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GEOMETRICAL DESCRIPTION OF SMOOTH PROJECTIVE SYMMETRIC VARIETIES WITH PICARD NUMBER ONE

Alessandro Ruzzi

Abstract

In [A. RUZZI, *Smooth projective symmetric varieties with Picard number equal to one*, to appear in Int. J. of Math.] we have classified the smooth projective symmetric G -varieties with Picard number one (and G semisimple). In this work we give a geometrical description of such varieties. In particular, we determine their automorphism group. When this group acts non-transitively on X , we describe a G -equivariant embedding of the variety X in a homogeneous variety (with respect to a larger group). We show that these varieties are all related to the exceptional groups.

keywords: Symmetric varieties, Fano varieties.

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Let G be a connected semi-simple group over \mathbb{C} and let θ be an involution of G . Let H be a subgroup of G such that $(G^\theta)^0 \subset H \subset N_G(G^\theta)$, then G/H is called a *symmetric space*. It is known that G/H is spherical, i.e. it has an open orbit under the action of an arbitrarily fixed Borel subgroup of G . Moreover, an open, normal equivariant embedding $G/H \hookrightarrow X$ of a symmetric space is called a *symmetric variety*. In [Ru07] we have classified the smooth complete symmetric varieties with Picard number one, by using the theory of Luna-Vust colored fans. In particular, we have proved that all these varieties are projective and actually Fano, i.e. their anticanonical bundle is ample. Indeed, their canonical bundle cannot be ample because all these varieties are rational. Furthermore, we have proved that, given a symmetric space G/H , there is at most one embedding of G/H with the previous properties.

There are some other known examples of smooth complete spherical varieties with Picard number one: i) the projective space is the unique toric variety with the previous properties; ii) the homogeneous varieties with the previous properties are the G/P , where G is a simple group and P is a maximal parabolic subgroup; iii) the horospherical ones are been classified by B. Pasquier in [Pa07]; iv) the wonderful varieties with Picard number one are classified in [Akh83]. There are also some partial results on the classification of Fano symmetric varieties with Picard number strictly greater than one in [Ru06].

The open equivariant normal embeddings of a fixed spherical space G/H are classified by certain combinatorial invariants, the so-called colored fans. These invariants are defined in terms of some invariants of G/H , the spherical roots

and the colors, which in the symmetric case are strongly related to some classical objects, naturally associated to the symmetric space: the restricted root system and the spherical weights.

In [Ru07] we have classified the smooth complete symmetric varieties with Picard number one by exhibiting the associated colored fan (and the open orbit). In this paper we prove the following geometrical description of such varieties.

Let V be a n -dimensional vector space, we denote by $\mathbb{G}_m(V)$ the *Grassmannian of the m -dimensional subspaces* of V . If $V = \mathbb{C}^n$ we denote $\mathbb{G}_m(V)$ by $\mathbb{G}_m(n)$. Let q be a non-degenerate symmetric bilinear form on V and let $SO(V, q)$ be the corresponding special orthogonal group. We say that a subspace of V is isotropic if the restriction of q to it is zero. Fix an integer m such that $2m \leq n$ and let $\mathbb{I}\mathbb{G}_m(V) \subset \mathbb{G}_m(V)$ be the algebraic subvariety whose points are identified with isotropic m -dimensional subspaces of V . The group $SO(V, q)$ acts on $\mathbb{I}\mathbb{G}_m(V)$ in the natural manner and the action is transitive if $2m < n$. Instead, if $2m = n$, $\mathbb{I}\mathbb{G}_m(V)$ consists of two isomorphic $SO(V, q)$ -orbits (each of them being a connected component of $\mathbb{I}\mathbb{G}_m(V)$); we denote such orbit by $\mathbb{S}_m(V)$ (or by \mathbb{S}_m if $V = \mathbb{C}^{2m}$ and q is the standard symmetric bilinear form). The variety $\mathbb{I}\mathbb{G}_m(V)$ is called the *isotropic Grassmannian of m -dimensional isotropic subspaces*, while $\mathbb{S}_m(V)$ is called the *spinorial variety of order m* . In an analogous manner, let q' be a non-degenerate skew-symmetric bilinear form on V and fix an integer m such that $2m \leq n$. We denote by $\mathbb{L}\mathbb{G}_m(V)$ the algebraic subvariety of $\mathbb{G}_m(V)$ whose points are identified with isotropic m -dimensional subspaces of V ; $\mathbb{L}\mathbb{G}_m(V)$ is called the *Lagrangian Grassmannian*.

Many properties of the smooth complete symmetric varieties with Picard number one depend only by the restricted root system. In particular, we have the following characterization of the non-homogeneous varieties (see §1.1.2 for the definition of the restricted root system and see §1.1.3 for the definition of the type of an involution):

Theorem 1 *Let X be a smooth equivariant completion of a symmetric space G/H with Picard number one (where G is semisimple and simply connected). Then $\text{Aut}(X)$ acts non-transitively on X if and only if: i) the restricted root system has type either A_2 or G_2 ; ii) H is the subgroup of invariants G^θ .*

The varieties with restricted root system G_2 have the following description:

Theorem 2 *Let G/G^θ be a symmetric space whose restricted root system has type G_2 . We have the following possibilities for the smooth equivariant completion X of G/G^θ with Picard number one:*

1. *The smooth equivariant completion with Picard number one of the symmetric space $G_2/(SL_2 \times SL_2)$ of type G . In this case, $\text{Aut}(X) = G_2$. The involution θ can be extended to an involution of SO_7 , whose invariant subgroup is $S(O_3 \times O_4)$. The unique smooth equivariant completion of $SO_7/S(O_3 \times O_4)$ with Picard number one is isomorphic to $\mathbb{G}_3(7) \subset \mathbb{P}^{34}$*

and X is the intersection of $\mathbb{G}_3(7)$ with a 27-dimensional G_2 -stable subspace of \mathbb{P}^{34} . In particular, if we interpret \mathbb{C}^7 as the subspace of imaginary elements of the complexified octonions $\mathbb{O}_{\mathbb{C}}$, then X parametrizes the subspaces W of \mathbb{C}^7 such that $W \oplus \mathbb{C}1$ is a subalgebra of $\mathbb{O}_{\mathbb{C}}$ isomorphic to the complexified quaternions.

2. The smooth equivariant completion with Picard number one of the symmetric $(G_2 \times G_2)$ -space G_2 . In this case, $\text{Aut}(X)$ is generated by $G_2 \times G_2$ and θ . The involution θ can be extended to an involution of $SO_7 \times SO_7$, with invariant subgroup equal to the diagonal. The unique smooth equivariant completion of SO_7 with Picard number one is isomorphic to the spinorial variety $\mathbb{S}_7 \subset \mathbb{P}^{63}$ and X is the intersection of \mathbb{S}_7 with a 49-dimensional G -stable subspace of \mathbb{P}^{63} .

The non-homogeneous varieties with restricted root system A_2 have the following description (see [LaMa01] or §1.3 for a definition of $\mathbb{L}\mathbb{G}_3(\mathbb{O}^6)$):

Theorem 3 *Let G/G^θ be a symmetric space whose restricted root system has type A_2 (and G is simply-connected). We have the following possibilities for the smooth equivariant completion X of G/G^θ with Picard number one:*

1. the smooth equivariant completion with Picard number one of the symmetric variety SL_3/SO_3 of type AI ; it is a hyperplane section of $\mathbb{L}\mathbb{G}_3(6)$.
2. the smooth equivariant completion with Picard number one of the symmetric variety SL_3 ; it is a hyperplane section of $\mathbb{G}_3(6)$.
3. the smooth equivariant completion with Picard number one of the symmetric variety SL_6/Sp_6 of type AII ; it is a hyperplane section of \mathbb{S}_{12} .
4. the smooth equivariant completion with Picard number one of the symmetric variety E_6/F_4 of type EII ; it is a hyperplane section of $E_7/P_7 \equiv \mathbb{L}\mathbb{G}_3(\mathbb{O}^6)$.

Moreover, $SL_3/SO_3 \subset SL_3 \subset SL_6/Sp_6 \subset E_6/F_4$ and also their smooth equivariant completions with Picard number one are contained nested in each other:

$$\begin{array}{ccccccc}
\overline{SL_3/SO_3} & \hookrightarrow & \overline{SL_3} & \hookrightarrow & \overline{SL_6/Sp_6} & \hookrightarrow & \overline{E_6/F_4} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{L}\mathbb{G}_3(6) & \hookrightarrow & \mathbb{G}_3(6) & \hookrightarrow & \mathbb{S}_{12} & \hookrightarrow & \mathbb{L}\mathbb{G}_3(\mathbb{O}^6)
\end{array}$$

The connected automorphism group of X is isomorphic to G , up to isogeny. If G/H is different from SL_3 , then the automorphism group of X is generated by $\text{Aut}^0(X)$ and θ . If, instead, $G/H = SL_3$ then $\text{Aut}^0(X)$ has index four in $\text{Aut}(X)$. Finally, $\text{Aut}(X)$ has always three orbits in X .

The completion of SL_3 was already studied by J. Buczyński (see [Bu07]). The second row of the previous diagram is the third row of the Freudenthal magic square (see [LaMa01]). Furthermore, the varieties in this row are Legendrian varieties (see [LaMa07] for the relative definition).

Finally, we have the following description of the homogeneous varieties. The smooth complete symmetric varieties with Picard number one and rank one are described by the following theorem of Akhiezer, which classifies the smooth complete G -varieties with two orbits, whose the closed one has codimension one. Indeed, the complete symmetric varieties with rank one have a unique non-open orbit which has codimension one (because the open orbit is affine). Moreover, they are smooth because they are normal and their singular locus is G -stable.

Theorem 4 (see [Akh83], Theorem 4) *Let G be a semisimple group and let H be a closed reductive subgroup. Let X be a smooth equivariant completion of G/H such that $X \setminus (G/H)$ is a G -orbit (of codimension one). Then X is a homogeneous space for a larger group. If G/H is a symmetric variety and G acts effectively on G/H , we have the following possibilities:*

1. G is $SL_2 \times SL_2$, H is SL_2 and X is $\{(x, t) \mid \det(x) = t^2\} \subset \mathbb{P}(M_2(\mathbb{C}) \oplus \mathbb{C})$;
2. G is $PSL_2 \times PSL_2$, H is PSL_2 and X is $\mathbb{P}(M_2(\mathbb{C}))$;
3. G is SL_n , $n \geq 2$, H is GL_{n-1} , θ has type AIV (or AI if $n = 2$) and X is $\mathbb{P}(\mathbb{C}^n) \times \mathbb{P}(\mathbb{C}^n)^*$;
4. G is PSL_2 , H is PSO_2 , θ has type AI and X is $\mathbb{P}(\mathfrak{sl}_2)$;
5. G is Sp_{2n} , $n \geq 3$, H is $Sp_2 \times Sp_{2n-2}$, θ has type CII and X is $\mathbb{G}_2(2n)$;
6. G is SO_n , $n \geq 5$, H is SO_{n-1} , θ has type AII (if $n = 6$), BII or DII and X is $\{(x, t) \mid q(x, x) = t^2\} \subset \mathbb{P}(\mathbb{C}^n \oplus \mathbb{C})$, where q is the standard symmetric bilinear form;
7. G is SO_n , $n \geq 5$, H is $S(O_1 \times O_{n-1})$, θ has type AII (if $n = 6$), BII or DII, and X is $\mathbb{P}(\mathbb{C}^n)$;
8. G is F_4 , H is $Spin_9$, θ has type FII and X is $E_6/P_{\omega_1} \cong \mathbb{P}^2 \circ_{\mathbb{C}}$.

We have the following cases when the rank is at least 2.

