(\underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet \right)$ the corresponding automorphism.

Proof. By applying the proper base change to the top square of the diagram (5.3.2) and Theorem 5.4.1 we find that

$$\pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathcal{V}_{\tilde{\omega}}}^\bullet \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \underline{IC}_{\mathcal{V}_{\tilde{\tau}}}^\bullet \right).$$

Assume that $\mathbb{K} = \mathbb{C}$. Since the quotient map $p_1 : \mathbb{V}_{L,P,\Sigma} \rightarrow \mathbb{Q}_{L,P,\Sigma}$ and $p_2 : \mathcal{V}_{\tilde{\tau}} \rightarrow \mathcal{Q}_{\tilde{\tau}}$ are smooth (they are principal PGL_n -bundle), we have $(p_2)^* \left(\underline{IC}_{\mathcal{Q}_{\tilde{\tau}}}^\bullet \right) \simeq \underline{IC}_{\mathcal{V}_{\tilde{\tau}}}^\bullet$ and $(p_1)^* \left(\underline{IC}_{\mathbb{Q}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet$. Applying Theorem 5.3.7 and the base change theorem to the bottom square of diagram (5.3.2) we find the required decomposition for ρ . \square

Recall that $d_{\tilde{\tau}}$ denotes the dimension of $\mathcal{Q}_{\tilde{\tau}}$. Let $r_{\tilde{\tau}} = (d_{\tilde{\tau}} - d_{\tilde{\omega}})/2$. By §6.4 we have:

Proposition 7.1.2. *If $\mathbb{K} = \overline{\mathbb{F}}_q$ and if $(\mathbf{L}, \mathbf{P}, \Sigma)$ is defined over \mathbb{F}_q , then*

$$\pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathcal{V}_{\tilde{\omega}}}^\bullet \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \underline{IC}_{\mathcal{V}_{\tilde{\tau}}}^\bullet(r_{\tilde{\tau}}) \right)$$

is defined over \mathbb{F}_q . In particular for $v \in W_M(\mathbf{L}, \mathbf{C})$, we have

$$\mathbf{X}_{\pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right)_{\theta_v \circ \tilde{\varphi}}} = \mathbf{X}_{\underline{IC}_{\mathcal{V}_{\tilde{\omega}}}^\bullet} + \sum_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} \mathrm{Tr}(v|A_{\tilde{\tau}}) q^{-r_{\tilde{\tau}}} \mathbf{X}_{\underline{IC}_{\mathcal{V}_{\tilde{\tau}}}^\bullet} \quad (7.1.3)$$

where $\tilde{\varphi} : F^* \left(\pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right) \right) \simeq \pi_* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right)$ is the canonical isomorphism induced by the unique isomorphism $\varphi : F^* \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right) \simeq \underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet$ which induces the identity on $\mathcal{H}_x^{-d_{\tilde{\omega}}} \left(\underline{IC}_{\mathbb{V}_{L,P,\Sigma}}^\bullet \right)$ with $x \in \mathbb{V}_{L,P,\Sigma}^o(\mathbb{F}_q)$.

We can proceed as in [11] to prove the following proposition from the mixed Hodge module version of the isomorphisms (7.1.1) and (7.1.2).

Proposition 7.1.3. *Assume $\mathbb{K} = \mathbb{C}$. Then*

$$IH_c^i(\mathbb{V}_{L,P,\Sigma}, \mathbb{Q}) \simeq IH_c^i(\mathcal{V}_{\tilde{\omega}}, \mathbb{Q}) \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \left(IH_c^{i+2r_{\tilde{\tau}}}(\mathcal{V}_{\tilde{\tau}}, \mathbb{Q}) \otimes \mathbb{Q}(r_{\tilde{\tau}}) \right) \right), \quad (7.1.4)$$

$$IH_c^i(\mathbb{Q}_{L,P,\Sigma}, \mathbb{Q}) \simeq IH_c^i(\mathcal{Q}_{\tilde{\omega}}, \mathbb{Q}) \oplus \left(\bigoplus_{\tilde{\tau} \triangleq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} A_{\tilde{\tau}} \otimes \left(IH_c^{i+2r_{\tilde{\tau}}}(\mathcal{Q}_{\tilde{\tau}}, \mathbb{Q}) \otimes \mathbb{Q}(r_{\tilde{\tau}}) \right) \right) \quad (7.1.5)$$

are isomorphisms of mixed Hodge structures.

7.2 A lemma

Assume that $\mathbb{K} = \overline{\mathbb{F}}_q$.

Let (O_1^F, \dots, O_k^F) be a generic tuple of adjoint orbits of \mathfrak{gl}_n^F of type $\omega \in (\mathbf{T}_n)^k$. The tuple (O_1, \dots, O_k) is of type $\tilde{\omega}$ (see Remark 6.7.1). However, the eigenvalues of O_i may not be in \mathbb{F}_q . To record this we prefer the notation \mathcal{V}_ω rather than $\mathcal{V}_{\tilde{\omega}}$. We denote again by F the Frobenius endomorphism on \mathcal{V}_ω .

