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# On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,  
Université François Rabelais, Tours, FRANCE

**Abstract** If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , we study conditions on a Radon measure  $\mu$  on  $\partial\Omega$  for solving the equation  $-\Delta u + e^u - 1 = 0$  in  $\Omega$  with  $u = \mu$  on  $\partial\Omega$ . The conditions are expressed in terms of Orlicz capacities.

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*Key words.* Elliptic equations, Orlicz capacities, reduced measures, boundary trace

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary and  $\mu$  a Radon measure on  $\partial\Omega$ . In this paper we consider first the problem of finding a function  $u$  solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in  $\Omega$  satisfying  $u = \mu$  on  $\partial\Omega$ . Let  $\rho(x) = \text{dist}(x, \partial\Omega)$ , then this problem admits a **weak formulation**: *find a function  $u \in L^1(\Omega)$  such that  $e^u \in L^1_\rho(\Omega)$  satisfying*

$$\int_{\Omega} (-u\Delta\zeta + (e^u - 1)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.2}$$

This equation has been initiated by Grillo and Véron [16] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with  $q > 1$ ), have been developed by Le Gall [19], Marcus and Véron [20], [21], Dynkin and Kuznetsov [10], [11], respectively by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the main features of the problem with power nonlinearities is the existence of a critical exponent  $q_c = \frac{N+1}{N-1}$  which is linked to the existence of boundary removable sets.

Existence of boundary removable points have been discovered by Gmira and Véron [15]. Let us recall briefly the main results for (1.3):

(i) If  $1 < q < q_c$ , then for any  $\mu \in \mathfrak{M}_+(\partial\Omega)$  there exists a unique function  $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$  which satisfies (1.3) in  $\Omega$  and takes the value  $\mu$  on  $\partial\Omega$  in the following weak sense

$$\int_{\Omega} (-u\Delta\zeta + u^q\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \quad (1.4)$$

(ii) If  $q \geq q_c$ , the above problem can be solved if and only if  $\mu$  vanishes on boundary Borel subsets with zero  $C_{\frac{2}{q},q}$ -Bessel capacity. Furthermore a boundary compact set is removable if and only if it has zero  $C_{\frac{2}{q},q}$ -capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \\ u &= \mu && \text{on } \partial\Omega. \end{aligned} \quad (1.5)$$

Following the terminology of [5] we say that a measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is **good** if (3.21) admits a weak solution. Let  $P^\Omega(x, y)$  (resp.  $G^\Omega(x, y)$ ) be the Poisson kernel (resp. the Green kernel) in  $\Omega$  and  $\mathbb{P}^\Omega[\mu]$  the Poisson potential of a boundary measure  $\mu$  (resp.  $\mathbb{G}^\Omega[\phi]$  the Green potential of a bounded measure  $\phi$  defined in  $\Omega$ ). A boundary measure  $\mu$  which satisfies

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx). \quad (1.6)$$

is called **admissible**. Since  $\mathbb{P}^\Omega[\mu]$  is a supersolution for (1.1), an admissible measure is good (see [26]). Our first result which extends a previous one obtained in [16] is the following.

**Theorem A.** *Suppose  $\mu \in \mathfrak{M}(\partial\Omega)$  admits Lebesgue decomposition  $\mu = \mu_S + \mu_R$  where  $\mu_S$  and  $\mu_R$  are mutually singular and  $\mu_R$  is absolutely continuous with respect to the  $(N-1)$ -dim Hausdorff measure  $dH^{N-1}$ . If*

$$\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx), \quad (1.7)$$

*then  $\mu$  is good.*

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modeled on the Legendre transform of  $r \mapsto p(r) := e^r - 1$ . These capacities have been studied by Aissaoui and Benkirane [2] and they inherit most of the properties of the Bessel capacities. The capacity  $C_{NL \ln L}$  associated to

the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this capacity framework we obtain the following types of results:

**Theorem B.** *Let  $\mu \in \mathfrak{M}_+(\partial\Omega)$  be a good measure, then  $\mu$  vanishes on boundary Borel subsets  $E$  with zero  $C_{NL\ln L}$ -capacity.*

We also give below a result of removability of boundary singularities.

**Theorem C.** *Let  $K \subset \partial\Omega$  be a compact subset with zero  $C_{NL\ln L}$ -capacity. Suppose  $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $K$ , then  $u$  is identically zero.*

In the last part of this paper we apply this approach to the problem

$$-\Delta u + e^u - 1 = \mu, \quad (1.8)$$

where  $\mu$  is a bounded measure, as well as removability questions for internal singularities of solutions of (1.1). In that case the natural capacity associated to the problem is

$$C_{\Delta L \ln L}(K) = \inf\{\|M[\Delta\eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\} \quad (1.9)$$

where  $M[\cdot]$  denotes Hardy-Littlewood's maximal function.