Theorem 5 *Let X be a smooth complete symmetric variety with Picard number one and rank at least 2. Denote its open orbit by G/H . If $Aut(X)$ acts transitively on X we have the following possibilities:*

1. The restricted root system has type $A_1 \times A_1$ and $G/N_G(G^\theta)$ is isomorphic to $SO_n/S(O_1 \times O_{n-1}) \times SO_m/S(O_1 \times O_{m-1})$ with $n, m \geq 3$. Then H has index 2 in $N_G(G^\theta)$ and X is isomorphic to $\mathbb{I}\mathbb{G}_1(n+m)$.
2. The restricted root system has type A_l , $l \geq 2$ and H is $N_G(G^\theta)$. Then X is isomorphic to the projectivization of an irreducible G -representation. More precisely we have the following varieties:

- G/H is isomorphic to PGL_{l+1} and X is isomorphic to $\mathbb{P}(M_{l+1}(\mathbb{C}))$ as $(PGL_{l+1} \times PGL_{l+1})$ -variety.
 - G/H is isomorphic to $SL_{l+1}/N_{SL_{l+1}}(SO_{l+1})$, (G, θ) has type AI and X is isomorphic to $\mathbb{P}(\text{Sym}^2(\mathbb{C}^{l+1}))$.
 - G/H is isomorphic to $SL_{2l+2}/N_{SL_{2l+2}}(Sp_{2l+2})$, (G, θ) has type AII and X is isomorphic to $\mathbb{P}(\wedge^2(\mathbb{C}^{2l+2}))$.
 - G/H is isomorphic to $E_6/N_{E_6}(F_4)$, (G, θ) has type EIV and X is isomorphic to $\mathbb{P}(\mathcal{J}_3(\mathbb{O}_{\mathbb{C}}))$.
3. G/H is isomorphic to SO_n and X is isomorphic to \mathbb{S}_n as $(SO_n \times SO_n)$ -variety ($n = 5$ or $n \geq 7$).
 4. G/H is isomorphic to $Spin_5$ and X is isomorphic to $\mathbb{L}\mathbb{G}_4(8)$ as $Spin_5 \times Spin_5$ -variety.
 5. G/H is isomorphic to Sp_{2l} , $l \geq 3$ and X is isomorphic to $\mathbb{L}\mathbb{G}_{2l}(4l)$ as $Sp_{2l} \times Sp_{2l}$ -variety.
 6. G/H is isomorphic to $SL_4/N_{SL_4}(S(GL_2 \times GL_2))$, (G, θ) has type AIII and X is isomorphic to $\mathbb{G}_2(6)$.
 7. G/H is isomorphic to $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$, (G, θ) has type CII and X is isomorphic to $E_6/P_1 \cong \mathbb{P}^2(\mathbb{O}_{\mathbb{C}})$.
 8. G/H is isomorphic to $SO_8/N_{SO_8}(GL_4)$, (G, θ) has type DIII and X is isomorphic to $\mathbb{G}_2(8)$.
 9. G/H is isomorphic to $SO_n/S(O_l \times O_{n-l})$, $n > 2l$ and (G, θ) has type either BI or DI ($n = 5$ or $n \geq 7$). Then X is isomorphic to $\mathbb{G}_l(n)$.
 10. G/H is isomorphic to $SO_{2l}/SO_l \times SO_l$ and (G, θ) has type DI ($l \geq 4$). Then X is isomorphic to $\mathbb{G}_l(2l)$.
 11. G/H is isomorphic to $Sp_{2n}/Sp_{2l} \times Sp_{2n-2l}$, $n \geq 3$, (G, θ) has type CII and X is isomorphic to $\mathbb{G}_{2l}(2n)$.

In the first section we introduce the notation and recall some general facts about symmetric varieties. In the second one we describe the non-homogeneous varieties, while in the third one we describe the homogeneous varieties. In particular, in §2.1 we will do some general considerations about the non-homogeneous varieties, in §2.2 we consider the non-homogeneous varieties with restricted root system G_2 and in §2.3 we consider the non-homogeneous varieties with restricted root system A_2 .

Notation. Throughout this work we say a s.c.s.v. with P.n.1 to mean a smooth complete symmetric variety with Picard number one. Similarly, we denote a smooth equivariant completion of G/H with Picard number one by s.e.c. of G/H with P.n.1.

1 Introduction and notation

1.1 Symmetric varieties and colored fans

In this section we introduce the necessary notation. The reader interested in the embedding theory of spherical varieties can see [Br97] or [Ti06]. In [Vu90] is explained such theory in the particular case of symmetric varieties.

1.1.1 First definitions

Let G be a connected semisimple algebraic group over \mathbb{C} and let θ be an involution of G . Let H be a closed subgroup of G such that $(G^\theta)^0 \subset H \subset N_G(G^\theta)$. We say that G/H is a *symmetric space*. *Unless otherwise specified, we always suppose G simply connected*. In this case the invariant subgroup G^θ is connected (see [Ti06], Remark 26.1). Notice that this assumption is not restrictive (see, for example, [Vu90], §2.1). An equivariant embedding of G/H is the data of a G -variety X together with an equivariant open immersion $G/H \hookrightarrow X$. We say that a normal equivariant embedding of a symmetric space is a *symmetric variety*. We say that a subtorus of G is split if $\theta(t) = t^{-1}$ for all its elements t . We say that a split torus of G of maximal dimension is a *maximal split torus* and that a maximal torus containing a maximal split torus is *maximally split*. One can prove that any maximally split torus is θ stable (see [Ti06], Lemma 26.5). We fix arbitrarily a maximal split torus T^1 and a maximally split torus T containing T^1 . Let R_G be the root system of G with respect to T and let R_G^0 be the root subsystem composed by the roots fixed by θ . We set $R_G^1 = R_G \setminus R_G^0$. We can choose a Borel subgroup B containing T and such that, if α is a positive root in R_G^1 , then $\theta(\alpha)$ is negative (see [dCoPr83], Lemma 1.2). A normal G -variety is called a *spherical variety* if it contains a dense orbit under the action of an arbitrarily chosen Borel subgroup of G . One can prove that BH is dense in G (see [dCoPr83], Proposition 1.3); thus the symmetric varieties are spherical.

1.1.2 Restricted root system

The classification of s.c.s.v. with P.n.1 depends on the restricted root system of G/H . In this section we define it. The subgroup $\chi(S)$ of $\chi(T^1)$ has finite index, so we can identify $\chi(T^1)_{\mathbb{R}}$ ($:= \chi(T^1) \otimes \mathbb{R}$) with $\chi(S)_{\mathbb{R}}$. Because T is θ -stable, θ induces an involution of $\chi(T)_{\mathbb{R}}$ which we call again θ . The inclusion $T^1 \subset T$ induces an isomorphism of $\chi(T^1)_{\mathbb{R}}$ with the (-1) -eigenspace of $\chi(T)_{\mathbb{R}}$ under the action of θ . Denote by W_G the Weyl group of G (with respect to T) and let (\cdot, \cdot) be the Killing form over $\text{span}_{\mathbb{R}}(R_G)$. We denote with the same symbol the restriction of (\cdot, \cdot) to $\chi(T^1)_{\mathbb{R}}$ and we identify $\chi(T^1)_{\mathbb{R}}$ with its dual $\chi_*(T^1)_{\mathbb{R}}$. The set $R_{G,\theta} := \{\beta - \theta(\beta) \mid \beta \in R_G\} \setminus \{0\}$ is a root system in $\chi(S)_{\mathbb{R}}$ (see [Vu90], §2.3 Lemme), which we call the *restricted root system* of (G, θ) ; we call the non-zero $\beta - \theta(\beta)$ the *restricted roots*. Notice that $R_{G,\theta}$ can be non-reduced. Usually we denote by β (respectively by α) a root of R_G (respectively of $R_{G,\theta}$); often

we denote by ϖ (respectively by ω) a weight of R_G (respectively of $R_{G,\theta}$). In particular, we denote by $\varpi_1, \dots, \varpi_n$ the fundamental weights of R_G (we have chosen the basis of R_G associated to B). Notice however that the weights of $R_{G,\theta}$ are weights of R_G . The involution $\iota := -v_0 \cdot \theta$ of $\chi(T)$ preserves the set of simple roots; moreover ι coincides with $-\theta$ modulo the lattice generated by R_G^0 (see [Ti06], p. 169). Here v_0 is the longest element of the Weyl group of R_G^0 . We denote by $\alpha_1, \dots, \alpha_s$ the elements of the basis $\{\beta - \theta(\beta) \mid \beta \in R_G \text{ simple}\} \setminus \{0\}$ of $R_{G,\theta}$. Let b_i be equal to $\frac{1}{2}$ if $2\alpha_i$ belongs to $R_{G,\theta}$ and equal to one otherwise; for each i we define α_i^\vee as the coroot $\frac{2b_i}{(\alpha_i, \alpha_i)}\alpha_i$. The set $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$ is a basis of the dual root system $R_{G,\theta}^\vee$, namely the root system composed by the coroots of the restricted roots. Let $\omega_1, \dots, \omega_s$ be the fundamental weights of $R_{G,\theta}$ with respect to $\{\alpha_1, \dots, \alpha_s\}$ and let $\omega_1^\vee, \dots, \omega_s^\vee$ be the fundamental weights of $R_{G,\theta}^\vee$ with respect to $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$. Let C^+ be the positive closed Weyl chamber of $\chi(S)_\mathbb{R}$; we call $-C^+$ the negative Weyl chamber. Given a dominant weight λ of G , we denote by $V(\lambda)$ the irreducible representation of highest weight λ . Given a basis of an irreducible root system, we order it as in [Hu72].

We say that a dominant weight $\varpi \in \chi(T)$ is a spherical weight if $V(\varpi)$ contains a non-zero vector fixed by G^θ . In this case, V^{G^θ} is one-dimensional and $\theta(\varpi) = -\varpi$, so ϖ belongs to $\chi(S)_\mathbb{R}$. The set of dominant weights of $R_{G,\theta}$ is the set of spherical weights. Moreover, supposing $\beta_j - \theta(\beta_j) = \alpha_i$ with $\beta_j \in R_G$ simple, ω_i is a positive multiple of $\varpi_j + \varpi_{\bar{\theta}(j)}$. More precisely, we have $\omega_i = \varpi_j + \varpi_{\bar{\theta}(j)}$ if $\bar{\theta}(j) \neq j$, $\omega_i = 2\varpi_j$ if $\bar{\theta}(j) = j$ and β_j is orthogonal to R_G^0 and $\omega_i = \varpi_j$ otherwise (see [ChMa03], Theorem 2.3 or [Ti06], Proposition 26.4). Thus the spherical lattice, i.e. the lattice generated by the spherical weights, coincides with the weight lattice of $R_{G,\theta}$ and C^+ is the intersection of $\chi(S)_\mathbb{R}$ with the positive closed Weyl chamber of R_G .

1.1.3 Classification of involutions

In this section we recall the classification of the involutions of a semisimple group (see also [Wa72], §1.1). We say that an involution (G, θ) is indecomposable if the unique θ -stable, normal, closed, connected subgroup of G is the trivial one. Moreover, (G, θ) is indecomposable if and only if $R_{G,\theta}$ is irreducible. Instead, if (G, θ) is not indecomposable, we say that it is decomposable. In this case G is a product of θ -stable, normal, closed, connected subgroups G_i and we write $(G, \theta) = \prod (G_i, \theta)$. Moreover G/G^θ is the product $\prod G_i/G_i^\theta$. We say that G/H is (in)decomposable if and only if (G, θ) is (in)decomposable. Finally, if (G, θ) is indecomposable we have two possibilities: i) G is simple; ii) $G = \dot{G} \times \dot{G}$ with \dot{G} simple and $\theta(x, y) := (y, x)$. In the last case, G/G^θ is isomorphic to \dot{G} on which G acts by left and right multiplication. The classification, up to conjugation, of the involutions of a simple group is as in figures 1 and 2. In the fourth column of these figures it is showed the Satake diagram of the involution. It is defined as follows: it is the Dynkin diagram of G with the following additional data: i) any vertex is either white or black; ii) a vertex is black if and only if

it corresponds to a simple root fixed by θ ; iii) two distinct white vertices are joined by a two-headed arrow if they corresponds to roots β_{i_1} and β_{i_2} such that $\bar{\theta}(\beta_{i_1}) = \beta_{i_2}$. In the sixth column we indicate the possibilities for the rank l of $R_{G,\theta}$ and for the rank n of G . In the last column we indicate with h.n.e. when the symmetric space G/G^θ is hermitian and non-exceptional. Similarly, we use the notation h.e. for hermitian exceptional symmetric spaces (see § 1.1.5 for the definitions). Remark that we allow G to be non-simply connected. If G is SO_n and θ has type BI , BII , DI or DII , then G^θ is $S(O_l \times O_{n-l})$ and is non-connected. Moreover, it is auto-normalizing and its identity component is $SO_l \times SO_{n-l}$.

Finally, if $G = \dot{G} \times \dot{G}$ (with \dot{G} simple) and $\theta(x, y) := (y, x)$, then R_G^0 is empty and the Satake diagram consists of two copies of the Dynkin diagram of \dot{G} . Moreover, each vertex of a copy of the Dynkin diagram of \dot{G} is joined to the corresponding vertex of the other copy. For example, if \dot{G} is SL_{l+1} , we have the following diagram:



1.1.4 Colored fans

We need the definition of colored fan associated to a symmetric variety only for the non-homogeneous case. Indeed, in the homogeneous case we need only to know that there is at most a s.e.c. of G/H with P.n.1.