Lemma 7.2.1. *We have*

$$|\mathrm{PGL}_n(\mathbb{F}_q)| \cdot \sum_{x \in \mathcal{Q}_\omega^F} \mathbf{X}_{IC_{\mathcal{Q}_\omega}^\bullet}(x) = \sum_{x \in \mathcal{V}_\omega^F} \mathbf{X}_{IC_{\mathcal{V}_\omega}^\bullet}(x) = \left\langle \Theta \otimes \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_1}^\bullet}) \otimes \cdots \otimes \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_k}^\bullet}), 1 \right\rangle_{\mathfrak{gl}_n^F}.$$

Proof. Let $f : \mathcal{V}_\omega \rightarrow \mathcal{Q}_\omega$ be the quotient map. Since $\mathrm{PGL}_n(\mathbb{F}_q)$ acts freely on \mathcal{V}_ω it induces an injective map $\mathcal{V}_\omega^F / \mathrm{PGL}_n(\mathbb{F}_q) \rightarrow \mathcal{Q}_\omega^F$. Since $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ is connected, any F -stable orbit of \mathcal{V}_ω has a rational point. Hence the above map is also surjective. As f is a principal PGL_n -bundle we have $f^*(IC_{\mathcal{Q}_\omega}^\bullet) \simeq IC_{\mathcal{V}_\omega}^\bullet$ and so $\mathbf{X}_{IC_{\mathcal{V}_\omega}^\bullet}(x) = \mathbf{X}_{IC_{\mathcal{Q}_\omega}^\bullet}(y)$ whenever $f(x) = y$. We thus deduce the first equality.

By Theorem 5.4.3 we have $IC_{\mathcal{V}_\omega}^\bullet = i^* IC_{\mathcal{O}_\omega}^\bullet = i^*(\kappa^{\boxtimes 2g} \boxtimes IC_{\mathcal{O}_1}^\bullet \boxtimes \cdots \boxtimes IC_{\mathcal{O}_k}^\bullet)$ where κ is the constant sheaf on GL_n and $\kappa^{\boxtimes 2g} := \kappa \boxtimes \cdots \boxtimes \kappa$ ($2g$ times). Hence for $x = (a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_k) \in \mathcal{V}_\omega^F$, we have

$$\mathbf{X}_{IC_{\mathcal{V}_\omega}^\bullet}(x) = \mathbf{X}_{IC_{\mathcal{O}_1}^\bullet}(x_1) \cdots \mathbf{X}_{IC_{\mathcal{O}_k}^\bullet}(x_k).$$

For $z \in \mathfrak{gl}_n^F$, put

$$\Xi(z) := \# \left\{ (a_1, b_1, \dots, a_g, b_g) \in (\mathfrak{gl}_n^F)^{2g} \mid \sum_i [a_i, b_i] = z \right\}.$$

Hence

$$\begin{aligned} \sum_{x \in \mathcal{V}_\omega^F} \mathbf{X}_{IC_{\mathcal{V}_\omega}^\bullet}(x) &= \sum_{(x_1, \dots, x_k) \in \overline{\mathcal{O}}_1 \times \cdots \times \overline{\mathcal{O}}_k} \Xi(-(x_1 + \cdots + x_k)) \mathbf{X}_{IC_{\mathcal{O}_1}^\bullet}(x_1) \cdots \mathbf{X}_{IC_{\mathcal{O}_k}^\bullet}(x_k) \\ &= (\Xi * \mathbf{X}_{IC_{\mathcal{O}_1}^\bullet} * \cdots * \mathbf{X}_{IC_{\mathcal{O}_k}^\bullet})(0). \end{aligned}$$

By [13, Proposition 3.2.1], we have

$$|\mathfrak{gl}_n^F| \cdot f(0) = \sum_{x \in \mathfrak{gl}_n^F} \mathcal{F}^{\mathfrak{gl}_n}(f)(x)$$

for any $f \in \mathrm{Fun}(\mathfrak{gl}_n^F)$. Since for any two functions $f, g \in \mathrm{Fun}(\mathfrak{gl}_n^F)$, we have $\mathcal{F}^{\mathfrak{gl}_n}(f * g) = \mathcal{F}^{\mathfrak{gl}_n}(f) \cdot \mathcal{F}^{\mathfrak{gl}_n}(g)$, we deduce that

$$\begin{aligned} \sum_{x \in \mathcal{V}_\omega^F} \mathbf{X}_{IC_{\mathcal{V}_\omega}^\bullet}(x) &= |\mathfrak{gl}_n^F|^{-1} \sum_{x \in \mathfrak{gl}_n^F} \mathcal{F}^{\mathfrak{gl}_n}(\Xi)(x) \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_1}^\bullet})(x) \cdots \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_k}^\bullet})(x) \\ &= \left\langle \mathcal{F}^{\mathfrak{gl}_n}(\Xi) \otimes \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_1}^\bullet}) \otimes \cdots \otimes \mathcal{F}^{\mathfrak{gl}_n}(\mathbf{X}_{IC_{\mathcal{O}_k}^\bullet}), 1 \right\rangle_{\mathfrak{gl}_n^F} \end{aligned}$$