**Theorem D.** *Let  $\mu \in \mathfrak{M}_+(\Omega)$  be a bounded good measure, then  $\mu$  vanishes on boundary Borel subsets  $E$  with zero  $C_{\Delta L \ln L}$ -capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets  $E$  with zero  $C_{NL\ln L}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

**Theorem E.** *Let  $K \subset \Omega$  be a compact subset with zero  $C_{\Delta L \ln L}$ -capacity. Suppose  $u \in C(\Omega \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega \setminus K$  which vanishes on  $\partial\Omega$ , then  $u$  is identically zero.*

This note is essentially derived from the preliminary report [27], written in 2004 and left escheated since this period.

## 2 Good measures

*Proof of Theorem A.* For  $k > 0$ , set  $\mu_{R,k} = \inf\{k, \mu_R\}$  and denote by  $u_k$  the solution of

$$\begin{aligned} -\Delta u_k + e^{u_k} - 1 &= 0 && \text{in } \Omega \\ u_k &= \mu_S + \mu_{R,k} && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

Such a solution exists because

$$\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}^\Omega[\mu_S])$$

by the maximum principle, and (1.7) implies that  $\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$ . The sequence  $u_k$  is nondecreasing. Since, for any  $\zeta \in C_c^{1,1}(\bar{\Omega})$ ,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function  $\zeta$  the solution  $\zeta_0$  of

$$\begin{aligned} -\Delta \zeta_0 &= 1 && \text{in } \Omega \\ \zeta_0 &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (2.2)$$

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \leq c \|\mu\|_{\mathfrak{M}}. \quad (2.3)$$

Thus  $u = \lim_{k \rightarrow \infty} u_k$  is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \leq c \|\mu\|_{\mathfrak{M}},$$

and the convergence of  $u_k$  and  $e^{u_k}$  to  $u$  and  $e^u$  hold respectively in  $L^1(\Omega)$  and  $L^1(\Omega; \rho dx)$  and  $u$  satisfies (1.2).  $\square$

The proof of the next result is directly inspired by [5] where nonlinear Poisson equations are treated.

**Proposition 2.1** *The following properties hold:*

- (i) *If  $\mu \in \mathfrak{M}_+(\partial\Omega)$  is a good measure, then any  $\tilde{\mu} \in \mathfrak{M}_+(\partial\Omega)$  smaller than  $\mu$  is good.*
- (ii) *Let  $\{\mu_n\}$  be an increasing sequence of good measures which converges to  $\mu$  in the weak sense of measures. Then  $\mu$  is good.*
- (iii) *If  $\mu \in \mathfrak{M}_+(\partial\Omega)$  is a good measure and  $f \in L_+^1(\partial\Omega)$ , then  $f + \mu$  is a good measure.*

*Proof.* (i) Let  $u = u_\mu$  be the solution of (3.21) and  $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$ . Since  $\mathbb{P}^\Omega[\tilde{\mu}]$  is a supersolution for (1.1),  $w$  is a supersolution too. Furthermore  $w$  is nonnegative and  $e^w - 1 \in L^1(\Omega; \rho dx)$ . By Doob's theorem  $w$  admits a boundary trace  $\mu^* \in \mathfrak{M}_+(\partial\Omega)$  and  $\mu^* \leq \tilde{\mu} \leq \mu$ . Let  $w^*$  be the solution of

$$\begin{aligned} -\Delta w^* + e^u - 1 &= 0 && \text{in } \Omega \\ w^* &= \tilde{\mu} && \text{on } \partial\Omega. \end{aligned}$$

then  $u \geq w \geq w^*$  and [22],

$$\lim_t \int_{\partial\Omega_t} w^*(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

(here we denote by  $\partial\Omega_t$  the set of  $x \in \Omega$  such that  $\rho(x) = t > 0$ ). This implies that the boundary trace of  $w^*$  is  $\tilde{\mu}$  and thus  $\mu^* = \tilde{\mu}$ . Set  $\Omega_t = \{x \in \Omega : \rho(x) > t\}$  and let  $v_t$  be the solution of

$$\begin{aligned} -\Delta v_t + e^{v_t} - 1 &= 0 && \text{in } \Omega_t \\ v_t &= w && \text{on } \partial\Omega_t. \end{aligned}$$

Then  $v_t \leq w$  in  $\Omega_t$ . Furthermore

$$0 < t' < t \implies v_{t'} \leq v_t \quad \text{in } \Omega_{t'}.$$

Then  $\tilde{u} = \lim_{t \rightarrow 0} v_t$  exists, the convergence holds in  $L^1(\Omega)$  and  $e^{v_t} \rightarrow e^{\tilde{u}}$  in  $L^1(\Omega; \rho dx)$  (here we use the fact that  $e^w \in L^1(\Omega; \rho dx)$ ). Because

$$\lim_{t \rightarrow 0} \int_{\partial\Omega_t} \tilde{w}(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

and  $v_t = \tilde{w}$  on  $\partial\Omega_t$ , it follows that  $\tilde{u}$  admits  $\tilde{\mu}$  for boundary trace and thus  $\tilde{u} = u_{\tilde{\mu}}$ .

