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SOS for sampled-data systems

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Abstract: This article proposes a new approach to stability analysis of linear systems with sampled-data inputs. The method, based on a variation of the discrete-time Lyapunov approach, provides stability conditions using functional variables subject to convex constraints. These stability conditions can be solved using the sum of squares methodology with little or no conservatism in both the case of synchronous and asynchronous sampling. Numerical examples are included to show convergence.

Keywords: Sampled-Data systems, Lyapunov function, Sum of squares.

1. INTRODUCTION

In recent years, much attention has been paid to Networked Control Systems (NCS) (see Hespanha et al. [2007], Zampieri [2008]). These systems contain several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation on a processor can corrupt the sampling period of a controller. On the other side, the sampling period can be included in the design in order to avoid this load. In both cases, the variations of the sampling period will affect the stability properties of the system. Another phenomenon, which has been widely investigated concerns stability under packet losses. In wireless networks, a transmission of data packets is not always guaranteed. The objective is to guarantee stability even if some packets are lost in the communication. It is thus an important issue to develop robust stability conditions with respect to the variations of sampling period.

Sampled-data systems have extensively been studied in the literature Chen and Francis [1995], Fridman et al. [2004], Fujioka [2009], Zhang and Branicky [2001], Zhang et al. [2001] and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However in the case of asynchronous sampling, there are still several open problems. For example, the practical situation where the difference between two successive sampling instants is not constant but time-varying. Recently, several articles have addressed the problem of time-varying periods based on a discrete-time approach, Suh [2008], Oishi and Fujioka [2009], Hetel et al. [2006]. Recent papers have considered the modeling of continuous-time systems with sampled-data control in the form of continuous-time systems with delayed control input. In Fridman et al. [2004], a Lyapunov-Krasovskii approach was introduced. Improvements were provided in Fujioka [2009], Mirkin [2007], using the small gain theorem, and in Naghshtabrizi et al. [2008], based on the analysis of impulsive systems. These approaches dealt with time-varying sampling periods as well as with uncertain systems (see Fridman et al. [2004] and Naghshtabrizi et al. [2008]). Nevertheless, these sufficient conditions are still conservative. This means that the sufficient conditions obtained by continuous time approaches are not able to guarantee asymptotic stability whereas the system is stable. Recently several authors Fridman [2010], Liu and Fridman [2009], Seuret [2009] refined those approaches and obtained tighter conditions.

The key insight of this paper is that once we have developed the discrete-continuous Lyapunov conditions sufficient for stability, then these conditions can be verified computationally using recently developed algorithms for the optimization of polynomial functions. In particular, we use the machinery developed in Peet et al. [2009] to reformulate the stability question as a convex optimization problem with polynomial variables. We then use the software package SOSTOOLS Prajna et al. [2002] to solve the optimization problem. As can be seen in the numerical examples, the result is a sequence of stability tests of increasing accuracy. Furthermore, in the numerical examples, the accuracy of the stability test approaches the analytical limit exponentially fast as a complexity of the algorithm increases.

This article is based on a Lyapunov approach introduced in Seuret [2011]. This result is based on the discrete-time Lyapunov theorem and expressed with the continuous-time model of sampled-data systems. More precisely, this article analyzes the link between the discrete-time Lyapunov theorem employed, for instance in Suh [2008], Oishi and Fujioka [2009], Hetel et al. [2006], and the continuous-time approach proposed in Fridman et al. [2004], Naghshtabrizi et al. [2008], Fridman [2010], Seuret [2009]. Asymptotic stability criteria are provided for both synchronous and asynchronous samplings. Those criteria were expressed in terms of linear matrix inequalities. The main contribution of this paper is the use of sum of

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squares tools to provide larger upper-bounds of the maximum allowable sampling period than the existing ones (based on the continuous-time modeling).

This article is organized as follows. The next section formulates the problem. Section 3 presents a result on asymptotic stability of sampled-data systems. Section 4 presents several theorems on asymptotic stability of sampled-data systems expressed in terms of sum of squares. Some examples and simulations are provided in Section 6 and show the efficiency of the method.

