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# PROJECTIVE NORMALITY OF COMPLETE TOROIDAL SYMMETRIC VARIETIES

ALESSANDRO RUZZI

## Abstract

In [CM II] Chirivì and Maffei have proved that the multiplication of sections of any two line bundles generated by global sections on a wonderful symmetric variety is surjective. We prove two criterions that allow us to reduce the same problem on a (smooth) complete toroidal symmetric variety to the analogous problem on the corresponding complete toric variety (respectively on the corresponding open toric variety). We have also studied in details some family of complete toroidal symmetric varieties, in particular those of rank 2.

In this work we study the projective normality of complete toroidal symmetric varieties. Let  $\overline{G}$  be an adjoint semisimple group over  $\mathbf{C}$  and let  $\theta$  be an involution of  $\overline{G}$ . We define  $\overline{H}$  as the subgroup of the elements fixed by  $\theta$  and we say that  $\overline{G}/\overline{H}$  is a homogeneous symmetric variety. De Concini and Procesi [CSV I] have defined a wonderful completion  $X$  of  $\overline{G}/\overline{H}$  and this is the unique wonderful completion of  $\overline{G}/\overline{H}$ . In this work we define a toroidal symmetric variety as a normal  $\overline{G}$ -variety with a dense open orbit isomorphic to  $\overline{G}/\overline{H}$  and a  $\overline{G}$ -equivariant map  $Y \rightarrow X$  extending the identity of  $\overline{G}/\overline{H}$ . They have been classified by De Concini and Procesi [CSV II]. In particular they showed that there is an equivalence of categories between the category of complete toroidal symmetric varieties and the category of toric varieties proper over an affine space  $\mathbf{A}^l$  considered as a  $(\mathbf{C}^*)^l$ -variety in the obvious way, where  $l$  is the rank of  $\overline{G}/\overline{H}$ . In this work we say that a toric variety proper over  $\mathbf{A}^l$  (considered as a  $(\mathbf{C}^*)^l$ -variety) is an open toric variety. Moreover there is a one-to-one correspondence between the completions  $Y$  of  $\overline{G}/\overline{H}$  which lie over  $X$  and the elements of a special class of complete toric varieties. One can show that the complete toric variety  $Z^c$  corresponding to a complete toroidal symmetric variety  $Y$  is a subvariety of  $Y$  and the open toric variety  $Z$  corresponding to  $Y$  is an open subvariety of  $Z^c$ .

In this work, unless explicitly stated otherwise, we shall always assume that the complete toroidal symmetric variety  $Y$  is smooth. Recall that by [CSV II] then it follows that: 1) any orbit closure in  $Y$  is also smooth; 2) the associated toric varieties  $Z$  and  $Z^c$  are both smooth. A complete toroidal symmetric variety is projective if and only if the corresponding complete toric variety is projective. Moreover they are projective if and only if the associated open toric variety is quasi-projective (see Proposition 1.5 and [Br] Corollary 3.3).

Chirivì and Maffei [CM II] have proved the following result which easily implies the projective normality of  $X$  with respect to any projective embedding by a complete linear system.

**Theorem 0.1** *Let  $L_1$  and  $L_2$  be any two line bundles generated by global sections on the wonderful symmetric variety  $X$ . Then the product of sections*

$$H^0(X, L_1) \otimes H^0(X, L_1) \longrightarrow H^0(X, L_1 \otimes L_2)$$

*is surjective.*

We try to generalize this result to any complete toroidal symmetric variety. First we prove that the surjectivity of the product of sections of two ample line bundles on a complete toroidal symmetric variety is equivalent to the surjectivity of the product of sections of the restrictions of the line bundles to the corresponding complete toric variety (see Theorem 2.1). Thus we have reduced the problem to a problem on toric varieties. Unfortunately, it is very difficult to verify the surjectivity of the product of sections of any two ample line bundles on a generic complete toric variety. However, we can simplify the problem for the special class of complete toric varieties which we are considering. Indeed we prove that the surjectivity of the product of sections of two ample line bundles on  $Z^c$ , say  $L_1$  and  $L_2$ , is equivalent to the surjectivity of the product of sections of the restrictions of the line bundles to  $Z$  (see Theorem 3.1 and Theorem 4.1). This problem is much simpler, because  $H^0(Z, L_1|Z)$  and  $H^0(Z, L_2|Z)$  are infinite dimensional vector spaces and it is sufficient to prove that a suitable finite dimensional subspace of  $H^0(Z, (L_1 \otimes L_2)|Z)$  is contained in the image of the product of sections. Indeed we prove that, given any ample line bundle  $L$  on  $Z^c$ ,  $H^0(Z, L|Z)$  is generated by  $H^0(Z^c, L)$  as an  $\mathcal{O}_Z(Z)$ -module.

In Section 5 we study the surjectivity of the product of sections of an ample line bundle on a toric variety proper over  $\mathbf{A}^l$ . If the variety has dimension 2, we prove that the product of sections of any two ample line bundles is surjective. If the dimension of the variety is larger than 2, we find a infinite number of varieties proper over  $\mathbf{A}^l$  such that, for any ample line bundle  $L$  on such a variety  $Z$ , the product of sections of  $L$

$$H^0(Z, L) \otimes H^0(Z, L) \longrightarrow H^0(Z, L \otimes L)$$

is surjective.

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## 1 Notations and background

Let  $G$  be a semisimple simply connected algebraic group over  $\mathbf{C}$  and let  $\theta$  be an involution of  $G$ . We define  $H$  as the normalizer  $N_G(G^\theta)$  of the subgroup of  $\theta$ -fixpoints. Let  $\overline{G}$  be the adjoint semisimple group associated to  $G$  and let  $\overline{H}$  be the subgroup of the elements fixed by the involution induced by  $\theta$ , then  $\overline{G}/\overline{H}$  is isomorphic to  $G/H$  through the map induced by the quotient map  $G \rightarrow \overline{G}$ .

**Definition 1.1** *We will say that  $G/H$  is a homogeneous symmetric variety (of adjoint type).*

Given any abelian group  $A$  we will denote by  $A_{\mathbf{R}}$  the corresponding real vector space  $A \otimes_{\mathbf{Z}} \mathbf{R}$ .

We can associate a not necessarily reduced root system to  $G/H$  (see [He] and [T]). Let  $T^1$  be a torus of  $G$  such that: 1)  $\theta(t) = t^{-1}$  for each  $t \in T^1$ ; 2) the dimension  $l$  of  $T^1$  is maximal. Let  $T$  be a maximal torus which contains  $T^1$ . One can show that  $T$  is stabilized by  $\theta$ , so  $\theta$  induces an involution on  $\chi(T)_{\mathbf{R}}$ , which we call again  $\theta$ . This involution stabilizes the root system  $R_G$  of  $G$  and it is orthogonal with respect to the Killing form. We can choose a Borel subgroup  $B$  of  $G$  such that the associated set of positive roots  $R_G^+$  has

the following property: for each  $\alpha \in R_G^+$  either  $\theta(\alpha)$  is equal to  $\alpha$  or  $\theta(\alpha)$  is a negative root. For each root  $\alpha$  we define  $\alpha^s = \alpha - \theta(\alpha)$ . The set  $R_{G,\theta} := \{\alpha^s \neq 0 : \alpha \in R_G\}$  is a not necessarily reduced root system of rank  $l$  called the *restricted root system*. We will say that the non-zero  $\alpha^s$  are restricted roots and that  $l$  is the rank of  $G/H$ . The restricted roots generate the  $(-1)$ -eigenspace of  $\chi(T)_{\mathbf{R}}$ . Let  $T^0$  be the connected component of the subgroup of the invariant of  $T$  and let  $S$  be the quotient of  $T^1$  by  $T^1 \cap T^0$ . The linear map  $\chi(S)_{\mathbf{R}} \longrightarrow \chi(T^1)_{\mathbf{R}}$  induced in an obvious way by the quotient map  $T^1 \twoheadrightarrow S$  is an isomorphism, while the linear map  $\chi(T)_{\mathbf{R}} \twoheadrightarrow \chi(T^1)_{\mathbf{R}}$  induced by the canonical injection  $T^1 \hookrightarrow T$  is surjective and its restriction to the  $(-1)$ -eigenspace is an isomorphism. Thus we can identify  $\chi(S)_{\mathbf{R}}$  with the space generated by the restricted roots. Moreover we can identify  $\chi(S)$  with the lattice generated by the restricted roots.

Now we want to describe the lattice of integral weights of  $R_{G,\theta}$ . Let  $\Lambda$  be the lattice of integral weights of  $R_G$ , let  $\Lambda^+$  be the set of dominant weights of  $R_G$  and let  $\Gamma$  be the basis of  $R_G$  associated to  $R_G^+$ . For each  $\alpha \in \Gamma$ , let  $\omega_\alpha$  be the fundamental weight associated to  $\alpha$ . For each dominant weight  $\lambda$ , let  $V_\lambda$  be the irreducible representation of  $G$  of highest weight  $\lambda$ . We will say that a dominant weight  $\lambda$  is *spherical* if  $V_\lambda$  contains a non-zero vector fixed by the Lie algebra  $\mathfrak{h}$  of  $H$ . Moreover we will say that a weight  $\mu$  is *special* if  $\theta(\mu) = -\mu$ . Let  $\Omega$  be the lattice generated by the spherical weights and let  $\Lambda^1$  be the lattice of the special weights. One can easily show that  $2\Lambda^1 \subset \Omega \subset \Lambda^1$ . We can describe  $\Omega$  more explicitly. Let  $R_G^0$  be the set of roots fixed by  $\theta$  and let  $R_G^1 = R_G - R_G^0$ . We set  $\Gamma_0 = \Gamma \cap R_G^0$  and  $\Gamma_1 = \Gamma \cap R_G^1$ . The set  $\Gamma^s = \{\alpha^s : \alpha \in \Gamma_1\}$  is a basis of  $R_{G,\theta}$ . The map  $\theta$  induces an involution  $\bar{\theta}$  of  $\Gamma_1$  such that, for each  $\alpha \in \Gamma_1$ ,  $\theta(\alpha) = -\bar{\theta}(\alpha) - \beta_\alpha$  where  $\beta_\alpha$  is a linear combination of simple roots in  $R_G^0$ . We can order the simple roots  $\alpha_1, \dots, \alpha_l, \alpha_{l+1}, \dots, \alpha_{l+s}, \alpha_{l+s+1}, \dots, \alpha_m$  so that  $\alpha_i$  is fixed by  $\theta$  if and only if  $i > l + s$ . Moreover we can suppose that  $\alpha_1^s, \dots, \alpha_l^s$  are distinct. For each  $i = 1, \dots, l$  we define  $\omega_i$  as follows: if  $\bar{\theta}(\alpha) = \alpha$  then  $\omega_i = \omega_{\alpha_i}$ , otherwise  $\omega_i = \omega_{\alpha_i} - \theta(\omega_{\alpha_i}) = \omega_{\alpha_i} - \omega_{\bar{\theta}(\alpha_i)}$ . A weight is special if and only if it is a linear combination of the weight  $\omega_1, \dots, \omega_l$ , so  $\{\omega_1, \dots, \omega_l\}$  is a basis of  $\Lambda^1$ . Moreover we can use these weights to give the following explicit description of  $\Omega$ .

