On domination of Cartesian product of directed cycles

Michel Mollard

On domination of Cartesian product of directed cycles

Michel Mollard*

Institut Fourier
100, rue des Maths
38402 St martin d'hères Cedex FRANCE

michel.mollard@ujf-grenoble.fr

Abstract

Let $\gamma(C_m \Box C_n)$ be the domination number of the Cartesian product of directed cycles C_m and C_n for $m, n \geq 2$. In [11] and [12] Liu and al. determined the value of $\gamma(C_m \Box C_n)$ when $m \leq 6$ and when both m and $n \equiv 0 \ (mod\, 3)$. In this article we give, in general, the value of $\gamma(C_m \Box C_n)$ when $m \equiv 2 \ (mod\, 3)$ and improve the known lower bound for most of the remaining cases. We also disprove the conjectured formula for the case $m \equiv 0 \ (mod\, 3)$ appearing in [12].

Keywords: Graph, Cartesian product, Domination number, Combinatorial problems.

1 Introduction and definitions

Let $G = (V, E)$ be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex u *dominates* a vertex v if $u = v$ or $uv \in E$. A set $S \subset V$ is a *dominating* set of G if any vertex of G is dominated by at least a vertex of S. The *domination number* of G , denoted by $\gamma(G)$ is the minimal cardinality of a dominating set. The set V is a dominating set thus $\gamma(G)$ is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two digraphs, the *Cartesian product* $G_1 \Box G_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ if and only

*CNRS Université Joseph Fourier

if $x_1y_1 \in E_1$ and $x_2 = y_2$ or $x_2y_2 \in E_2$ and $x_1 = y_1$. Note that $G\Box H$ is isomorph to $H\Box G$.

In two articles published in this journal ([11], [12]) Liu and al. consider the domination number of Cartesian product of two directed cycles. They give the value of $\gamma(C_m\Box C_n)$ when $m \leq 6$ and when both m and $n \equiv 0 \ (mod 3)$. Furthermore they propose lower and upper bounds for the general case.

In this paper we are able to give, in general, the value of $\gamma(C_m\Box C_n)$ when $m \equiv 2 \ (mod\, 3)$ and we improve the lower bounds for most of the still unknown cases. We also disprove the conjectured formula appearing in [12] for the case $m \equiv 0 \ (mod\, 3)$.

We denote the vertices of a directed cycle C_n by the integers $\{0, 1, \ldots, n-1\}$ considered modulo n. Thus, when used for vertices labelling, $a+b$ and $a-b$ will denote the vertices $a+b$ and $a-b \ (modn)$. Furthermore there is an arc xy from x to y in C_n if and only if $y = x+1 \ (modn)$. For any i in $\{0, 1, \ldots, n-1\}$ we will denote by C_m^i the subgraph of $C_m\Box C_n$ induced by the vertices $\{(k, i) \ / \ k \in \{0, 1, \ldots, m-1\}\}$. Note that C_m^i is isomorph to C_m.

2 Main results

Proposition 1 *Let S be a dominating set of $\gamma(C_m\Box C_n)$ then for all i in $\{0, 1, \ldots, n-1\}$ considered modulo n we have $\left|S \cap C_m^{i-1}\right| + 2\left|S \cap C_m^i\right| \geq m$.*

Proof : The m vertices of C_m^i can only be dominated by vertices of $S \cap C_m^i$ and $S \cap C_m^{i-1}$. Each of the vertices of $S \cap C_m^i$ and $S \cap C_m^{i-1}$dominates respectively two and one vertex in C_m^i. The result follows.

□

Theorem 2 *Let $m, n \geq 2$ and $k_1 = \left\lfloor \frac{m}{3} \right\rfloor$ then*

- *if $m \equiv 0 \ (mod\, 3)$ then $\gamma(C_m\Box C_n) \geq nk_1$*
- *if $m \equiv 1 \ (mod\, 3)$ then $\gamma(C_m\Box C_n) \geq nk_1 + \frac{n}{2}$*
- *if $m \equiv 2 \ (mod\, 3)$ then $\gamma(C_m\Box C_n) \geq nk_1 + n$.*

Proof :

Let S be a dominating set of $C_m\Box C_n$ and for any i in $\{0, 1, \ldots, n-1\}$ let $a_i = |S \cap C_m^i|$.

