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# On domination of Cartesian product of directed cycles

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## Abstract

Let  $\gamma(C_m \square C_n)$  be the domination number of the Cartesian product of directed cycles  $C_m$  and  $C_n$  for  $m, n \geq 2$ . In [11] and [12] Liu and al. determined the value of  $\gamma(C_m \square C_n)$  when  $m \leq 6$  and when both  $m$  and  $n \equiv 0 \pmod{3}$ . In this article we give, in general, the value of  $\gamma(C_m \square C_n)$  when  $m \equiv 2 \pmod{3}$  and improve the known lower bound for most of the remaining cases. We also disprove the conjectured formula for the case  $m \equiv 0 \pmod{3}$  appearing in [12].

**Keywords:** Graph, Cartesian product, Domination number, Combinatorial problems.

## 1 Introduction and definitions

Let  $G = (V, E)$  be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex  $u$  *dominates* a vertex  $v$  if  $u = v$  or  $uv \in E$ . A set  $S \subset V$  is a *dominating* set of  $G$  if any vertex of  $G$  is dominated by at least a vertex of  $S$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$  is the minimal cardinality of a dominating set. The set  $V$  is a dominating set thus  $\gamma(G)$  is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

For  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  two digraphs, the *Cartesian product*  $G_1 \square G_2$  is the digraph with vertex set  $V_1 \times V_2$  and  $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$  if and only

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if  $x_1y_1 \in E_1$  and  $x_2 = y_2$  or  $x_2y_2 \in E_2$  and  $x_1 = y_1$ . Note that  $G \square H$  is isomorph to  $H \square G$ .

In two articles published in this journal ([11], [12]) Liu and al. consider the domination number of Cartesian product of two directed cycles. They give the value of  $\gamma(C_m \square C_n)$  when  $m \leq 6$  and when both  $m$  and  $n \equiv 0 \pmod{3}$ . Furthermore they propose lower and upper bounds for the general case.

In this paper we are able to give, in general, the value of  $\gamma(C_m \square C_n)$  when  $m \equiv 2 \pmod{3}$  and we improve the lower bounds for most of the still unknown cases. We also disprove the conjectured formula appearing in [12] for the case  $m \equiv 0 \pmod{3}$ .

We denote the vertices of a directed cycle  $C_n$  by the integers  $\{0, 1, \dots, n-1\}$  considered modulo  $n$ . Thus, when used for vertices labelling,  $a+b$  and  $a-b$  will denote the vertices  $a+b$  and  $a-b \pmod{n}$ . Furthermore there is an arc  $xy$  from  $x$  to  $y$  in  $C_n$  if and only if  $y = x+1 \pmod{n}$ . For any  $i$  in  $\{0, 1, \dots, n-1\}$  we will denote by  $C_m^i$  the subgraph of  $C_m \square C_n$  induced by the vertices  $\{(k, i) / k \in \{0, 1, \dots, m-1\}\}$ . Note that  $C_m^i$  is isomorph to  $C_m$ .

## 2 Main results

**Proposition 1** *Let  $S$  be a dominating set of  $\gamma(C_m \square C_n)$  then for all  $i$  in  $\{0, 1, \dots, n-1\}$  considered modulo  $n$  we have  $|S \cap C_m^{i-1}| + 2|S \cap C_m^i| \geq m$ .*

**Proof :** The  $m$  vertices of  $C_m^i$  can only be dominated by vertices of  $S \cap C_m^i$  and  $S \cap C_m^{i-1}$ . Each of the vertices of  $S \cap C_m^i$  and  $S \cap C_m^{i-1}$  dominates respectively two and one vertex in  $C_m^i$ . The result follows. □

**Theorem 2** *Let  $m, n \geq 2$  and  $k_1 = \lfloor \frac{m}{3} \rfloor$  then*

- *if  $m \equiv 0 \pmod{3}$  then  $\gamma(C_m \square C_n) \geq nk_1$*
- *if  $m \equiv 1 \pmod{3}$  then  $\gamma(C_m \square C_n) \geq nk_1 + \frac{n}{2}$*
- *if  $m \equiv 2 \pmod{3}$  then  $\gamma(C_m \square C_n) \geq nk_1 + n$ .*

**Proof :**

Let  $S$  be a dominating set of  $C_m \square C_n$  and for any  $i$  in  $\{0, 1, \dots, n-1\}$  let  $a_i = |S \cap C_m^i|$ .