**Proposition 1.1 (Lemma 2.1 and Theorem 2.3 in [CM I])** *Let  $\Omega^+$  be the set of spherical weights, then  $\Omega \cap \Lambda^+ = \Omega^+$ . Moreover  $\Omega = \bigoplus_{i=1}^l \mathbf{Z}a_i\omega_i$  where  $a_i \in \{1, 2\}$  for each  $i$ . The constant  $a_i$  is equal to 2 if  $\theta(\alpha_i) = -\alpha_i$ , while it is equal to 1 if  $\theta(\alpha_i) \neq -\alpha_i$ . In particular  $a_i = 1$  if  $\bar{\theta}(\alpha_i) \neq \alpha_i$ . For each  $i$  and  $j$  we have  $\langle a_i\omega_i, (\alpha_j^s)^\vee \rangle = b_i\delta_{i,j}$  where  $(\alpha_j^s)^\vee$  is the coroot associated to  $\alpha_j^s$  and  $b_i \in \{1, 2\}$ . Moreover  $b_i = 2$  if and only if  $2\alpha_i^s \in R_{G,\theta}$ .*

Notice that the proposition implies that the fundamental Weyl chamber  $C^+$  of  $R_{G,\theta}$  is the intersection of  $\chi(S)_{\mathbf{R}}$  with the fundamental Weyl chamber of  $R_G$ . We will say that a special weight  $\sum n_i\omega_i$  is *regular* if  $n_i > 0$  for each  $i$ . Thus a spherical weight is regular if and only if it is a strongly dominant weight of the restricted root system (with respect to the basis  $\Gamma^s$ ). The Weyl group  $W_{L,\theta}$  of the restricted root system is called the restricted Weyl group and it has the following description.

**Proposition 1.2 (See 1.8 in [CS])** *One can identify the restricted Weyl group  $W_{L,\theta}$  with the group  $\{w \in W : w \cdot \mathfrak{t}_1 \subset \mathfrak{t}_1\}/W_0$ , where  $W$  is the Weyl group of  $R_G$  and  $W_0$  is the Weyl group of the root system  $R_G^0$ .*

Let  $\lambda$  be a regular spherical weight and let  $k_\lambda$  be a non-zero vector of  $V_\lambda$  fixed by  $\mathfrak{h}$ . One can show that  $k_\lambda$  is unique up to a non-zero scalar. Let  $x_0$  be the class of  $k_\lambda$  in  $\mathbf{P}(V_\lambda)$ . De Concini and Procesi have defined the wonderful completion  $X$  of  $G/H$  as the closure of  $Gx_0$  in  $\mathbf{P}(V_\lambda)$ .

Now we want to give a local description of  $X$ . Choose a basis of  $V_\lambda$  formed by weight vectors and consider the affine open set  $\tilde{A}$  of  $\mathbf{P}(V_\lambda)$  where the coordinate corresponding to the highest weight  $v_\lambda$  is not zero. Notice that  $\tilde{A} \cap X$  is  $U^-$  stable, where  $U^-$  is the unipotent group associated to  $R_G^1 \cap (-R_G^+)$ , namely  $U^- = \prod_{\alpha \in R_G^1 \cap (-R_G^+)} U_\alpha$  as a variety. One can show that the closure of  $Tx_0$  in  $\tilde{A}$  is an affine space  $\mathbf{A}^l$  with coordinates  $-\alpha_1^s, \dots, -\alpha_l^s$ . Moreover the map  $\varphi : U^- \times \mathbf{A}^l \rightarrow \tilde{A} \cap X$  given by  $\varphi(g, v) = g \cdot v$  is an isomorphism. For each  $i$ , let  $X_{-\alpha_i^s}$  be the divisor of  $X$  whose intersection with  $U^- \times \mathbf{A}^l$  is the locus of zeroes of  $-\alpha_i^s$ .

De Concini and Procesi have proved that  $X$  does not depend on  $\lambda$ . Moreover  $X$  is wonderful according to the definition of Luna:

**Theorem 1.1 (Theorem 3.1 in [CSV I])** *Let  $X$  be as before, then:*

1.  $X$  is a smooth projective  $G$ -equivariant embedding of  $G/H$ . In particular the stabilizer of  $x_0$  is  $H$ ;
2.  $X \setminus (Gx_0)$  is a divisor with normal crossings. It has irreducible components  $X_{-\alpha_1^s}, \dots, X_{-\alpha_l^s}$  and they are smooth subvarieties of  $X$ ;
3. the  $G$ -orbits of  $X$  correspond to the subsets of  $\{1, 2, \dots, l\}$ , so that the orbit closures are the intersections  $X_{-\alpha_{i_1}^s} \cap \dots \cap X_{-\alpha_{i_k}^s}$  with  $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, l\}$ ;
4. there is an unique closed orbit  $\bigcap_{i=1}^l X_{-\alpha_i^s}$  and it is isomorphic to  $G/P$ , where  $P$  is the parabolic subgroup of  $G$  associated to  $\Gamma_0$ , i.e. the Lie algebra of  $P$  is  $\mathfrak{t} \oplus \bigoplus_{\alpha \in R_G^1 \cup R_G^+} \mathfrak{g}_\alpha$ .

We will say that a  $G$ -variety  $Y$  with a open orbit isomorphic to  $G/H$  is a *symmetric variety* if it is normal. All symmetric varieties are spherical varieties, i.e. they have a dense  $B$ -orbit; indeed  $BH/H$  is a dense open  $B$ -orbit in  $G/H$  (see Proposition 1.3 in [CSV I]). We will say that a symmetric variety  $Y$  with an open orbit isomorphic to  $G/H$  is *toroidal* if there is a  $G$ -equivariant map  $\pi : Y \rightarrow X$  extending the identity map of  $G/H$ . The toroidal symmetric varieties have been classified by De Concini and Procesi (see [CSV II]). Later on, Luna and Vust have classified the spherical varieties with a fixed open  $G$ -orbit (see [LV]); in particular they have classified the symmetric varieties with a fixed open  $G$ -orbit (see also [V]). In the literature a spherical variety is called *toroidal* if every  $B$ -stable prime divisor which contains a  $G$ -orbit is  $G$ -stable. We want to remark that a symmetric variety is toroidal (according to our definition) if and only if it is a toroidal spherical variety (see [LV] and [V]).

Let  $Z_0^c$  be the closure of  $\mathbf{A}^l$  in  $X$ ; it is the complete toric variety whose fan is composed by the Weyl chambers and their faces.

**Theorem 1.2 (Theorem 5.2 and 5.3 in [CSV II])** *There is an equivalence between the category of complete toroidal symmetric varieties and the category of toric varieties proper over  $\mathbf{A}^1$ . Given a complete toroidal symmetric variety  $\pi : Y \rightarrow X$  the corresponding open toric variety  $Z$  is the inverse image of  $\mathbf{A}^1$  in  $Y$ . Moreover there is a one-to-one correspondence between complete toroidal symmetric varieties and complete toric  $S$ -varieties over  $Z_0^c$  whose fan is  $W_{L,\theta}$ -invariant. The variety  $Z^c$  corresponding to the symmetric variety  $Y$  is the inverse image of  $Z_0^c$ , moreover the variety  $Z^c$  is the closure of  $Z$  in  $Y$  and the variety  $Y$  is smooth if and only if  $Z$  (or equivalently  $Z^c$ ) is smooth.*

The inverse image of the open set  $U^- \times \mathbf{A}^1$  is isomorphic to  $U^- \times \pi^{-1}(\mathbf{A}^1)$  in a  $U^- \cdot T$  equivariant way. Moreover the  $G$ -orbits of  $Y$  are in one-to-one correspondence with the  $S$ -orbits of  $Z$ . In this work, unless explicitly stated otherwise, we shall always assume that the complete toroidal symmetric variety  $Y$  is smooth. In this case any orbit closure in  $Y$  is also smooth.

Now we want to study the line bundles on a complete toroidal symmetric variety. First of all, we can identify the Picard group of the wonderful symmetric variety  $X$  with a subgroup  $\Lambda_X$  of the lattice  $\Lambda$  of integral weights of  $G$ .

**Proposition 1.3 (Proposition 8.1 in [CSV I])** *The map  $\text{Pic}(X) \rightarrow \text{Pic}(G/P)$  induced by the canonical inclusion is injective.*

Recall that we can identify  $\text{Pic}(G/P)$  with a sublattice of the lattice of weights. Moreover  $\text{Pic}(G/P) \cong \text{Pic}_G(G/P)$  because  $G$  is simply connected. Thus to any linearized line bundle  $L \in \text{Pic}_G(G/P)$  we can associate the opposite  $\lambda$  of the character  $-\lambda$  with which  $T$  acts on the fibre over  $P/P$ . We denote a line bundle on  $X$  by  $L_\lambda$  if its image is the weight  $\lambda$ . Let  $\lambda$  be a dominant weight such that  $\mathbf{P}(V_\lambda)$  contains a line  $r$  fixed by  $H$ , for example  $\lambda \in \Omega^+$ . One can show that the map  $G/H \ni gH \rightarrow g \cdot r$  can be extended to a morphism  $\psi_\lambda : X \rightarrow \mathbf{P}(V_\lambda)$ . The line bundle  $\psi_\lambda^* \mathcal{O}(1)$  is  $L_\lambda$ . Thus, given a line bundle  $L_\lambda$  on  $X$  such that  $\lambda$  is dominant, there is a sub-representation of  $H^0(X, L_\lambda)$  isomorphic to  $V_\lambda^*$  and obtained by pullback of  $H^0(\mathbf{P}(V_\lambda), \mathcal{O}(1))$  to  $X$  (this representation is unique because  $X$  is a spherical variety).

**Definition 1.2** *We will say that a root  $\alpha \in \Gamma_1$  is an EXCEPTIONAL ROOT if  $\bar{\theta}(\alpha) \neq \alpha$  and  $\langle \alpha, \theta(\alpha) \rangle \neq 0$ . Moreover we will say that a normal  $G$ -equivariant embedding of  $G/H$  is an EXCEPTIONAL SYMMETRIC VARIETY if there is an exceptional root.*

**Theorem 1.3 (Theorem 4.8 in [CS])**  *$\text{Pic}(X)$  is generated by the spherical weights and by the fundamental weights corresponding to the exceptional roots.*

Let  $X$  be the wonderful symmetric variety and let  $Y$  be the complete toroidal symmetric variety over  $X$  associated to a toric variety  $Z$ . We will denote by  $\Delta$  the fan of  $Z$  and by  $\Delta^c$  the fan of  $Z^c$ . Let  $o_\gamma$  be the  $S$ -orbit of  $Z$  associated to  $\gamma \in \Delta$ . We will denote by  $O_\tau$  the corresponding  $G$ -orbit of  $Y$ . We shall denote by  $Z_\gamma$  the stable subvariety of  $Z$  associated to  $\gamma \in \Delta$ , by  $Z_\gamma^c$  the stable subvariety of  $Z^c$  associated to  $\gamma \in \Delta^c$  and by  $Y_\gamma$  the stable subvariety of  $Y$  associated to  $\gamma \in \Delta$ . We set  $\Delta(i) = \{\gamma \in \Delta : \dim \gamma = i\}$  and  $\Delta^c(i) = \{\gamma \in \Delta^c : \dim \gamma = i\}$ .

The closed orbits  $O_\sigma$  of  $Y$  are in one-to-one correspondence with the maximal cones of the fan  $\Delta$  associated to  $Z$  and they are all isomorphic to the unique

closed orbit of  $X$  through the restriction of the projection, so we can identify  $Pic(O_\sigma)$  with  $\Lambda_X$  for each  $\sigma \in \Delta(l)$ . The group  $Pic(Z)$  is freely generated by the line bundles  $O(Z_\tau)$  where  $\tau$  varies in the set  $\Delta(1) \setminus \Delta_0(1)$ . Given a cone  $\gamma \subset (\chi(S)_*)_{\mathbf{R}}$  we will call  $\gamma^\perp$  the subspace of  $\chi(S)_{\mathbf{R}}$  consisting of the vectors which vanishes on  $\gamma$ .