Assume first $m = 3k_1$. Let J be the set of $i \in \{0, 1, \ldots, n-1\}$ such that $a_i \leq k_1 - 1$. Let $J' = \{i \mid i+1 \ (mod\, n) \in J\}$. By proposition 1, we have $a_{i-1} + 2a_i \geq 3k_1$ thus, if i belongs to J, $a_{i-1} + a_i \geq 2k_1 + 1$. A first consequence is that $J \cap J' = \emptyset$. Furthermore $\sum_{i\in J\cup J'} a_i \geq |J|\,(2k_1+1)$. If $i \notin J \cup J'$ we have $a_i \geq k_1$ therefore $\sum_{i\in\{0,1,\ldots,n-1\}} a_i \geq |J|(2k_1+1) + (n-2|J|)k_1 = |J| + nk_1 \geq nk_1$.

Assume now $m = 3k_1+1$. Let J be the set of $i \in \{0, 1, \ldots, n-1\}$ such that $a_i \leq k_1$. Let $J' = \{i \mid i+1 \ (mod\, n) \in J\}$. If i belongs to J we have $a_{i-1} + 2a_i \geq 3k_1 + 1$ thus $a_{i-1} + a_i \geq 2k_1 + 1$. Then $J \cap J' = \emptyset$ and $\sum_{i\in J\cup J'} a_i \geq |J|\,(2k_1+1)$. Therefore $\sum_{i\in\{0,1,\ldots,n-1\}} a_i \geq |J|(2k_1+1) + (n-2|J|)(k_1+1) = nk_1 + n - |J| \geq nk_1 + \frac{n}{2}$.

Assume then $m = 3k_1 + 2$. Let J be the set of $i \in \{0, 1, \ldots, n-1\}$ such that $a_i \leq k_1$. Let $J' = \{i \mid i+1 \ (mod\, n) \in J\}$. If $i \in J$ we have $a_{i-1} + 2a_i \geq 3k_1 + 2$

thus $a_{i-1}+a_i \geq 2k_1+2$. Then $J \cap J' = \emptyset$ and $\sum_{i \in J \cup J'} a_i \geq |J|\,(2k_1+2)$. Therefore $\sum_{i \in \{0,1,\ldots,n-1\}} a_i \geq |J|(2k_1+2)+(n-2|J|)(k_1+1) \geq n(k_1+1)$.

$\square$

Let us now study in details the case $m \equiv 2\ (mod\,3)$. Assume $m = 3k_1+2$. Let A be the set of k_1+1 vertices of C_m defined by $S = \{0\} \cup \{2+3p \mid p = 0,1,\ldots,k_1-1\} = \{0\} \cup \{2,5,\ldots,m-6,m-3\}$. For any i in $\{0,1,\ldots,m-1\}$ let A_i be the translate, considered modulo m, of A by i; thus $A_i = \{i\} \cup \{i+2, i+5, \ldots, i-6, i-3\}$(see Figure 1).

We will call a set S of vertices of $C_m \square C_n$ a A-set if for any j in $\{0,1,\ldots,n-1\}$ we have $S \cap C_m^j = A_{i_j}$ for some i_j. We have $|S|= n(k_1+1)$, thus if a set is both a A-set and a dominating set, by theorem 2, it is minimum and we have $\gamma(C_m \square C_n) = n(k_1+1)$.

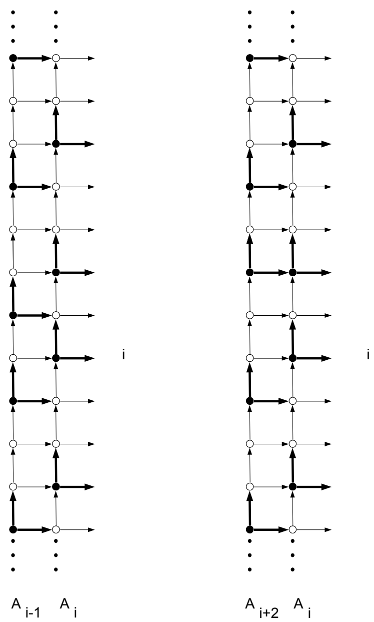

Figure 1: A_i,A_{i-1} and A_{i+2}

Lemma 3 *Let $m = 3k_1+2$. Let S be a A-set and for any j in $\{0,1,\ldots,n-1\}$ let i_j such that $S \cap C_m^j = A_{i_j}$. Assume that:*