Notation : Throughout the article, the sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the set of positive integers, positive scalars, the set of n-dimensional vectors, the set of $n\times n$ matrices and the set of symmetric matrices of $\mathbb{R}^{n\times n}$. The superscript 'T' stands for the matrix transposition. The notation $P>0$ for $P\in\mathbb{S}^n$ means that P is positive definite. The symbols I and 0 represent the identity and the zero matrices of appropriate dimension.

2. PROBLEM FORMULATION

Consider the linear system with a sampled-data input

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are of appropriate dimension. They are assumed to be constant and known. The proposed control law for this system is a piecewise-constant state feedback of the form $u(t) = u_d(t_k),\ t_k\le t<t_{k+1}$, where u_d is a discrete-time control signal and $0=t_0<t_1<\ldots<t_k<\ldots$ are the sampling instants. Note that t_k tends to infinity as k tends to infinity. The objective is to ensure the stability of the system together with a state-feedback controller of the form

$$u(t) = Kx(t_k),\ t_k\le t<t_{k+1}. \tag{2}$$

where the gain K in $\mathbb{R}^{n\times m}$ is given. Assume that there exists a positive scalar T such that the difference between two successive sampling instants $T_k = t_{k+1}-t_k$ satisfies

$$\forall k\ge 0,\quad 0<T_k\le T. \tag{3}$$

Several authors investigated stability analysis of such systems. In Fridman et al. [2004], a continuous-time approach to modeling sampled-data systems was developed. This paper accounts for sampling effects by using a time-varying delay of the form $\tau(t) = t-t_k$, for $t\in[t_k,t_{k+1}]$, $k=1,\ldots$. From (3), it follows that $\tau(t)\le T$ since $\tau(t)\le t_{k+1}-t_k$. In this approach, the differential equation (1) with the control law (2) is integrated over a sampling period. Then for $t\in[t_k,\ t_{k+1}]$, we have the following discrete-time system

$$x_{k+1} = \Gamma(T_k)x_k, \tag{4}$$

where we define the function

$$\Gamma(s) = \left[e^{As} + \int_0^s e^{A(s-\theta)}d\theta BK\right].$$

The continuous solution between sampling points is given by

$$x(t) = \Gamma(t-t_k)x_k \qquad \text{for } t\in[t_k,t_{k+1}].$$

Notation: Taking a cue from time-delay systems theory, we denote the segment of solution on $t\in[t_k,t_{k+1}]$ by x_{Tk}, so that

$$x_{Tk}(s) = \Gamma(s)x(t_k) \qquad \text{for } s\in[0,T_k].$$

We use $\mathscr{K}^n$ to denote the space of continuous maps from $[0,\ T]\to\mathbb{R}^n$, where recall T is the upper-bound of the T_k's.

If the matrices A and BK and sampling period are constant, the discrete dynamics become $x_{k+1} = \Gamma(T)x_k$, where T is the sampling period. A simple method to check the stability of the system is to ensure that $\Gamma(T)$ has all eigenvalues inside the unit circle. If the sampling period is time-varying, then we must verify that $\Gamma(T_k)$ has eigenvalues inside the unit circle for all $T_k\in[0\ T]$ which corresponds to an infinite dimensional problem. This is obviously complicated and even more so when the system is uncertain or nonlinear. Several authors have investigated this approach Hetel [2007], Oishi and Fujioka [2009], Suh [2008].

Based on the fact that sampled-data systems with uncertain sampling period can be seen as an infinite dimensional system, a time-delay approach to represent the sampled-data systems appears to be well-suited. Sufficient conditions for stability of sampled-data systems based were indeed designed in Fridman et al. [2004] by analyzing stability of a class of systems with time-varying delay. However, these results were somewhat conservative in that they did not account for the unique structure of the delay in a sampled-data system. In Naghshtabrizi et al. [2008], the authors introduce a new type of Lyapunov-Krasovskii functional which depends more explicitly on the delay function. In particular, they use the fact the $\dot{\tau}=1$ in their formulation. This led to improvement in the accuracy of the stability conditions. In the present article , we take a different approach which does not model the hold as a delay, but rather uses a new type of sampled-data Lyapunov Theorem introduced in Seuret [2011] and inspired by Peet et al. [2009]. The conditions are enforced using sum-of-squares optimization.