**Theorem 1.4 (Theorem 2.4 in [Bi])** *Let  $Y$  be the complete toroidal symmetric variety associated to  $Z$ . Then*

1. *The maps  $Z \xrightarrow{i} Y \xrightarrow{\pi} X$  induce the split exact sequence*

$$0 \longrightarrow Pic(X) \xrightarrow{\pi^*} Pic(Y) \xrightarrow{i^*} Pic(Z) \longrightarrow 0,$$

*so  $Pic(Y)$  is (not canonically) isomorphic to  $Pic(X) \oplus Pic(Z)$ .*

2. *A section of the previous split short exact sequence is given by sending the free generators  $O(Z_\tau)$ , with  $\tau \in \Delta(1) \setminus \Delta_0(1)$ , to  $O(Y_\tau)$ .*
3. *The morphism given by the restriction to the closed orbits*

$$c_1^G : Pic(Y) \rightarrow \prod_{\sigma \in \Delta(l)} Pic(O_\sigma)$$

*is injective and its image can be identified with the lattice*

$$\Lambda_Y = \{h = (h|\sigma) \in \prod_{\sigma \in \Delta(l)} \Lambda_X \subset \prod_{\sigma \in \Delta(l)} \Lambda : h|\sigma - h|\sigma' \in \chi(S) \cap (\sigma \cap \sigma')^\perp \\ \forall \sigma, \sigma' \in \Delta(l).\}$$

We will indicate with  $L_h$  the line bundle whose image is  $h$ . We know that  $Pic(Y)$  is isomorphic to the group of equivariant line bundles  $Pic_G(Y)$  because  $G$  is simply connected and  $Y$  is complete. Moreover, given a line bundle  $L_h$ ,  $-h_\sigma$  is the character of the action of  $T$  on the fibre over the  $T$ -stable point  $O_\sigma \cap Z$ . In a similar way, we define  $h^c$  as the set  $(h|\sigma)$  where  $\sigma$  varies in  $\Delta^c(l)$  and  $-h_\sigma$  is the character of the action of  $T$  on the fibre over the  $T$ -stable point of  $Z^c$  corresponding to  $\sigma$ . In some case we can give a useful interpretation of  $h$  and  $h^c$ . Let  $M$  be a lattice in  $\chi(S)_{\mathbf{R}}$  which contains  $\chi(S)$ .

**Definition 1.3** *A real valued function  $h : |\Delta| \rightarrow \mathbf{R}$  on the support of  $\Delta$  is called a  $(\Delta, M)$ -LINEAR FUNCTION if  $h$  is linear on each  $\sigma \in \Delta$ . Let  $h|\sigma$  be the unique linear function which coincide with  $h$  on  $\sigma$ . We request moreover that  $h|\sigma$  belongs to  $M$  and that  $h|\sigma_1 - h|\sigma_2$  belongs to  $\chi(S)$  for each  $\sigma, \sigma_1$  and  $\sigma_2$  in  $\Delta(l)$ . Let  $SF(\Delta, M)$  be the additive group of the  $(M, \Delta)$ -linear functions.*

Sometimes we say that an element  $h$  of  $SF(\Delta, M)$  is a  $\Delta$ -linear function and a  $\Delta$ -linear function  $h$  may be thought of as a function  $h : \chi_*(S)_{\mathbf{R}} \rightarrow \mathbf{R} \cup \{-\infty\}$ . We say that a  $\Delta$ -linear function  $h$  is convex if  $h(v + v') \geq h(v) + h(v')$  for each  $v, v'$ . We say that  $h$  is strictly convex on  $\Delta$  if moreover  $h|\sigma \neq h|\sigma'$  for each  $\sigma, \sigma'$ . If a torus  $S'$  is an étale cover of  $S$ , then we can identify  $SF(\Delta, \chi(S'))$  with the group of  $S'$ -linearized line bundles on  $Z$ . We can suppose that  $-h|\sigma$  is the character of  $S'$  on the fibre  $L_h(x_\sigma)$  over the  $S'$ -stable point  $x_\sigma$  associated to any maximal cone  $\sigma$  of  $\Delta$ .

**Definition 1.4** Let  $L_h$  be a line bundle on  $Y$ . We will say that  $L_h$  and  $h$  are **ALMOST SPHERICAL** if  $h|\sigma$  belongs to the lattice generated by the spherical weights for each  $\sigma \in \Delta(l)$ . Moreover we will say that  $L_h$  and  $h$  are **SPHERICAL** if  $h|\sigma$  is a spherical weight for each  $\sigma \in \Delta(l)$ .

If  $L_h$  is an almost spherical line bundle, the element  $h$  may be thought of as a  $(\Delta, \Omega)$ -linear function and, accordingly, the element  $h^c$  may be thought of as a  $(\Delta^c, \Omega)$ -linear function.

Now we want to describe the sections of a line bundle over  $Y$ . Observe that the space of sections is multiplicity-free because  $Y$  is a spherical variety.

**Definition 1.5** Given  $h \in \Lambda_Y$  let

$$\Pi(Z, h) = \left\{ \mu \in \bigcap_{\sigma \in \Delta(l)} (h|\sigma + (\chi(S) \cap \sigma)) \right\}, \quad \Pi(Y, h) = \Pi(Z, h) \cap \Lambda^+,$$

$$\text{and} \quad \Pi(Z^c, h) = \left\{ \mu \in \bigcap_{\sigma \in \Delta^c(l)} (h|\sigma + (\chi(S) \cap \sigma)) \right\}.$$

Observe that a weight  $\lambda$  in  $\Lambda_X$  belongs to  $\Pi(Z, h)$  (respectively to  $\Pi(Z^c, h)$ ) if and only if  $\lambda \geq h$  (respectively  $\lambda \geq h^c$ ) as function on  $\chi_*(S)_{\mathbf{R}}$ .

**Theorem 1.5 (Theorem 3.4 in [Bi])** Let  $L_h$  be a line bundle on  $Y$ . Then

$$H^0(Y, L_h) = \bigoplus_{\mu \in \Pi(Y, h)} V_{\mu}^*.$$

In particular  $H^0(Y, L_h) \neq 0$  if and only if  $\Pi(Y, h)$  is not empty.

We want to give an idea of a possible construction of  $H^0(Y, L_h)$ . We will say that a vector  $v \in \chi_*(S) - \{0\}$  is primitive if an equality  $v = av'$ , where  $v'$  is a vector in  $\chi_*(S)$  and  $a$  is a positive integer, implies  $a = 1$ . Given any one-dimensional rational cone  $\tau$ , we denote by  $\varrho(\tau)$  the unique primitive vector contained in  $\tau$ .

**Lemma 1.1 (Lemma 2.7 in [Bi])** The line bundle  $O(Y_{\tau})$  is almost spherical and the associated  $\Delta$ -linear function  $d^{\tau}$  satisfies the equalities  $d^{\tau}(\varrho(\tau')) = -\delta_{\tau, \tau'}$  for each  $\tau' \in \Delta(1)$ . Moreover, if we define  $s_{\tau} \in H(Y, L_{d^{\tau}})$  as the unique, up to a non-zero scalar, section whose divisor is  $Y_{\tau}$ , then we have that  $s_{\tau}$  is  $G$ -invariant.

Observe that, given  $\lambda \in \Lambda_X$ , then  $\lambda \in \Pi(Z, h)$  if and only if  $h = \lambda + \sum_{\tau \in \Delta(1)} a_{\tau} d^{\tau}$  for appropriate positive integers  $a_{\tau}$ . Let  $\lambda \in \Pi(Y, h)$  and recall that  $H^0(X, L_{\lambda})$  contains  $V_{\lambda}^*$ . We have  $H^0(X, L_{\lambda}) \subset H^0(Y, \pi^* L_{\lambda})$ , so  $H^0(Y, \pi^* L_{\lambda})$  contains a lowest weight vector  $v_{-\lambda}$  of weight  $-\lambda$ . Making use of the definition of  $\Pi(Z, h)$ , we can find positive constants  $a_{\tau}$  such that  $h - \lambda = \sum_{\tau \in \Delta(1)} a_{\tau} d^{\tau}$ . The Lemma 1.1 implies that  $v_{-\lambda} \cdot \prod s_{\tau}^{a_{\tau}}$  is a non-zero section of  $H^0(Y, L_h)$  with weight  $-\lambda$ . Moreover  $v_{-\lambda} \cdot \prod s_{\tau}^{a_{\tau}}$  is invariant by the unipotent part of the opposite  $B^-$  of the fixed Borel group of  $G$ . Thus  $H^0(Y, L_h) \supseteq \bigoplus_{\mu \in \Pi(Y, h)} V_{\mu}^*$ . Because of the previous theorem we give the following definition:

**Definition 1.6** Given  $h$  in  $\Lambda_Y$  and  $\lambda$  in  $\Pi(Y, h)$ , write  $h = \lambda + \sum_{\tau \in \Delta(1)} a_{\tau} d^{\tau}$  for suitable  $a_{\tau} \in \mathbf{Z}^+$ . We define  $s^{h-\lambda}$  as the section  $\prod s_{\tau}^{a_{\tau}}$  of  $H^0(Y, L_{h-\lambda})$ .



We want to describe also the sections over  $Z$ , respectively over  $Z^c$ .

**Proposition 1.4** *Let  $L_h$  be a line bundle on  $Y$ . Then*

1.

$$H^0(Z, L_h|Z) = \bigoplus_{\mu \in \Pi(Z, h)} \mathcal{C}\chi^\mu,$$

where  $\chi^\mu$  is a  $T$ -seminvariant section of weight  $-\mu$ .

2.

$$H^0(Z^c, L_h|Z^c) = \bigoplus_{\mu \in \Pi(Z^c, h)} \mathcal{C}\chi^\mu.$$

*Remark.* Let  $\pi : Y \rightarrow Y'$  be an  $G$ -equivariant morphism between two complete toroidal symmetric varieties and let  $L_h$  be a line bundle on  $Y'$ . Then the pullback  $\pi^*(L_h)$  is the line bundle on  $Y$  associated to  $h$  and  $H^0(Y, \pi^*(L_h)) = H^0(Y', L_h)$ .

Now we want to explain some relations between the previous sets.

**Corollary 1.1 (Corollary 4.1, Proposition 4.2 and Theorem 4.2 in [Bi])**

1. Given  $h \in \Lambda_Y$ , we have the equality

$$\Pi(Y, h) = \Pi(Z^c, h) \cap \Lambda^+.$$

2. If  $h \in \Lambda_Y$  is almost spherical, then

$$\Pi(Z^c, h) = \bigcup_{w \in W_{L, \theta}} w \cdot \Pi(Y, h).$$

Moreover the restriction map  $H^0(Y, L_h) \rightarrow H^0(Z^c, L_h|Z^c)$  is surjective.

We want to make some remarks about the case of an almost spherical  $h$ . Let  $w \cdot \mu \in \Pi(Z^c, h)$  with  $\mu \in \Pi(Y, h)$ . Any section  $s \in V_\mu^* \subset H^0(Y, L_h)$  of weight  $-\mu$  has non-zero restriction to  $Z^c$  because  $U^- \cdot Z^c$  is dense in  $Y$ . Moreover, up to choose another basis of the root system, we can suppose that it is a lowest weight vector.

Recall that there is a one-to-one correspondence between the convex functions on  $(\chi(S)_*)_{\mathbf{R}}$  with values in  $\mathbf{R} \cup \{-\infty\}$  and the convex sets in  $\chi(S)_{\mathbf{R}}$ , which send a convex function  $h$  to the convex set  $Q_h = \{m \in \chi(S)_{\mathbf{R}} : m(v) \geq h(v), \forall v \in (\chi(S)_*)_{\mathbf{R}}\}$ . Moreover  $h(v) = \inf\{m(v); m \in Q_h\}$  for each  $v \in (\chi(S)_*)_{\mathbf{R}}$ . Recall that a polyhedron is the intersection of a finite number of semispaces.