Assume first  $m = 3k_1$ . Let  $J$  be the set of  $i \in \{0, 1, \dots, n-1\}$  such that  $a_i \leq k_1 - 1$ . Let  $J' = \{i \mid i+1 \pmod{n} \in J\}$ . By proposition 1, we have  $a_{i-1} + 2a_i \geq 3k_1$  thus, if  $i$  belongs to  $J$ ,  $a_{i-1} + a_i \geq 2k_1 + 1$ . A first consequence is that  $J \cap J' = \emptyset$ . Furthermore  $\sum_{i \in J \cup J'} a_i \geq |J|(2k_1 + 1)$ . If  $i \notin J \cup J'$  we have  $a_i \geq k_1$  therefore  $\sum_{i \in \{0, 1, \dots, n-1\}} a_i \geq |J|(2k_1 + 1) + (n - 2|J|)k_1 = |J| + nk_1 \geq nk_1$ .

Assume now  $m = 3k_1 + 1$ . Let  $J$  be the set of  $i \in \{0, 1, \dots, n-1\}$  such that  $a_i \leq k_1$ . Let  $J' = \{i \mid i+1 \pmod{n} \in J\}$ . If  $i$  belongs to  $J$  we have  $a_{i-1} + 2a_i \geq 3k_1 + 1$  thus  $a_{i-1} + a_i \geq 2k_1 + 1$ . Then  $J \cap J' = \emptyset$  and  $\sum_{i \in J \cup J'} a_i \geq |J|(2k_1 + 1)$ . Therefore  $\sum_{i \in \{0, 1, \dots, n-1\}} a_i \geq |J|(2k_1 + 1) + (n - 2|J|)(k_1 + 1) = nk_1 + n - |J| \geq nk_1 + \frac{n}{2}$ .

Assume then  $m = 3k_1 + 2$ . Let  $J$  be the set of  $i \in \{0, 1, \dots, n-1\}$  such that  $a_i \leq k_1$ . Let  $J' = \{i \mid i+1 \pmod{n} \in J\}$ . If  $i \in J$  we have  $a_{i-1} + 2a_i \geq 3k_1 + 2$

70 thus  $a_{i-1} + a_i \geq 2k_1 + 2$ . Then  $J \cap J' = \emptyset$  and  $\sum_{i \in J \cup J'} a_i \geq |J| (2k_1 + 2)$ . Therefore  
 72  $\sum_{i \in \{0, 1, \dots, n-1\}} a_i \geq |J| (2k_1 + 2) + (n - 2|J|)(k_1 + 1) \geq n(k_1 + 1)$ . □

74 Let us now study in details the case  $m \equiv 2 \pmod{3}$ . Assume  $m = 3k_1 + 2$ . Let  $A$   
 be the set of  $k_1 + 1$  vertices of  $C_m$  defined by  $S = \{0\} \cup \{2 + 3p \mid p = 0, 1, \dots, k_1 - 1\} =$   
 76  $\{0\} \cup \{2, 5, \dots, m - 6, m - 3\}$ . For any  $i$  in  $\{0, 1, \dots, m - 1\}$  let  $A_i$  be the translate,  
 considered modulo  $m$ , of  $A$  by  $i$ ; thus  $A_i = \{i\} \cup \{i + 2, i + 5, \dots, i - 6, i - 3\}$  (see  
 Figure 1).

78 We will call a set  $S$  of vertices of  $C_m \square C_n$  a  $A$ -set if for any  $j$  in  $\{0, 1, \dots, n - 1\}$   
 we have  $S \cap C_m^j = A_{i_j}$  for some  $i_j$ . We have  $|S| = n(k_1 + 1)$ , thus if a set is  
 80 both a  $A$ -set and a dominating set, by theorem 2, it is minimum and we have  
 $\gamma(C_m \square C_n) = n(k_1 + 1)$ .