3. ASYMPTOTIC STABILITY OF SAMPLED-DATA SYSTEMS

3.1 Main theorem

In this section we introduce a new Lyapunov theorem which applies to general nonlinear sampled-data systems. This theorem accounts for the interaction between continuous and discrete element of a sampled-data system. A version of this result was introduced in Seuret [2011] and was partially inspired by the concept of spacing functions introduced in Peet et al. [2009]. Essentially, the theorem says that if there exists a Lyapunov function which has a net decrease over every sampling interval, then there exists a storage function which is continuously decreasing for all time. Consider the following system.

$$\dot{x}(t) = f(x(t),x(t_k)), \qquad t\in[t_k,t_{k+1}],\quad k=1,\ldots,\infty. \tag{5}$$

We assume global existence and continuity of solutions.

Theorem 1. Seuret [2011] Suppose $V:\mathbb{R}^n\to\mathbb{R}^+$ satisfies the following for $\mu_1>\mu_2>0$ and $p>0$

$$\mu_1|x|^p\le V(x)\le\mu_2|x|^p, \qquad \text{for all } x\in\mathbb{R}^n. \tag{6}$$

The two following statements are equivalent.

(i) If x is a solution of Equation (5), then

$$V(x(t_{k+1}))-V(x(t_k))<0, \qquad \text{for all } k\ge 0.$$

(ii) There exist continuous functions $Q_k:\mathbb{R}\times\mathscr{K}\to\mathbb{R}$, differentiable over $[t_k\ t_{k+1}[$ which satisfy the following

$$Q_k(T_k,z) = Q_k(0,z), \qquad \text{for all } k\ge 0 \text{ and } z\in\mathscr{K}, \tag{7}$$

and such that if x is a solution of Equation (5), then

$$\frac{d}{dt}\left[V(x(t))+Q_k(t-t_k,x_{Tk})\right]<0, \qquad \text{for all } t\in[t_k,t_{k+1}]. \tag{8}$$

Moreover, if either of these statements is satisfied, then solutions of system (5) are asymptotically stable.

Proof.

Assume (ii) is satisfied. Define the storage function

$$W(t) = [V(x(t)) + Q_k(t - t_k, x_{Tk})].$$

Then $\frac{d}{dt}W(t) < 0$ for $t \in [t_k, t_{k+1}]$ and

$$\begin{aligned}
V(x(t_{k+1})) - V(x(t_k)) &= \int_{t_k}^{t_{k+1}} \frac{d}{ds}V(x(s))ds \\
&= \int_{t_k}^{t_{k+1}} \frac{d}{ds}V(x(s))ds + Q_k(T_k, x_{Tk}) - Q_k(0, x_{Tk}) \\
&= \int_{t_k}^{t_{k+1}} \frac{d}{ds}V(x(s))ds + \int_{t_k}^{t_{k+1}} \frac{d}{ds}Q_k(s - t_k, x_{Tk})ds \\
&= \int_{t_k}^{t_{k+1}} \frac{d}{ds}\left(V(x(s)) + Q_k(s - t_k, x_{Tk})\right)ds \\
&= \int_{t_k}^{t_{k+1}} \frac{d}{ds}W(s)ds < 0.
\end{aligned}$$

Hence (i) is satisfied.

Now assume (i) is satisfied. Define, for all functions $z \in \mathscr{K}$,

$$Q_k(s, z) := -V(z(s)) + \frac{s}{T_k}\left(V(z(T_k)) - V(z(0))\right).$$

Then

$$\begin{aligned}
Q_k(0, z) &= -V(z(0)), \\
Q_k(T_k, z) &= -V(z(T_k)) + (V(z(T_k)) - V(z(0))) \\
&= -V(z(0)).
\end{aligned}$$

Consequently, this leads to $Q_k(T_k, z) = Q_k(0, z)$ which ensures that condition (7) is satisfied. Furthermore, by considering $s = t - t_k$ and $z = x_{Tk}$,

$$\begin{aligned}
&\frac{d}{dt}[V(x(t)) + Q_k(t - t_k, x_{Tk})] \\
&= \frac{d}{dt}\Bigg[V(x(t)) - V(x_{Tk}(t - t_k)) \\
&\qquad + \frac{t - t_k}{T_k}\left(V(x_{Tk}(T_k)) - V(x_{Tk}(0))\right)\Bigg] \\
&= \frac{d}{dt}\left[V(x(t)) - V(x(t)) + \frac{t - t_k}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right)\right] \\
&= \frac{d}{dt}\left[\frac{t - t_k}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right)\right] \\
&= \frac{1}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right) < 0.
\end{aligned}$$

This proves the equivalence between (i) and (ii).