Let $D(G/H)$ be the set of B -stable prime divisors of G/H ; its elements are called *colors*. Since BH/H is an affine open set, the colors are the irreducible components of $(G/H) \setminus (BH/H)$. We say that a *spherical variety* is *simple* if it contains a unique closed orbit. Let X be a simple symmetric variety with closed orbit Y . Let $D(X)$ be the subset of $D(G/H)$ consisting of the colors whose closure in X contains Y . We say that $D(X)$ is the set of *colors of X* . To each prime divisor D of X , we can associate the normalized discrete valuation v_D of $\mathbb{C}(G/H)$ whose ring is the local ring $\mathcal{O}_{X,D}$. One can prove that D is G -stable if and only if v_D is G -invariant, i.e. $v_D(s \cdot f) = v_D(f)$ for each $s \in G$ and $f \in \mathbb{C}(G/H)$. Let N be the set of all G -invariant valuations of $\mathbb{C}(G/H)$ taking value in \mathbb{Z} and let $N(X)$ be the set of the valuations associated to the G -stable prime divisors of X . Given $\omega \in N(X)$, we denote by E_ω the associated G -stable divisor of X . Observe that each irreducible component of $X \setminus (G/H)$ has codimension one, because G/H is affine. Let $S := T/T \cap H \simeq T \cdot x_0$, where $x_0 = H/H$ denotes the base point of G/H . One can show that the group $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is isomorphic to the character group $\chi(S)$ of S (see [Vu90], §2.3); in particular, it is a free abelian group. We define the *rank of G/H* as the rank of $\chi(S)$. We can identify the dual group $Hom_{\mathbb{Z}}(\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*, \mathbb{Z})$

type	G	G^θ	Satake diagram	$R_{G,\theta}$		
AI	SL_{l+1}	SO_{l+1}	$\circ - \cdots - \circ$	A_l	$l \geq 1$	h.n.e. if $l = 1$
AII	SL_{2l+2}	Sp_{2l+2}	$\bullet - \circ - \bullet - \cdots - \bullet - \circ - \bullet$	A_l	$l \geq 1$	
$AIII$	SL_{n+1}	$S(GL_l \times GL_{n+1-l})$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet - \cdots - \circ$ with curved arrows from the first \circ to the last \bullet and from the last \bullet to the first \circ	BC_l	$\frac{n}{2} \geq l \geq 2$	h.e.
$AIII$	SL_{2l}	$S(GL_l \times GL_l)$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet - \cdots - \circ$ with curved arrows from the first \circ to the last \bullet and from the last \bullet to the first \circ	C_l	$l \geq 2$	h.n.e.
AIV	SL_{n+1}	GL_n	$\circ - \cdots - \circ - \bullet - \cdots - \bullet - \cdots - \circ$ with curved arrows from the first \circ to the last \bullet and from the last \bullet to the first \circ	BC_1	$n > l = 1$	h.e.
BI	SO_{2n+1}	$S(O_l \times O_{2n+1-l})$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet - \cdots - \bullet$	B_l	$n \geq l \geq 2$	h.n.e. if $l = 2$
BII	SO_{2n+1}	$S(O_1 \times O_{2n})$	$\circ - \bullet - \cdots - \bullet - \cdots - \bullet$	A_1	$n - 1 \geq l = 1$	
CI	Sp_{2l}	GL_l	$\circ - \cdots - \circ - \bullet - \cdots - \bullet$	C_l	$l \geq 3$	h.n.e.
CII	Sp_{2n}	$Sp_{2l} \times Sp_{2n-2l}$	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet$	BC_l	$\frac{n-1}{2} \geq l \geq 1$	
CII	Sp_{4l}	$Sp_{2l} \times Sp_{2l}$	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet$	C_l	$l \geq 2$	
DI	SO_{2n}	$S(O_l \times O_{2n-l})$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet$ with two branches from the last \bullet	B_l	$n-2 \geq l \geq 2$	h.n.e. if $l = 2$
DI	SO_{2l+2}	$S(O_l \times O_{l+2})$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet$ with two branches from the last \bullet	B_l	$l \geq 3$	
DI	SO_{2l}	$S(O_l \times O_l)$	$\circ - \cdots - \circ - \bullet - \cdots - \bullet$ with two branches from the last \bullet	D_l	$l \geq 4$	
DII	SO_{2n}	$S(O_1 \times O_{2n-1})$	$\circ - \bullet - \cdots - \bullet - \cdots - \bullet$ with two branches from the last \bullet	A_1	$n \geq 4$	
$DIII$	SO_{4l}	GL_{2l}	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet$ with two branches from the last \bullet	C_l	$l \geq 2$	h.n.e.
$DIII$	SO_{4l+2}	GL_{2l+1}	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet$ with two branches from the last \bullet	BC_l	$l \geq 2$	h.e.
EI	E_6	C_4	$\circ - \circ - \circ - \circ - \circ - \circ$ with a vertical branch from the 4th node	E_6	$l = 6$	
EII	E_6	$A_5 \times A_1$	$\circ - \circ - \circ - \circ - \circ - \circ$ with a vertical branch from the 3rd node and curved arrows from the 1st to 5th and 5th to 1st nodes	F_4	$l = 4$	
$EIII$	E_6	$D_5 \times \mathbb{C}^*$	$\circ - \bullet - \cdots - \bullet - \cdots - \circ$ with a vertical branch from the 3rd node and curved arrows from the 1st to 5th and 5th to 1st nodes	BC_2	$l = 2$	h.e.
EIV	E_6	F_4	$\circ - \bullet - \cdots - \bullet - \cdots - \circ$ with a vertical branch from the 3rd node	A_2	$l = 2$	

Figure 1: Involutions of simple groups

type	G	G^θ	Satake diagram	$R_{G,\theta}$		
EV	E_7	A_7		E_7	$l = 7$	
EVI	E_7	$D_6 \times A_1$		F_4	$l = 4$	
$EVII$	E_7	$E_6 \times \mathbb{C}^*$		C_3	$l = 3$	h.n.e.
$EVIII$	E_8	D_8		E_8	$l = 8$	
EIX	E_8	$E_7 \times A_1$		F_4	$l = 4$	
FI	F_4	$C_3 \times A_1$		F_4	$l = 4$	
FII	F_4	B_4		BC_1	$l = 1$	
G	G_2	$A_1 \times A_1$		G_2	$l = 2$	

Figure 2: Involutions of simple groups (cont.)

with the group $\chi_*(S)$ of one-parameter subgroups of S ; so we can identify $\chi_*(S)_{\mathbb{R}}$ with $\text{Hom}_{\mathbb{Z}}(\chi(S), \mathbb{R})$. The restriction map to $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is injective over N (see [Br97], §3.1 Corollaire 3), so we can identify N with a subset of $\chi_*(S)_{\mathbb{R}}$. We say that N is the *valuation semigroup* of G/H . Indeed, N is the semigroup constituted by the vectors in the intersection of the lattice $\chi_*(S)$ with an appropriate rational polyhedral convex cone \mathcal{CN} , called the *valuation cone*. For each color D , we define $\rho(D)$ as the restriction of v_D to $\chi(S)$. In general, the map $\rho : D(G/H) \rightarrow \chi_*(S)_{\mathbb{R}}$ is not injective.

Let $C(X)$ be the cone in $\chi_*(S)_{\mathbb{R}}$ generated by $N(X)$ and $\rho(D(X))$. We denote by $\text{cone}(v_1, \dots, v_r)$ the cone generated by the vectors v_1, \dots, v_r . Given a cone C in $\chi_*(S)_{\mathbb{R}}$ and a subset D of $D(G/H)$, we say that (C, D) is a colored cone if:

- C is generated by $\rho(D)$ and by a finite number of vectors in N ;
- the relative interior of C intersects \mathcal{CN} .

The map $X \rightarrow (C(X), D(X))$ is a bijection from the set of simple symmetric varieties to the set of colored cones (see [Br97], §3.3 Théorème).

Given a symmetric variety \tilde{X} (not necessarily simple), let $\{Y_i\}_{i \in I}$ be the set of G -orbits. Observe that \tilde{X} contains a finite number of G -orbits, thus $\tilde{X}_i := \{x \in \tilde{X} \mid \overline{G \cdot x} \supset Y_i\}$ is open in \tilde{X} and is a simple symmetric variety whose closed orbit is Y_i . We define $D(\tilde{X})$ as the set $\bigcup_{i \in I} D(\tilde{X}_i)$. The family $\{(C(\tilde{X}_i), D(\tilde{X}_i))\}_{i \in I}$ is called the colored fan of \tilde{X} and determines completely \tilde{X} (see [Br97], §3.4 Théorème 1). Moreover \tilde{X} is complete if and only if $\mathcal{CN} \subset \bigcup_{i \in I} C(\tilde{X}_i)$ (see [Br97], §3.4 Théorème 2).

Thus, there is a bijection between the orbit closures of a symmetric variety X and the colored cones in the colored fan of X ; moreover, this bijection reverses the inclusions. In particular, the whole X corresponds to $(0, \emptyset)$, while the closed orbits correspond to the maximal colored cones (see [Br97], §3.4 Proposition).

Furthermore, the 1-codimensional orbit closures correspond to the 1-dimensional cones (because G/H is affine).

Let X and X' be two symmetric varieties with the same open orbit G/H and associated respectively to the colored fans $\{(C, F)\}$ and $\{(C', F')\}$. Then the identity map $G/H \rightarrow G/H$ can be extended to a map $X \rightarrow X'$ if and only if for each colored cone (C, F) of X there is a colored cone (C', F') of X' such that $C \subset C'$ and $F \subset F'$ (see [Br97], §3.4 Théorème 2).

1.1.5 The sets N and $D(G/H)$

The set N is equal to $-C^+ \cap \chi_*(S)$; in particular, it consists of the lattice vectors of the rational polyhedral convex cone $CN = -C^+$. The set $\rho(D(G/H))$ is equal to $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$. Observe that $\rho(D(G/H))$ is a basis of $\chi_*(S)_\mathbb{R}$, so $\text{rank } G/H = \text{rank } R_{G,\theta}$. For each i , the fiber $\rho^{-1}(\alpha_i^\vee)$ contains at most 2 colors (see [Vu90], §2.4, Proposition 1 and Proposition 2). We say that a simple restricted root α_{i_0} is *exceptional* if there are two distinct simple roots β_{i_1} and β_{i_2} in R_G such that: 1) $\beta_{i_1} - \theta(\beta_{i_1}) = \beta_{i_2} - \theta(\beta_{i_2}) = \alpha_{i_0}$; 2) either $\theta(\beta_{i_1}) \neq -\beta_{i_2}$ or, if $\theta(\beta_{i_1}) = -\beta_{i_2}$, $(\beta_{i_1}, \beta_{i_2}) \neq 0$. In this case we say that also $\alpha_{i_0}^\vee$, θ and all the equivariant embeddings of G/H are exceptional. If G/H is exceptional, then ρ is not injective. We say that G/H contains a *Hermitian factor* if the center of G^θ has positive dimension. If G/H contains a Hermitian factor and is indecomposable, we would rather say that G/H is Hermitian. In this case $R_{G,\theta}$ has type BC_l , C_l , B_2 or A_1 . Moreover, an indecomposable symmetric space is exceptional if and only if it is Hermitian and its restricted root system has type BC_l . If G/H does not contain a Hermitian factor, then ρ is injective (see [Vu90], §2.4, Proposition 1). Furthermore, given an Hermitian G/H , ρ is injective if and only if G/H is not exceptional and $H = N_G(H)$. If ρ is injective, we denote by D_{α^\vee} the unique color contained in $\rho^{-1}(\alpha^\vee)$. If G/G^θ does not contain a Hermitian factor, G is simply-connected and V is an irreducible G -representation which contains a line fixed by G^θ , then the highest weight of V is spherical, because G^θ is connected and semisimple.

Given an involution θ of G , to choose a subgroup $G^\theta \subset H \subset N_G(G^\theta)$ is equivalent to give a lattice included between the restricted root lattice and the spherical weight lattice; this lattice would be $\chi(S)$. By [Vu90], Lemme 3.1, $\chi(S)$ is the restricted root lattice (resp. the spherical weight lattice) if and only if H is $N_G(G^\theta)$ (resp. G^θ).

1.1.6 Local description of symmetric varieties

Given a root β , let U_β be the unipotent one-dimensional subgroup of G corresponding to β . Given $\mu \in \chi_*(T)_\mathbb{Q} \equiv \chi(T)_\mathbb{Q}$, we denote by $P(\mu)$ the parabolic subgroup of G generated by T and by the subgroups U_β corresponding to the roots β such that $(\beta, \mu) \geq 0$. Given a parabolic subgroup $P = P(\mu)$, sometimes we denote by P^- the opposite standard parabolic subgroup, namely $P(-\mu)$.

Let X be a symmetric variety and let Y be a closed orbit of X . One can show that there is a unique B -stable affine open set X_B that intersects Y and is minimal for this property. Moreover, the complement $X \setminus X_B$ is the union of the B -stable prime divisors not containing Y (see [Br97], §2.2, Proposition). The stabilizer P of X_B is a parabolic subgroup containing B . Suppose X smooth or non-exceptional and let L be the standard Levi subgroup of P . Then L is θ -stable, $\theta(P) = P^-$ and the P -variety X_B is the product $R_u P \times Z$ of the unipotent radical of P and of an affine L -symmetric variety Z containing x_0 (see [Br97], §2.3, Théorème and [Ru07], Lemma 2.1). Moreover, $C(Z)$ is equal to $C(X')$, where X' is the simple subvariety of X with closed orbit Y . If Y is projective, then Z contains a L -fixed point whose stabilizer in G is P^- .

Remark. Suppose that G/H does not contain a Hermitian factor and let X be a simple embedding of G/H corresponding to a colored cone (C, F) with $\dim C = \text{rank } G/H$. Then the closed orbit of X is a homogeneous variety G/P^- , with P^- standard (with respect to B^-). Moreover, P^- is associated to the set of simple roots $\{\beta\}$ such that either $\beta = \theta(\beta)$ or $(\beta - \theta(\beta))^\vee \in F$ (see [Ru07] pages 6-9). Moreover, the dual root system of $R_{L,\theta}$ has basis $\rho(F)$. We need these properties in §2.2 to give an explicit description of some non-homogeneous s.c.s.v. with P.n.1.

1.2 Combinatorial classification of smooth complete symmetric varieties with Picard number one

In [Ru07] we have proved the following combinatorial classification of the s.c.s.v. with P.n.1. Here we describe the colored fan of X only in the case in which we will need it, namely when X is non-homogeneous.