$$\begin{cases} \textit{for any } j \in \{1,\ldots,n-1\}\ \ i_j = i_{j-1}+1\ (mod\,m)\ \textit{ or } i_j = i_{j-1}-2\ (mod\,m) \\ \textit{and} \\ i_0 = i_{n-1}+1\ (mod\,m)\ \textit{ or } i_0 = i_{n-1}-2\ (mod\,m). \end{cases}$$

then S is a dominating set of $C_m \square C_n$.

Proof :

Note first that for any i in $\{0, 1, \ldots, m-1\}$ the set of non dominated vertices of C_m by A_i is $T = \{i+4, i+7, \ldots, i-4, i-1\}$. Note also that $A_{i+2} = \{i+2\} \cup \{i+4, i+7, \ldots, i-4, i-1\}$ and $A_{i-1} = \{i-1\} \cup \{i+1, i+4, \ldots, i-7, i-4\}$. Thus $T \subset A_{i+2}$ and $T \subset A_{i-1}$.

Let j in $\{1, \ldots, n-1\}$. Let us prove that the vertices of C_m^j are dominated. Indeed, by the previous remark and the lemma hypothesis, the vertices non dominated by $S \cap C_m^j$ are dominated by $S \cap C_m^{j-1}$(see Figure 1). For the same reasons the vertices of C_m^0 are dominated by those of $S \cap C_m^0$ and $S \cap C_m^{n-1}$.

$\square$

Lemma 4 *Let $m = 3k_1 + 2$. If there exist integers $a, b \geq 0$ such that*

$$\begin{cases} a + b = n - 1 \\ \textit{and} \\ a - 2b \equiv 2\ (mod\, m) \textit{ or } a - 2b \equiv m - 1\ (mod\, m) \end{cases}$$

then $\gamma(C_m \Box C_n) = n(k_1 + 1)$.

Proof : Consider a word $w = w_1 \ldots w_{n-1}$ on the alphabet $\{1, -2\}$ with a occurrences of 1 and b of -2. Such a word exists, for example $w = 1^a(-2)^b$. We can associate to w a set S of vertices of $C_m \Box C_n$ by the following algorithm:

- $S \cap C_m^0 = A_0$
- For $i = 1$ to $n-1$ do
  begin
  Let k such that $S \cap C_m^{i-1} = A_k$
  If $w_i = 1$ let $k' = k + 1\ (mod\, m)$ else $k' = k - 2\ (mod\, m)$
  $S \cap C_m^i := A_{k'}$
  end

By construction S is a A-set. Notice that we have $S \cap C_m^{n-1} := A_{i_{n-1}}$ where $i_{n-1} \equiv \sum_{k=1}^{n-1} w_k \equiv a - 2b(mod\, m)$. Thus $i_{n-1} = 2$ or $i_{n-1} = m - 1(mod\, m)$. By lemma 3 S is a dominating set. Furthermore, because S is a A-set, $|S| = n(k_1 + 1)$,; thus by theorem 2 it is minimum and we have $\gamma(C_m \Box C_n) = n(k_1 + 1)$. $\square$

Theorem 5 *Let $m, n \geq 2$ and $m \equiv 2 \bmod 3$. Let $k_1 = \left\lfloor \frac{m}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{n}{3} \right\rfloor$.*

- *if $n = 3k_2$ then $\gamma(C_m \Box C_n) = n(k_1 + 1)$*
- *if $n = 3k_2 + 1$ and $2k_2 \geq k_1$ then $\gamma(C_m \Box C_n) = n(k_1 + 1)$*
- *if $n = 3k_2 + 1$ and $2k_2 < k_1$ then $\gamma(C_m \Box C_n) > n(k_1 + 1)$*
- *if $n = 3k_2 + 2$ and $n \geq m$ then $\gamma(C_m \Box C_n) = n(k_1 + 1)$.*
- *if $n = 3k_2 + 2$ and $n \leq m$ then then $\gamma(C_m \Box C_n) = m(k_2 + 1)$.*