Now, from the discrete-time Lyapunov theorem, we have $\lim_{k\to\infty} x(t_k) = 0$. To show $\lim_{t\to\infty} x(t)$, we note that by assumption of the existence and continuity of solutions, the solution map is continuous and thus bounded on the interval $[0, T]$. Thus $\lim_{k\to\infty}\|x_{Tk}\|_\infty \to 0$ where $\|\cdot\|_\infty$ is the supremum norm. We conclude that $\lim_{t\to\infty} x(t)$. ■

A graphical depiction of the proof of Theorem 1 is shown in Figure 1. The main contribution of the theorem is the introduction of a new kind of Lyapunov functional for sampled-data systems.

There are several articles in the literature which use related approaches (see for instance Naghshtabrizi et al. [2008], Fridman [2010]). Typically, however, these results are expressed as positivity of a Lyapunov-Krasovskii functional which is positive definite. In the above result, positivity is relaxed through the use of the spacing function Q.

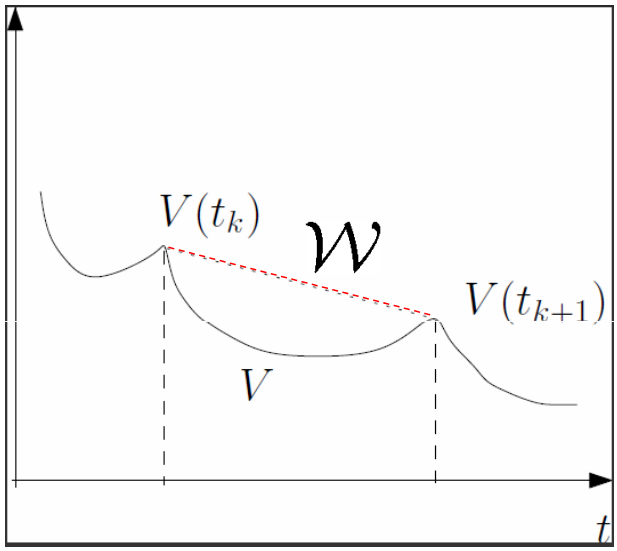

Fig. 1. Illustration of the proof of Theorem 1

The following sections show how the conditions of Theorem 1 can be enforced using sum-of-squares optimization in both the synchronous and asynchronous case. This is similar to the approach taken in Peet et al. [2009].

3.2 Stability under synchronous sampling

Recall the sampled-data system.

$$\dot{x}(t) = Ax(t) + BKx(t_k), \qquad \text{for } t \in [t_k, t_k + T_k],\ k \geq 0. \tag{9}$$

The following theorem gives conditions for stability. The conditions of the theorem can be enforced using sum-of-squares, as will be described shortly.

Theorem 2. Consider system (9) with $T_k = T$ for some given $T > 0$. If there exist $P \in \mathbb{S}^n$, positive definite and a polynomial matrix, of degree N, $M : [0,\ T] \to \mathbb{S}^{2n}$ such that

$$P > 0, \tag{10}$$

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^T M(0) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \tag{11}$$

$$M(T) = 0, \tag{12}$$

and such that for all $\tau \in [0,\ T]$, the following inequality holds

$$\begin{aligned}
\Psi(\tau) = &\begin{bmatrix} 0 \\ I_n \end{bmatrix} P\,[BK\ A] + \begin{bmatrix} K^T B^T \\ A^T \end{bmatrix} P\,[0\ I_n] \\
&+ \dot{M}(\tau) + M(\tau)\begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T + \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T M(\tau) < 0.
\end{aligned} \tag{13}$$

Then closed loop system is asymptotically stable for the constant sampling period T.