**Definition 1.7** *Let  $Y$  be a complete toroidal symmetric variety and let  $L_h$  be an almost spherical line bundle on  $Y$ . We define the polytope associated to  $L_h$  as*

$$P_h = \{m \in \chi(S)_{\mathbf{R}} : m(v) \geq h^c(v) \forall v \in |\Delta^c|\}.$$

Moreover we define the polyhedron associated to  $L_h$  as

$$Q_h = \{m \in \chi(S)_{\mathbf{R}} : m(v) \geq h(v) \forall v \in |\Delta|\}.$$

Observe that  $P_h = \{m \in \chi(S)_{\mathbf{R}} : m(\varrho(\tau)) \geq h^c(\varrho(\tau)) \forall \tau \in \Delta^c(1)\}$  and  $Q_h = \{m \in \chi(S)_{\mathbf{R}} : m(\varrho(\tau)) \geq h(\varrho(\tau)) \forall \tau \in \Delta(1)\}$ .

Brion (see Theorem 3.3 in [Br]) has found a characterization of the ample line bundles (respectively the line bundles generated by global sections) on a spherical variety. Using the previous characterization, one can obtain the following proposition about the line bundles on a complete toroidal symmetric variety. See Section 7 in [Ru] for a direct proof.

**Proposition 1.5** *Let  $L_h$  be a line bundle on  $Y$ . Then*

1.  $L_h$  is generated by global sections if and only if  $h$  is convex and  $h|\sigma$  is dominant for each  $\sigma \in \Delta(l)$ .
2.  $L_h$  is ample if and only if  $h$  is strictly convex on  $\Delta$  and  $h|\sigma$  is a regular weight for each  $\sigma \in \Delta(l)$ .

*Remark.* 1) One can also prove that  $L_h$  is ample if and only if it is very ample (see Proposition 7.1 in [Ru]).

2) A complete toroidal symmetric variety  $Y$  is projective if and only if  $Z^c$  is projective. Moreover  $Y$  is projective if and only if  $Z$  is quasi-projective. Indeed, given  $h \in SF(\Delta, \chi(S))$ , the associated line bundle on  $Z$  is ample if and only if  $h$  is strictly convex over  $\Delta$  (see Theorem 3.3 in [Br]).

3) One can reformulate the Proposition 1.5 using  $h^c$  instead of  $h$ . Let  $h$  be an almost spherical  $\Delta$ -linear function. One can prove that  $h^c$  is convex on  $\Delta^c$  if and only if  $h$  is convex on  $\Delta$  and  $h|\sigma$  is dominant for each  $\sigma \in \Delta(l)$ . Moreover  $h^c$  is strictly convex on  $\Delta^c$  if and only if  $h$  is strictly convex on  $\Delta$  and  $h|\sigma$  is a regular weight for each  $\sigma \in \Delta(l)$ .

## 2 Reduction to the complete toric variety

*In the following we will always suppose that  $L_h$  is an almost spherical line bundle, unless we will explicitly say otherwise.*

We start to study the multiplication of sections of two line bundles on  $Y$ . First of all, we want to show that this problem is equivalent to the similar problem on the complete toric variety  $Z^c$  associated to  $Y$ . Let  $L_h$  and  $L_k$  be any two line bundles on  $Y$  generated by global sections. Let

$$M_{h,k} : H^0(Y, L_h) \otimes H^0(Y, L_k) \longrightarrow H^0(Y, L_{h+k})$$

be the product of sections on  $Y$  and let

$$m_{h,k}^c : H^0(Z^c, L_h|Z^c) \otimes H^0(Z^c, L_k|Z^c) \longrightarrow H^0(Z^c, L_{h+k}|Z^c)$$

be the product of sections of the restrictions to  $Z^c$  of these line bundles.

**Theorem 2.1** *Let  $L_h$  and  $L_k$  be two almost spherical line bundles on  $Y$  generated by global sections. Then  $M_{h,k}$  is surjective if and only if  $m_{h,k}^c$  is surjective.*

*Proof.* The necessity of the condition is implied by the Corollary 1.1. Indeed, if  $i : Z^c \rightarrow Y$  is the canonic inclusion, then  $m_{h,k}^c \circ (i^* \otimes i^*) = i^* \circ M_{h,k}$ . Now suppose that  $m_{h,k}^c$  is surjective. It is sufficient to show that the image of  $M_{h,k}$  contains a basis of semi-invariant sections. If  $h$  and  $k$  are linear then  $M_{h,k}$

is surjective by the Theorem 0.1. In general, given  $\nu \in \Pi(Y, h+k)$  there are  $\lambda \in \Pi(Z^c, h)$  and  $\mu \in \Pi(Z^c, k)$  such that  $\nu = \lambda + \mu$ . Moreover there are elements  $w_1$  and  $w_2$  of  $W_{L, \theta}$  such that  $w_1 \cdot \lambda$  and  $w_2 \cdot \mu$  are dominant weights. Observe that  $\nu \geq w_1 \cdot \lambda + w_2 \cdot \mu$  on  $|\Delta|$ . Moreover  $w_1 \cdot \lambda \geq h$  and  $w_2 \cdot \mu \geq k$  because  $h^c$  and  $k^c$  are convex and invariant for the action of  $W_{L, \theta}$ . Thus  $s^{h-w_1 \cdot \lambda} H^0(Y, L_{w_1 \cdot \lambda}) \subset H^0(Y, L_h)$  and  $s^{k-w_2 \cdot \mu} H^0(Y, L_{w_2 \cdot \mu}) \subset H^0(Y, L_k)$ . We know that  $\text{Im } M_{w_1 \cdot \lambda, w_2 \cdot \mu}$  contains a lowest weight vector  $\varphi \in H^0(Y, L_{w_1 \cdot \lambda + w_2 \cdot \mu})$  of weight  $-\nu$ . Thus  $s^{h+k-w_1 \cdot \lambda - w_2 \cdot \mu} \varphi$  is contained in  $s^{h+k-w_1 \cdot \lambda - w_2 \cdot \mu} \text{Im } M_{w_1 \cdot \lambda, w_2 \cdot \mu} \subset \text{Im } M_{h, k}$  and it is not zero.  $\square$

We can prove the following proposition without assuming the surjectivity of  $m_{h, k}^c$ . Given two convex  $(\Omega, \Delta)$ -linear functions, say  $h$  and  $k$ , let  $\Pi(Y, h, k)$  be the set of the weights of the lowest weight vectors contained in  $\text{Im } M_{h, k}$ .

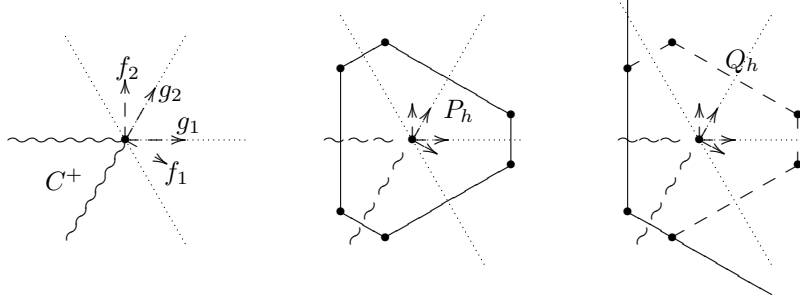
**Proposition 2.1**  $\Pi(Y, h, k)$  is saturated with respect to the dominant order of the roots in  $R_{G, \theta}$ .

*Proof.*  $\Pi(Y, h+k)$  is saturated because the simple restricted roots have negative values on  $|\Delta|$ . Given  $\nu \in \Pi(Y, h, k)$  there are two weights  $\lambda \in \Pi(Z^c, h)$  and  $\mu \in \Pi(Z^c, k)$  such that  $\nu = \lambda + \mu$ . Moreover there are elements  $w_1, w_2$  in the Weyl group  $W_{L, \theta}$  such that  $w_1 \cdot \lambda$  and  $w_2 \cdot \mu$  are dominant weights. Observe that  $\nu \geq w_1 \cdot \lambda + w_2 \cdot \mu$  on  $|\Delta|$ , so  $\nu \in \prod(Y, w_1 \cdot \lambda + w_2 \cdot \mu)$ . Let  $\nu'$  be a spherical weight dominated by  $\nu$ , then  $\nu' \in \Pi(Y, w_1 \cdot \lambda + w_2 \cdot \mu)$  because this set is saturated. Let  $\varphi$  be a lowest weight vector of weight  $\nu'$ . Making use of the surjectivity of  $M_{w_1 \cdot \lambda, w_2 \cdot \mu}$  we have  $\varphi \in s^{h+k-w_1 \cdot \lambda - w_2 \cdot \mu} \text{Im } M_{w_1 \cdot \lambda, w_2 \cdot \mu} \subset \text{Im } M_{h, k}$ .  $\square$

### 3 Reduction to the open toric variety

In this section we want to show that, given two ample line bundles on  $Y$ , the product of their sections on  $Z^c$  is surjective if and only if the product of sections on  $Z$  is surjective. Moreover we will study the relation between the sections of  $L|_Z$  and the sections of  $L|_{Z^c}$  for any ample line bundle  $L$  on  $Y$ .

Let us define some notations. We fix a cone  $\sigma \in \Delta(l)$  and we set  $v_h = h|\sigma$  for each  $h \in SF(\Delta, \Omega)$ , so  $\prod(Z, h) = Q_h \cap (\chi(S) + v_h)$  and  $\Pi(Z^c, h) = P_h \cap (\chi(S) + v_h)$ . Write  $\Gamma^s = \{f_1, \dots, f_l\}$ ; observe that it is a basis of  $\chi(S)_{\mathbf{R}}$  and let  $\{e_1, \dots, e_l\}$  be the dual basis of  $(\chi(S)_*)_{\mathbf{R}}$ . We have to define a second basis  $\{g_1, \dots, g_l\}$  of  $\chi(S)_{\mathbf{R}}$  because the fundamental Weyl chamber  $C^+$  is more easily defined using the basis of fundamental weights than the basis of the simple roots. The vector  $g_i$  is a positive multiple of  $-\omega_i$ , more precisely  $-g_i$  is the  $i$ -th fundamental weight of the unique reduced root system contained in  $R_{G, \theta}$  which share a basis with  $R_{G, \theta}$ . The vectors  $g_1, \dots, g_l$  generate a lattice which contains  $\chi(S)$ . Let  $\{\check{g}_1, \dots, \check{g}_l\}$  be the dual basis of  $\{g_1, \dots, g_l\}$ . Observe that  $C^+ = \sigma(-g_1, \dots, -g_l)$ . Given a point  $p$  in  $\chi(S)_{\mathbf{R}}$  we will use the following notations:  $p = \sum x_i f_i = \sum \dot{x}_i g_i$ , using the “normal” coordinates for the basis  $\{f_1, \dots, f_l\}$  and the “dotted” coordinates for the basis  $\{g_1, \dots, g_l\}$ . (In the following figures we consider the case where the restricted root system is of type  $A_2$  and  $Z$  is  $\mathbf{A}^2$ ).



**Proposition 3.1** *Let  $L_h$  be an ample spherical line bundle on  $Y$ . Then  $Q_h \cap C^+ = P_h \cap C^+$  and  $Q_h = P_h \cap C^+ + \sigma(f_1, \dots, f_l)$ .*

*Proof.* The equations of  $Q_h$  are of the form  $\sum b_i x_i \geq b$  for appropriate constants  $b_i$ . Thus  $Q_h$  is stable by translation with respect to vectors in  $\sigma(f_1, \dots, f_l)$ , i.e.  $Q_h + \bigoplus \mathbf{R}^+(f_i) \subset Q_h$ . Let  $H_j$  be the hyperplane of  $\chi(S)_{\mathbf{R}}$  generated by  $g_1, \dots, \hat{g}_j, \dots, g_l$  and let  $s_j$  be the orthogonal reflection with respect to  $H_j$ . Observe that, if  $P_h$  contains a point  $p$ , then it contains all the translates of  $p$  by  $W_{L,\theta}$ , so it contains the orthogonal projections  $\frac{1}{2}(p + s_j p)$  of  $p$  to the hyperplane  $H_j$ . Moreover there is no vertex of  $P_h$  contained in  $H_j$ , because  $h^c$  is strictly convex on  $\Delta^c$ .