(ii) Let  $u_n = u_{\mu_n}$  be the solutions of (3.21) with boundary value  $\mu_n$ . The sequence  $\{u_n\}$  is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \leq - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \quad (2.4)$$

we conclude as in the proof of Theorem 1, that  $u_n$  increases and converges to a solution  $u = u_\mu$  of (3.21) with boundary value  $\mu$ .

(iii) In the proof of (i) we have actually used the following result : *Let  $w$  be a nonnegative supersolution of (1.1) such that  $e^w \in L^1(\Omega; \rho dx)$  and let  $\mu \in \mathfrak{M}_+(\partial\Omega)$  be the boundary trace of  $w$ . Then  $\mu$  is good.* Let  $f \in L^1_+(\partial\Omega)$  and  $\mu$  be a good measure. We denote by  $u = u_\mu$  the solution of (3.21). For  $k > 0$ , set  $f_k = \min\{k, f\}$ . The function  $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$  is a nonnegative supersolution, and, since  $\mathbb{P}^\Omega[f_k] \leq k$ ,  $e^{w_k} \in L^1(\Omega; \rho dx)$ . Furthermore the boundary trace of  $w_k$  is  $\mu + f_k$ . Therefore  $\mu + f_k$  is good. We conclude by II that  $\mu + f$  is good  $\square$

*Remark.* The assertions (i) and (ii) in Theorem 1 are still valid if we replace  $r \mapsto e^r - 1$  by any continuous nondecreasing function  $f$  vanishing at 0.

### 3 The Orlicz space framework

#### 3.1 Orlicz capacities

The set of nonnegative measures  $\mu$  on  $\partial\Omega$  such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx) \quad (3.1)$$

is not a linear space, but it is a convex subset of  $\mathfrak{M}_+(\partial\Omega)$ . The role of this set comes from the fact that any measure in  $M^{exp}(\partial\Omega)$  is good. Put

$$p(t) = \operatorname{sgn}(s)(e^s - 1), \quad P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s| + 1), \quad P^*(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$

Then  $P$  and  $P^*$  are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \leq P(x) + P^*(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if  $x = \bar{p}(y)$  or  $y = p(x)$ . It is classical to define

$$M_P(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx)\}, \quad (3.2)$$

$$M_{P^*}(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx)\}. \quad (3.3)$$

The Orlicz spaces  $L_P(\Omega; \rho dx)$  and  $L_{P^*}(\Omega; \rho dx)$  are the vector spaces spanned respectively by  $M_P(\Omega; \rho dx)$  and  $M_{P^*}(\Omega; \rho dx)$ . They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_\rho}} = \inf \left\{ k > 0 : \int_{\Omega} P\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.4)$$

and

$$\|\phi\|_{L_{P^*_\rho}} = \inf \left\{ k > 0 : \int_{\Omega} P^*\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.5)$$

Furthermore the Hölder-Young inequality asserts [17]

$$\left| \int_{\Omega} \phi \psi \rho dx \right| \leq \|\phi\|_{L_{P_\rho}} \|\psi\|_{L_{P^*_\rho}} \quad \forall (\phi, \psi) \in L_P(\Omega; \rho dx) \times L_{P^*}(\Omega; \rho dx). \quad (3.6)$$

Since  $P^*$  satisfies the  $\Delta_2$ -condition,  $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$  and  $L_P(\Omega; \rho dx)$  is the dual space of  $L_{P^*}(\Omega; \rho dx)$ , (see [13], [2]). Furthermore, since

$$\frac{|a| \ln(1 + |a|)}{2} \leq P^*(a) \leq |a| \ln(1 + |a|) \quad \forall a \in \mathbb{R},$$

the space  $L_{P^*}(\Omega; \rho dx)$  is associated with the class  $L \ln L(\Omega; \rho dx)$  and to the Hardy-Littlewood maximal function (see [13]). We recall its definition: we consider a cube  $Q_0$  containing  $\Omega$ , with sides parallel to the axes. If  $f \in L^1(\Omega)$  we denote by  $\tilde{f}$  its extension by 0 in  $Q_0 \setminus \Omega$  and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f|(y) dy : Q \in \mathcal{Q}_x \right\}$$

where  $\mathcal{Q}_x$  denotes the set of all cubes containing  $x$  and contained in  $Q_0$ , with sides parallel to the axes. Thus

$$\|f\|_{L \ln L_\rho} := \int_{Q_0} M_{Q_0}[f](x) \rho dx \approx \|f\|_{L_{P^*}}. \quad (3.7)$$

**Definition 3.1** *The space of all measures on  $\partial\Omega$  such that  $\mathbb{P}^\Omega[\mu] \in L_P(\Omega; \rho dx)$  is denoted by  $B^{\text{exp}}(\partial\Omega)$  and endowed with the norm*

$$\|\mu\|_{B^{\text{exp}}} = \|\mathbb{P}^\Omega[\mu]\|_{L_{P\rho}}. \quad (3.8)$$

*The subset of measures such that  $\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx)$  is denoted by  $\mathfrak{M}^{\text{exp}}(\partial\Omega)$ .*

The following result follows immediately from the definition of the Luxemburg norm.