Proof. Consider the classical quadratic Lyapunov function for linear continuous-time systems. Define $V : \mathbb{R}^n \to \mathbb{R}^+$ as

$$V(x) = x^T P x,$$

where $P > 0$ is in $\mathbb{S}$. This function V satisfies condition (6) from Theorem 2. Now define the functional for all $s \in [0,\ T]$ and all functions $z \in \mathscr{K}$

$$Q(s, z) = \begin{bmatrix} z(0) \\ z(s) \end{bmatrix}^T M(s) \begin{bmatrix} z(0) \\ z(s) \end{bmatrix}.$$

First, from (12), we note that

$$Q(0,z) = \begin{bmatrix} z(0) \\ z(0) \end{bmatrix}^T M(0) \begin{bmatrix} z(0) \\ z(0) \end{bmatrix}$$
$$= z(0)^T \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T M(0) \begin{bmatrix} I_n \\ I_n \end{bmatrix} z(0) = 0.$$

Furthermore,

$$Q(T,z) = \begin{bmatrix} z(0) \\ z(T) \end{bmatrix}^T M(T) \begin{bmatrix} z(0) \\ z(T) \end{bmatrix} = 0.$$

Therefore, we have $Q(T,z) = Q(0,z) = 0$ and hence condition (7) is satisfied.

Computing the derivative term (8), we get

$$\begin{aligned}
&\frac{d}{dt}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] \\
&= \frac{d}{dt}\left[x(t)^T P x(t) + \begin{bmatrix} x_{Tk}(0) \\ x_{Tk}(t-t_k) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x_{Tk}(0) \\ x_{Tk}(t-t_k) \end{bmatrix}\right] \\
&= \frac{d}{dt}\left[x(t)^T P x(t) + \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}\right] \\
&= x^T(t) P \dot{x}(t) + \dot{x}^T(t) P x(t) + \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&+ \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \frac{d}{dt} M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} + \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix}.
\end{aligned}$$

Recalling that $\dot{x}(t) = Ax(t) + BKx(t_k)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] \\
&= \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix}^T P\,[BK \;\; A] + \begin{bmatrix} K^T B^T \\ A^T \end{bmatrix} P\,[0 \;\; I_n] \right) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&+ \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&+ \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \dot{M}(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&+ \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix} \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \Psi(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix},
\end{aligned}$$

for all $t \in [t_k, t_k + T]$. Thus if there exists a solution of inequality (13), and by virtue of Theorem 1, the closed loop system is asymptotically stable for the constant sampling period T. ■

It is important to highlight that Theorem 2 only guarantees the stability of the solutions of system (9) for a given sampling period T. By virtue of Theorem 1, the previous theorem only ensures that:

$$\Gamma^T(T) P \Gamma(T) - P < 0.$$

for a given T. As it was mentioned in the introduction, the stability of sampled-data systems with a constant sampling period can be dealt by checking if the eigenvalues of the matrix $\Gamma(T)$ are included in the unit circle. This is only possible if the matrices (A,B) which characterize the system are known. Thus, in the case of uncertain systems where the matrices (A,B) lies in a polytope, it is not easy to derived the eigenvalues of the matrix $\Gamma(T)$. Noting that, in Theorem 2, the stability condition linearly depends on the matrices A and B, it is possible to extend the previous results to the case of systems with a polytopic type of uncertainties. This correspond to the main improvement of this paper.

3.3 Asynchronous sampling

The case of asynchronous sampling, where T_k is unknown but bounded in some range, is clearly more realistic than the synchronous case in a networked control scenario. However, the stability conditions for this case are not significantly more complex than for the synchronous case. We simply allow the function M to vary with the sampling period, T_k.

