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On domination of Cartesian product of directed cycles

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Abstract

Let $\gamma(\vec{C}_m \square \vec{C}_n)$ be the domination number of the Cartesian product of directed cycles \vec{C}_m and \vec{C}_n for $m, n \geq 2$. Shaheen [13] and Liu and al.[11], [12] determined the value of $\gamma(\vec{C}_m \square \vec{C}_n)$ when $m \leq 6$ and [12] when both m and $n \equiv 0 \pmod{3}$. In this article we give, in general, the value of $\gamma(\vec{C}_m \square \vec{C}_n)$ when $m \equiv 2 \pmod{3}$ and improve the known lower bounds for most of the remaining cases. We also disprove the conjectured formula for the case $m \equiv 0 \pmod{3}$ appearing in [12].

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1 Introduction and definitions

Let $D = (V, A)$ be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex u *dominates* a vertex v if $u = v$ or $uv \in A$. A set $W \subset V$ is a *dominating* set of D if any vertex of V is dominated by at least a vertex of W . The *domination number* of D , denoted by $\gamma(D)$ is the minimum cardinality of a dominating set. The set V is a dominating set thus $\gamma(D)$ is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

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For two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ the *Cartesian product* $D_1 \square D_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in A(D_1 \square D_2)$ if and only if $x_1 y_1 \in A_1$ and $x_2 = y_2$ or $x_2 y_2 \in A_2$ and $x_1 = y_1$. Note that $D_2 \square D_1$ is isomorphic to $D_1 \square D_2$. In [13] Shaheen determined the domination number of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ for $m \leq 6$ and arbitrary n . In two articles [11], [12] Liu and al. considered independently the domination number of Cartesian product of two directed cycles. They gave also the value of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ when $m \leq 6$ and when both m and $n \equiv 0 \pmod{3}$ [12]. Furthermore they proposed lower and upper bounds for the general case.

In this paper we are able to give, in general, the value of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ when $m \equiv 2 \pmod{3}$ and we improve the lower bounds for most of the still unknown cases. We also disprove the conjectured formula appearing in [12] for the case $m \equiv 0 \pmod{3}$.

We denote the vertices of a directed cycle $\overrightarrow{C_n}$ by $C_n = \{0, 1, \dots, n-1\}$, the integers considered modulo n . Thus, when used for vertices labelling, $a+b$ and $a-b$ will denote the vertices $a+b$ and $a-b \pmod{n}$. Notice that there exists an arc xy from x to y in $\overrightarrow{C_n}$ if and only if $y \equiv x+1 \pmod{n}$, thus with our convention, if and only if $y = x+1$. For any i in $\{0, 1, \dots, n-1\}$ we will denote by $\overrightarrow{C_m^i}$ the subgraph of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ induced by the vertices $\{(k, i) / k \in \{0, 1, \dots, m-1\}\}$. Note that $\overrightarrow{C_m^i}$ is isomorphic to $\overrightarrow{C_m}$. We will denote by C_m^i the set of vertices of $\overrightarrow{C_m^i}$.

2 General bounds and the case $m \equiv 2 \pmod{3}$

Proposition 1 *Let W be a dominating set of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ then for all i in $\{0, 1, \dots, n-1\}$ considered modulo n we have $|W \cap C_m^{i-1}| + 2|W \cap C_m^i| \geq m$.*

Proof : The m vertices of C_m^i can only be dominated by vertices of $W \cap C_m^i$ and $W \cap C_m^{i-1}$. Each of the vertices of $W \cap C_m^i$ dominates two vertices in C_m^i . Similarly each of the vertices of $W \cap C_m^{i-1}$ dominates one vertex in C_m^i . The result follows. \square

Theorem 2 *Let $m, n \geq 2$ and $k_1 = \lfloor \frac{m}{3} \rfloor$ then*

- *if $m \equiv 0 \pmod{3}$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq nk_1$*
- *if $m \equiv 1 \pmod{3}$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq nk_1 + \frac{n}{2}$*
- *if $m \equiv 2 \pmod{3}$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq nk_1 + n$.*

Proof :

Let W be a dominating set of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ and for any i in $\{0, 1, \dots, n-1\}$ let $a_i = |W \cap C_m^i|$. Notice first that each of the vertices of W dominates three vertices of $\overrightarrow{C_m} \square \overrightarrow{C_n}$. Thus $|W| \geq \frac{mn}{3}$. This general bound give the announced result for $m = 3k_1$, $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq nk_1 + \frac{n}{3}$ for $m = 3k_1 + 1$ and $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) \geq nk_1 + 2\frac{n}{3}$ for $m = 3k_1 + 2$. We will improve these two last results.