**Proposition 3.2** *If  $\mu \in B^{\text{exp}}(\partial\Omega)$  there exists  $a_0 > 0$  such that  $a\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$  for all  $0 \leq a < a_0$ . Conversely, if  $\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$ , then  $a\mu \in B^{\text{exp}}(\partial\Omega)$  for all  $a > 0$ .*

The analytic characterization of  $B^{\text{exp}}(\partial\Omega)$  can be done in introducing the space of normal derivatives of Green potentials of  $L \ln L$  functions:

$$N^{L \ln L}(\partial\Omega) = \{ \eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx) \}. \quad (3.9)$$

where  $\rho^*$  is a smooth positive function with value  $\rho$  in a neighborhood of  $\partial\Omega$ . Then

$$\left| \int_{\partial\Omega} \eta d\mu \right| = \left| \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \right| \leq \|\mathbb{P}^\Omega[\mu]\|_{L_{P\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P^*}}. \quad (3.10)$$

We take for norm on  $N^{L \ln L}(\partial\Omega)$

$$\|\eta\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P^*}}, \quad (3.11)$$

and define the  $C_{N^{L \ln L}}$ -capacity of a compact subset  $K$  of  $\partial\Omega$  by

$$C_{N^{L \ln L}}(K) = \inf \{ \|\eta\|_{N^{L \ln L}} : \eta \in C^2(\partial\Omega), 0 \leq \eta \leq 1, \eta \geq 1 \text{ in a neighborhood of } K \}. \quad (3.12)$$



Considering the bilinear form  $\mathcal{H}$  on  $L_{P_\rho^*}(\partial\Omega) \times L_{P_\rho}(\partial\Omega)$

$$\mathcal{H}(\eta, \mu) := - \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \quad (3.13)$$

then

$$\begin{aligned} \mathcal{H}(\eta, \mu) &= - \int_{\Omega} \int_{\partial\Omega} P^\Omega(x, y) d\mu(y) \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) dx \\ &= - \int_{\partial\Omega} \int_{\Omega} \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) P^\Omega(x, y) dx d\mu(y). \end{aligned} \quad (3.14)$$

It is classical to define

$$C_{NL \ln L}^*(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(K^c) = 0, \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}} \leq 1\}. \quad (3.15)$$

The following result due to Fuglede [14] (and to Aissaoui-Benkirane in the Orlicz space framework [2]) is a consequence of the Kneser-Fan min-max theorem.

**Proposition 3.3** *For any compact set  $K \subset \partial\Omega$ , there holds*

$$C_{NL \ln L}^*(K) = C_{NL \ln L}(K). \quad (3.16)$$

As a direct consequence of (3.10), we have the following

**Proposition 3.4** *If  $\mu \in B_+^{\text{exp}}(\partial\Omega)$ , it does not charge Borel subsets with  $C_{NL \ln L}$ -capacity zero.*

### 3.2 Good measures and removable sets

*Proof of Theorem B.* Let  $K$  be a compact subset with  $C_{NL \ln L}$ -capacity zero. There exist a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of  $K$  and

$$\lim_{n \rightarrow \infty} \|\eta_n\|_{NL \ln L} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}} = 0. \quad (3.17)$$

Take  $\rho^* \mathbb{P}^\Omega[\eta_n]$  as a test function, then

$$\int_{\Omega} (-u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) + (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n]) dx = - \int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu$$

Since  $\frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} = \eta_n$  and  $\mu > 0$ , there holds  $-\int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu \geq \mu(K)$ .

Furthermore

$$\left| \int_{\Omega} u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) dx \right| \leq \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}. \quad (3.18)$$

Then

$$\mu(E) \leq \int_{\Omega} (e^u - 1) \rho^* \mathbb{P}^{\Omega}[\eta_n] dx + \|u\|_{L^{p_{\rho}}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L^{p_{\rho}^*}}.$$

By the same argument as in [5],  $\lim_{n \rightarrow \infty} \rho^* \mathbb{P}^{\Omega}[\eta_n] = 0$ , a.e. in  $\Omega$ , and there exists a nonnegative  $L^1_{\rho}$ -function  $\Phi$  such that  $0 \leq \rho^* \mathbb{P}^{\Omega}[\eta_n] \leq \Phi$ . By (3.17), (3.18) and Lebesgue's theorem,  $\mu(E) = 0$ .  $\square$