Theorem 1.1 (see [Ru07], Theorem 3.1) *Let G be a semisimple, simply connected group and let G/H be a symmetric space. Suppose that there is a s.e.c. X of G/H with P.n.1. Then:*

- *The embedding X is unique, up to equivariant isomorphism.*
- *The number of colors of G/H is equal to the rank l of G/H ; in particular there are no exceptional roots.*
- *We have exactly the following possibilities for $R_{G,\theta}$ and $\chi(S)$ (supposing ρ injective):*
 1. *$R_{G,\theta}$ has type $A_1 \times A_1$ and $\chi(S)$ has basis $\{2\omega_1, \omega_1 + \omega_2\}$;*
 2. *$R_{G,\theta}$ has type A_1 and $\chi(S)$ is the restricted root lattice;*
 3. *$R_{G,\theta}$ has type either A_1 or A_2 and $\chi(S)$ is the spherical weight lattice;*
 4. *$R_{G,\theta}$ has type B_1 and $\chi(S)$ is the restricted root lattice;*
 5. *$R_{G,\theta}$ has type B_2 and $\chi(S)$ is the spherical weight lattice;*
 6. *$R_{G,\theta}$ has type C_l and $\chi(S)$ is the spherical weight lattice;*

7. $R_{G,\theta}$ has type BC_l , so $\chi(S)$ coincides both with the restricted root lattice and the spherical root lattice;
 8. $R_{G,\theta}$ has type D_l with $l > 4$ and $\chi_*(S)$ is freely generated by $\omega_1^\vee, \dots, \omega_{l-2}^\vee, \omega_{l-1}^\vee + \omega_l^\vee, 2\omega_l^\vee$; in particular H has index two in $N_G(G^\theta)$;
 9. $R_{G,\theta}$ has type D_4 and $\chi_*(S) = \mathbb{Z}\omega_i^\vee \oplus \mathbb{Z}\omega_j^\vee \oplus \mathbb{Z}(\omega_j^\vee + \omega_k^\vee) \oplus \mathbb{Z}2\omega_k^\vee$, where $\{i, j, k, 2\}$ is a permutation of $\{1, 2, 3, 4\}$;
 10. $R_{G,\theta}$ has type G_2 , so $\chi(S)$ coincides both with the restricted root lattice and the spherical root lattice.
- If $R_{G,\theta} = A_2$ and $\chi(S)$ is the spherical weight lattice, then the s.e.c. of G/H with P.n.1 has two closed orbits. Its colored fan has maximal colored cones $(\text{cone}(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_1^\vee}\})$ and $(\text{cone}(\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_2^\vee}\})$.
 - If $R_{G,\theta} = G_2$, then the s.e.c. of G/H with P.n.1 is simple and is associated to $(\text{cone}(\alpha_2^\vee, -\omega_2^\vee), \{D_{\alpha_2^\vee}\})$.

1.3 Description of some exceptional groups via composition algebras

We need a description of some exceptional groups via complex composition algebras and Jordan algebras. The interested reader can see [LaMa01] and [Ad96] for a detailed exposition of the facts which we recall here.

Let \mathbb{A} be a *complex composition algebra*, i.e. $\mathbb{A} = \mathbb{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathbb{A}_{\mathbb{R}}$ is a real division algebra (namely $\mathbb{A}_{\mathbb{R}} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O}). If $a \in \mathbb{A}$, let \bar{a} be its conjugate; we denote by $Im \mathbb{A}$ the subspace of pure imaginary elements, i.e. the elements a such that $\bar{a} = -a$. Let $\mathcal{J}_3(\mathbb{A})$ be the space of \mathbb{A} -Hermitian matrices of order three:

$$\mathcal{J}_3(\mathbb{A}) = \left\{ \begin{pmatrix} r_1 & \bar{x}_3 & \bar{x}_2 \\ x_3 & r_2 & \bar{x}_1 \\ x_2 & x_1 & r_3 \end{pmatrix}, r_i \in \mathbb{C}, x_i \in \mathbb{A} \right\}.$$

$\mathcal{J}_3(\mathbb{A})$ has the structure of a Jordan algebra with the multiplication $A \circ B := \frac{1}{2}(AB + BA)$, where AB is the usual matrix multiplication. There is a well defined cubic form on $\mathcal{J}_3(\mathbb{A})$, which we call the determinant. Given $P \in \mathcal{J}_3(\mathbb{A})$, its comatrix is defined by

$$com P = P^2 - (\text{trace } P)P + \frac{1}{2}((\text{trace } P)^2 - \text{trace } P^2)I$$

and characterized by the identity $com(P)P = \det(P)I$. In particular, P is invertible if and only if $\det(P)$ is different from 0.

Let $SL_3(\mathbb{A}) \subset GL_{\mathbb{C}}(\mathcal{J}_3(\mathbb{A}))$ be the subgroup preserving the determinant; $\mathcal{J}_3(\mathbb{A})$ is an irreducible $SL_3(\mathbb{A})$ -representation. We let $SO_3(\mathbb{A})$ denote the group of complex linear transformations preserving the Jordan multiplication; it is also the subgroup of $SL_3(\mathbb{A})$ preserving the quadratic form $Q(A) = \text{trace}(A^2)$.

We call $\mathcal{Z}_2(\mathbb{A}) := \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}^*$ the *space of Zorn matrices*. The space $\mathfrak{sp}_6(\mathbb{A}) := \mathbb{C}^* \oplus \mathcal{J}_3(\mathbb{A})^* \oplus (\text{Lie}(SL_3(\mathbb{A}))) \oplus \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathbb{C}$ has a structure

of Lie algebra and \mathcal{Z}_2 has a natural structure of (simple) $\mathfrak{sp}_6(\mathbb{A})$ -module. There is a unique closed connected subgroup of $GL_{\mathbb{A}}(\mathcal{Z}_2(\mathbb{A}))$ with Lie algebra $\mathfrak{sp}_6(\mathbb{A})$; we denote it by $Sp_6(\mathbb{A})$. Moreover, there is an $Sp_6(\mathbb{A})$ -invariant symplectic form on $\mathcal{Z}_2(\mathbb{A})$.

The closed $Sp_6(\mathbb{A})$ -orbit in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ is the image of the $Sp_6(\mathbb{A})$ -equivariant rational map:

$$\begin{aligned} \phi : \mathbb{P}(\mathbb{C} \oplus \mathcal{J}_3(\mathbb{A})) & \dashrightarrow & \mathbb{P}(\mathcal{Z}_2(\mathbb{A})) \\ (x, P) & \rightarrow & (x^3, x^2 P, x \operatorname{com}(P), \det(P)). \end{aligned}$$

Furthermore, if \mathbb{C} is interpreted as a space of diagonal matrices (in $\mathcal{J}_3(\mathbb{A})$) and (I, P) is interpreted as a matrix of three row vectors in \mathbb{A}^6 , then the previous map is the usual Plucker map. The condition $P \in \mathcal{J}_3(\mathbb{A})$ can be interpreted as the fact that the three vectors defined by the matrix (I, P) are orthogonal with respect to the Hermitian symplectic two-form $w(x, y) = {}^t x J \bar{y}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Therefore, it is natural to see the closed $Sp_6(\mathbb{A})$ -orbit in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ as the Grassmannian $\mathbb{L}\mathbb{G}_3(\mathbb{A}^6)$ of isotropic 3-planes in \mathbb{A}^6 .

Explicitly, we have the following possibilities:

$$\begin{array}{l} \mathbb{A}_R : \quad \mathbb{R}^{\mathbb{C}} \longrightarrow \mathbb{C}^{\mathbb{C}} \longrightarrow \mathbb{H}^{\mathbb{C}} \longrightarrow \mathbb{O} \\ \\ SO_3(\mathbb{A}) : \quad \begin{array}{ccccccc} SO_3^{\mathbb{C}} & \longrightarrow & SL_3^{\mathbb{C}} & \longrightarrow & Sp_6^{\mathbb{C}} & \longrightarrow & F_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ SL_3(\mathbb{A}) : \quad & SL_3^{\mathbb{C}} & \longrightarrow & SL_3 \times SL_3^{\mathbb{C}} & \longrightarrow & SL_6^{\mathbb{C}} & \longrightarrow & E_6 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ Sp_6(\mathbb{A}) : \quad & Sp_6^{\mathbb{C}} & \longrightarrow & SL_6^{\mathbb{C}} & \longrightarrow & Spin_{12}^{\mathbb{C}} & \longrightarrow & E_7 \end{array} \end{array}$$

2 Non-homogeneous varieties

In this section we describe the non-homogeneous s.c.s.v. with P.n.1. The most part of the proof that these varieties are really non-homogeneous can be done in a general way. We will do it in §2.1 together with some lemmas which will allow ourselves to determine the automorphism group of such varieties. In §2.2 and §2.3 we will complete the proofs by a case-by-case analysis.

2.1 Some general considerations

First, we show that if $\operatorname{Aut}^0(X)$ acts non-transitively on X , then it stabilizes a closed G -orbit.

Lemma 2.1 *Let X be a s.c.s.v. with P.n.1 and let Y be a proper G -stable closed subvariety of X . If Y is smooth, then it is a G -orbit.*

Proof. Recall that the minimal B -stable affine open set X_B which intersects Y is a product $P_u \times Z$, where Z is an affine L -symmetric variety (see §1.1.6). One can easily show that Z is an irreducible L -representation (see [Ru07], pages 9 and 17 or [Ru07], Theorem 2.2). Observe that Y is smooth if and only if $Y \cap Z$ is smooth. But $Y \cap Z$ is a cone because the center of L acts non-trivially on it. Thus $Y \cap Z$ is smooth if and only if it is a subrepresentation of Z . Since Z is irreducible, $Y \cap Z$ is smooth if and only if it is a point. In such case Y is a closed orbit. \square

We use the previous lemma only to prove the following corollary.

Corollary 2.1 *Let X be a s.c.s.v. with P.n.1. If $\text{Aut}^0(X)$ acts non-transitively on X , then $\text{Aut}^0(X)$ stabilizes a closed G -orbit.*

Proof. Let Y be a minimal closed G -stable subvariety stabilized by $\text{Aut}^0(X)$. Suppose by contradiction that Y is not a G -orbit, then it is singular by the Lemma 2.1. Thus the singular locus of Y is not empty and is stabilized by $\text{Aut}^0(X)$. In particular, its irreducible components are $\text{Aut}^0(X)$ -stable, closed subvarieties of X properly contained in Y ; a contradiction. \square

Now, we explain how to prove that a closed G -orbit is stabilized by $\text{Aut}^0(X)$.

Lemma 2.2 *Let G/P^- be a closed orbit of X and let Z be as in §1.1.6. If the highest weight of Z as L -representation is not dominant as weight of G , then G/P^- is stabilized by $\text{Aut}^0(X)$.*

One can prove also the vice versa, but we do not need it. Indeed, when a s.c.s.v. X with P.n.1 is isomorphic to a homogeneous variety, we will construct an explicit isomorphism between X and such variety.

Proof. A closed G -orbit G/P^- of X is stabilized by $\text{Aut}^0(X)$ if $H^0(G/P^-, N_{G/P^-, X}) = 0$, where $N_{G/P^-, X}$ is the normal bundle to G/P^- (see [Akh95], §2.3). Recall that X_B is $P_u \times Z$; moreover, the intersection of G/P^- with X_B is $P_u \times \{0\}$. Thus we can identify Z with the fiber of the normal bundle $N_{G/P^-, X}$ over P^-/P^- . We know by the Borel-Weil theorem (see [Akh95], §4.3) that $H^0(G/P^-, N_{G/P^-, X}) = 0$ if and only if the highest weight of Z as L -representation is not dominant as weight of G . \square

We will prove that if X is not homogeneous, then it has rank 2. Now, we give some combinatorial conditions to identify the highest weight of the L -representation Z .

Lemma 2.3 *Let X be a s.c.s.v. with P.n.1 and with rank 2. Let Z be as in §1.1.6 and let ω be the highest weight of Z as L -representation. Suppose that G/H does not contain a Hermitian factor and that there is a generator ϖ^\vee of $\chi_*(Z(L)^0/(Z(L)^0 \cap H))$ such that $(\lambda, \varpi^\vee) \geq 0$ for every $\lambda \in W_{L, \theta} \cdot C(Z)^\vee$. Let α be the simple restricted root of $R_{[L, L], \theta}$. Then ω is determined by the two conditions: 1) $(\omega, \alpha^\vee) = 1$; 2) $(\omega, \varpi^\vee) = 1$.*

Here, $W_{L, \theta}$ is the Weyl group of $R_{L, \theta}$.

Proof. By the remark of §1.1.6 the center of L has dimension one and $R_{[L,L],\theta}$ has rank one. By [Ru07], Theorem 2.2 we have $(\omega, \alpha^\vee) = 1$. Let σ be the cone in $\chi_*(S)_\mathbb{R}$ corresponding to the closure of $T \cdot x_0$ in Z . In proof of Lemma 2.8 in [Ru07] is proved that dual cone σ^\vee of σ is equal to $W_{L,\theta} \cdot C(Z)^\vee$. There exists $x' := \lim_{t \rightarrow 0} \varpi^\vee(t) \cdot x_0$ in the closure of $T \cdot x_0$ in Z because $(\lambda, \varpi^\vee) \geq 0$ for every $\lambda \in \sigma^\vee$. Thus $(\omega, \varpi^\vee) = 1$ because $(Z(L)^0 / (Z(L)^0 \cap H)) \cup x'$ is contained in Z (and Z has a unique isotypic component as $Z(L)^0$ -representation). To conclude it is sufficient to observe that $\chi(S)_\mathbb{R}$ has dimension 2. \square

Remark that, given a s.c.s.v. with P.n.1, with rank 2 and such that G/H does not contain a Hermitian factor, there is always a ϖ^\vee as in the previous hypotheses because of [Ru07], Lemma 2.14 (with the notation of such lemma ϖ^\vee is $\sum_{i=1}^l f_i$, where $\{f_1, \dots, f_l\}$ is the dual basis of the basis $\{e_1, \dots, e_l\}$ generating σ^\vee).