Theorem 3. Consider system (5). For given $0 \le T_1 < T_2 < \infty$, if there exist $P \in \mathbb{S}^n$, positive definite and a bi-polynomial matrix $\bar{M} : [0,\ T_2] \times [T_1,\ T_2] \to \mathbb{S}^{2n}$ such that for all $T \in [T_1,\ T_2]$,

$$P > 0, \tag{14}$$

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^T \bar{M}(0,T) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \tag{15}$$

$$\bar{M}(T,T) = 0, \tag{16}$$

and such that for all $s \in [0,\ T]$, the following inequality holds

$$\begin{aligned}
\bar{\Psi}(\tau,T) = &\begin{bmatrix} 0 \\ I_n \end{bmatrix} P\,[BK \;\; A] + \begin{bmatrix} K^T B^T \\ A^T \end{bmatrix} P\,[0 \;\; I_n] + \frac{d}{ds}\bar{M}(s,T) \\
&+ \bar{M}(s,T) \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T \bar{M}(s,T) < 0.
\end{aligned} \tag{17}$$

Then if $T_k \in [T_1, T_2]$ for all $k \ge 0$, the closed loop system is asymptotically stable.

Proof. The proof is a trivial extension of Theorem 2. ■

By virtue of Theorem 1, if the conditions of Theorem 3 are satisfied, then $V(x) = x^T P x$ is decreasing across every time interval T_k. That is

$$\forall T \in [T_1,\ T_2], \quad \Gamma^T(T) P \Gamma(T) - P < 0.$$

Another important remark concerns the functional Q introduced in Theorem 2. In Fridman [2010] or Seuret [2011], an integral term of the form

$$\tilde{Q}(t-t_k, x_{KT}) = (t_{k+1} - t) \int_{t_k}^{t} \dot{x}^T(s) R \dot{x}(s) ds,$$

is employed. This term has an important role in reducing the conservatism of the stability conditions. However it unavoidably leads to the use of the Jensen inequality to compute an upper bound of the derivative of $V + Q + \tilde{Q}$. This inequality unavoidably introduces conservatism in the stability criteria. In the present paper, an exact expression of the derivative of $V + Q$ is provided since the following equality is obtained

$$\frac{d}{dt}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] = \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \bar{\Psi}(t-t_k, T_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}$$

Then the stability criterion from Theorem 3 is less conservative then the ones from Fridman [2010] or Seuret [2011].

4. SUM OF SQUARES AS ALGORITHMIC TOOL

4.1 General presentation of SOS

The methodology we use to implement the conditions of Theorems 2 and 3 is based on the sum-of-squares decomposition of positive polynomials. When applying this methodology we assume that all matrix functions are polynomial, can be approximated by polynomials, or there is a change of coordinates that renders them polynomial.

Denote by $\mathbb{R}[y]$ the ring of polynomials in $y = (y_1, \ldots, y_n)$ with real coefficients. Denote by Σ_s the cone of polynomials that

admits a SOS decomposition, i.e., those $p \in \mathbb{R}[y]$ for which there exist $h_i \in \mathbb{R}[y], i = 1, \ldots, M$ so that

$$p(y) = \sum_{i=1}^{M} h_i^2(y).$$

If $p(y) \in \Sigma_s$, then clearly $p(y) \geq 0$ for all y. The converse is not always true, although the converse does hold for univariate matrix-valued polynomials. The advantage of SOS is that the problem of testing if $p(y) \geq 0$ is known to be NP-hard, whereas testing if $p(y) \in \Sigma_s$ is equivalent to an SDP(Parrilo [2000]), and hence is worst-case polynomial-time verifiable. SOS results apply to matrix-valued polynomials as well as scalars, although in this case the inequality means positive semidefinite. The SDPs related to SOS can be formulated efficiently and the solution can be retrieved using SOSTOOLS(Prajna et al. [2002]), which interfaces with semidefinite solvers such as SeDuMi(Sturm [1999]).

Consider now the conditions in Theorem 2 which take the form.

$$L(s) \leq 0, \quad s \in \mathscr{S}, \tag{18}$$

where $L(s) \in \mathbb{R}^{n \times n}$ and $\mathscr{S}$ is a semialgebraic set described by polynomial inequalities:

$$\mathscr{S} = \{s \in R \quad | \quad g_i(s) \geq 0, \quad i = 1, \ldots, M\},$$

where $g_i(s)$ are polynomial functions. In order to test condition (18), we can apply Positivstellensatz results such as Putinar [1993] which allow us to test positivity on a semialgebraic set using SOS. Specifically, Condition (18) holds if there exists SOS polynomials $P_i(s,y)$, such that

$$L(s) + \sum_{i=1}^{M} g_i(s) P_i(s,y) = P_0(s).$$

Intuitively, the above condition guarantees that when $s \in \mathscr{S}$, we have $L(s) \leq -\sum_{i=1}^{M} g_i(s) p_i(s,y) \leq 0$ since $g_i \geq 0$ and $p_i \geq 0$, and therefore $L(s) \leq 0$ for those s.