**Definition 3.5** *A subset  $E \subset \partial\Omega$  is said removable for equation (1.1), if and only if any positive solution  $u \in C^2(\Omega)$  of (1.1) in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus E$  and vanishes on  $\partial\Omega \setminus E$ , is identically zero.*

*Proof of Theorem C.* Let  $u \in C(\overline{\Omega} \setminus K)$  be a solution of (1.1) which is zero on  $\partial\Omega \setminus K$ . Let  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of  $K$  and (3.17) holds. Put  $\theta_n = 1 - \eta_n$ . Put  $\rho_K(x) = \text{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate and the fact that  $u$  vanishes on  $K^c$ , there holds

$$u(x) \leq C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function  $\zeta_n = \rho^* \mathbb{P}^{\Omega}[\theta_n]$  is an admissible test function for  $u$ , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

Clearly  $\mathbb{P}^{\Omega}[\theta_n] = 1 - \mathbb{P}^{\Omega}[\eta_n]$  and

$$\Delta \zeta_n = \Delta \rho^* - \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])$$

Inasmuch we can modify  $\rho^*$  in order to have  $-\Delta \rho^* \geq 0$ , in which case  $\rho^* = \rho$  near  $\partial\Omega$  is replaced by  $\rho^* \approx \rho$ , we derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u \zeta_n dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \zeta_n dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])) \zeta_n dx, \end{aligned}$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \leq C|r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx, \quad (3.19)$$

since  $\zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|$ . Furthermore

$$\begin{aligned} \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) &= -\ln \rho + \ln(\rho + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \\ &\leq -\ln \rho + \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \end{aligned}$$

But (we can assume  $\rho \leq 1$ )

$$\begin{aligned} &\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx \\ &\leq -\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx, \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} dx \\ &= \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| > 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} dx \\ &\leq \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx = 0 \quad (3.20)$$

Using (4.12), we derive  $u = 0$ .

Conversely, assume that  $C_{NL \ln L}(K) > 0$ . By Proposition 3.3 there exists a non negative non-zero measure  $\mu \in \mathfrak{M}_+(\partial\Omega)$  such that  $\mu(K^c) = 0$  in the space  $B_+^{exp}(\partial\Omega)$ . This means that  $\theta\mu \in M_+^{exp}(\partial\Omega)$  for some  $\theta > 0$ . Thus problem (3.21) admits a solution.  $\square$

Several questions can be adressed

- 1- If a measure  $\mu$  is good does there exist an increasing sequence of measures  $\{\mu_n\}$  which converges to  $\mu$  such that  $\theta_n \mu_n$  is admissible for some  $\theta_n > 0$ ?
- 2- If a measure  $\mu$ , singular with respect to  $\mathcal{H}^{N-1}$  is good does it exist an increasing sequence of admissible measures  $\{\mu_n\}$  converging to  $\mu$  ?
- 3- If a measure  $\mu$  does not charge Borel sets with  $C^{L \ln L}$ -capacity zero, does it exist  $\theta > 0$  such that  $\theta\mu$  is admissible?
- 4- If a singular measure  $\mu$  is good, then  $(1 - \delta)\mu$  is admissible for any  $\delta \in (0, 1)$  ?

### 3.3 More general nonlinearities

$$\begin{aligned} -\Delta u + P(u) &= 0 && \text{in } \Omega \\ u &= \mu && \text{on } \partial\Omega, \end{aligned} \quad (3.21)$$

where  $P$  is a convex increasing function vanishing at 0 and such that  $\lim_{r \rightarrow \infty} P(r)/r = \infty$ : In Theorem A- $P$ , (1.7) should be replaced by

$$P(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx). \quad (3.22)$$

In Proposition 2.1- $P$ , (i), (ii) and (iii) still hold. For simplicity we assume that  $P$  is a  $N$ -function in the sense of Orlicz spaces i.e.

$$P(r) = \int_0^r p(s) ds$$

where  $p$  is increasing, vanishes at 0 and tends to infinity at infinity. Let  $P^*$  be the conjugate  $N$ -function,  $L_P(\Omega; \rho dx)$  and  $L_{P^*}(\Omega; \rho dx)$  the corresponding Orlicz spaces endowed with the Luxemburg norms. Then Proposition 3.4- $P$  is valid, provided the space

$$B^P(\partial\Omega) := \{\mu \in \mathfrak{M}(\partial\Omega) : \mathbb{P}^\Omega[\mu] \in L_P(\Omega; \rho dx)\}$$

endowed with its natural norm replaces  $B^{\text{exp}}(\partial\Omega)$  with the norm (4.10). We set

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega; \rho dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P^*}}$$

and the corresponding capacity  $C_{N^{P^*}}$ . The proof of Proposition 3.4- $P$ , consequence of Young inequality between Orlicz space is valid with modification. **However**, it appears that the full characterization of removable sets cannot be adapted without further properties of the function  $P^*$  like the  $\Delta_2$ -condition. Some results in this directions have been obtained in [18] where a necessary and sufficient condition for removability of boundary set is given, under a very restrictive growth condition on  $P$  which reduces the nonlinearity to power-like with limited exponent.