The function  $h_K$  associated to a polyhedron  $K$  has always finite values if and only if  $K$  is compact. Moreover, there is a decomposition in convex cones of the convex set  $\{n \in (\chi(S)_*)_{\mathbf{R}} : h_K(n) \in \mathbf{R}\}$  such that there is a one-to-one correspondence between the cones of such decomposition and the faces of  $K$ . Hence there is a one-to-one correspondence between the 1-dimensional cones of such decomposition and the semi-spaces that define  $K$ . Given such a cone  $\tau$  the associated semi-space is  $\{m \in \chi(S)_{\mathbf{R}} : m(\varrho(\tau)) \geq h_K(\varrho(\tau))\}$ .

Let  $m$  be any point in  $Q_h \cap C^+ \cap \chi(S)_{\mathbf{Q}}$ . Given any  $\tau \in \Delta^c(1)$ , there are  $w \in W_{L,\theta}$  and  $\tau' \in \Delta(1)$  such that  $\varrho(\tau) = w \cdot \varrho(\tau')$ , because of the symmetry of  $\Delta^c$ . Observe that  $w^{-1} \cdot m - m$  is a linear combination  $\sum c_i f_i$  of the  $f_i$  with positive coefficients, so  $m(\varrho(\tau)) = (w^{-1} \cdot m)(\varrho(\tau')) \geq m(\varrho(\tau')) \geq h(\varrho(\tau')) = h^c(\varrho(\tau))$ . Thus  $m$  belongs to  $P_h$ , hence  $P_h \cap C^+ = Q_h \cap C^+$  because  $P_h$  is closed. The decomposition in cones of  $(\chi(S)_*)_{\mathbf{R}}$  associated to  $h_{P_h \cap C^+}$  has 1-dimensional cones  $\{\sigma(\tilde{g}_1), \dots, \sigma(\tilde{g}_l)\} \cup \Delta(1)$ . The function  $h_{P_h \cap C^+}$  has finite values on all  $(\chi(S)_*)_{\mathbf{R}}$ , it is equal to  $h$  on  $|\Delta|$  and vanishes on the vectors  $\tilde{g}_1, \dots, \tilde{g}_l$ . The function associated to  $\sigma(f_1, \dots, f_l)$  vanishes on  $|\Delta|$  and has value  $-\infty$  on the complementary set. The sum  $h_{P_h \cap C^+} + h_{\sigma(f_1, \dots, f_l)}$  is equal to the function  $h$  associated to  $Q_h$ , thus the proposition follows by the fact that  $h_Q + h_{Q'} = h_{Q+Q'}$  for each polyhedrons  $Q$  and  $Q'$ .  $\square$

We can prove a stronger statement on the “rational” points of  $Q_h$ , respectively of  $P_h$ .

**Proposition 3.2** *Let  $L_h$  be an ample spherical line bundle on  $Y$ , then  $Q_h \cap (v_h + \chi(S)) = P_h \cap C^+ \cap (v_h + \chi(S)) + \sum_{i=1}^l \mathbf{Z}^+ f_i$ .*

*Remark.* Observe that  $H^0(Z, L|Z)$  is a  $O_{Z^c}(Z^c)$ -module through the restriction map  $O_{Z^c}(Z^c) \rightarrow O_Z(Z)$  and  $H^0(Z^c, L|Z^c)$  is a  $O_{Z^c}(Z^c)$ -submodule of

$H^0(Z, L|Z)$ . This proposition implies that  $H^0(Z, L|Z)$  is generated by  $H^0(Z^c, L|Z^c)$  as an  $O_Z(Z)$ -module.

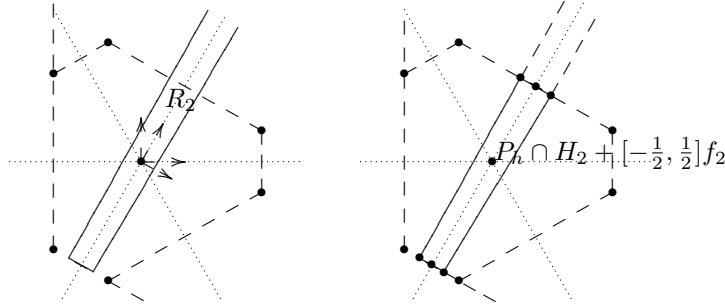
*Proof.* Observe that  $f_j$  is orthogonal to  $H_j$  and the vector  $\tilde{f}_i = \frac{1}{2}(f_i + s_j f_i)$  belongs to  $H_j$  for each  $i \neq j$ . The vectors  $\tilde{f}_i$  with  $i \neq j$  are pairwise distinct and form a basis of  $H_j$ . Moreover  $\tilde{f}_i = f_i + \frac{1}{2}d_i f_j$  for a suitable positive integer  $d_i$ , because the  $-f_i$  are the simple restricted roots. We have the following easy consequence of the Proposition 3.1.

**Lemma 3.1**  $Q_h \cap H_j = P_h \cap H_j \cap C^+ + \bigoplus_{i \neq j} \mathbf{R}^+ \tilde{f}_i$ .

*Proof.* Given  $p = p' + \sum r_i f_i \in Q_h \cap H_j$  with  $p' \in P_h \cap C^+$  and  $r_i$  positive constants, we have  $p = \frac{1}{2}(p' + s_j p') + \sum r_i \frac{1}{2}(f_i + s_j f_i)$ .  $\square$

Let  $R_j = \{p + a f_j \mid p \in Q_h \cap H_j \text{ and } -1/2 \leq a \leq 1/2\}$ . First of all we want to describe the conditions for a point  $p \in \chi(S)_{\mathbf{R}}$  to belong to  $R_j$ . Fixed any  $j$ , we define another basis  $\{u_1, \dots, u_l\}$  of  $\chi(S)_{\mathbf{R}}$  such that  $u_j = f_j$  and  $u_i = g_i$  if  $i \neq j$ . The conditions for a point  $p = \sum y_i u_i$  to belong to  $Q_h \cap H_j$  are  $y_j = 0$  plus conditions of the form  $\sum_{i \neq j} n_i y_i \geq n$ . Thus the conditions for a point  $p = \sum y_i u_i$  to belong to  $R_j$  are the inequalities of the form  $\sum_{i \neq j} n_i y_i \geq n$  that define  $Q_h \cap H_j$  plus the inequalities  $-1/2 \leq y_j \leq 1/2$ . The following fundamental lemma is the unique part of the proof in which we will use the strictly convexity of  $h^c$  i.e. the ampleness of  $L_h$ .

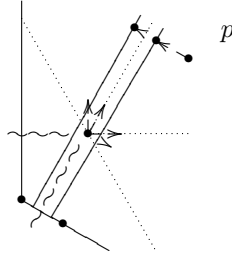
**Lemma 3.2**  $R_j$  is contained in  $Q_h$  for each  $j$ .



*Proof.* Making use of the Lemma 3.1, it is sufficient to show that  $P_h \cap H_j \cap C^+ + [-1/2, 1/2]f_j \subset Q_h$ . Because of the convexity of  $Q_h$ , it is sufficient to show that  $Q_h$  contains the points  $p' \pm (1/2)f_j$  for each vertex  $p'$  of  $P_h \cap H_j$ . Given such a vertex  $p'$ , let  $p = p' + a f_j$  be the intersection of  $p' + \mathbf{R}^{>0} f_j$  with the border of  $P_h$  ( $p$  exists because  $h^c$  is strictly convex). The point  $p$  should be a vertex of  $P_h$ , otherwise  $p$  is an interior point of a segment  $I$  contained in  $P_h$ . In this last case the symmetry of  $P_h$  implies that  $p'$  is an interior point of the orthogonal projection of  $I$  to  $H_j$ , a contradiction. Since  $p$  belongs to  $\chi(S)$  and  $s_j(p)$  is equal to  $p' - a f_j$ , we have  $2a \in \mathbf{Z}$ . Observe that  $p$  and  $s_j(p)$  belong to  $P_h$ , hence also the points  $p' + (1/2)f_j$  and  $p' - (1/2)f_j$  belong to  $P_h$  by convexity.  $\square$

Now we can conclude the proof of the Proposition 3.2 (look to the following figure). Let  $p = \sum x_i f_i = \sum \hat{x}_i g_i$  be a point contained in  $Q_h \cap (M + v_h)$ . If  $\hat{x}_i \leq 0$  for each  $i$ , then  $p \in P_h \cap C^+$ . Otherwise there is an index  $j$  such that  $\hat{x}_j > 0$ . We know that  $p = p' + \sum a_i f_i$  where  $p' \in P \cap C^+$  and the  $a_i$  are positive

constants. Observe that the point  $p - [a_j]f_j = p' + (a_j - [a_j])f_j + \sum_{i \neq j} a_i f_i$  belongs to  $Q_h \cap (M + v_h)$  and it has  $j$ -th coordinate with respect to  $\{g_1, \dots, g_l\}$  strictly less than 2 ( $[a_j]$  is the integral part of  $a_j$ ). Moreover, this coordinate can be at most 1 because  $p - [a_j]f_j$  is a weight. We can suppose that it is exactly 1, so  $p - [a_j]f_j - (1/2)f_j$  belongs to  $Q_h \cap H_j$  and it is the projection of  $p - [a_j]f_j$  to  $H_j$ . Thus  $p - [a_j]f_j$  belongs to  $R_j$ , so also  $p - [a_j]f_j - f_j$  belongs to  $R_j$  and its  $j$ -th coordinate with respect to  $\{g_1, \dots, g_l\}$  is negative. Moreover  $p - (p - [a_j]f_j - f_j)$  is a linear combination of the  $f_i$  with positive integral coefficients. If there is an index  $k$  such that  $p - [a_j]f_j - f_j$  has negative  $k$ -th coordinate with respect to  $\{g_1, \dots, g_l\}$ , then we reiterate the process. This process ends in a finite number of steps because  $Q_h$  is contained in the semi-space  $\{\sum x_i \geq h(e_1 + \dots + e_l)\}$ .  $\square$



Now we can prove the most important theorem of this work.

**Theorem 3.1** *Let  $L_h$  and  $L_k$  be two spherical ample line bundles on  $Y$ . Then  $m_{h,k}$  is surjective if and only if  $m_{h,k}^c$  is surjective.*

*Proof.* One can easily show that the theorem is equivalent to the following more combinatorial statement:

$$Q_h \cap (v_h + \chi(S)) + Q_k \cap (v_k + \chi(S)) = Q_{h+k} \cap (v_{h+k} + \chi(S))$$

if and only if

$$P_h \cap (v_h + \chi(S)) + P_k \cap (v_k + \chi(S)) = P_{h+k} \cap (v_{h+k} + \chi(S)).$$

The sufficiency of the condition is easy to show. Given a point  $p \in Q_{h+k} \cap (\chi(S) + v_{h+k})$  we know that  $p = p' + \sum c_i f_i$  where  $p' \in P_{h+k} \cap C^+ \cap (\chi(S) + v_{h+k})$  and the  $c_i$  are positive integers. Moreover there are  $p_h \in P_h \cap (\chi(S) + v_h)$  and  $p_k \in P_k \cap (\chi(S) + v_k)$  such that  $p' = p_h + p_k$ . Thus  $p = (p_h + \sum c_i f_i) + p_k$  and  $p_h + \sum c_i f_i$  belongs to  $Q_h \cap (\chi(S) + v_h)$ .