We conclude by noticing that $\mathcal{F}^{\mathfrak{gl}_n}(\Xi) = \Theta$, see [13, Proposition 3.2.2]. \square

Proposition 7.2.2. *Assume that Σ is a reduced to a point and that $(\mathbf{L}, \mathbf{P}, \Sigma)$ is defined over \mathbb{F}_q . The variety $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is polynomial count. Moreover, the number of F -stable PGL_n -orbits of $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ equals*

$$\#\{\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}(\mathbb{F}_q) / \mathrm{PGL}_n(\mathbb{F}_q)\} = \frac{|\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}(\mathbb{F}_q)|}{|\mathrm{PGL}_n(\mathbb{F}_q)|}.$$

Proof. The second assertion follows from the fact that PGL_n is connected and acts freely on $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$, see beginning of the proof of Lemma 7.2.1. The variety $\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is nonsingular by Theorem 5.3.5. By Formula (7.1.3) applied with $\nu = 1$, we have

$$\mathbf{X}_{\pi_\ast(\kappa)} = \mathbf{X}_{IC_{V_{\tilde{\omega}}}} + \sum_{\tilde{\tau} \leq \tilde{\omega}, \tilde{\tau} \neq \tilde{\omega}} (\dim A_{\tilde{\tau}}) q^{-r_{\tilde{\tau}}} \mathbf{X}_{IC_{V_{\tilde{\tau}}}}. \quad (7.2.1)$$

By Grothendieck trace formula we have

$$\sum_{x \in \mathcal{V}_{\tilde{\omega}}^F} \mathbf{X}_{\pi_\ast(\kappa)}(x) = \#\{\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}(\mathbb{F}_q)\}.$$

By Lemma 7.2.1, Theorem 6.8.1 and Theorem 6.9.1, we see that there exists a rational function $Q \in \mathbb{Q}(T)$ such that for any $r \in \mathbb{Z}_{>0}$

$$\sum_{x \in \mathcal{V}_{V_{\tilde{\tau}}}^{F^r}} \mathbf{X}_{IC_{V_{\tilde{\tau}}}}(x) = Q(q^r).$$

By integrating Formula (7.2.1) over $\mathcal{V}_{\tilde{\omega}}^F$, we deduce that

$$\#\{\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}(\mathbb{F}_{q^r})\} = P(q^r)$$

for some $P \in \mathbb{Q}(T)$. Since $P(q^r)$ is an integer for all $r \in \mathbb{Z}_{>0}$, the rational function P must be a polynomial with rational coefficients. \square

7.3 The split case

Here we assume that $\mathbb{K} = \mathbb{C}$. Let (O_1, \dots, O_k) be a generic tuple of adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$ of type $\tilde{\omega}$. As in [13, Appendix 7.1], we may define a finitely generated ring extension R of \mathbb{Z} and a k -tuple of R -schemes $(\mathfrak{D}_1, \dots, \mathfrak{D}_k)$ such that \mathfrak{D}_i is a spreading out of O_i and such that for any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$ into a finite field \mathbb{F}_q , the tuple $(\mathfrak{D}_1^\varphi(\mathbb{F}_q), \dots, \mathfrak{D}_k^\varphi(\mathbb{F}_q))$ is a generic tuple of adjoint orbits of $\mathfrak{gl}_n(\mathbb{F}_q)$ of type $\tilde{\omega}$. We denote by $\mathfrak{Q}_{\tilde{\omega}}^\varphi$ the \mathbb{F}_q -schemes obtained from $\mathfrak{Q}_{\tilde{\omega}}$ by base change. We first prove the following theorem.

Theorem 7.3.1. *The cohomology group $IH_c^i(\mathfrak{Q}_{\tilde{\omega}}, \mathbb{C})$ vanishes if i is odd. For any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$ we have*

$$P_c(\mathfrak{Q}_{\tilde{\omega}}, q) = \sum_{x \in \mathfrak{Q}_{\tilde{\omega}}^\varphi(\mathbb{F}_q)} \mathbf{X}_{IC_{\mathfrak{Q}_{\tilde{\omega}}^\varphi(\mathbb{F}_q)}}(x)$$

where $P_c(X, q) := \sum_i \dim(IH_c^{2i}(X, \mathbb{C})) q^i$.