We define the *canonical completion of G/H* as the simple symmetric variety associated to the cone $(-C^+, \emptyset)$. We say that a symmetric variety is *toroidal* if $D(X) = \emptyset$; the canonical completion of G/H is the minimal toroidal completion of G/H . The orbit closures of a toroidal symmetric variety are the intersections of G -stable divisors. Given any symmetric variety X , there is a minimal toroidal variety \tilde{X} with a proper map $\pi : \tilde{X} \rightarrow X$ which extends the identity over G/H : if X has colored fan $\{(C_i, F_i)\}_{i \in I}$, then \tilde{X} has colored fan $\{(-C^+ \cap C_i, \emptyset)\}_{i \in I}$; we call \tilde{X} the *decoloration of X* . We say that the canonical completion of G/H is *wonderful* if it is smooth (see [Lu96], page 249 for the general definition of a wonderful variety).

In the following of this section, let X be a s.c.s.v. with P.n.1, with rank 2 and such that $\text{Aut}^0(X)$ stabilizes all the closed G -orbits. The decoloration \tilde{X} of X is the blow-up of X along the closed orbits (see Theorem 3.3 in [Br91]); in particular, \tilde{X} is smooth and the G -stable prime divisor of X is the image of a G -stable prime divisor of \tilde{X} .

Lemma 2.4 *Let X be an indecomposable s.c.s.v. with P.n.1 and with rank 2. Suppose that $\text{Aut}^0(X)$ stabilizes all the closed G -orbits in X . Then $\text{Aut}^0(\tilde{X})$ stabilizes all the G -orbits in \tilde{X} . Moreover, $\text{Aut}^0(\tilde{X})$ is isomorphic to $\text{Aut}^0(X)$.*

Proof. $\text{Aut}^0(X)$ is contained in $\text{Aut}^0(\tilde{X})$ because the closed orbits are stable under the action of $\text{Aut}^0(X)$. Moreover, by a result of A. Blanchard, $\text{Aut}^0(\tilde{X})$ acts on X in such a way that the projection is equivariant (see [Akh95], §2.4). Thus $\text{Aut}^0(\tilde{X})$ is isomorphic to $\text{Aut}^0(X)$. Since \tilde{X} is complete and $\text{Pic}(\tilde{X})$ is discrete, the group $\text{Aut}^0(\tilde{X})$ is linear algebraic and its Lie algebra is the space of global vector fields, namely $H^0(\tilde{X}, \mathcal{T}_{\tilde{X}})$.

First suppose X simple, then \tilde{X} is a wonderful G -variety. Moreover, $\text{Aut}^0(\tilde{X})$ is semisimple and \tilde{X} is a wonderful $\text{Aut}^0(X)$ -variety with set of colors $D(G/H)$ (see [Br07], Theorem 2.4.2). In [Br07] it is also determined the automorphism group of the wonderful completion of a simple adjoint group \bar{G} (see [Br07], Example 2.4.5); in particular, it is proved that, if $\bar{G} \neq PSL(2)$, the connected

automorphism group of the wonderful completion of \bar{G} is $\bar{G} \times \bar{G}$. Coming back to our problem, the wonderful G -variety \tilde{X} has two stable prime divisors: the exceptional divisor E and the strict transform \tilde{D} of the G -stable divisor D of X . The divisor \tilde{D} corresponds to a simple restricted root (see [dCoPr83], Lemma 2.2), which is not a dominant weight because the restricted root system is irreducible. Thus \tilde{D} is fixed by $Aut^0(\tilde{X})$ (see [Br07], Theorem 2.4.1), so $Aut^0(\tilde{X})$ fixes all the G -orbits in \tilde{X} .

We can proceed similarly in the case where X is non-simple. Indeed, using [Br89], Proposition 3.3, [Vu90], §2.4 Proposition 1 and Proposition 2, one can easily prove that $H^0(\tilde{X}, \mathcal{O}(F)) = \mathbb{C}s_F$ for each G -stable divisor F of \tilde{X} (here s_F is a global section with divisor F). Since $\xi \cdot s_F$ is a scalar multiple of s_F for any $\xi \in H^0(\tilde{X}, \mathcal{T}_{\tilde{X}}) = Lie(Aut^0(\tilde{X}))$, F is stabilized by $Aut^0(\tilde{X})$ as in the proof of Theorem 2.4.1 in [Br07] (see also Proposition 4.1.1 in [BiBr96]). \square

Lemma 2.5 *With the same hypotheses of the Lemma 2.4, the automorphism group $Aut(\tilde{X})$ of the decoloration \tilde{X} of X coincides with $Aut(G/H)$.*

Proof. The group $Aut(\tilde{X})$ is contained in $Aut(G/H)$ because it stabilizes G/H . Moreover, we can extend every automorphism of G/H to an automorphism of \tilde{X} , because \tilde{X} is normal and toroidal (these facts imply that each (non-open) G -orbit \mathcal{O} is contained in the closure of an orbit of dimension equal to $\dim \mathcal{O} + 1$). \square

Lemma 2.6 *Let X be an indecomposable s.c.s.v. with P.n.1 and with rank 2. Suppose that:*

- $Aut^0(X)$ stabilizes all the closed G -orbits in X ;
- $Aut(G/\tilde{P}^-) = G/Z(G)$ for a closed orbit G/\tilde{P}^- of \tilde{X} .

Then $Aut^0(X)$ is $G/(Z(G) \cap H)$.

Proof. Since $Aut^0(\tilde{X})$ stabilizes all the G -orbits in \tilde{X} , $Aut^0(\tilde{X})$ is reductive by a result of M. Brion (see [Br07], Theorem 4.4.1).

Notice that an element of $Z(G)$ acts trivially on G/H (and on \tilde{X}) if and only if it belongs to H , thus $Aut^0(\tilde{X})$ contains $G/(Z(G) \cap H)$. The restriction $\psi : Aut^0(\tilde{X}) \rightarrow Aut^0(G/\tilde{P}^-) = G/Z(G)$ is an isogeny, because $Aut^0(\tilde{X})$ stabilizes all the G -orbits. Indeed, $\ker \psi$ centralizes $G/(Z(G) \cap H)$, because $Aut^0(\tilde{X})$ is reductive, and stabilizes G/H , so it is contained in the finite group $Aut_G(G/H) \cong N_G(H)/H$. Therefore $Aut^0(X)$ is $G/(Z(G) \cap H)$. \square

Supposing the hypotheses of the Lemma 2.6 satisfied, we are going to study the full automorphism group $Aut(X)$. This group permutes the $Aut^0(X)$ -orbits; in particular it stabilizes the open orbit and the 1-codimensional orbit. Therefore $Aut(X)$ is contained in $Aut(G/H)$. We want to prove the converse. Notice that $Aut(X)$ is contained in $Aut(\tilde{X})$, because \tilde{X} is the blow-up of X along an $Aut(X)$ -stable (not necessarily connected) subvariety. Hence, we have to determine the automorphisms of \tilde{X} which descend to X .

Let $\tilde{\cdot} : \text{Aut}(\tilde{X}) \rightarrow \text{Aut}_{\text{alg}}(\text{Aut}^0(\tilde{X})) \cong \text{Aut}_{\text{alg}}(G)$ be the action of $\text{Aut}(\tilde{X})$ on $\text{Aut}^0(\tilde{X})$ by conjugation:

$$\tilde{\varphi}(g)(x) = \varphi(g \cdot \varphi^{-1}(x)), \quad \forall \varphi \in \text{Aut}(\tilde{X}), g \in \text{Aut}^0(\tilde{X}), x \in \tilde{X}.$$

Let T'' be a maximal torus of H , then its centralizer $T' := C_G(T'')$ is a maximal torus of G (see [Ti06], Lemma 26.2); moreover T'' contains a regular one-parameter subgroup λ of G . Thus $B' := P(\lambda)$ is a Borel subgroup of G and $B'' := (B' \cap H)^0$ is a Borel subgroup of H . The group $\text{Aut}_{\text{alg}}(G)$ is generated by $\overline{G} := G/Z(G)$ and $E = \{\psi \in \text{Aut}_{\text{alg}}(G) : \psi(T') = T' \text{ and } \psi(B') = B'\}$; the intersection of E with \overline{G} is $\overline{T}' := T'/Z(G)$, so $\text{Aut}_{\text{alg}}(G)$ is the semidirect product of \overline{G} and $E' := \{\psi \in E : \psi(t) = t \ \forall t \in T'\}$. Observe that every $\psi \in E$ induces an automorphism of the Dynkin diagram (with respect to T' and B'), moreover such automorphism is trivial if and only if ψ belongs to \overline{T}' .

Lemma 2.7 *Let X be as in Lemma 2.6. Let K be $\{\varphi \in \text{Aut}(G/H) : \varphi(x_0) = x_0 \text{ and } \tilde{\varphi} \in E\}$. Suppose that, if $N(H)/H$ is non-trivial, then: i) $N(H)/H$ is simple and ii) $Z(G)/(Z(G) \cap H)$ is non-trivial. Then $\text{Aut}(\tilde{X})$ is generated by $\text{Aut}^0(\tilde{X})$ and K . Moreover, the kernel of $\tilde{\cdot}$ is contained in $\text{Aut}^0(\tilde{X})$.*

Proof. $\text{Aut}(\tilde{X})$ is generated by $G/(Z(G) \cap H)$ and by the stabilizer of x_0 in $\text{Aut}(\tilde{X})$, so we have only to consider the elements of $(\text{Aut}(\tilde{X}))_{x_0}$. The kernel of $\tilde{\cdot}$ is the group of G -equivariant automorphisms of \tilde{X} (and G/H), thus it is isomorphic to $N_G(H)/H$. Moreover, it contains $Z(G)/(Z(G) \cap H)$, so it is $Z(G)/(Z(G) \cap H)$ by the hypotheses. In particular, it is contained in $\text{Aut}^0(\tilde{X})$. Notice that, given $\varphi \in \text{Aut}_{x_0}(\tilde{X})$, then $\tilde{\varphi}$ belongs to E up to composing φ with an element of $H/(Z(G) \cap H)$. Thus $\text{Aut}(G/H)$ is generated by $\text{Aut}^0(G/H) = G/(Z(G) \cap H)$ and by K . \square

The restriction of $\tilde{\cdot}$ to K is injective because $Z(\text{Aut}(G/H))$ does not fix x_0 . Observe that any automorphism of G stabilizing H induces an automorphism of G/H , so K is isomorphic to $K' := \{\psi \in E : \psi(H) = H\}$. Thus, to study $\text{Aut}(\tilde{X})$ it is sufficient to determine K' . Remark that the involution θ is always contained $\text{Aut}(X)$ by [Ti06], Lemma 26.2. We will prove that, if G/H is not isomorphic to SL_3 , then $K' \cap E'$ is E' ; in this case the map $\varphi \rightarrow \tilde{\varphi}$ is surjective.

Lemma 2.8 *Let X be as in Lemma 2.6. The automorphism group $\text{Aut}(\tilde{X})$ coincides with $\text{Aut}(X)$ if the following conditions are verified:*

- *If $N(H)/H$ is non-trivial, then it is simple and $Z(G)/(Z(G) \cap H)$ is non-trivial.*
- *If X is simple, then $R_{G,\theta}$ has type G_2 . Moreover, given any $\varphi \in K$, $\tilde{\varphi}$ induces a trivial automorphism of the Dynkin diagram (associated to T' and B').*

- If X is non-simple then $R_{G,\tilde{\theta}}$ has type A_2 . Moreover, any φ in $K \setminus \text{Aut}^0(G/H)$ exchanges $E_{-\omega_1^\vee}$ with $E_{-\omega_2^\vee}$. Furthermore, $\tilde{\varphi}$ induces a non-trivial automorphism of the Dynkin diagram (associated to T' and B').

Proof. We must prove that every automorphism φ in K descends to an automorphism of X . Observe that every automorphism of G/H can be extended to the 1-codimensional orbit of X , so we have to show that φ can be defined on the closed G -orbits of X .

First suppose \tilde{X} simple; in particular, φ stabilizes the closed orbit of \tilde{X} . Let x_1 be a point (in the closed orbit of \tilde{X}) fixed by B' and let \tilde{P}' be the stabilizer of x_1 in G . Then φ fixes x_1 because $\tilde{\varphi} \in E$. Let G/P' be the image of $G/\tilde{P}' \subset \tilde{X}$ in X . But φ induces the trivial automorphism of the Dynkin diagram (with respect to T' and B'), so $\tilde{\varphi}(P')$ is P' .

Finally, suppose that X contains two closed orbits. In this case $N(\tilde{X}) = \{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ and the closed G -orbits of X are $G/P(-\omega_1)$ and $G/P(-\omega_2)$; observe that $P(-\omega_1)$ and $P(-\omega_2)$ are not conjugated. The closed G -orbit \mathcal{O}_i of \tilde{X} which dominates $G/P(-\omega_i)$ is the intersection of $E_{-\omega_i^\vee}$ with $E_{-\omega_1^\vee - \omega_2^\vee}$. Let x_i be a point in \mathcal{O}_i stabilized by B' , let P'_i be the stabilizer of $\pi(x_i)$ and let P' be the stabilizer of x_1 ; one can prove that P' is also the stabilizer of x_2 .