Suppose now that  $Q_h \cap (v_h + \chi(S)) + Q_k \cap (v_k + \chi(S)) = Q_{h+k} \cap (v_{h+k} + \chi(S))$ . Let  $p = \sum z_i f_i = \sum z_i g_i$  be a point in  $P_{h+k} \cap (\chi(S) + v_{h+k})$ . We can suppose that  $p$  belongs to  $C^+$  by the symmetry of the polytopes  $P_h$  and  $P_k$ . By hypothesis there are two points  $p'_0 \in Q_h \cap (\chi(S) + v_h)$  and  $q'_0 \in Q_k \cap (\chi(S) + v_k)$  such that  $p'_0 + q'_0 = p$ . First, we will show that we can choose  $p'_0$  and  $q'_0$  such that  $p'_0$  belongs to  $P_h$ . Indeed we know that  $p'_0 = p_0 + v$  where  $p'_0 \in P_h \cap C^+ \cap (\chi(S) + v_h)$  and  $v \in \bigoplus \mathbf{Z}^+ f_i$ , so  $p = p_0 + q_0$  where  $q_0 := q'_0 + v$  belongs to  $Q_k \cap (\chi(S) + v_k)$ .

Proceeding as in the Proposition 3.2, we can define a sequence of pairs of points  $\{(p_i, q_i)\}_{i=0, \dots, r}$  with the following properties: 1)  $p_i \in Q_h \cap (\chi(S) + v_h)$  for each  $i$ ; 2)  $q_i \in Q_k \cap (\chi(S) + v_k)$  for each  $i$ ; 3)  $p = p_i + q_i$  for each  $i$ ; 4)  $(p_0, q_0)$  is as before; 5)  $(p_{i+1}, q_{i+1}) = (p_i + f_{j_i}, q_i - f_{j_i})$  for a suitable  $j_i$  and 6)  $q_r \in P_k$ . Indeed we can define the  $\{q_i\}$  as in the Proposition 3.2 and then we set  $p_i = p - q_i$ .

Now it is sufficient to show by induction that we can choose the indices  $j_i$  so that  $p_i$  belongs to  $P_h$  for each  $i$ . We know that  $p_0 \in P_h$ . Now suppose that  $p_n$  belongs to  $P_h$  by inductive hypothesis. Suppose that  $p_n = \sum x_i f_i = \sum \dot{x}_i g_i$  and  $q_n = \sum y_i f_i = \sum \dot{y}_i g_i$ . If  $q_n \in P_k$  we define  $r = n$  and there is nothing to prove. Otherwise there is an index  $j_n$  such that  $\dot{y}_{j_n} > 0$  and it is sufficient to prove that  $p_n + f_{j_n}$  belongs to  $P_h$ . Observe that  $-\dot{x}_{j_n} > 0$ , so  $-\dot{x}_{j_n} \geq 1$  because  $\dot{x}_{j_n}$  is an integer. Moreover  $s_{j_n} p_n = p_n - \dot{x}_{j_n} f_{j_n}$  belongs to  $P_h$ . Thus  $P_h$  contains  $p_n + f_{j_n}$  because it is convex. Thus we can choose  $p_{n+1} = p_n + f_{j_n}$ .  $\square$

*Remark.* 1) The previous theorem is valid with the weaker hypotheses that  $h, k$  are convex and that  $h|\sigma, k|\sigma$  are regular spherical weights for each  $\sigma \in \Delta(l)$ . Indeed these hypotheses imply that no vertex of  $P_h$  (respectively of  $P_k$ ) is contained in a Weyl wall.

2) Suppose that  $h = k$  is convex and that  $h|\sigma$  is a regular spherical weight for each  $\sigma \in \Delta(l)$ . In this case one can show that  $L_h|Z$  is the pullback of an ample line bundle on a possibly singular toric variety  $Z'$  over  $\mathbf{A}^l$ . This suggests to consider only ample line bundles.

## 4 Line bundles on an exceptional complete toroidal symmetric variety

Let  $Y$  be an exceptional complete toroidal symmetric variety, let  $Z$  be the associated open toric variety and let  $\Delta$  be the fan of  $Z$ . Given an ample spherical line bundle  $L_h$  over  $Y$ , we know that the multiplication  $M_{h,h}$  of sections on  $Y$  is surjective if and only if the multiplication  $m_{h,h}$  of sections on  $Z$  is surjective. In this section we want to generalize this fact to the non spherical line bundles. Recall that  $Pic(X)$  is generated by the spherical weights and by the fundamental weights  $\omega_{\alpha_1}, \dots, \omega_{\alpha_s}$  corresponding to the exceptional roots  $\alpha_1, \dots, \alpha_s$ .

**Proposition 4.1** *Let  $L_{h'}$  be an ample line bundle on  $Y$  such that  $M_{h',h'}$  is surjective and let  $a_1, \dots, a_i$  be positive integers. If we define  $h = h' + \sum a_i \omega_{\alpha_i}$  then the product  $M_{h,h}$  of sections of  $L_h$  over  $Y$  is surjective.*

*Proof.* Observe that  $L_h$  is an ample bundle on  $Y$ . We will prove the proposition by induction on  $\sum a_i$ . The map  $M_{h,h}$  is trivially surjective if  $\sum a_i = 0$ . We need a lemma on the maps  $M_{h,\omega_{\alpha_i}}$ .

**Lemma 4.1** *Let  $L_h$  be an ample line bundle on  $Y$  and let  $\omega \in \{\omega_{\alpha_1}, \dots, \omega_{\alpha_s}\}$ . Then  $M_{h,\omega}$  is surjective.*

*Proof.* In the following  $V_\lambda^*$  is the unique subrepresentation of  $H^0(Y, L_\lambda)$  which contains a lowest weight vector  $v_\lambda$  of weight  $-\lambda$ . We have  $H^0(Y, L_h) = \bigoplus_{\lambda \in \Pi(Y,h)} s^{h-\lambda} V_\lambda^*$ ,  $H^0(Y, L_{h+\omega}) = \bigoplus_{\lambda \in \Pi(Y,h)} s^{h-\lambda} V_{\omega+\lambda}^*$  and  $H^0(Y, L_\omega) = V_\omega^*$ . The lemma is implied by the fact that, for each  $\lambda \in \Pi(Y,h)$ , the vector  $M_{h,\omega}(s^{h-\lambda} v_\lambda \otimes v_\omega)$  is a lowest weight vector of weight  $-\lambda - \omega$ .  $\square$

Now we go back to the proposition. Let  $j$  be an index such that  $a_j > 0$  and define  $\tilde{h} = h - \omega_{\alpha_j}$ . We have the following commutative diagram

$$\begin{array}{ccc}
H^0(Y, L_{\tilde{h}}) \otimes H^0(Y, L_{\tilde{h}}) \otimes H^0(Y, L_{\omega_{\alpha_j}}) \otimes H^0(Y, L_{\omega_{\alpha_j}}) & \longrightarrow & H^0(Y, L_h) \otimes H^0(Y, L_h) \\
\downarrow m_1 & & \downarrow M_{h,h} \\
H^0(Y, L_{2\tilde{h}}) \otimes H^0(Y, L_{\omega_{\alpha_j}}) \otimes H^0(Y, L_{\omega_{\alpha_j}}) & & \\
\downarrow m_2 & & \\
H^0(Y, L_{2\tilde{h}+\omega_{\alpha_j}}) \otimes H^0(Y, L_{\omega_{\alpha_j}}) & \xrightarrow{M_{2\tilde{h}+\omega_{\alpha_j}, \omega_{\alpha_j}}} & H^0(Y, L_{2h}),
\end{array}$$

where  $m_1$  is surjective by induction,  $m_2$  and  $M_{2\tilde{h}+\omega_{\alpha_j}, \omega_{\alpha_j}}$  are surjective by the previous lemma; thus  $M_{h,h}$  is surjective.  $\square$

**Theorem 4.1** *Let  $L_h$  be an ample line bundle on  $Y$ . If  $m_{h,h}$  is surjective then  $M_{h,h}$  is surjective.*

*Proof.* We know that, up to exchange  $\alpha_i$  with  $\bar{\theta}(\alpha_i)$  for some  $i$  in  $\{1, \dots, l\}$ , there are positive integers  $a_1, \dots, a_l$  such that the line bundle  $L_{h'}$ , with  $h' = h - \sum a_i w_i$ , is spherical and ample. The restriction of  $L_h$  to  $Z$  is isomorphic to the restriction of  $L_{h'}$  to  $Z$ , so  $m_{h',h'}$  is surjective. Thus  $M_{h',h'}$  is surjective because of the Theorem 3.1. Hence  $M_{h,h}$  is surjective by the previous proposition.  $\square$

## 5 Open projectively normal toric varieties

Now we want to describe some families of open toric varieties such that, if  $L_h$  is any ample line bundle on such a variety, then the product  $m_{h,h}$  of sections is surjective. One family is formed by all the varieties of dimension 2 proper over  $\mathbf{A}^2$ . Moreover we will find an infinite number of varieties that have such property for every given dimension. *In the following we will identify  $\chi(S)$  with  $\mathbf{Z}^l$ .* Given a real number  $x$ , we denote the integral part of  $x$  by  $[x]$ .

### 5.1 Blow-ups of $\mathbf{A}^l$

Now we study the class of varieties that are blow-ups of  $\mathbf{A}^l$  along an irreducible stable closed subvariety.

**Proposition 5.1** *Let  $Z$  be the blow-up of  $\mathbf{A}^l$  along the irreducible stable closed subvariety associated to  $\sigma(e_1, \dots, e_r)$ . Let  $L_h$  and  $L_k$  be two line bundles generated by global sections on  $Z$ , then the product of sections  $m_{h,k}$  is surjective.*

*Proof.* The inequalities for  $Q_h$  (respectively for  $Q_k$ ) are  $z_i \geq a_i$  for each  $i = 1, \dots, l$  and  $z_1 + \dots + z_r \geq b$  (respectively  $z_i \geq c_i$  for each  $i = 1, \dots, l$  and  $z_1 + \dots + z_r \geq d$ ). Here the  $a_i, b$ , the  $c_i$  and  $d$  are suitable integers. Let  $m$  be any point in  $Q_{h+k} \cap M$ , then there are  $\tilde{m}_1 = (x_1, \dots, x_l)$  in  $Q_h$  and  $\tilde{m}_2 = (y_1, \dots, y_l)$  in  $Q_k$  such that  $\tilde{m}_1 + \tilde{m}_2 = m$ , but they may have non integral coordinates. We want to translate  $\tilde{m}_1$  with respect to a “little” vector  $v$  so that  $\tilde{m}_1 + v$  will belongs to  $Q_h \cap \mathbf{Z}^l$  and  $\tilde{m}_2 - v$  will belongs to  $Q_k \cap \mathbf{Z}^l$ . Let  $\epsilon_i = -([x_i] - x_i)$  and observe that  $\epsilon_i$  is also equal to  $-([y_i] - y_i)$ . We have  $[x_i] + \epsilon_i \geq x_i \geq [x_i] \geq a_i$  and  $[y_i] +$