By Remark 3.3.3, Theorem 7.3.1 is a consequence of the following two results.

Theorem 7.3.2. *The variety $\mathfrak{Q}_{\tilde{\omega}}$ is pure.*

Proof. Let ξ and \mathbf{v} be as in Proposition 5.2.4 and let θ be generic with respect to \mathbf{v} . Since the natural projective map $\mathfrak{M}_{\xi, \theta}(\mathbf{v}) \rightarrow \mathfrak{Q}_{\tilde{\omega}}$ is a resolution of singularities, the group $IH_c^i(\mathfrak{Q}_{\tilde{\omega}}, \mathbb{C})$ is a direct summand of $H_c^i(\mathfrak{M}_{\xi, \theta}(\mathbf{v}), \mathbb{C})$ as a mixed Hodge structure. It is thus enough to prove that $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is pure. But the proof of the purity of $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ is completely similar to the proof of [13, Proposition 2.2.6]. \square

Theorem 7.3.3. *There exists a polynomial $P(t) \in \mathbb{Z}[t]$ such that for any ring homomorphism $\varphi : R \rightarrow \mathbb{F}_q$*

$$\sum_{x \in \mathfrak{Q}_{\tilde{\omega}}^\varphi(\mathbb{F}_q)} \mathbf{X}_{IC_{\mathfrak{Q}_{\tilde{\omega}}^\varphi(\mathbb{F}_q)}}(x) = P(q).$$

Moreover,

$$E^{ic}(\mathfrak{Q}_{\tilde{\omega}}; x, y) = P(xy).$$

Proof. By §7.1 and Proposition 7.2.2, the variety $\mathcal{V}_{\tilde{\omega}}$ satisfies the condition of Theorem 3.3.2. Hence

$$E^{ic}(\mathcal{V}_{\tilde{\omega}}; \sqrt{q}, \sqrt{q}) = \sum_{x \in \mathfrak{B}_{\tilde{\omega}}^{\varphi}(\mathbb{F}_q)} \mathbf{X}_{IC_{\mathfrak{B}_{\tilde{\omega}}^{\varphi}(\mathbb{F}_q)}}(x) \quad (7.3.1)$$

where $\mathfrak{B}_{\tilde{\omega}}^{\varphi}$ is the \mathbb{F}_q -scheme obtained from $\mathcal{V}_{\tilde{\omega}}$ by base change.

If the adjoint orbits O_1, \dots, O_k are semisimple, then the right hand side reads $\#\{\mathfrak{B}_{\tilde{\omega}}^{\varphi}(\mathbb{F}_q)\}$ and so $\mathcal{V}_{\tilde{\omega}}$ is polynomial count. Hence $Q_{\tilde{\omega}}$ is also polynomial count and so we apply Theorem 3.3.1 to see that Theorem 7.3.3 holds when O_1, \dots, O_k are semisimple.

By Proposition 7.2.2, the variety $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is also polynomial count when Σ is reduced to a point.

Hence using the decomposition (7.1.5), we can easily see by induction on $\tilde{\tau} \trianglelefteq \tilde{\omega}$ that the E-polynomial $E^{ic}(Q_{\tilde{\omega}}; x, y)$ depends only on the product xy (see proof of Theorem 3.3.2), i.e., there exists a polynomial $P \in \mathbb{Z}[T]$ such that $E^{ic}(Q_{\tilde{\omega}}; x, y) = P(xy)$.

It remains to prove that $P(q) = E^{ic}(Q_{\tilde{\omega}}; \sqrt{q}, \sqrt{q})$ satisfies the first assertion of Theorem 7.3.3.

By the first equality of Lemma 7.2.1 and (7.3.1), we are reduced to prove that

$$E^{ic}(Q_{\tilde{\omega}}; \sqrt{q}, \sqrt{q}) = \frac{1}{|\mathrm{PGL}_n(\mathbb{F}_q)|} E^{ic}(\mathcal{V}_{\tilde{\omega}}; \sqrt{q}, \sqrt{q}). \quad (7.3.2)$$

If O_1, \dots, O_k are semisimple then both $Q_{\tilde{\omega}}$ and $\mathcal{V}_{\tilde{\omega}}$ are nonsingular and so by Theorem 3.3.1 this identity reads

$$\#\{\mathfrak{S}_{\tilde{\omega}}^{\varphi}(\mathbb{F}_q)\} = \frac{1}{|\mathrm{PGL}_n(\mathbb{F}_q)|} \#\{\mathfrak{B}_{\tilde{\omega}}^{\varphi}(\mathbb{F}_q)\}$$

and so it is true in this case. By the same argument we also have

$$E(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; \sqrt{q}, \sqrt{q}) = \frac{1}{|\mathrm{PGL}_n(\mathbb{F}_q)|} E(\mathbb{V}_{\mathbf{L}, \mathbf{P}, \Sigma}; \sqrt{q}, \sqrt{q})$$

when Σ is reduced to a point by Theorem 3.3.1 and Proposition 7.2.2. Hence the identity (7.3.2) can be proved by induction on $\tilde{\tau} \trianglelefteq \tilde{\omega}$ using both (7.1.4) and (7.1.5). □

We now have the following relation between multiplicities and the betti numbers of quiver varieties.