The divisor $E_{-\omega_1^\vee - \omega_2^\vee}$ is the strict transform of D , while $E_{-\omega_1^\vee}$ and $E_{-\omega_2^\vee}$ are the exceptional divisors of \tilde{X} . Any automorphism φ in K exchanges \mathcal{O}_1 with \mathcal{O}_2 , because it exchanges $E_{-\omega_1^\vee}$ with $E_{-\omega_2^\vee}$. Moreover, $\varphi(x_1) = x_2$ because $\tilde{\varphi}$ belongs to E . But $\tilde{\varphi}$ is associated to a non-trivial automorphism of the Dynkin diagram, so $\tilde{\varphi}(P'_1)$ is a parabolic subgroup of G containing B' , distinct by P'_1 and with the same dimension of P'_1 ; thus it is P'_2 . \square

2.2 Restricted root systems of type G_2

Now, we study the s.c.s.v. with P.n.1 whose restricted root system has type G_2 . First of all, we determine their connected automorphism group; in particular, we prove that such varieties are non-homogeneous.

Lemma 2.9 *Suppose that $R_{G,\theta}$ has type G_2 . Then the connected automorphism group $\text{Aut}^0(X)$ is isomorphic to G .*

Proof of Lemma 2.9. First, we prove that $\text{Aut}^0(X)$ does not act transitively on X . The simple variety X is associated to the colored cone $(\text{cone}(\alpha_2^\vee, -\omega_2^\vee), \{D_{\alpha_2^\vee}\})$ and its closed orbit is isomorphic to $G/P(-\omega_1)$. Let ω be the highest weight of Z with respect to L (see §1.1.6), then $(\omega, \alpha_2^\vee) = 1$ because $R_{[L,L],\theta} = \{\pm\alpha_2\}$ (see Lemma 2.3). Moreover, $Z(L)^0/(Z(L)^0 \cap H)$ is the one-parameter subgroup of $T/T \cap H$ corresponding to $-\omega_1^\vee$; indeed $(-\omega_1, \omega_1^\vee) < 0$ and $-\omega_1$ belongs to $C(Z)^\vee = \text{cone}(\alpha_2^\vee, -\omega_2^\vee)^\vee$. Hence $1 = (\omega, -\omega_1^\vee) = (a\omega_1 + \omega_2, -2\alpha_1 - \alpha_2) = -2a - 1$, so ω is $-\omega_1 + \omega_2$. Thus $\text{Aut}^0(X)$ is isomorphic to G by the Lemmas 2.2 and 2.6. Observe that the connected automorphism group of the closed orbit of \tilde{X} is G , while the connected automorphism group of the closed orbit of X is either SO_7 or $SO_7 \times SO_7$. \square

Proposition 2.1 *The point 1) of Theorem 2 holds. More precisely, X is the intersection in $\mathbb{P}(\bigwedge^3 V(\omega_1))$ of $\mathbb{G}_3(7)$ with $\mathbb{P}(V(2\omega_1) \oplus \mathbb{C})$.*

Remark that the G_2 -representation $\bigwedge^3 V(\omega_1)$ is isomorphic to $V(\omega_1) \oplus V(2\omega_1) \oplus \mathbb{C}$ (and that $\dim V(\omega_1) = 7$).

Proof of Proposition 2.1. The center of G_2 is trivial, thus $\text{Aut}(X)$ is contained in $\text{Aut}_{\text{alg}}(G_2)$ (see the proof of Lemma 2.7); such group is connected, so also $\text{Aut}(X)$ is connected.

Now, we prove that X is “contained” in $\mathbb{G}_3(7)$. There is an involution of SO_7 that extends θ ; again, we denote it by θ . We have $SO_7^\theta = S(O_3 \times O_4)$, thus $G_2/(SL_2 \times SL_2)$ is a closed subvariety of $SO_7/S(O_3 \times O_4)$. There is a unique s.e.c. of $SO_7/S(O_3 \times O_4)$ with P.n.1 and is isomorphic to $\mathbb{G}_3(7)$ (see the point 9 of Theorem 5).

We have to introduce some notation. Let V be a 7-dimensional vector space and let $\{e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3\}$ be a basis of V . Let q be the symmetric bilinear form associated to the quadratic form $(e_0^*)^2 + e_1^* e_{-1}^* + e_2^* e_{-2}^* + e_3^* e_{-3}^*$ and let ϖ be the trilinear form $e_0^* \wedge e_1^* \wedge e_{-1}^* + e_0^* \wedge e_2^* \wedge e_{-2}^* + e_0^* \wedge e_3^* \wedge e_{-3}^* + 2e_1^* \wedge e_2^* \wedge e_3^* + 2e_{-1}^* \wedge e_{-2}^* \wedge e_{-3}^* \in \bigwedge^3 V^*$. The subgroup G of $SL(V)$ composed by the linear transformations which preserve q and ϖ is the simple group of type G_2 ; moreover, we can realize SO_7 as $SO(V, q)$. The vector space V is the standard representation $V(\omega_1)$ of G and we can suppose that $\{e_{-3}, e_{-2}, e_1, e_0, e_1, e_2, e_3\}$ is a basis of weight vectors for an appropriate maximal torus T of G . Moreover, we can choose a Borel subgroup B of G so that the weight of e_i is a positive root (respectively is 0) if and only if $i > 0$ (respectively $i = 0$). The Grassmannian $\mathbb{G}_3(7)$ is contained in $\mathbb{P}(\bigwedge^3 V)$ ($\cong \mathbb{P}(V \oplus V(2\omega_1) \oplus \mathbb{C})$ as G_2 -space). The G_2 -subrepresentation of $\bigwedge^3 V$ isomorphic to V has the following basis of T -weights: $\{e_{-2} \wedge e_{-3} \wedge e_0 - e_1 \wedge e_2 \wedge e_{-2} - e_1 \wedge e_3 \wedge e_{-3}, e_2 \wedge e_{-2} \wedge e_{-3} - e_1 \wedge e_2 \wedge e_0 + e_1 \wedge e_{-1} \wedge e_{-3}, e_3 \wedge e_{-2} \wedge e_{-3} - e_1 \wedge e_3 \wedge e_0 - e_1 \wedge e_{-1} \wedge e_{-2}, e_1 \wedge e_2 \wedge e_3 - e_{-1} \wedge e_{-2} \wedge e_{-3}, e_2 \wedge e_3 \wedge e_{-3} - e_1 \wedge e_2 \wedge e_{-1} + e_0 \wedge e_{-1} \wedge e_{-3}, -e_2 \wedge e_3 \wedge e_{-2} - e_1 \wedge e_3 \wedge e_{-1} - e_0 \wedge e_{-1} \wedge e_{-2}, -e_1 \wedge e_{-1} \wedge e_{-2} - e_2 \wedge e_3 \wedge e_0 - e_3 \wedge e_{-1} \wedge e_{-3}\}$. Let X'' be the closure of $G_2/(SL_2 \times SL_2)$ in $\mathbb{G}_3(7)$ and let X' be the intersection of $\mathbb{G}_3(7)$ with $\mathbb{P}(V(2\omega_1) \oplus \mathbb{C})$; observe that X'' is contained in X' because ω_1 is not spherical (see §1.1.2). We claim that X is the normalization of X'' . Indeed $\mathbb{P}(V(2\omega_1) \oplus \mathbb{C})$ has two closed G_2 -orbits: one is isomorphic to $G_2/P(\omega_1)$ and the other one is the point $\mathbb{P}(\mathbb{C})$. Therefore X' contains one closed G_2 -orbit, otherwise $\mathbb{G}_3(7)$ would contain a G_2 -fixed point, in particular V would be reducible as G_2 -representation. Thus, the normalization of X'' is a simple, complete variety with closed orbit $G/P(\omega_1)$, so it is X .

We want to prove that X'' coincides with X' and is smooth (so it coincides also with X). Notice that X' is connected, because it is G_2 -stable and contains a unique closed G_2 -orbit. Hence, it is sufficient to prove that X' has dimension 8 and is smooth in a neighborhood of a point belonging to $G_2/P(\omega_1)$. One can verify that $e_1 \wedge e_{-2} \wedge e_{-3}$ is a highest weight vector of $V(2\omega_1) \subset \bigwedge^3 V$. Let A be the affine open subset of $\mathbb{G}_3(7)$ composed by the subspaces generated by

vectors $\{e_j + \sum a_{i,j} e_i\}_{j=1,-2,-3; i=2,3,0,-1}$, where the $a_{i,j}$ are arbitrary constants. Thus A is isomorphic to the space of matrices $\{(I_3, M) : M \in M_{3,4}(\mathbb{C})\} \subset M_{3,7}(\mathbb{C})$ and the matrix $(I, \{a_{i,j}\})$ corresponds to the subspace generated by $\{e_j + \sum a_{i,j} e_i\}_{j=1,-2,-3; i=2,3,0,-1}$ (notice that the columns of (I_3, M) are indexed by a reordering of $\{-3, \dots, 3\}$). In particular, $(I, 0)$ corresponds to the point $[e_1 \wedge e_{-2} \wedge e_{-3}] \in G_2/P(\omega_1)$.

The equations of $A \cap \mathbb{P}(V(2\omega_1 \oplus \mathbb{C}))$ in A are given by the vanishing of the coordinates associated to the basis vectors of V given before. More precisely, given such vector $\sum_i e_{j_1,i} \wedge e_{j_2,i} \wedge e_{j_3,i}$, the associated equation is $\sum_i T_{(1,2,3),(j_1,i,j_2,i,j_3,i)}$ where $T_{(1,2,3),(j_1,i,j_2,i,j_3,i)}$ is the 3×3 minor of (I_3, M) extracted by the j_1,i -th, the j_2,i -th and the j_3,i -th row. Thus the subvariety $A \cap X' = A \cap \mathbb{P}(V(2\omega_1 \oplus \mathbb{C}))$ of A has equations:

$$\begin{aligned} a_{1,0} &= -a_{3,2} + a_{2,3}, \\ a_{2,-1} &= -a_{1,2} + T_{(2,3),(2,0)}, \\ a_{3,-1} &= -a_{1,3} + T_{(2,3),(3,0)}, \\ a_{1,-1} &= T_{(2,3),(2,3)}, \\ T_{(1,2),(2,3)} - T_{(2,3),(2,-1)} + T_{(1,2),(0,-1)} &= 0, \\ T_{(1,3),(2,3)} - T_{(2,3),(3,-1)} + T_{(1,3),(0,-1)} &= 0, \\ a_{3,-1} + T_{(1,2,3),(2,3,0)} + T_{(1,2),(3,-1)} &= 0 \end{aligned}$$

(where $T_{(h,k),(n,m)}$ is the minor of $\{a_{i,j}\}$ extracted by the h -th and k -th row and by n -th and m -th column). The closed subset A' of A defined by the first four equations is the graph of a polynomial map, thus it is smooth of dimension 8. Hence the last three equations are identically verified on A' , because X'' has dimension 8 (and $A \cap X'' \subset A \cap X' \subset A'$). Therefore $A \cap X'$ coincides with A' and is smooth.

Now, we prove the last statement of the point 1). Identify V with $Im \otimes_{\mathbb{C}} \mathbb{C}$ and define the associator $[\cdot, \cdot, \cdot] : \bigwedge^3(\mathbb{O}_{\mathbb{C}}) \rightarrow \mathbb{O}_{\mathbb{C}}$ as the linear map such that $[a, b, c] = (ab)c - a(bc)$. This map is G_2 -equivariant and its restriction to $\bigwedge^3 Im(\mathbb{O}_{\mathbb{C}})$ has kernel $V(2\omega_1) \oplus \mathbb{C}$. Thus X parameters the 3-dimensional subspaces W of $Im \otimes_{\mathbb{C}} \mathbb{C}$ over which $[\cdot, \cdot, \cdot]$ is zero. Furthermore, $[1, \cdot, \cdot]$ is identically zero. Let W be a subspace associated to a point in X and let \overline{W} be the subalgebra of $\mathbb{O}_{\mathbb{C}}$ generated by W and 1. It can be either the entire algebra $\mathbb{O}_{\mathbb{C}}$ or a subalgebra of dimension four. But, if \overline{W} is the whole algebra, then $\mathbb{O}_{\mathbb{C}}$ is generated by four elements which associates between them, a contradiction (recall that $\mathbb{O}_{\mathbb{C}}$ and \overline{W} are composition algebras). Thus, \overline{W} is a composition algebra of dimension four, so it is isomorphic to $\mathbb{H}_{\mathbb{C}}$. \square

Let (V_1, q_1) and (V_2, q_2) be copies respectively of (V, q) and $(V, -q)$, where V is as in the previous proposition; we can suppose $G_2 \times G_2 \subset SO(V_1, q_1) \times SO(V_2, q_2)$. Moreover, let W_i be a maximal isotropic subspace of V_i for each i . Let W be a maximal isotropic subspace of $V_1 \oplus V_2$ which contains $W_1 \oplus W_2$.

Proposition 2.2 *The point 2) of Theorem 2 holds. More precisely, X is the intersection in $\mathbb{P}(\Lambda^{\text{even}}(W)) \cong \mathbb{P}((V_1 \otimes V_2) \oplus V_1 \oplus V_2 \oplus \mathbb{C})$ of \mathbb{S}_7 with $\mathbb{P}((V_1 \otimes V_2) \oplus \mathbb{C})$*

Proof of Proposition 2.2. The automorphism group of X is determined in [Br07], Example 2.4.5. Alternatively, one can study it in a very similar way to the Proposition 2.1.