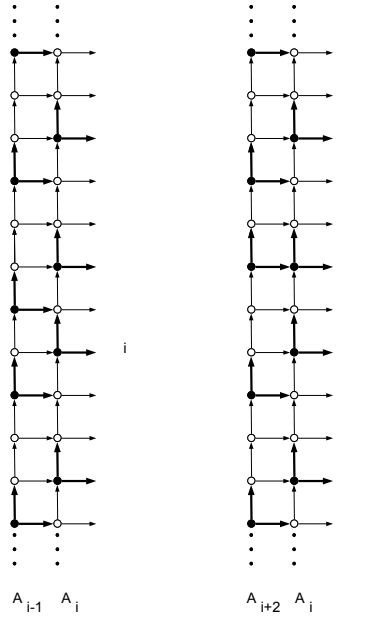


Figure 1:  $A_i, A_{i-1}$  and  $A_{i+2}$

**Lemma 3** Let  $m = 3k_1 + 2$ . Let  $S$  be a  $A$ -set and for any  $j$  in  $\{0, 1, \dots, n - 1\}$  let  $i_j$  such that  $S \cap C_m^j = A_{i_j}$ . Assume that:

$$\begin{cases} \text{for any } j \in \{1, \dots, n - 1\} \ i_j = i_{j-1} + 1 \pmod{m} \text{ or } i_j = i_{j-1} - 2 \pmod{m} \\ \text{and} \\ i_0 = i_{n-1} + 1 \pmod{m} \text{ or } i_0 = i_{n-1} - 2 \pmod{m}. \end{cases}$$

82 then  $S$  is a dominating set of  $C_m \square C_n$ .

**Proof :**

84 Note first that for any  $i$  in  $\{0, 1, \dots, m-1\}$  the set of non dominated vertices  
of  $C_m$  by  $A_i$  is  $T = \{i+4, i+7, \dots, i-4, i-1\}$ . Note also that  $A_{i+2} = \{i+2\} \cup$   
86  $\{i+4, i+7, \dots, i-4, i-1\}$  and  $A_{i-1} = \{i-1\} \cup \{i+1, i+4, \dots, i-7, i-4\}$ . Thus  
 $T \subset A_{i+2}$  and  $T \subset A_{i-1}$ .

88 Let  $j$  in  $\{1, \dots, n-1\}$ . Let us prove that the vertices of  $C_m^j$  are dominated.  
Indeed, by the previous remark and the lemma hypothesis, the vertices non domi-  
90 nated by  $S \cap C_m^j$  are dominated by  $S \cap C_m^{j-1}$  (see Figure 1). For the same reasons  
the vertices of  $C_m^0$  are dominated by those of  $S \cap C_m^0$  and  $S \cap C_m^{-1}$ .

92  $\square$

**Lemma 4** *Let  $m = 3k_1 + 2$ . If there exist integers  $a, b \geq 0$  such that*

$$\begin{cases} a + b = n - 1 \\ \text{and} \\ a - 2b \equiv 2 \pmod{m} \text{ or } a - 2b \equiv m - 1 \pmod{m} \end{cases}$$

then  $\gamma(C_m \square C_n) = n(k_1 + 1)$ .

94 **Proof :** Consider a word  $w = w_1 \dots w_{n-1}$  on the alphabet  $\{1, -2\}$  with  $a$   
occurrences of 1 and  $b$  of  $-2$ . Such a word exists, for example  $w = 1^a(-2)^b$ . We  
96 can associate to  $w$  a set  $S$  of vertices of  $C_m \square C_n$  by the following algorithm:

- $S \cap C_m^0 = A_0$
- For  $i = 1$  to  $n - 1$  do  
begin  
100 Let  $k$  such that  $S \cap C_m^{i-1} = A_k$   
If  $w_i = 1$  let  $k' = k + 1 \pmod{m}$  else  $k' = k - 2 \pmod{m}$   
102  $S \cap C_m^i := A_{k'}$   
end