## 4 Internal measures

Several above techniques can be extended to the following types of problem in which  $\mu \in \mathfrak{M}_+^b(\Omega)$ :

$$\begin{aligned} -\Delta u + e^u - 1 &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (4.1)$$

For this specific problem many interesting results can be found in [3] where the analysis of  $\mu$  is made by comparison with the Hausdorff measure in dimension  $(N-2)$ ,  $\mathcal{H}^{N-2}$ . It is proved in particular that if a measure  $\mu$  satisfies  $\mu \leq 4\pi\mathcal{H}^{N-2}$ , then problem (3.21) admits a solution, while if  $\mu$  charges some Borel set  $A$  with Hausdorff dimension less than  $N - 2$ , no solution exists. The results we provide are different and in the Orlicz capacities framework.

We define the classes  $M_P(\Omega)$  and  $M_{P^*}(\Omega)$  similarly to  $M_P(\Omega; \rho dx)$  and  $M_{P^*}(\Omega; \rho dx)$  except that the measure  $\rho dx$  is replaced by the Lebesgue measure  $dx$ . The Orlicz spaces  $L_P(\Omega)$  and  $L_{P^*}(\Omega)$  are defined from  $M_P(\Omega)$  and  $M_{P^*}(\Omega)$  and endowed with the respective Luxemburg norms  $\|\cdot\|_P$  and  $\|\cdot\|_{P^*}$ . We put

$$\Delta^{L \ln L}(\Omega) := \{\eta \in W_0^{1,1}(\Omega) : \Delta\eta \in L_{P^*}(\Omega)\}, \quad (4.2)$$

with natural norm

$$\|\eta\|_{\Delta^{L \ln L}} := \|\eta\|_{L^1} + \|\Delta\eta\|_{L_{P^*}}. \quad (4.3)$$

The norm in  $M_{P^*}(\Omega)$  can be characterized using the Hardy-Littlewood maximal function  $f \mapsto M_{Q_0}[f]$  since

$$\|f\|_{L \ln L} := \int_{Q_0} M_{Q_0}[f](x) dx \approx \|f\|_{L_{P^*}}. \quad (4.4)$$

Since  $P^*$  satisfies the  $\Delta_2$ -condition,  $C_0^\infty(\Omega)$  is dense in  $\Delta^{L \ln L}(\Omega)$ . Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta \mathbb{G}^\Omega[\mu] dx \right| = \left| \int_{\Omega} \mathbb{G}^\Omega[\mu] \Delta \eta dx \right| \leq \|\mathbb{G}^\Omega[\mu]\|_{L_P} \|\Delta\eta\|_{L_{P^*}}, \quad (4.5)$$

for  $\eta \in C_c^{1,1}(\bar{\Omega})$ . We define the  $C_{\Delta^{L \ln L}}$ -capacity of a compact subset  $K$  of  $\partial\Omega$  by

$$C_{\Delta^{L \ln L}}(K) = \inf\{\|\Delta\eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), \eta \geq 0, \eta \geq 1 \text{ in a neighborhood of } K\}, \quad (4.6)$$

By the min-max theorem there holds

$$C_{\Delta^{L \ln L}}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+^b(\Omega), \mu(K^c) = 0, \|\mathbb{G}^\Omega[\mu]\| \leq 1\}. \quad (4.7)$$

*Remark.* The characterization of the  $C_{\Delta^{L \ln L}}$ -capacity is not simple, however, by a result of [7, Th1], there holds

$$\|D^2\eta\|_{L^{1,\infty}} \leq C \|\Delta\eta\|_{L \ln L} \quad \forall \eta \in C_c^{1,1}(\bar{\Omega}) \quad (4.8)$$

where  $L^{1,\infty}(\Omega)$  denotes the weak  $L^1$ -space, that is the space of all measurable functions  $f$  defined in  $\Omega$  satisfying

$$\text{meas}(\{x \in \Omega : |f(x)| > t\}) \leq \frac{c}{t}, \quad \forall t > 0 \quad (4.9)$$

and  $\|f\|_{L^{1,\infty}}$  is the smallest constant such that (4.9) holds.