Clearly the involution of $G_2 \times G_2$ can be extended to an involution of $SO(V_1) \times SO(V_2)$ which we denote again by θ ; in particular we have $G_2 \cong G_2 \times G_2 / (G_2 \times G_2)^\theta \subset SO(V_1) \times SO(V_2) / (SO(V_1) \times SO(V_2))^\theta \cong SO(V, q)$. Let X'' be the closure of G_2 in the unique s.e.c. of $SO(V, q)$ with P.n.1, which is isomorphic to \mathbb{S}_7 (see the point 3 of Theorem 5). We claim that X is the normalization of X'' (one could show that X'' is normal by [Ti03], Proposition 9).

Let $\varphi : \mathbb{S}_7 \rightarrow \mathbb{P}(\Lambda^{\text{even}} W)$ be the $Spin_{14}$ -equivariant embedding of \mathbb{S}_7 in the projectivitation of a half-spin representation of $Spin_{14}$. Write $V_i = W_i \oplus \widetilde{W}_i \oplus \mathbb{C}_i$ and $V = V_1 \oplus V_2 = W \oplus \widetilde{W}$, where $\widetilde{W}_i, \widetilde{W}$ are maximal isotropic subspaces such that $\widetilde{W}_1 \oplus \widetilde{W}_2 \subset \widetilde{W}$. The representation $\Lambda^{\text{even}} W$ is isomorphic to $\Lambda^\bullet W_1 \otimes \Lambda^\bullet W_2 \cong \Lambda^\bullet W_1 \otimes (\Lambda^\bullet W_2)^*$ as $(Spin_7 \times Spin_7)$ -representation (see the highest weights and the dimensions). Moreover $\Lambda^\bullet W_i$ is isomorphic to $V_i \oplus \mathbb{C}_i$ as G_2 -representation, so $\Lambda^\bullet W_1 \otimes \Lambda^\bullet W_2$ is isomorphic to $(V_1 \otimes V_2) \oplus V_1 \oplus V_2 \oplus \mathbb{C}$ as G_2 -representation. Let \mathbb{P} be the projective subspace of $\mathbb{P}(\Lambda^{\text{even}} W)$ isomorphic to $\mathbb{P}((V_1 \otimes V_2) \oplus \mathbb{C})$. Observe that X'' is contained in $X' := \mathbb{S}_7 \cap \mathbb{P} \subset \mathbb{P}(\Lambda^{\text{even}} W)$ because $V_1 \oplus V_2$ does not contain a line fixed by G^θ (see §1.1.2).

Observe that X' contains one closed G -orbit, in particular X' is connected. Indeed \mathbb{P} contains two closed G -orbits: one isomorphic to $G/P(\omega_1)$ and the other one isomorphic to the G -stable point $\mathbb{P}(\mathbb{C})$. On the other hand, there is not a G -stable maximal isotropic subspace of $V_1 \oplus V_2$, so $\mathbb{P}(\mathbb{C})$ is not contained in \mathbb{S}_7 . Thus the normalization of X'' is the simple, complete symmetric variety with closed orbit $G/P(\omega_1)$, so it is X . We want to prove that X'' is smooth and coincides with X' ; it is sufficient to prove that X' is smooth of dimension 12; in this case X' is irreducible, so it coincides with X'' (and X). Moreover, it is sufficient to study X' in a open neighborhood of an arbitrarily fixed point of $G/P(\omega_1)$, for example $x = [e_1 \wedge e_{-2} \wedge e_{-3} \wedge e_0 + f_0 \wedge f_2 \wedge f_3 \wedge f_{-1}]$, where $\{e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3\}$ is a basis of V_1 as before and $\{f_{-3}, f_{-2}, f_1, f_0, f_1, f_2, f_3\}$ is the corresponding basis of V_2 .

Let u be $e_0 + f_0$; we can suppose that W is generated by $e_1, e_2, e_3, f_1, f_2, f_3$ and u . The trivial subrepresentation \mathbb{C}_1 of $\Lambda^\bullet W_1$ is spanned by $2\sqrt{-2} \cdot 1_{W_i} + e_1 \wedge e_2 \wedge e_3$; moreover, $W_1 \oplus \Lambda^2 W_1$ is contained in the G_2 -stable subspace of $\Lambda^\bullet W_1$ isomorphic to V_1 . An open neighborhood of x in \mathbb{S}_7 is given by $U^- \cdot x$, where U^- is the unipotent radical of the standard parabolic subgroup opposite to $Stab_{Spin_{14}}(x)$. Notice that, as algebraic variety, $U^- \cdot x$ is isomorphic to $Lie(U^-)$ by the exponential map. The coordinates of $exp(p) \cdot x$ are the Pfaffians of the diagonal minors of p . Let $x_{i,j}$ be the coordinates of the space $M_{14}(\mathbb{C})$ of matrices

of order 14 with respect to the basis $\{e_1, e_2, e_3, u, f_1, f_2, f_3, e_{-1}, e_{-2}, e_{-3}, \frac{1}{2}(e_0 - f_0), f_{-1}, f_{-2}, f_{-3}\}$. We claim that a open neighborhood of x in $X' \cap (U^- \cdot x)$ is the graph of a polynomial map. Given a skew-symmetric matrix, let $[i_1, i_2, \dots, i_{2k}]$ be the Pfaffian of the principal minor extracted from the rows and the columns of indices $i_1 < i_2 < \dots < i_{2k}$.

The three equations corresponding to the vectors $2\sqrt{-2}e_i \wedge e_j + e_i \wedge e_j \wedge u \wedge f_1 \wedge f_2 \wedge f_3$ ($\in V_1$) with $i, j \in \{1, 2, 3\}$ are, respectively, $0 = x_{i,j} - [i, j, 4, 5, 6, 7] = x_{i,j} - (x_{4,7}p_{i,j} + q_{i,j})$, where the $p_{i,j}$ and the $q_{i,j}$ are homogeneous polynomials in the $x_{h,k}$ such that : 1) $(h, k) \neq (4, 7)$; 2) either $h > 3$ or $k > 3$. Finally, we consider the equation $0 = x_{4,7} - [1, 2, 3, 4] = x_{4,7} - (x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{2,3}x_{1,4})$ associated to $f_4 \wedge u + f_4 \wedge e_1 \wedge e_2 \wedge e_3$ ($\in V_2$). Substituting the first three equations in the last one, we obtain $x_{4,7}(1 + x_{1,2}f_{1,2} + x_{1,3}f_{1,3} + x_{2,3}f_{2,3}) = g$ where the $f_{i,j}$ and g are polynomial in the $x_{4,j}$ with $j < 4$.

Therefore, there is an open neighborhood A of x in $U^- \cdot x$ where the previous four equations become $x_{1,2} = h_{1,2}$, $x_{1,3} = h_{1,3}$, $x_{2,3} = h_{2,3}$ and $x_{4,7} = h_{4,7}$ (here the $h_{i,j}$ are polynomials in the coordinates different from $x_{1,2}$, $x_{1,3}$, $x_{2,3}$ and $x_{4,7}$). Observe that the previous equations are independent; on the other hand X' has dimension at least 12, because it contains X'' . Therefore, the subvariety A' of A obtained imposing the previous four equations is smooth with dimension 12, thus it is equal to $A \cap X''$. Hence $X' = X'' (= X)$. \square

2.3 Restricted root system of type A_2 (with $H = G^\theta$)

In this section G/H is always a symmetric space with restricted root system A_2 and $H = G^\theta$. We prove that its s.e.c. with P.n.1 is a hyperplane section of a variety of the subexceptional series of subadjoint varieties. These last varieties constitute the third line of the geometric version of Freudenthal's magic square and are Legendrian varieties (see [LaMa01] and [LaMa07]). For the completion of SL_3 this result is due to Buczyński (see [Bu07]). These varieties are contained nested in each other. First, we prove that such varieties are hyperplane sections of Legendrian varieties. Then, we study their connected automorphism group; in particular we show that these varieties are not homogeneous. Finally, we study their automorphism group.

The quotient $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ is a symmetric variety isomorphic to the image of $SL_3(\mathbb{A}) \cdot [1, I]$ by ϕ ; in particular it is contained in the hyperplane section $X' := \{[x_1, x_2, x_3, x_4] \in \mathbb{L}\mathbb{G}_3(\mathbb{A}^6) \subset \mathbb{P}(\mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}^*) : x_1 = x_4\}$ (see §1.3). Such section is irreducible because it is the image by ϕ of $\{[x, P] : x^3 = \det(P)\}$; moreover, it has the same dimension as $SL_3(\mathbb{A})/SO_3(\mathbb{A})$, thus it is the closure of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$.

Lemma 2.10 *The variety X' is isomorphic to the s.e.c. X of $SL_3(\mathbb{A})/SO_3(\mathbb{A})$ with P.n.1.*

Proof of Lemma 2.10. The variety X' and its normalization X'' have the same number of orbits by [Ti03], Proposition 1; moreover, each orbit of X''

have the same dimension as its image in X' . Observe that no orbit of X'' has dimension 0; thus X' has two closed orbits, namely $G/P(\omega_1)$ and $G/P(\omega_2)$. One can easily show that X'' has a unique orbit \mathcal{O} of codimension one (otherwise, by the theory of spherical embeddings, X' would contain a closed orbit isomorphic to $G/P(\omega_1 + \omega_2)$). Thus $X'' = G/H \cup \mathcal{O} \cup G/P(\omega_1) \cup G/P(\omega_2)$. By Proposition 8.2 in [LaMa01], we know the possible dimensions for the singular locus of X' . Because it is $SL_3(\mathbb{A})$ -stable, one can easily see that X' is smooth (and coincides with X''). Studying the colored fan of X'' , one can show easily that X'' has Picard number one, so it is isomorphic to X . \square

Observe that we have proved the existence of the commutative diagram of Theorem 3.

Lemma 2.11 *Aut⁰(X) is isomorphic to $G/(Z(G) \cap H)$.*

Proof of Lemma 2.11. The character group $\chi(S)$ of S is the lattice generated by spherical weights. Let X' be the simple subvariety corresponding to the colored cone $(\text{cone}(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_1^\vee}\})$; its closed orbit is isomorphic to $G/P(\omega_2)$. Let Z be as §1.1.6 and let $\omega = a_1\omega_1 + a_2\omega_2$ be the highest weight of Z (with respect to L). The simple restricted root of $R_{[L,L],\theta}$ is α_1 , so $a_1 = (\omega, \alpha_1^\vee) = 1$ by the Lemma 2.3. Moreover, $Z(L)^0/(Z(L)^0 \cap H)$ is the one-parameter subgroup of $T/T \cap H$ corresponding to $-3\omega_2^\vee$; indeed $(-\omega_2, \omega_2^\vee) < 0$ and $-\omega_2 \in C(Z)^\vee = \text{cone}(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee)^\vee$. Thus $1 = (\omega, -3\omega_2^\vee) = -1 - 2a_2$, so ω is $\omega_1 - \omega_2$. We can study the simple variety corresponding to the colored cone $(\text{cone}(\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee), \{D_{\alpha_2^\vee}\})$ in a similar way. Thus $\text{Aut}^0(X)$ is isomorphic $G/(Z(G) \cap H)$ by the Lemma 2.6. \square

To finish the proof of Theorem 3 we have only to determine $\text{Aut}(X)$.

Proposition 2.3 *We have:*

- If G/H is SL_3/SO_3 , then $\text{Aut}(X)$ is generated by SL_3 and θ .
- If G/H is SL_3 , then $\text{Aut}^0(X)$ is $(SL_3 \times SL_3)/\{(g, g) : g \in Z(SL_3)\}$. Moreover, $\text{Aut}(X)$ is generated by $\text{Aut}^0(X)$, θ and (φ, φ) , where φ is the automorphism of SL_3 corresponding to the non-trivial automorphism of the Dynkin diagram.
- If G/H is SL_6/Sp_6 , then $\text{Aut}(X)$ is generated by $SL_6/\{\pm id\}$ and θ .
- If G/H is E_6/F_4 then $\text{Aut}^0(X)$ is simply-connected and $\text{Aut}(X)$ is generated by E_6 and θ .

Proof of Proposition 2.3. By the Lemma 2.11, $\text{Aut}^0(X)$ is $G/(Z(G) \cap H)$. Notice that $Z(G)$ acts non-trivially on X because H is strictly contained in $N_G(G^\theta)$. Moreover, if G is different from $SL_3 \times SL_3$ and SL_6 , then the center of G is a simple group, thus $\text{Aut}^0(X)$ is G . If G is $SL_3 \times SL_3$, then its center is $C_3 \times C_3$, where C_n is the group of n -th roots of the unit. Thus the intersection of the center of G with G^θ is the diagonal of $C_3 \times C_3$. If G is SL_6 then its center

is C_6 and its intersection with $Sp_6 (= G^\theta)$ is $\{\pm id\}$. Observe that $N_G(H)/H$ is simple because the type of $R_{G,\theta}$ is A_2 .