Assume first $m = 3k_1 + 1$. Let J be the set of $i \in \{0, 1, \dots, n-1\}$ such that $a_i \leq k_1$. If $J = \emptyset$ then $|W| \geq n(k_1 + 1) \geq nk_1 + \frac{n}{2}$ and we are done. Otherwise let $J' = \{i \mid i+1 \pmod{n} \in J\}$. By proposition 1, we have $a_{i-1} + 2a_i \geq 3k_1 + 1$. Then

if i belongs to J , $a_{i-1} + a_i \geq 2k_1 + 1$. Thus $J \cap J' = \emptyset$ and $\sum_{i \in J \cup J'} a_i \geq |J|(2k_1 + 1)$.
 68 Therefore $\sum_{i \in \{0, 1, \dots, n-1\}} a_i \geq |J|(2k_1 + 1) + (n - 2|J|)(k_1 + 1) = nk_1 + n - |J| \geq nk_1 + \frac{n}{2}$.
 Assume now $m = 3k_1 + 2$. Let J be the set of $i \in \{0, 1, \dots, n-1\}$ such that
 70 $a_i \leq k_1$. If $J = \emptyset$ then we are done. Otherwise let $J' = \{i \mid i + 1 \pmod{n} \in J\}$.
 If $i \in J$ we have $a_{i-1} + 2a_i \geq 3k_1 + 2$ thus $a_{i-1} + a_i \geq 2k_1 + 2$. Then $J \cap J' = \emptyset$
 72 and $\sum_{i \in J \cup J'} a_i \geq |J|(2k_1 + 2)$. Therefore $\sum_{i \in \{0, 1, \dots, n-1\}} a_i \geq |J|(2k_1 + 2) + (n - 2|J|)(k_1 + 1) \geq n(k_1 + 1)$.

74 □

Let us now study in details the case $m \equiv 2 \pmod{3}$. Assume $m = 3k_1 + 2$. Let A
 76 be the set of $k_1 + 1$ vertices of \overrightarrow{C}_m defined by $A = \{0\} \cup \{2 + 3p \mid p = 0, 1, \dots, k_1 - 1\} =$
 $\{0\} \cup \{2, 5, \dots, m - 6, m - 3\}$. For any i in $\{0, 1, \dots, m - 1\}$ let A_i be the translate,
 78 considered modulo m , of A by i ; thus $A_i = \{i\} \cup \{i + 2, i + 5, \dots, i - 6, i - 3\}$ (see
 Figure 1).

80 We will call a set S of vertices of $\overrightarrow{C}_m \square \overrightarrow{C}_n$ a A -set if for any j in $\{0, 1, \dots, n - 1\}$
 we have $S \cap C_m^j = A_{i_j}$ for some i_j . Therefore $|S| = n(k_1 + 1)$; thus if a set is
 82 both a A -set and a dominating set, by Theorem 2, it is minimum and we have
 $\gamma(\overrightarrow{C}_m \square \overrightarrow{C}_n) = n(k_1 + 1)$.