Theorem 7.3.4. *We have*

$$P_c(Q_{\tilde{\omega}}, q) = q^{d_{\tilde{\omega}}/2} \langle \Lambda \otimes R_{\tilde{\omega}}, 1 \rangle$$

where in the notation $R_{\tilde{\omega}}$ we read $\tilde{\omega} = \omega^1 \cdots \omega^r \in \tilde{\mathbf{T}}$ as $(1, \omega^1) \cdots (1, \omega^r) \in \mathbf{T}$.

Proof. This is a consequence of Theorem 7.3.1, Lemma 7.2.1 and Theorem 6.8.1. □

By Theorem 6.9.1 we have:

Corollary 7.3.5.

$$P_c(Q_{\tilde{\omega}}, q) = q^{d_{\tilde{\omega}}/2} \mathbb{H}_{\tilde{\omega}}(q).$$

7.4 The general case

We keep the notation of §5.3 and §7.1.

7.4.1 Preliminaries

We use the notation of §4.1.

Let P be a parabolic subgroup of GL_n , L a Levi factor of P and let $\Sigma = \sigma + C$ where C is a nilpotent orbit of the Lie algebra \mathfrak{l} of L and where σ is in the center $z_{\mathfrak{l}}$. We may assume that $L = \prod_{j=1}^r (\mathrm{GL}_{n_{j,1}} \times \cdots \times \mathrm{GL}_{n_{j,s_j}})$ so that $M := C_{\mathrm{GL}_n}(\sigma) = \prod_{i=1}^r \mathrm{GL}_{m_i}$ and $\mathrm{GL}_{n_{j,1}} \times \cdots \times \mathrm{GL}_{n_{j,s_j}} \subset \mathrm{GL}_{m_j}$. Then the nilpotent orbit C may be written as

$$C = \prod_{j=1}^r (C_{j,1} \times \cdots \times C_{j,s_j})$$

with $C_{j,l}$ a nilpotent orbit of $\mathfrak{gl}_{n_{j,l}}$. Let $\tau^{j,l}$ be the partition of $n_{j,l}$ given by the size of the Jordan blocks of $C_{j,l}$. Re-ordering if necessary, we may assume that $\tau^{j,l} \geq \tau^{j,l+1}$ for all j, l . Then put $\tilde{\tau}_j := \tau^{j,1} \tau^{j,2} \cdots \tau^{j,s_j} \in \tilde{\mathbf{T}}_{m_j}$ for each $j = 1, \dots, r$.

Consider the map $\tilde{\mathfrak{F}}_r : \tilde{\mathbf{T}}_{m_1} \times \cdots \times \tilde{\mathbf{T}}_{m_r} \rightarrow \tilde{\mathbf{T}}_n$ where $\tilde{\mathfrak{F}}_r(\tilde{\tau}_1, \dots, \tilde{\tau}_r)$ is defined by re-ordering the partitions in the concatenation of the types $\tilde{\tau}_1, \dots, \tilde{\tau}_r$. Similarly we define $\mathfrak{F}_r : \mathbf{T}_{m_1} \times \cdots \times \mathbf{T}_{m_r} \rightarrow \mathbf{T}_n$.

We denote by $S : \tilde{\mathbf{T}} \rightarrow \mathcal{P}$ the map which assigns to a type $\tilde{\tau} = \tau^1 \cdots \tau^r \in \tilde{\mathbf{T}}$ the partition $\tilde{\tau}^+ = \sum_{i=1}^r \tau^i$.

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathbf{T}_{m_1} \times \cdots \times \mathbf{T}_{m_r} & \xrightarrow{\mathfrak{S}^r} & \tilde{\mathbf{T}}_{m_1} \times \cdots \times \tilde{\mathbf{T}}_{m_r} & \xrightarrow{S^r} & \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_r} \\ \downarrow \mathfrak{F}_r & & \downarrow \tilde{\mathfrak{F}}_r & & \downarrow \tilde{\mathfrak{F}}_r \\ \mathbf{T}_n & \xrightarrow{\mathfrak{S}} & \tilde{\mathbf{T}}_n & & \tilde{\mathbf{T}}_n \end{array}$$

where \mathfrak{S} is defined as in §4.1.

Let \bar{O} be the image of the projection $\mathbb{X}_{L,P,\Sigma} \rightarrow \mathfrak{gl}_n$. The adjoint orbit O is then of type $\tilde{\mathfrak{F}}_r \circ S^r(\tilde{\tau}_1, \dots, \tilde{\tau}_r)$.