**Definition 4.1** *The space of all bounded measures in  $\Omega$  such that  $\mathbb{G}^\Omega[\mu] \in L^p(\Omega)$  is denoted by  $B^{\text{exp}}(\Omega)$ , with norm*

$$\|\mu\|_{B^{\text{exp}}} = \|\mathbb{G}^\Omega[\mu]\|_{L^p}. \quad (4.10)$$

*The subset of measures such that  $\exp(\mathbb{G}^\Omega[\mu]) \in L^1(\Omega)$  is denoted by  $\mathfrak{M}^{\text{exp}}(\Omega)$ .*

Thus Proposition 3.4 and Theorem B are valid under the form

**Proposition 4.2** *If  $\mu \in B_+^{\text{exp}}(\Omega)$ , it does not charge Borel subsets with  $C_{\Delta L \ln L}$ -capacity zero.*

**Theorem 4.3** *Let  $\mu \in \mathfrak{M}_+(\Omega)$  be a good measure, then  $\mu$  vanishes on Borel subset  $E$  with zero  $C_{\Delta L \ln L}$ -capacity.*

Theorem C has the following counter part

**Theorem 4.4** *Let  $K \subset \Omega$  be compact. Any solution of*

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \setminus K \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (4.11)$$

*vanishes identically in  $\Omega$  if and only if  $C_{\Delta L \ln L}(K) = 0$ .*

*Proof.* Let  $u \in C(\Omega \setminus K)$  be a solution of (1.1) which is zero on  $\partial\Omega$ . Let  $\{\eta_n\} \subset C^2(\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of  $K$  and (3.17) holds. Put  $\rho_K(x) = \text{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate for this type of nonlinearity (see [25]), there holds

$$u(x) \leq C \ln(2/\rho_K(x)) + D.$$

Put  $\theta_n = 1 - \eta_n$ . Then the function  $\zeta_n = \phi_1 \theta_n$  ( $\phi_1$  being the first eigenfunction of  $-\Delta$ ) is an admissible test function for  $u$ , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

We derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) dx - \int_{\Omega} Q(\Delta(\zeta_n)) dx. \end{aligned}$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta \zeta_n| \ln(1 + |\Delta \zeta_n|) dx, \quad (4.12)$$

Since the right-hand side goes to zero when  $n \rightarrow \infty$ , the conclusion follows.  $\square$

## 4.1 More on good measures

The main characterization of good measures is the following

**Theorem 4.5** *Assume  $\mu$  is a positive good measure, then there exists an increasing sequence  $\{\mu_n\} \subset B_+^{exp}(\Omega)$  which converges weakly to  $\mu$ .*

The proof will necessitate several intermediate results which are classical in the framework of Lebesgue measure or Bessel capacities, but appear to be new for Orlicz capacities.

**Lemma 4.6** *Let  $K \subset \Omega$ , then  $C_{\Delta^{L \ln L}}(K) = 0$  if and only if there exists  $\eta \in \Delta^{L \ln L}(\Omega)$  such that  $\eta \geq 0$  and  $K \subset \{y \in \Omega : \eta(y) = \infty\}$ .*

*Proof.* By the definition of the capacity, for any  $\lambda > 0$  and  $\eta \in \Delta^{L \ln L}(\Omega)$ ,  $\eta \geq 0$ ,

$$C_{\Delta^{L \ln L}}(\{y \in \Omega : \eta(y) \geq \lambda\}) \leq \frac{1}{\lambda} \|\eta\|_{\Delta^{L \ln L}}. \quad (4.13)$$

This implies

$$C_{\Delta^{L \ln L}}(\{y \in \Omega : \eta(y) = \infty\}) = 0.$$

□

**Lemma 4.7** *Suppose  $\{\eta_j\}$  is a Cauchy sequence in  $\Delta^{L \ln L}(\Omega)$ . Then there exist a subsequence  $\{\eta_{j_\ell}\}$  and  $\eta \in \Delta^{L \ln L}(\Omega)$  such that*

$$\lim_{i_\ell \rightarrow \infty} \eta_{j_\ell} = \eta,$$

*uniformly outside an open subset of arbitrary small  $C_{\Delta^{L \ln L}}$ -capacity.*

*Proof.* By Lemma 4.6,  $\eta_j$  and  $\eta$  are finite outside a set  $F$  with zero  $C_{\Delta^{L \ln L}}$ -capacity. There exists a subsequence  $\{\eta_{j_\ell}\}$  such that

$$\|\eta_{j_\ell} - \eta\|_{\Delta^{L \ln L}} \leq 2^{-2^\ell}.$$

Put  $E_\ell = \{y \in \Omega : \eta_{j_\ell} - \eta(y) \geq 2^{-\ell}\}$ . By (4.13)  $C_{\Delta^{L \ln L}}(E_\ell) \leq 2^{-\ell}$ , and if  $G_m = \cup_{\ell \geq m} E_\ell$ , there holds  $C_{\Delta^{L \ln L}}(G_m) \leq 2^{1-m}$ . Therefore

$$C_{\Delta^{L \ln L}}(\cap_{m \geq 1} G_m) = 0.$$

Since for any  $y \notin G_m \cup F$ , there holds

$$|(\eta_{j_\ell} - \eta)(y)| \leq 2^{-\ell},$$

the claim follows. □

**Lemma 4.8** *If  $\eta \in \Delta^{L \ln L}(\partial\Omega)$  it has a unique quasi-continuous representative with respect to the capacity  $C_{\Delta^{L \ln L}}$ .*