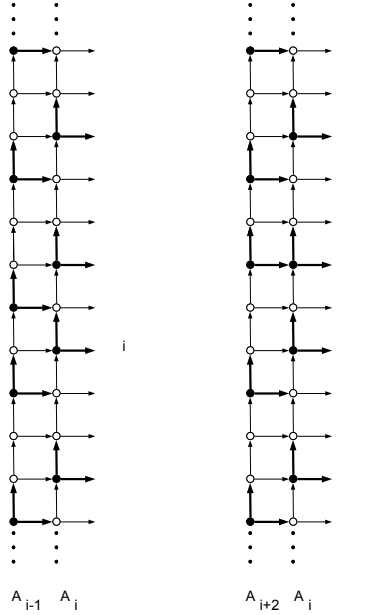


Figure 1: A_i, A_{i-1} and A_{i+2}

Lemma 3 Let $m = 3k_1 + 2$. Let S be a A -set and for any j in $\{0, 1, \dots, n - 1\}$ let
 i_j such that $S \cap C_m^j = A_{i_j}$. Assume that:

$$\left\{ \begin{array}{l} \text{for any } j \in \{1, \dots, n - 1\} \ i_j \equiv i_{j-1} + 1 \pmod{m} \text{ or } i_j \equiv i_{j-1} - 2 \pmod{m} \\ \text{and} \\ i_0 \equiv i_{n-1} + 1 \pmod{m} \text{ or } i_0 \equiv i_{n-1} - 2 \pmod{m}. \end{array} \right.$$

84 then S is a dominating set of $\overrightarrow{C_m} \square \overrightarrow{C_n}$.

Proof :

86 Note first that for any i in $\{0, 1, \dots, m-1\}$ the set of non dominated vertices
of C_m by A_i is $T = \{i+4, i+7, \dots, i-4, i-1\}$. Note also that $A_{i+2} = \{i+2\} \cup$
88 $\{i+4, i+7, \dots, i-4, i-1\}$ and $A_{i-1} = \{i-1\} \cup \{i+1, i+4, \dots, i-7, i-4\}$. Thus
 $T \subset A_{i+2}$ and $T \subset A_{i-1}$.

90 Let j in $\{1, \dots, n-1\}$. Let us prove that the vertices of C_m^j are dominated.
Indeed, by the previous remark and the lemma hypothesis, the vertices non domi-
92 nated by $S \cap C_m^j$ are dominated by $S \cap C_m^{j-1}$ (see Figure 1). For the same reasons
the vertices of C_m^0 are dominated by those of $S \cap C_m^0$ and $S \cap C_m^{n-1}$.

94 □

Lemma 4 Let $m = 3k_1 + 2$. If there exist integers $a, b \geq 0$ such that

$$\begin{cases} a + b = n - 1 \\ \text{and} \\ a - 2b \equiv 2 \pmod{m} \text{ or } a - 2b \equiv m - 1 \pmod{m} \end{cases}$$

then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$.

96 **Proof :** Consider a word $w = w_1 \dots w_{n-1}$ on the alphabet $\{1, -2\}$ with a
occurrences of 1 and b of -2 . Such a word exists, for example $w = 1^a(-2)^b$. We
98 can associate to w a set S of vertices of $\overrightarrow{C_m} \square \overrightarrow{C_n}$ by the following algorithm:

- $S \cap C_m^0 = A_0$
- For $i = 1$ to $n - 1$ do
begin
102 Let k such that $S \cap C_m^{i-1} = A_k$
If $w_i = 1$ let $k' \equiv k + 1 \pmod{m}$ else $k' \equiv k - 2 \pmod{m}$
104 $S \cap C_m^i := A_{k'}$
end

106 By construction S is a A -set. Notice that we have $S \cap C_m^{n-1} := A_{i_{n-1}}$ where $i_{n-1} \equiv$
 $\sum_{k=1}^{n-1} w_k \equiv a - 2b \pmod{m}$. Thus $i_{n-1} \equiv 2 \pmod{m}$ or $i_{n-1} \equiv m - 1 \pmod{m}$. By
108 lemma 3 S is a dominating set. Furthermore, because S is a A -set, $|S| = n(k_1 + 1)$,
thus by theorem 2 it is minimum and we have $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$. □

110 **Theorem 5** Let $m, n \geq 2$ and $m \equiv 2 \pmod{3}$. Let $k_1 = \lfloor \frac{m}{3} \rfloor$ and $k_2 = \lfloor \frac{n}{3} \rfloor$.