We have (cf. §4.1)

$$W_M(L, C) \simeq \prod_{j=1}^r W_{\tilde{\tau}_j}. \quad (7.4.1)$$

The fiber $(\mathfrak{S}^r)^{-1}(\tilde{\tau}_1, \dots, \tilde{\tau}_r)$ is canonically in bijection with the conjugacy classes of $W_M(L, C)$. For $\nu \in (\mathfrak{S}^r)^{-1}(\tilde{\tau}_1, \dots, \tilde{\tau}_r)$, we denote by $\sigma_\nu \in W_M(L, C)$ a representative of the corresponding conjugacy class.

7.4.2 Relations between multiplicities

Let $(\mathbf{L}, \mathbf{P}, \mathbf{\Sigma})$ be given by a k -tuple $((L_1, P_1, \Sigma_1), \dots, (L_k, P_k, \Sigma_k))$ of triples as in §5.3 with $\Sigma_i = \sigma_i + C_i$, and let $M_i = C_{\mathrm{GL}_{n_i}}(\sigma_i)$. Put

$$W_{\mathbf{M}}(\mathbf{L}, \mathbf{C}) := \prod_{i=1}^k W_{M_i}(L_i, C_i).$$

As in Formula (7.4.1), for each $i = 1, \dots, k$, there exist $\tilde{\tau}_{i,1}, \dots, \tilde{\tau}_{i,r_i} \in \tilde{\mathbf{T}}$ (defined from C_i) such that

$$W_{M_i}(L_i, C_i) \simeq \prod_{j=1}^{r_i} W_{\tilde{\tau}_{i,j}}.$$

Then for each $i = 1, \dots, k$, $\tilde{\mathfrak{F}}_{r_i}(\tilde{\tau}_{i,1}^+, \dots, \tilde{\tau}_{i,r_i}^+) \in \tilde{\mathbf{T}}_n$ is the type $\tilde{\omega}_i$ of O_i .

For each $i = 1, \dots, k$, pick $\nu_i \in (\mathfrak{S}^{r_i})^{-1}(\tilde{\tau}_{i,1}, \dots, \tilde{\tau}_{i,r_i}) \subset \mathbf{T}^{r_i}$, let $\tau_i = \mathfrak{F}_{r_i}(\nu_i)$ and put $\boldsymbol{\tau} := (\tau_1, \tau_2, \dots, \tau_k) \in (\mathbf{T}_n)^k$ and $\nu := (\nu_1, \dots, \nu_k)$.

Put $\sigma_\nu := (\sigma_{\nu_1}, \dots, \sigma_{\nu_k}) \in W_{\mathbf{M}}(\mathbf{L}, \mathbf{C})$.

Proposition 7.4.1. *Assume that both $\text{char}(\mathbb{F}_q)$ and q are large enough such that generic k -tuples of irreducible characters of $\text{GL}_n(\mathbb{F}_q)$ exist in the types which occur below. We have*

$$\langle \Lambda \otimes R_\tau, 1 \rangle = \sum_{\tilde{\mu} \leq \tilde{\omega}} \text{Tr}(\sigma_\nu | A_{\tilde{\mu}}) \langle \Lambda \otimes R_{\tilde{\mu}}, 1 \rangle$$

with

$$A_{\tilde{\mu}} = \text{Hom}_{W_M}(\text{Ind}_{W_L}^{W_M}(V_C), V_{\chi^{\tilde{\mu}}}).$$

In the notation $R_{\tilde{\mu}}$, we read $\tilde{\mu} = \mu^1 \mu^2 \cdots \mu^r \in \tilde{\mathbf{T}}$ as $(1, \mu^1) \cdots (1, \mu^r) \in \mathbf{T}$.

Proof. Let $\tilde{\tau} = \overbrace{\tau^1 \cdots \tau^1}^{a_1} \cdots \overbrace{\tau^r \cdots \tau^r}^{a_r} \in \tilde{\mathbf{T}}_m$ (with $\tau^i \neq \tau^j$ for $i \neq j$), $\nu \in \mathfrak{S}^{-1}(\tilde{\tau})$.