Proof : We will use lemma 4 considering the following integer solutions of

$$\begin{cases} a, b \geq 0 \\ a + b = n - 1 \\ a - 2b \equiv 2\ (mod\, m) \text{ or } a - 2b \equiv m - 1\ (mod\, m). \end{cases}$$

- if $n = 3k_2$ then $k_2 \geq 1$. Take $a = 2k_2$ and $b = k_2 - 1$
- if $n = 3k_2 + 1$ and $2k_2 \geq k_1$ then take $a = 2k_2 - k_1$ and $b = k_2 + k_1$
- if $n = 3k_2 + 1$ and $2k_2 < k_1$ then $\gamma(C_m \square C_n) = \gamma(C_n \square C_m) \geq \frac{(2k_2+1)m}{2}$ by theorem 2. Furthermore $\frac{(2k_2+1)m}{2} - n(k_1 + 1) = \frac{k_1}{2} - k_2 > 0$
- if $n = 3k_2 + 2$ and $k_2 \geq k_1$ then take $a = 2k_2 - 2k_1$ and $b = k_2 + 2k_1 + 1$
- if $n = 3k_2 + 2$ and $k_2 \leq k_1$ then use $\gamma(C_m \square C_n) = \gamma(C_n \square C_m)$.

□

3 The case $m \equiv 0 \ (mod\, 3)$

In [12] Liu and al. conjectured the following formula:

Conjecture 6 *Let $k \geq 2$. Then $\gamma(C_{3k} \square C_n) = k(n+1)$ for $n \not\equiv 0 \ (mod\, 3)$.*

Our theorem 5 confirms the conjecture when $n \equiv 2 \ (mod\, 3)$. Unfortunately the formula is not always valid when $n \equiv 1 \ (mod\, 3)$.

Indeed consider $C_{3k} \square C_4$. In [11] the following result is proved:

Theorem 7 *Let $n \geq 2$. Then $\gamma(C_4 \square C_n) = \frac{3n}{2}$ if $n \equiv 0 \ (mod\, 8)$ and $\gamma(C_4 \square C_n) = n + \lceil \frac{n+1}{2} \rceil$ otherwise*

We have thus $\gamma(C_{3k} \square C_4) = \gamma(C_4 \square C_{3k}) = 3k + \lceil \frac{3k+1}{2} \rceil$ when $k \not\equiv 0 \ (mod\, 8)$. On the other side the conjecture 6 proposes the value $\gamma(C_{3k} \square C_4) = 5k$. These two numbers are different when $k \geq 3$.

References

[1] T.Y. Chang, W.E. Clark: "The Domination numbers of the $5 \times n$ and $6 \times n$ grid graphs", *J.Graph Thory*, **17** (1993) 81-107.

[2] M. El-Zahar, C.M. Pareek: "Domination number of products of graphs", *Ars Combin.*, **31** (1991) 223-227.

[3] M. El-Zahar, S. Khamis, Kh. Nazzal: "On the Domination number of the Cartesian product of the cycle of length n and any graph", *Discrete App. Math.*, **155** (2007) 515-522.

[4] R.J. Faudree, R.H. Schelp: 'The Domination number for the product of graphs", *Congr. Numer.*, **79** (1990) 29-33.

[5] S. Gravier, M. Mollard: "On Domination numbers of Cartesian product of paths", *Discrete App. Math.*, **80** (1997) 247-250.

[6] B. Hartnell, D. Rall: "On dominating the Cartesian product of a graph and K_2", *Discuss. Math. Graph Theory*, **24(3)** (2004) 389-402.

[7] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater: *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater eds.: *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.

[9] M.S. Jacobson, L.F. Kinch: "On the Domination number of products of graphs I", *Ars Combin.*, **18** (1983) 33-44.

[10] S. Klavžar, N. Seifter: "Dominating Cartesian products of cycles", *Discrete App. Math.*, **59** (1995) 129-136.

[11] J. Liu, X.D. Zhang, X. Chen, J.Meng: "The Domination number of Cartesian products of directed cycles", *Inf. Process. Lett.*, **110(5)** (2010) 171-173.

[12] J. Liu, X.D. Zhang, X. Chen, J.Meng: "On Domination number of Cartesian product of directed cycles", *Inf. Process. Lett.*, **111(1)** (2010) 36-39.