104 By construction  $S$  is a  $A$ -set. Notice that we have  $S \cap C_m^{n-1} := A_{i_{n-1}}$  where  $i_{n-1} \equiv$   
 $\sum_{k=1}^{n-1} w_k \equiv a - 2b \pmod{m}$ . Thus  $i_{n-1} = 2$  or  $i_{n-1} = m - 1 \pmod{m}$ . By lemma 3  
106  $S$  is a dominating set. Furthermore, because  $S$  is a  $A$ -set,  $|S| = n(k_1 + 1)$ ; thus by  
theorem 2 it is minimum and we have  $\gamma(C_m \square C_n) = n(k_1 + 1)$ .  $\square$

108 **Theorem 5** *Let  $m, n \geq 2$  and  $m \equiv 2 \pmod{3}$ . Let  $k_1 = \lfloor \frac{m}{3} \rfloor$  and  $k_2 = \lfloor \frac{n}{3} \rfloor$ .*

- if  $n = 3k_2$  then  $\gamma(C_m \square C_n) = n(k_1 + 1)$
- if  $n = 3k_2 + 1$  and  $2k_2 \geq k_1$  then  $\gamma(C_m \square C_n) = n(k_1 + 1)$
- if  $n = 3k_2 + 1$  and  $2k_2 < k_1$  then  $\gamma(C_m \square C_n) > n(k_1 + 1)$
- if  $n = 3k_2 + 2$  and  $n \geq m$  then  $\gamma(C_m \square C_n) = n(k_1 + 1)$ .
- if  $n = 3k_2 + 2$  and  $n \leq m$  then  $\gamma(C_m \square C_n) = m(k_2 + 1)$ .

**Proof :** We will use lemma 4 considering the following integer solutions of

$$\begin{cases} a, b \geq 0 \\ a + b = n - 1 \\ a - 2b \equiv 2 \pmod{m} \text{ or } a - 2b \equiv m - 1 \pmod{m}. \end{cases}$$

- 114 • if  $n = 3k_2$  then  $k_2 \geq 1$ . Take  $a = 2k_2$  and  $b = k_2 - 1$
- if  $n = 3k_2 + 1$  and  $2k_2 \geq k_1$  then take  $a = 2k_2 - k_1$  and  $b = k_2 + k_1$
- 116 • if  $n = 3k_2 + 1$  and  $2k_2 < k_1$  then  $\gamma(C_m \square C_n) = \gamma(C_n \square C_m) \geq \frac{(2k_2+1)m}{2}$  by  
theorem 2. Furthermore  $\frac{(2k_2+1)m}{2} - n(k_1 + 1) = \frac{k_1}{2} - k_2 > 0$
- 118 • if  $n = 3k_2 + 2$  and  $k_2 \geq k_1$  then take  $a = 2k_2 - 2k_1$  and  $b = k_2 + 2k_1 + 1$
- if  $n = 3k_2 + 2$  and  $k_2 \leq k_1$  then use  $\gamma(C_m \square C_n) = \gamma(C_n \square C_m)$ .

120 □

### 3 The case $m \equiv 0 \pmod{3}$

122 In [12] Liu and al. conjectured the following formula:

**Conjecture 6** *Let  $k \geq 2$ . Then  $\gamma(C_{3k} \square C_n) = k(n + 1)$  for  $n \not\equiv 0 \pmod{3}$ .*

124 Our theorem 5 confirms the conjecture when  $n \equiv 2 \pmod{3}$ . Unfortunately the  
formula is not always valid when  $n \equiv 1 \pmod{3}$ .

126 Indeed consider  $C_{3k} \square C_4$ . In [11] the following result is proved:

**Theorem 7** *Let  $n \geq 2$ . Then  $\gamma(C_4 \square C_n) = \frac{3n}{2}$  if  $n \equiv 0 \pmod{8}$  and  $\gamma(C_4 \square C_n) =$   
128  $n + \lceil \frac{n+1}{2} \rceil$  otherwise*

130 We have thus  $\gamma(C_{3k} \square C_4) = \gamma(C_4 \square C_{3k}) = 3k + \lceil \frac{3k+1}{2} \rceil$  when  $k \not\equiv 0 \pmod{8}$ .  
On the other side the conjecture 6 proposes the value  $\gamma(C_{3k} \square C_4) = 5k$ . These two  
numbers are different when  $k \geq 3$ .

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