By Proposition 6.2.5, we have

$$s_\nu(\mathbf{x}) = \sum_{\mu \leq \tilde{\tau}^+} \text{Tr}(\sigma_\nu | A_{\nu, \mu}) s_\mu(\mathbf{x})$$

where for $\mu \in \mathcal{P}_m$

$$A_{\nu, \mu} := \text{Hom}_{S_m}(\text{Ind}_{S_m}^{S_m}(V_\nu), V_\mu)$$

with $S := (S_{m_1})^{a_1} \times \cdots \times (S_{m_s})^{a_s}$ and $m_i = |\tau^i|$. Since σ_ν is in the same conjugacy class of $W_{\tilde{\tau}'} = W_{\tilde{\tau}}$ as $\sigma_{\nu'}$, we have

$$\begin{aligned} s_{\nu'}(\mathbf{x}) &= \sum_{\mu \leq \tilde{\tau}^+} \text{Tr}(\sigma_\nu | A_{\nu', \mu}) s_\mu(\mathbf{x}) \\ &= \sum_{\mu \leq \tilde{\tau}^+} \text{Tr}(\sigma_\nu | A_{\nu', \mu'}) s_{\mu'}(\mathbf{x}) \\ &= (-1)^{r(\nu)} \sum_{\mu \leq \tilde{\tau}^+} \text{Tr}(\sigma_\nu | A_{\nu, \mu}) s_{\mu'}(\mathbf{x}) \end{aligned}$$

where $r(\nu)$ is as in Formula (6.9.1).

Fix $i = 1, \dots, k$ and let $\nu_{i,j} := \mathfrak{S}^{-1}(\tilde{\tau}_{i,j})$ so that $\nu_i = (\nu_{i,1}, \dots, \nu_{i,r_i})$. Then $s_{\tau_i}(\mathbf{x}_i) = \prod_{j=1}^{r_i} s_{\nu_{i,j}}(\mathbf{x}_i)$ and so

$$\begin{aligned} s_{\tau_i}(\mathbf{x}_i) &= (-1)^{r(\tau_i)} \sum_{(\mu_1, \dots, \mu_{r_i}) \leq (\tilde{\tau}_1^+, \dots, \tilde{\tau}_{r_i}^+)} \text{Tr}(\sigma_{\nu_i} | A_{\nu_{i,1}, \mu_1} \otimes \cdots \otimes A_{\nu_{i,r_i}, \mu_{r_i}}) s_{\mu_1}(\mathbf{x}_i) \cdots s_{\mu_{r_i}}(\mathbf{x}_i) \\ &= (-1)^{r(\tau_i)} \sum_{\tilde{\mu} \leq \tilde{\omega}_i} \text{Tr}(\sigma_{\nu_i} | A_{\tilde{\mu}}) s_{\tilde{\mu}'}(\mathbf{x}_i). \end{aligned}$$

We thus have

$$\mathbb{H}_\tau(q) = \sum_{\tilde{\mu} \leq \tilde{\omega}} \text{Tr}(\sigma_\nu | A_{\tilde{\mu}}) \mathbb{H}_{\tilde{\mu}}(q).$$

The result follows from Theorem 6.9.1. □

7.4.3 The main theorem

Consider the quiver variety $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ over \mathbb{C} . Let $\pi : \mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma} \rightarrow \mathcal{Q}_{\tilde{\omega}}$ be the canonical map. For $\mathbf{w} \in W_{\mathbf{M}}(\mathbf{L}, \mathbb{C})$ put

$$P_c^{\mathbf{w}}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; q) := \sum_i \text{Tr}(\mathbf{w} | IH_c^{2i}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, \mathbb{C})) q^i.$$

We assume that the characteristic of \mathbb{F}_q and q are large enough so that the results of §7.3 and §7.4.2 apply.

Consider the notation of §7.4.2.

Theorem 7.4.2. *We have*

$$P_c^{\sigma_v}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; q) = q^{d_{\tilde{\omega}}/2} \langle \Lambda \otimes R_{\tau}, 1 \rangle.$$

Proof. By (7.1.5) we have

$$P_c^{\sigma_v}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; q) = \sum_{\tilde{\tau} \leq \tilde{\omega}} \text{Tr}(\sigma_v | A_{\tilde{\tau}}) q^{r(\tilde{\tau})} P_c(\mathbb{Q}_{\tilde{\tau}}; q). \quad (7.4.2)$$

The result follows from Proposition 7.4.1 and Theorem 7.3.4. \square

Corollary 7.4.3. (a) *The multiplicity $\langle \Lambda \otimes R_{\tau}, 1 \rangle$ is a polynomial in q with integer coefficients. If moreover the degrees of the types τ_1, \dots, τ_k are all equal to 1, then the coefficients are positive.*

(b) *The coefficient of the highest power of q in $\langle \Lambda \otimes R_{\tau}, 1 \rangle$ equals 1.*

Proof. From Formula (7.4.2) we see that $P_c^{\sigma_v}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; q)$ is a polynomial in q which satisfies the properties mentioned in (a) and (b). Indeed $\text{Tr}(\sigma_v | A_{\tilde{\omega}}) = 1$ and if the degrees of the τ_i 's are all equal to 1 then $\sigma_v = 1$. The assertions (a) and (b) are now consequences of Theorem 7.4.2. \square

Corollary 7.4.4. *We have*

$$P_c^{\sigma_v}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}; q) = q^{d_{\tilde{\omega}}/2} \mathbb{H}_{\tau}(q).$$