4.2 Application to the stability theorem

In this brief subsection, we identify the functions g_i's corresponding to sets $[0,T]$ and $[T_1, T_2]$ used in theorems 2 and 3. The function for Theroem 2 is

$$g_1(s) = -(T-s)s,$$

which represents $s \in [0,T]$ and for Theorem 3, we use

$$g_1(s,T) = -(T-s)s \text{ and } g_2(T) = -(T_2 - T)(T - T_1).$$

where g_2 represents $T \in [T_1, T_2]$.

5. EXAMPLES

Consider system (1) with several matrix definitions

- Example 1 from Fridman et al. [2004], Naghshtabrizi et al. [2008]:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix},$$

- Example 2 from Fridman [2010]:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

- and Example 3 from Gu et al. [2003], Michiels et al. [2004]:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Theorems	T_2 for Ex.1	T_2 for Ex.2
Fridman et al. [2004]	0.869	0.99
Naghshtabrizi et al. [2008]	1.113	1.99
Fridman [2010]	1.695	2.03
Liu and Fridman [2009]	1.695	2.53
Seuret [2011]	1.723	2.62
Th.2 $N = 1$	0.702	2.319
Th.2 $N = 3$	1.729	3.219
Th.2 $N = 5$	1.729	3.269
Th.3 $N = 1$	0.701	2.310
Th.3 $N = 3$	1.729	3.218
Th.3 $N = 5$	1.729	3.269

Table 1. Maximum allowable sampling period T_2 for examples 1, 2, with $T_1 = 0$.

Theorems for Ex.3	
Seuret [2011]	$[0.201,\ 1.623]$
Th.2 $N = 1$	$\varnothing$
Th.2 $N = 3$	$[0.2007,\ 2.016] \bigcup [2.606,\ 3.055]$
Th.2 $N = 5$	$[0.2007,\ 2.020] \bigcup [2.470,\ 3.694]$
Seuret [2011]	$[0.400,\ 1.251]$
Th.3 $N = 1$	$\varnothing$
Th.3 $N = 3$	$[0.4,\ 1.820]$ or $[2.680,\ 3.005]$
Th.3 $N = 5$	$[0.4,\ 1.828]$ or $[2.520,\ 3.550]$

Table 2. Interval of allowable asynchronous samplings of the form $[T_1,\ T_2]$ for example 3.

Tables 1 and 2 summarize the results obtained in the literature and using the theorems provided in the present paper for examples $1, 2$ and 3. One can see that the obtained results are less conservative then existing ones. More precisely, for each example, Theorem 3 allows guaranteeing the stability of the solutions of the sampled-data systems with the same Lyapunov matrix P.

Another important remark deals with Example 3. This system is well known in the time-delay literature because the delay has a stabilizing effect. This means that its solutions are not stable for sufficiently small delay but become stable for sufficiently large delay. The method proposed in this article is able to take into account this phenomena and is also able to isolate several intervals of possible values for the length of the sampling interval where the system is stable. Note that Theorem 3 (with $N = 5$) guarantees the stability for all asynchronous sampling lying in $[0.4,\ 1.828]$ or $[2.520,\ 3.550]$. However no guarantee of stability can be provided if the sampling interval switched from one interval to the other. This recalls the classical behavior of switched systems. A resulting system of a switched systems defined by two stable systems is not necessary stable. In the present situation this comes from the fact that the Lyapunov matrices obtained for each interval are different.

6. CONCLUSION

In this article, a novel analysis of continuous linear systems under asynchronous sampling is provided. This approach is based on the discrete-time Lyapunov Theorem applied to the continuous-time model of the sampled-data systems. Numerical results compare favorably with result in the literature. Perhaps the most important feature of the method presented in this paper is that it is expressed using the sum-of-squares framework and is thus easily extended to nonlinear systems and systems with parametric uncertainty.

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