- if $n = 3k_2$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$
- if $n = 3k_2 + 1$ and $2k_2 \geq k_1$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$
- if $n = 3k_2 + 1$ and $2k_2 < k_1$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) > n(k_1 + 1)$
- if $n = 3k_2 + 2$ and $n \geq m$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = n(k_1 + 1)$.
- if $n = 3k_2 + 2$ and $n \leq m$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = m(k_2 + 1)$.

Proof : We will use lemma 4 considering the following integer solutions of

$$\begin{cases} a, b \geq 0 \\ a + b = n - 1 \\ a - 2b \equiv 2 \pmod{m} \text{ or } a - 2b \equiv m - 1 \pmod{m}. \end{cases}$$

- 116 • if $n = 3k_2$ then $k_2 \geq 1$. Take $a = 2k_2$ and $b = k_2 - 1$
 - if $n = 3k_2 + 1$ and $2k_2 \geq k_1$ then take $a = 2k_2 - k_1$ and $b = k_2 + k_1$
 - 118 • if $n = 3k_2 + 1$ and $2k_2 < k_1$ then $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = \gamma(\overrightarrow{C_n} \square \overrightarrow{C_m}) \geq \frac{(2k_2+1)m}{2}$ by theorem 2. Furthermore $\frac{(2k_2+1)m}{2} - n(k_1 + 1) = \frac{k_1}{2} - k_2 > 0$
 - 120 • if $n = 3k_2 + 2$ and $k_2 \geq k_1$ then take $a = 2k_2 - 2k_1$ and $b = k_2 + 2k_1 + 1$
 - if $n = 3k_2 + 2$ and $k_2 \leq k_1$ then use $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n}) = \gamma(\overrightarrow{C_n} \square \overrightarrow{C_m})$.
- 122 □

3 The case $m \equiv 0 \pmod{3}$

124 In [12] Liu and al. conjectured the following formula:

Conjecture 6 *Let $k \geq 2$. Then $\gamma(\overrightarrow{C_{3k}} \square \overrightarrow{C_n}) = k(n + 1)$ for $n \not\equiv 0 \pmod{3}$.*

126 Our theorem 5 confirms the conjecture when $n \equiv 2 \pmod{3}$. Unfortunately the formula is not always valid when $n \equiv 1 \pmod{3}$.

128 Indeed consider $C_{3k} \square C_4$. In [11] the following result is proved:

Theorem 7 *Let $n \geq 2$. Then $\gamma(\overrightarrow{C_4} \square \overrightarrow{C_n}) = \frac{3n}{2}$ if $n \equiv 0 \pmod{8}$ and $\gamma(\overrightarrow{C_4} \square \overrightarrow{C_n}) = n + \lceil \frac{n+1}{2} \rceil$ otherwise*

130

We have thus $\gamma(\overrightarrow{C_{3k}} \square \overrightarrow{C_4}) = \gamma(\overrightarrow{C_4} \square \overrightarrow{C_{3k}}) = 3k + \lceil \frac{3k+1}{2} \rceil$ when $k \not\equiv 0 \pmod{8}$.
 132 On the other side the conjecture 6 proposes the value $\gamma(\overrightarrow{C_{3k}} \square \overrightarrow{C_4}) = 5k$. These two numbers are different when $k \geq 3$.

4 Conclusion

134

136 Consider the possible reminder of m, n modulo 3. In some of the nine possible values we are able to determine $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$. The remaining cases are

- $m \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$
- 138 • the symmetrical case $m \equiv 1 \pmod{3}$ and $n \equiv 0 \pmod{3}$.
- m and $n \equiv 1 \pmod{3}$.

140 The case m or $n \equiv 2 \pmod{3}$ is not completely solved by Theorem 5. The following sub cases are still open

- 142 • $m \equiv 2 \pmod{3}$ and $n \equiv 1 \pmod{3}$ with $m > 2n + 1$

- the symmetrical case $m \equiv 1 \pmod{3}$ and $n \equiv 1 \pmod{3}$ with $n > 2m + 1$.

For these values of m, n there does not always exist a dominating set reaching the bound stated in Theorem 2 and thus the determination of $\gamma(\overrightarrow{C_m} \square \overrightarrow{C_n})$ seems to be a more difficult problem.

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