For $i = 1, \dots, k$, let C_i^1 be an adjoint orbit of $\mathfrak{gl}_n(\mathbb{C})$ of type $\tilde{\tau}_i = \tilde{\mathfrak{F}}_{r_i}(\tilde{\tau}_{i,1}, \dots, \tilde{\tau}_{i,r_i}) \in \tilde{\mathbf{T}}_n$ such that the tuple (C_1^1, \dots, C_k^1) is generic. Since the tuple (O_1, \dots, O_k) is assumed to be generic, such a choice is always possible. Indeed, with the notation of §7.4.1, the integers m_1, \dots, m_r are the multiplicities of the eigenvalues of the semisimple part σ of an element in O while $\{n_{i,j}\}$ are the multiplicities of the eigenvalues of the semisimple part, say t , of an element of an orbit C^1 of type $\tilde{\mathfrak{F}}_r(\tilde{\tau}_1, \dots, \tilde{\tau}_r)$. Then each m_i is a sum of $n_{i,j}$ with $j = 1, \dots, s_i$. Let $t_i \in I_i$ be the semisimple part of an element in C_i^1 . It is then easy to see that if d divides all the multiplicities of the eigenvalues of the t_i 's for all $i = 1, \dots, k$, then d will divide the multiplicities of all the eigenvalues of the σ_i , for all $i = 1, \dots, k$.

Put $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_k)$. We have

Corollary 7.4.5.

$$P_c(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, q) = P_c(\mathbb{Q}_{\tilde{\tau}}, q).$$

Proof. Follows from Theorem 7.3.4 and Theorem 7.4.2. \square

Remark 7.4.6. Identify $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ with $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$. Then $\mathbb{Q}_{\tilde{\tau}}$ is of the form $\mathfrak{M}_{\theta}(\mathbf{v}) \simeq \mathfrak{M}_{\theta, \theta}(\mathbf{v})$. Then it is known [6] that the two quiver varieties $\mathfrak{M}_{\xi, \theta}(\mathbf{v})$ and $\mathfrak{M}_{\theta, \theta}(\mathbf{v})$ are fibres of the same locally trivial fibration (for the complex topology) $\mathfrak{M} \rightarrow \mathbb{C}$, hence have isomorphic intersection cohomology. This is compatible with the above corollary.

Corollary 7.4.7. *We have $\langle \Lambda \otimes R_{\tau}, 1 \rangle \neq 0$ if and only if $\mathbf{v} \in \Phi(\Gamma)$.*

Proof. It follows from Theorem 7.4.2 that $\langle \Lambda \otimes R_{\tau}, 1 \rangle \neq 0$ if and only if $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is not empty. But $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is not empty if and only if $\mathcal{V}_{\tilde{\omega}}$ is not empty. Hence the corollary follows from Proposition 5.3.6. \square

Corollary 7.4.8. *If $g = 0$, then \mathbf{v} is a real root if and only if $\langle R_{\tau}, 1 \rangle = 1$. If $g \geq 1$, then \mathbf{v} is always an imaginary root (in particular the inner product $\langle \Lambda \otimes R_{\tau}, 1 \rangle$ never vanishes).*

Proof. Assume that $g = 0$. By Proposition 5.3.6, the variety is $\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}$ is reduced to a point (namely $P_c^{\sigma_v}(\mathbb{Q}_{\mathbf{L}, \mathbf{P}, \Sigma}, q) = \text{Tr}(\sigma_v | A_{\tilde{\omega}}) = 1$) if and only if \mathbf{v} is a real root. The first assertion of the corollary is now a direct consequence of Theorem 7.4.2. Assume that $g \geq 1$. For $i = 1, \dots, k$, and $j = 1, \dots, s_i, s_i + 1$ define $m_j^i = v_{[i, j-1]} - v_{[i, j]}$ (with the convention that $v_{[i, 0]} = n$ and $v_{[i, s_i+1]} = 0$). Then $\sum_j m_j^i = n$ and so the multiset $\{m_j^i\}_{j=1, \dots, s_i+1}$ defines a unique partition $\mu^i = (\mu_1^i, \dots, \mu_{s_i+1}^i)$ of n . We now define a dimension vector \mathbf{f} of Γ by the requirement that $f_0 = n$, $f_{[i, j]} = n - \sum_{r=1}^j \mu_r^i$. Then $(\mathbf{e}_0, \mathbf{f}) = (2 - 2g)n - \sum_{i=1}^k f_{[i, 1]} \leq 0$, and $(\mathbf{e}_{[i, j]}, \mathbf{f}) = \mu_{j+1}^i - \mu_j^i \leq 0$. Hence \mathbf{f} is in the fundamental set of imaginary roots [17]. Since \mathbf{f} can be obtained from \mathbf{v} by an element in the Weyl group of Γ (see above Proposition 5.3.6), we conclude that \mathbf{v} is always an imaginary root of Γ . □

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