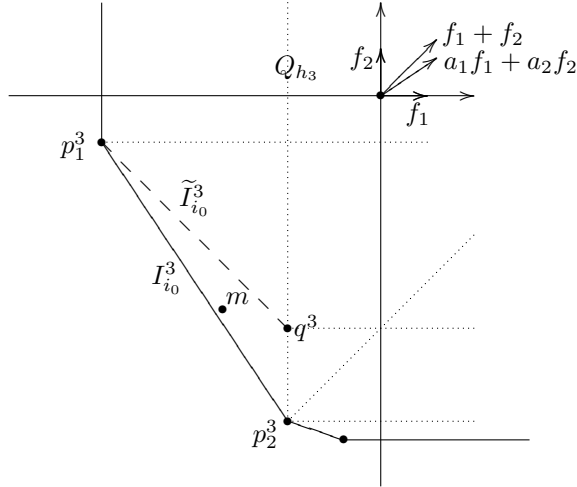
$\epsilon_i \geq y_i \geq [y_i] \geq c_i$  because the  $a_i$  and the  $c_i$  are integers. If  $([y_1], \dots, [y_l])$  belongs to  $Q_k$ , then we define  $m_1 = ([x_1] + \epsilon_1, \dots, [x_l] + \epsilon_l)$  and  $m_2 = ([y_1], \dots, [y_l])$ . In the same way, if  $([x_1], \dots, [x_l])$  belongs to  $Q_h$ , then we define  $m_1 = ([x_1], \dots, [x_l])$  and  $m_2 = ([y_1] + \epsilon_1, \dots, [y_l] + \epsilon_l)$ . Thus we can suppose that  $\sum_{i=1}^r [x_i] \leq b$  and  $\sum_{i=1}^r [y_i] \leq d$ . We define  $m_1 = ([x_1] + \epsilon_1, \dots, [x_s] + \epsilon_s, [x_{s+1}], \dots, [x_l])$  for an index  $s$  less than  $r$  and such that  $h(e_1 + \dots + e_r) = \sum_{i=1}^r [x_i] + \sum_{i=1}^s \epsilon_i = m_1(e_1 + \dots + e_r)$ . Finally set  $m_2 = m - m_1$  and notice that  $m_2$  belongs to  $Q_k$  since  $(m - m_1)(e_1 + \dots + e_r) \geq (h + k)(e_1 + \dots + e_r) - h(e_1 + \dots + e_r)$ .  $\square$

## 5.2 Open toric varieties of dimension 2

Now we consider the family of (smooth) toric varieties proper over  $\mathbf{A}^2$ .

**Theorem 5.1** *Let  $Z$  be any smooth toric variety proper over  $\mathbf{A}^2$ . Let  $L_{h_1}$  and  $L_{h_2}$  be two linearized line bundles generated by global sections and suppose that  $h_1$  and  $h_2$  are strictly convex on the same fan, then the product of sections  $m_{h_1, h_2}$  is surjective.*

The hypotheses mean that there is a toric variety  $Z'$  and two ample line bundles  $L'_h$  and  $L'_k$  over  $Z'$  such that  $L_h$  is the pullback of  $L'_h$  and  $L_k$  is the pullback of  $L'_k$ . We want to remark that  $Z'$  may be singular. In the following figure we consider the case where the fan of  $Z'$  contains four 1-dimensional cones.



*Proof.* Define a scalar product  $(, )$  such that  $\{f_1, f_2\}$  is an orthonormal basis. Let  $h_3 = h_1 + h_2$  and let  $\Delta$  be the fan of  $Z$ . It is obviously sufficient to prove that  $Q_{h_3} \cap \chi(S) = Q_{h_1} \cap \chi(S) + Q_{h_2} \cap \chi(S)$ . We can write the polyhedrons  $Q_{h_i}$  as unions  $\bigcup (I_i^j + \sigma(f_1, f_2))$ , where the  $I_i^j$  are rational polytopes of dimension at most equal to one. Moreover we can suppose that: 1)  $(I_i^1 + \sigma(f_1, f_2)) + (I_i^2 + \sigma(f_1, f_2)) = (I_i^3 + \sigma(f_1, f_2))$ ; 2)  $\dim I_i^1 = \dim I_i^2 = \dim I_i^3$ .

Let  $m = (x_1, x_2)$  be any point in  $Q_{h_3} \cap \chi(S)$  and suppose that  $m$  is contained in  $I_{i_0}^3 + \sigma(f_1, f_2)$ . If  $I_{i_0}^3$  is a point then we have reduced ourselves to the case of the polyhedrons associated to the pullback of line bundles on  $\mathbf{A}^2$ . Thus we can suppose that  $I_{i_0}^3$  is a segment line. Let  $a_1 f_1 + a_2 f_2$  be a vector orthogonal

to this segment. We can suppose that it is a primitive vector contained in  $\sigma(f_1, f_2)$  and that  $a_1 \geq a_2$  (up to exchange  $f_1$  with  $f_2$ ). We want to prove that  $(I_{i_0}^1 + \sigma(f_2, f_2)) \cap \chi(S) + (I_{i_0}^2 + \sigma(f_1, f_2)) \cap \chi(S) = (I_{i_0}^3 + \sigma(f_1, f_2)) \cap \chi(S)$  by induction on  $a_1 + a_2$ .

For each  $j$  let  $p_1^j = (z_1^{1,j}, z_2^{1,j})$  and  $p_2^j = (z_1^{2,j}, z_2^{2,j})$  be the vertices of  $I_{i_0}^j$  and let  $q^j = (z_1^{2,j}, z_2^{1,j} + z_1^{1,j} - z_1^{2,j})$ . Let  $\tilde{I}_{i_0}^j$  be the convex hull of  $p_1^j$  and  $q^j$ . The polyhedron  $I_{i_0}^j + \sigma(f_1, f_2)$  is the union of  $I_{i_0}^j + \sigma(f_1 + f_2, f_2)$ ,  $\tilde{I}_{i_0}^j + \sigma(f_1, f_2)$  and  $p_2^j + \sigma(f_1, f_2)$ . Moreover  $(\tilde{I}_{i_0}^1 + \sigma(f_1, f_2)) \cap \chi(S) + (\tilde{I}_{i_0}^2 + \sigma(f_1, f_2)) \cap \chi(S) = (\tilde{I}_{i_0}^3 + \sigma(f_1, f_2)) \cap \chi(S)$  by the Proposition 5.1. Finally  $(I_{i_0}^1 + \sigma(f_1 + f_2, f_2)) \cap \chi(S) + (I_{i_0}^2 + \sigma(f_1 + f_2, f_2)) \cap \chi(S) = (I_{i_0}^3 + \sigma(f_1 + f_2, f_2)) \cap \chi(S)$  because of the inductive hypothesis (indeed  $\{f_1, f_1 + f_2\}$  is a basis of  $\chi(S)$ ,  $a_1 f_1 + a_2 f_2 = (a_1 - a_2)f_1 + a_2(f_1 + f_2)$  and  $a_1 < a_1 + a_2$ ).  $\square$

### 5.3 Stable subvarieties

In some case we can reduce the study of the product of sections of two ample line bundles to the study of the product of sections of the restrictions of these line bundles to irreducible stable closed subvarieties. First of all, Brion has proved the following proposition.

**Proposition 5.2 (See Theorem 4 in [BrI])** *Let  $L$  be any line bundle generated by global sections on a smooth quasi-projective toric variety  $Z$ . Suppose that either  $Z$  is complete or it is proper over  $\mathbf{A}^1$ . Then, given two cones  $\gamma \subset \gamma'$  in  $\Delta$ , the restriction*

$$H^0(Z_\gamma, L|_{Z_\gamma}) \longrightarrow H^0(Z_{\gamma'}, L|_{Z_{\gamma'}})$$

*is surjective.*

**Proposition 5.3** *Let  $L_h$  and  $L_k$  be two linearized line bundles generated by global sections on a smooth quasi-projective toric variety  $Z$ . Suppose that either  $Z$  is complete or it is proper over  $\mathbf{A}^1$ . Let  $\tau$  be a cone in  $\Delta(1)$  and let  $s$  be a global section of  $L_{h+k}$  which does not vanish on  $Z_\tau$ . If  $s|_{Z_\tau}$  belongs to the image of the product  $m_{h,k}^\tau$  of sections of the restrictions of  $L_h$  and  $L_k$  to  $Z_\tau$ , then  $s$  belongs to the image of the product  $m_{h,k}$  of sections of  $L_h$  and  $L_k$ .*

*Proof.* Since there is a basis of semi-invariant sections, we can suppose that  $s$  is a semi-invariant section of weight  $\mu$ , so  $\mu(\varrho(\tau)) = (h+k)(\varrho(\tau))$ . Because of the previous proposition, there are sections  $s'_i \in \Gamma(Z_\tau^c, L_h)$  and  $s''_i \in \Gamma(Z_\tau^c, L_k)$  such that  $m_{h,k}^\tau(\sum s'_i|_{Z_\tau^c} \otimes s''_i|_{Z_\tau^c})$  is  $s|_{Z_\tau^c}$ , so  $m_{h,k}(\sum s'_i \otimes s''_i) = s$ .  $\square$

### 5.4 Two families of open toric varieties of dimension at least 3

Now we want to show that there is an infinite number of open toric varieties of any fixed dimension (greater than 2) such that the product of sections of any two ample line bundles is surjective. The main tool in what follows is the Proposition 5.3. We will consider a very special class of varieties. Indeed, given any ample line bundle  $L$  on a variety of this family, then  $H^0(Z, L)$  is generated as an  $O_Z(Z)$ -module by the sections which do not vanish on a suitable divisor.

**Proposition 5.4** *Let  $Z^n$  be the open toric variety obtained from  $A^l$  through the sequence of blow-ups along the irreducible stable subvarieties associated respectively to  $\sigma(e_1, \dots, e_l)$ ,  $\sigma(e_1, \dots, e_{l-1}, (\sum_{i=1}^{l-1} e_i) + e_l)$ ,  $\sigma(e_1, \dots, e_{l-1}, 2(\sum_{i=1}^{l-1} e_i) + e_l), \dots, \sigma(e_1, \dots, e_{l-1}, i(\sum_{i=1}^{l-1} e_i) + e_l), \dots, \sigma(e_1, \dots, e_{l-1}, (n-1)(\sum_{i=1}^{l-1} e_i) + e_l)$ . Let  $L_h$  and  $L_k$  be any two line bundles generated by global sections on  $Z^n$ , then the product of sections  $m_{h,k}$  is surjective.*

*Proof.* We have already considered the case  $n = 1$  in Proposition 5.1, so we can suppose  $n \geq 2$ . Up to changing the linearizations of the line bundles we can suppose that  $h(e_j) = k(e_j) = 0$  for each  $j$ . Observe that, if  $(Q_h \cap M) + (Q_k \cap M)$  contains a weight  $p$ , then it contains any weight  $p + \sum a_i f_i$  where the  $a_i$  are positive integers. So we can consider only the “minimal” weights.

**Lemma 5.1** *Let  $p$  be any weight in  $Q_{h+k} \cap M$  and suppose that there is not a weight  $p'$  in  $Q_{h+k} \cap M$  such that  $p \in p' + \sigma(f_1, \dots, f_l)$ . Then there is a cone  $\tau \in \Delta(1)$  such that  $p(\varrho(\tau)) = (h+k)(\varrho(\tau))$ .*

*Proof.* The weight  $p - f_l$  does not belong to  $Q_{h+k}$ , thus there is an index  $i$  such that  $(p - f_l)(i(\sum_{i=1}^{l-1} e_i) + e_l) < (h+k)(i(\sum_{i=1}^{l-1} e_i) + e_l)$ , so  $p(i(\sum_{i=1}^{l-1} e_i) + e_l) = (h+k)(i(\sum_{i=1}^{l-1} e_i) + e_l)$ .  $\square$

Hence it is sufficient to prove the surjectivity of the product of sections of the restrictions of  $L_h$  and  $L_k$  to the divisors  $Z_i^n$  associated to  $\sigma(i(\sum_{i=j}^{l-1} e_j) + e_l)$  with  $i = 0, \dots, n$ . Observe that  $Z_0$  is the blow-up of  $A^{l-1}$  in the stable point;  $Z_n^n$  is the projective space of dimension  $l-1$  while the other  $Z_i^n$  are isomorphic to the blow-up of  $Z_n^n$  in a stable point (associated to a cone  $\sigma(\tilde{e}_1, \dots, \tilde{e}_{l-1})$ ). Since  $Z_1^n$  dominates  $Z_n^n$  it is sufficient to study the product of sections of any two line bundles generated by global sections on  $Z_1^n$ . We will proceed in a similar way to the proof of Proposition 5.1.