Now, we determine K (defined as in the Lemma 2.7). It is sufficient to determine the subset K' of E . Let v be the longest element of W and let v_0 be the longest element of the Weyl group of R_G^0 (here we consider the root system of G with respect to T). Then $\theta' := v\omega_0\theta$ fixes T and B . One can easily show that θ' exchange ω_1 with ω_2 (see the figures 1,2 and [dCoPr83], §1.4), thus it exchanges $E_{-\omega_1^\vee}$ with $E_{-\omega_2^\vee}$. Hence, also θ exchanges the previous two G -stable divisors; in particular θ exchanges the two closed orbits of \tilde{X} . Notice that θ belongs to K' and induces the non-trivial automorphism of the Dynkin diagram of G , with respect to T' and B' (see page 493 in [LaMa01] and §26 in [Ti06]). If G is different from $SL_3 \times SL_3$, then E/\bar{T}' contains exactly two elements. Thus $K'/N_{\bar{T}'}(H)$ coincides with E/\bar{T}' . In particular, $Aut(G/H)$ is generated by $Aut^0(G/H)$ and θ . Moreover, $Aut(G/H)$ coincides with $Aut(X)$ by Lemma 2.8 and by [Ti06], Lemma 26.2.

Finally, suppose $G = SL_3 \times SL_3$. Let \dot{T} be a maximal torus of SL_3 , let $\dot{B} (\supset \dot{T})$ be a Borel subgroup of SL_3 and let φ be the equivariant automorphism of SL_3 associated to the non-trivial automorphism of the Dynkin diagram (with respect to \dot{T} and \dot{B}). We can set $T = T' = \dot{T} \times \dot{T}$, $B = \dot{B} \times \dot{B}^-$ and $B' = \dot{B} \times \dot{B}$. One can easily see that K' is generated by $N_{T'}(H)$, θ and (φ, φ) . (Notice that E/\bar{T}' has eight elements: id , θ , (φ, id) , (id, φ) , (φ, φ) , $\theta \circ (\varphi, \varphi)$, $\theta \circ (\varphi, id)$ and $\theta \circ (id, \varphi)$). Observe that (φ, φ) stabilizes both B and B^- . We have $\omega_i = \varpi_i^1 - \varpi_i^2$, where $\{\varpi_1^j, \varpi_2^j\}$ are the fundamental weights of the j -th copy of SL_3 in G (with respect to \dot{T} and \dot{B}). Thus (φ, φ) exchange ω_1 with ω_2 , so (φ, φ) exchange $E_{-\omega_1^\vee}$ with $E_{-\omega_2^\vee}$. Furthermore, (φ, φ) induces a non-trivial automorphism of the Dynkin diagram of G (with respect to T' and B'). Therefore, $Aut(X)$ coincides with $Aut(G/H) (= Aut(\tilde{X}))$. \square

3 Homogeneous varieties

We begin with the s.c.s.v. with P.n.1 and rank 1. Fixed any symmetric space G/H with rank 1, the canonical completion X is the unique completion of G/H because the valuation cone is a half-line. Thus X has a unique non-open orbit which has codimension one. Moreover, X is smooth because the singular locus is G -stable and X is normal; in particular it is wonderful. The Picard number of a wonderful (not necessarily symmetric) variety is equal to number of colors and it is at least equal to the rank of the variety. Thus the complete symmetric varieties with rank 1 have Picard number at most equal to 2.

More generally, the wonderful (eventually non-symmetric) varieties with Picard number 1 have rank 1. Moreover, the wonderful varieties with rank 1 are exactly the G -varieties with two orbits, whose the closed one has codimension one (we request also that G is reductive). They are classified by the Theorem 4 in [Akh83] due to Akhiezer. He has showed that they are all homogeneous

(under the action of the automorphism group).

In the following we always suppose that G/H has rank l strictly greater than one. Now we prove the Theorem 5. We exhibit an open equivariant embedding of every symmetric space G/H appearing in Theorem 5 into an appropriate homogenous variety with Picard number one and the same dimension of G/H . Then, we conclude using the fact that the s.e.c. of G/H with P.n.1 is unique.

Given a linear endomorphism φ of a vector space V , let $graph(\varphi)$ be the subspace $\{(v, \varphi(v)) : v \in V\}$ of $V \oplus V$. Observe that $graph(\varphi)$ has the same dimension of V . Given a (skew-)symmetric bilinear form on V , then we can define a (skew-)symmetric bilinear q' form on $V \oplus V$ such that $q'(v_1 + v_2, w_1 + w_2) = q(v_1, w_1) - q(v_2, w_2)$ for each $v_1 + v_2, w_1 + w_2 \in V \oplus V$.

Remark. Given an irreducible spherical representation V and a non-zero vector v in V^{G^θ} , then $\mathbb{C}v$ is stabilized by $N_G(G^\theta)$ because $\dim V^{G^\theta} = 1$. Moreover, the stabilizer of $\mathbb{C}v$ is exactly $N_G(G^\theta)$ because of [dCoPr83], Lemma 1.7.

Proof of the point 1 of Theorem 5. We begin with the decomposable s.c.s.v. with P.n.1. First of all, we want to describe their open orbit. If $R_{G,\theta}$ has type A_1 , then $G/N_G(G^\theta)$ is isomorphic to $SO_n/S(O_1 \times SO_{n-1})$ for an appropriate $n \geq 3$. If $n = 3$, the type of (G, θ) is AI ; if $n = 4$ then $G/N_G(G^\theta)$ is isomorphic to PSL_2 ; if $n = 6$, the type of (G, θ) is AII ; if $n = 2k$ with $k \geq 4$, the type of (G, θ) is DII ; if $n = 2k + 1$ with $k \geq 2$, the type of (G, θ) is BII . Given a s.c.s.v. with P.n.1 and $R_{G,\theta} = A_1 \times A_1$, write $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and let $g_j \in N_{G_j}(G_j^\theta)$ be a representant of the non-trivial element of $N_{G_j}(G_j^\theta)/G_j^\theta$ for each j . Then H is generated by G^θ and (g_1, g_2) .

The group $SO_n \times SO_m$ is contained in SO_{n+m} . Let $\{e_1, \dots, e_{n+m}\}$ be an orthonormal basis of \mathbb{C}^{n+m} such that $SO_n \subset GL(\text{span}_{\mathbb{C}}\{e_1, \dots, e_n\})$ and $SO_m \subset GL(\text{span}_{\mathbb{C}}\{e_{n+1}, \dots, e_{n+m}\})$. The identity component of $Stab_{SO_n \times SO_m}(e_1 + ie_{n+1})$ is $SO_{n-1} \times SO_{m-1}$ (see also [dCoPr83], Lemma 1.7). Furthermore, $SO(\mathbb{C}^{n+m}) \cdot [e_1 + ie_{n+1}] \subset \mathbb{P}(\mathbb{C}^{n+m})$ is $\mathbb{I}\mathbb{G}_1(n+m)$ because $e_1 + ie_{n+1}$ is an isotropic vector. Finally, there is a unique subgroup $(G^\theta)^0 \subset H \subset N_G(G^\theta)$ such that exists a s.e.c. of G/H with P.n.1. \square

Proof of the point 2 of Theorem 5. It is sufficient to exhibit an irreducible spherical representation with dimension equal to $\dim G/H + 1$.

i) If G/H is PGL_{l+1} , then X is isomorphic to $\mathbb{P}(M_{l+1}(\mathbb{C}))$ as $(PGL_{l+1} \times PGL_{l+1})$ -variety. Indeed, $PGL_{l+1} \times PGL_{l+1}$ acts on $\mathbb{P}(M_{l+1}(\mathbb{C}))$ and the stabilizer of $[1]$ is the diagonal, namely $(PGL_{l+1} \times PGL_{l+1})^\theta$. Moreover $\mathbb{P}(M_{l+1}(\mathbb{C}))$ has dimension equal to $l^2 + 2l$.

ii) If G/H is $SL_{l+1}/N_{SL_{l+1}}(SO_{l+1})$ and (G, θ) has type AI , then X is isomorphic to $\mathbb{P}(Sym^2(\mathbb{C}^{l+1}))$ (see the description of spherical weights in §1.1.2).

iii) If G/H is $SL_{2l+2}/N_{SL_{2l+2}}(Sp_{2l+2})$ and (G, θ) has type AII , then X is isomorphic to $\mathbb{P}(\bigwedge^2(\mathbb{C}^{2l+2}))$ (see the description of spherical weights in §1.1.2).

iv) If G/H is $E_6/N_{E_6}(F_4)$ and (G, θ) has type EIV , then X is $\mathbb{P}(\mathcal{J}_3(\mathbb{O}_{\mathbb{C}}))$. Indeed, $\mathcal{J}_3(\mathbb{O}_{\mathbb{C}})$ is the 27-dimensional irreducible representation of E_6 corresponding to the first fundamental weight. The subgroup F_4 of E_6 is isomorphic

to the group of automorphism of $\mathcal{J}_3(\mathbb{O}_{\mathbb{C}})$; in particular F_4 fixes the identity matrix. Thus $\mathbb{P}(\mathcal{J}_3(\mathbb{O}_{\mathbb{C}}))$ contains the 26-dimensional variety G/H . \square

Proof of the points 3, 4 and 5 of Theorem 5. Observe that $Spin_5 \times Spin_5$ is isomorphic to $Sp_4 \times Sp_4$. We have an inclusion of SO_l (resp. of Sp_{2l}) into $\mathbb{I}\mathbb{G}_l(2l)$ (resp. into $\mathbb{L}\mathbb{G}_{2l}(4l)$) given by the morphism $\phi : g \rightarrow graph(g)$. It is easy to show that this map is compatible with the action of G on the domain and on the codomain of ϕ . Now, it is sufficient to observe that G has the same dimension of the codomain of ϕ . \square

Proof of the point 6 of Theorem 5. The group SL_4 acts on the six-dimensional space $\bigwedge^2(\mathbb{C}^4)$, thus it acts on $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$. The stabilizer of the space generated by $e_1 \wedge e_2$ and $e_3 \wedge e_4$ is $N_{SL_4}(S(GL_2 \times GL_2))$, thus $SL_4/N_{SL_4}(S(GL_2 \times GL_2))$ is contained in $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$. Moreover $SL_4/N_{SL_4}(S(GL_2 \times GL_2))$ has the same dimension of $\mathbb{G}_2(\bigwedge^2 \mathbb{C}^4)$, so the Grassmannian is the unique s.e.c. of $SL_4/N_{SL_4}(S(GL_2 \times GL_2))$ with P.n.1. \square

Proof of the point 7 of Theorem 5. Let (G', σ) be an involution of type EI , where G' is the simply connected, simple group of type E_6 and G'^{σ} has type C_4 . Choose a maximally σ -split torus and a Borel subgroup of G' as in §1.1.1; then σ acts as $-id$ over $R_{G'}$ and the parabolic subgroup $P := P(\varpi_1)$ of G' is opposed to $\sigma(P)$. Furthermore, $P \cap \sigma(P)$ is a Levi subgroup of P containing P^{σ} ; the derived subgroup of $P \cap \sigma(P)$ has type D_5 while $(P^{\sigma})^0$ has type $B_2 \times B_2$. Hence G'^{σ}/P^{σ} is isomorphic to $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$, up to quotient by a finite group. Notice that G'/P is a s.e.c. of G'^{σ}/P^{σ} with P.n.1. Moreover, G'/P is embedded in $\mathbb{P}(\mathcal{J}_3(\mathbb{O}_{\mathbb{C}}))$ and $\mathcal{J}_3(\mathbb{O}_{\mathbb{C}})$ is a 27-dimensional irreducible representation of E_6 . But the smallest spherical representation of Sp_8 has dimension 27, thus $\mathcal{J}_3(\mathbb{O}_{\mathbb{C}})$ is an irreducible Sp_8 -representation (see the figure 1). Thus P^{σ} is $N_{Sp_8}(Sp_4 \times Sp_4)$ by the previous remark. \square

Proof of the points 8, 9, 10 and 11 of Theorem 5. First, observe that $SO_8/N_{SO_8}(GL_4)$ is isomorphic to $SO_8/S(O_2 \times O_6)$. Indeed the involution of SO_8 of type $DIII$ (and rank two) is conjugated by an equivariant automorphism of SO_8 to the involution of SO_8 of type DI (and rank two). In fact, these involutions have the same Satake diagrams up to an automorphism of the Dynkin diagram of SO_8 .

Now, suppose G/H equal to $SO_{n'}/S(O_{l'} \times O_{n'-l'})$ (resp. to $Sp_{n'}/(Sp_{l'} \times Sp_{n'-l'})$) and let q be the non-degenerate bilinear form on $\mathbb{C}^{n'}$ with respect to which is defined $SO_{n'}$ (resp. $Sp_{n'}$). Let W be a l' -dimensional subspace of $\mathbb{C}^{n'}$ over which q is non-degenerate. We denote by W^{\perp} the orthogonal complement of W . The stabilizer of W contains $SO(W) \times SO(W^{\perp})$ (resp. $Sp(W) \times Sp(W^{\perp})$). By [dCoPr83], Lemma 1.7, the variety $\mathbb{G}_l(n)$ is a s.e.c. with P.n.1 of $SO_{n'}/Stab_{SO_{n'}}(W)$ (resp. of $Sp_{n'}/Stab_{Sp_{n'}}(W)$). Notice that G/H cannot be $Spin_{n'}/(Spin_2 \times Spin_{n'-2})$ because this last variety is Hermitian and ρ is not injective. Moreover, G/H cannot be $Sp_8/N_{Sp_8}(Sp_4 \times Sp_4)$ because of the point 7 of Theorem 5. So we can conclude because of the Theorem 1.1. \square

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