**Lemma 5.2** *Let  $L_{h'}$  and  $L_{k'}$  be any two line bundles on  $Z_1^n$  generated by global sections. Then the multiplication of sections is surjective.*

*Proof.* We can suppose that  $h'(\tilde{e}_i) = k'(\tilde{e}_i) = 0$  for each  $i$ . In the following we identify  $\mathbf{Z}^{l-1}$  with  $\bigoplus_{i=1}^{l-1} \mathbf{Z}\tilde{e}_i$ . Given any point  $m = (z_1, \dots, z_{l-1})$  in  $Q_{h'+k'}$  with integral coordinates, there are  $\tilde{m}_1 = (x_1, \dots, x_{l-1}) \in Q_{h'}$  and  $\tilde{m}_2 = (y_1, \dots, y_{l-1}) \in Q_{k'}$  such that  $\tilde{m}_1 + \tilde{m}_2 = m$ . Now we want to simplify the notations. Let  $\epsilon_i = -\lceil [x_i] - x_i \rceil$  and set  $m_1 = ([x_1] + \epsilon_1, \dots, [x_{\bar{r}}] + \epsilon_{\bar{r}}, [x_{\bar{r}+1}], \dots, [x_{l-1}])$  and  $m_2 = ([y_1], \dots, [y_{\bar{r}}], [y_{\bar{r}+1}] + \epsilon_{\bar{r}+1}, \dots, [y_{l-1}] + \epsilon_{l-1})$  for a suitable  $\bar{r}$ . Let  $t = \sum_{i=1}^{l-1} \epsilon_i$ ,  $r = \sum_{i=1}^{\bar{r}} \epsilon_i$ ,  $[x]' = \sum_{i=1}^{l-1} [x_i]$ ,  $x = \sum_{i=1}^{l-1} x_i = \tilde{m}_1(\sum_{i=1}^{l-1} \tilde{e}_i)$ ,  $[y]' = \sum_{i=1}^{l-1} [y_i]$ ,  $y = \sum_{i=1}^{l-1} y_i = \tilde{m}_2(\sum_{i=1}^{l-1} \tilde{e}_i)$ ,  $a = h'(\sum_{i=1}^{l-1} \tilde{e}_i)$ ,  $b = -h'(-\sum_{i=1}^{l-1} \tilde{e}_i)$ ,  $c = k'(\sum_{i=1}^{l-1} \tilde{e}_i)$  and  $d = -k'(-\sum_{i=1}^{l-1} \tilde{e}_i)$ . We have  $m_1(\sum_{i=1}^{l-1} \tilde{e}_i) = [x] + r$ ,  $m_2(\sum_{i=1}^{l-1} \tilde{e}_i) = [y] + t - r$  and  $a + c \leq [x] + [y] + t = x + y \leq b + d$ . Notice that  $r$  takes all the values between 0 and  $t$  when  $\bar{r}$  varies between 0 and  $l-1$ .

1) If  $t + [x]' \leq b$  we define  $r$  as  $\min\{[y]' + t - c, t\}$ . If  $[y]' \geq c$  then  $r = t$ , so  $b \geq [x]' + t = [x]' + r \geq x \geq a$  and  $c \leq [y]' \leq y \leq d$ . If  $c \geq [y]'$  then  $b \geq [x]' + t \geq [x]' + r = [x]' + [y]' + t - c \geq a + c - c = a$  and  $c = [y]' + t - ([y]' + t - c) = [y]' + t - r \leq d$ .

2) Suppose now that  $[y]' + [x]' + t \leq b + c$ . If  $c - [y]'$  is positive then we define  $r = t + [y]' - c$ , so  $t - r = c - [y]'$  ( $t + [y]' \geq y \geq c$  so  $r \geq 0$ ). In this case  $c = [y]' + t - r \leq d$  and  $a \leq [x]' + [y]' + t - c = [x]' + r \leq b$ . If

$c - [y]'$  is negative then we define  $r = t$ , so  $c \leq [y]' = [y]' + t - r \leq d$  and  $a \leq x \leq [x]' + t = [x]' + r \leq c + b - [y]' \leq b$ .

3) Finally suppose that  $t + [x]' > b$  and  $[y]' + [x]' + t > b + c$ . We define  $r = b - [x]'$ , so  $a \leq [x]' + r = b$  and  $d \geq [y]' + [x]' + t - b = [y]' + t - r \geq c$ .  $\square$

The varieties  $Z^n$  have the following remarkable property: the “minimal” weights of  $\prod(Z^n, h + k)$  come from semi-invariant sections that do not vanish on a suitable divisor. This property allows us to prove the theorem without assuming the ampleness of both  $L_h$  and  $L_k$ . Unfortunately this property does not hold if we consider varieties whose fans are a little less symmetric. Notice that the fans of the varieties  $Z^n$  are invariant for any automorphism of  $\chi(S)_*$  which permutes the vectors of the basis and fixes  $e_l$ . In the following we define a class of varieties without such symmetry and obtained by blow-up from varieties of the previous family.

**Theorem 5.2** *Let  $\tilde{Z}^n$  be the blow-up of  $Z^n$  along the stable subvariety associated to  $\sigma(\sum_{j=1}^l e_j, e_2, \dots, e_l)$ . Let  $L_h$  be any ample line bundles on  $\tilde{Z}^n$ , then the product of sections  $m_{h,h}$  is surjective.*

*Proof.* We introduce some notations to simplify the counts:  $w := e_1 + 2 \sum_{j=2}^l e_j$ ,  $v_i := i(\sum_{i=1}^{l-1} e_i) + e_l$ ,  $a_i = h(v_i)$  and  $b = h(w)$  for each  $i$ . Moreover we suppose that  $h(e_j) = 0$  for each  $j$ . In the proof we allow  $L_h$  to be the pullback of an ample linearized line bundle on  $Z^n$ , i.e.  $h(w) = h(v_1)$ . We want to prove the proposition by induction on  $h(w)$  and on the dimension of  $\tilde{Z}^n$ . We can suppose that  $h(w) > h(v_1)$  because of the previous proposition. We want to prove that  $Q_{2h} \cap \chi(S) = Q_h \cap \chi(S) + Q_h \cap \chi(S)$  in a similar way to the previous proposition. As before, we have  $Q_h \cap \chi(S) + Q_h \cap \chi(S) \supset Q_h \cap \chi(S) + Q_h \cap \chi(S) + \bigoplus \mathbf{Z}^{\geq 0} f_i$ . Hence we can suppose that  $m - f_i$  does not belong to  $Q_{2h} \cap \chi(S)$ . Thus either there is an  $i$  such that  $m(v_i) = a_i$  or  $m(w) - 2h(w) \in \{0, 1\}$ .

Either we have to study a divisor or  $m(w) = 2h(w) + 1$ . One can easily verify that the only divisor that we do not have already considered is  $\tilde{Z}_{\sigma(v_1)}^n$ . This one can be studied in a very similar way to the divisor  $Z_{\sigma(v_1)}^n$  of  $Z^n$ , so we leave the details to the readers. Thus we can suppose that  $m(w) = 2b + 1$ . The strictly convexity of  $h$  on the fan  $\Delta$  implies the following conditions:  $a_i + (i - 1)b < (2i - 1)a_1 \forall i$ ,  $b > a_1 > 0$ ,  $2a_1 > b$  and  $ia_1 > a_i$ .

Let  $\Delta$  be the fan of  $\tilde{Z}^n$ , let  $\Delta'$  be the fan of  $Z^n$  and let  $h'$  be the  $\Delta$ -linear function such that  $h'(e_i) = 0$ ,  $h'(v_i) = h(v_i)$  and  $h'(w) = h(w) - 1$ . One can easily prove that  $h'$  is convex. Moreover  $h'$  is strictly convex either on  $\Delta$  or on  $\Delta'$ .

By induction we can suppose that  $m_{h',h'}$  is surjective, so we can suppose that there are two points  $m_1 = (x_1, \dots, x_l) \in Q_h \cap \chi(S)$  and  $m_2 = (y_1, \dots, y_l) \in Q_{h'} \cap \chi(S)$  such that  $m_1 + m_2 = m$ . We can suppose that  $m_2$  does not belong to  $Q_h$ , thus  $m_2(w) = b - 1$  and  $m_1(w) = b + 2$ .

We can suppose that  $m_1 - f_l \notin Q_h$  because  $m_2 + f_l \in Q_h$ . Thus there is  $i$  such that  $m_1(v_i) = a_i$ . Moreover we can suppose that  $(m_1 + f_1 - f_j, m_2 - f_1 + f_j)$  does not belong to  $Q_h \times Q_{h'}$  for any  $j = 2, \dots, l - 1$ , so  $x_j = 0$  or  $y_1 = 0$ . If  $y_1 = 0$  then  $2a_1 - 1 \geq m_2(w) = 2m_2(v_1) \geq 2a_1$ , a contradiction. Hence  $y_1 \neq 0$  and  $x_j = 0$  for each  $j = 2, \dots, l - 1$ . If there is  $i > 1$  such that  $m_1(v_i) = ix_1 + x_l = a_i$ , then

$$\begin{aligned}(2i-1)a_1 &\leq (2i-1)(x_1+x_l) = m_1(v_i) + (i-1)m_1(w) = \\ &= a_i + (i-1)(b+2) \leq (2i-1)a_1 + 2i-2,\end{aligned}$$

so  $0 \leq (2i-1)(x_1+x_l-a_1) \leq 2i-2$ . We have  $x_1+x_l = a_1$  because  $x_1+x_l-a_1$  is an integer. We have showed that  $m_1(v_1) = x_1+x_l = a_1$  or  $m_1(e_l) = x_l = 0$ . In the last case we have  $x_2 = \dots = x_l = 0$  and  $x_1 = b+2$ . We can suppose that  $(m_1 - f_1, m_2 + f_1)$  does not belong to  $Q_h \times Q_{h'}$ , so there is  $s > 0$  such that  $m_1(v_s) - a_s < s$ . Observe that  $m_1(v_s) = sx_1 = sb + 2s$ , so  $a_s \leq sb = m_1(v_s) - 2s < a_s - s < a_s$ , a contradiction.

Finally suppose that  $x_j = 0$  for each  $j = 2, \dots, l-1$ ,  $x_1+x_l = a_1$  and  $x_1+2x_l = b+2$ , so  $x_1 = 2a_1 - b - 2$  and  $x_l = b + 2 - a_1$ . We can suppose that  $(m_1 + f_1 - f_l, m_2 - f_1 + f_l)$  does not belong to  $Q_h \times Q_{h'}$ , so there is  $i > 1$  such that  $\varepsilon := m_2(v_i) - a_i < i$ . We have  $a_i \leq m_1(v_i) = (2i-1)a_1 - (i-1)b - 2(i-1)$ , so  $(2i-1)a_1 \geq a_i + (i-1)b + 2(i-1)$ . Finally

$$\begin{aligned}(2i-1)a_1 &\leq (2i-1)(\sum y_j) \leq m_2(v_i) + (i-1)m_2(w) \\ &= (i-1)(b-1) + a_i + \varepsilon \leq (2i-1)a_1 - 3(i-1) + \varepsilon,\end{aligned}$$

so  $3(i-1) \leq \varepsilon \leq i-1$ , a contradiction.  $\square$

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