*Proof.* Uniqueness is clear as in the Bessel capacity case [1, Chap 6]. Let  $\{\eta_j\} \subset C_0^2(\Omega)$  be a sequence which converges to  $\eta$  in  $\Delta^{L \ln L}(\Omega)$ . Then there exists a subsequence  $\{\eta_{j_\ell}\}$  such that  $\eta_{j_\ell}$  converges to  $\eta$  uniformly on the complement of an open set of arbitrarily small  $C_{\Delta^{L \ln L}}$ -capacity. This is the claim.  $\square$

*Proof of Theorem 4.5.* The method is adapted from [12, Th 8], [4, Lemma 4.2]. By Lemma 4.8 we can define the functional  $h$  on  $\Delta^{L \ln L}(\Omega)$  by

$$h(\eta) = \int_{\Omega} \bar{\eta}_+ d\mu \quad \forall \eta \in \Delta^{L \ln L}(\Omega),$$

where  $\bar{\eta}$  stands for the  $C_{\Delta^{L \ln L}}$ -quasi-continuous representative of  $\eta$ . Notice that we can write

$$h(\eta) = - \int_{\Omega} \Delta \mathbb{G}^{\Omega}[\mu] \eta dx = - \int_{\Omega} \mathbb{G}^{\Omega}[\mu] \Delta \eta dx$$

The following steps are similar to the previous proofs:

*Step 1-* The functional  $h$  is convex, positively homogeneous and l.s.c. The convexity and the homogeneity are clear. If  $\eta_n \rightarrow \eta$  in  $\Delta^{L \ln L}(\partial\Omega)$ , then by Lemma 4.8 we can extract a subsequence which is converging everywhere except for a set with zero capacity. The conclusion follows from Fatou's lemma.

*Step 2-* Since  $L_P(\Omega)$  is the dual space of  $L_{P^*}(\Omega)$ , for any continuous linear form  $\alpha$  on  $\Delta^{L \ln L}(\Omega)$  there exists  $\beta \in L_P(\Omega)$  such that

$$\alpha(\eta) = - \int_{\Omega} \beta \Delta \eta dx \quad \forall \eta \in \Delta^{L \ln L}(\Omega).$$

Therefore, in the sense of distributions there holds

$$\alpha(\eta) = - \langle \Delta \beta, \eta \rangle \quad \forall \eta \in C_0^{\infty}(\Omega).$$

*Step 3-* By the geometric Hahn-Banch theorem,  $h$  is the upper convex hull of the continuous linear functionals on  $\Delta^{L \ln L}(\partial\Omega)$  it dominates. Fix a function  $\eta_0 \in C_0^{\infty}(\Omega)$  and  $\epsilon > 0$ , there exists a continuous linear form  $\alpha$  on  $\Delta^{L \ln L}(\Omega)$  and constants  $a, b$  such that

$$a + bt + \alpha(\eta) \leq 0 \quad \forall (\eta, t) \in \mathcal{E} := \{(\eta, t) \in \Delta^{L \ln L}(\Omega) \times \mathbb{R} : h(\eta) \leq t\},$$

and

$$a + b(h(\eta_0) - \epsilon) + \alpha(\eta_0) > 0.$$



The same ideas as in [4, Lemma 4.2] yields successively to  $a = 0$  and  $b < 0$ . If we put  $\sigma(\eta) = -b^{-1}\alpha(\eta)$  we derive  $\sigma(\eta) \leq h(\eta)$  for all  $\eta \in \Delta^{L \ln L}(\Omega)$ . This implies in particular that  $\sigma(\eta) \leq 0$  if  $\eta \leq 0$ , thus  $\sigma$  is a positive linear form on  $\Delta^{L \ln L}(\Omega)$ . Therefore there exists a Radon measure  $\nu$  on  $\Omega$  and  $\beta \in L_P(\Omega)$  such that  $-\Delta\beta = \nu$ ,  $0 \leq \nu \leq \mu$  and

$$\int_{\Omega} \eta_0 d\mu \leq \epsilon + \int_{\Omega} \eta_0 d\nu.$$

*Step 4-* Considering an increasing sequence of compact sets  $K_j$  such that  $K_j \subset \overset{\circ}{K}_{j+1}$  and  $\cup_j K_j = \Omega$ , we construct for each  $j \in \mathbb{N}^*$  a Radon measure  $\nu_j$  and  $\beta_j \in L_P(\Omega)$  such that  $-\Delta\beta_j = \nu_j$ ,  $0 \leq \nu_j \leq \mu$  and

$$\int_{K_j} d\mu \leq j^{-1} + \int_{K_j} d\nu_j.$$

At last we can assume that the sequence  $\{\nu_j\}$  is increasing since if  $-\Delta\beta_j = \nu_j$  for  $j = 1, 2$ , then

$$-\Delta\beta_{1,2} = \sup\{\nu_1, \nu_2\} \leq \nu_1 + \nu_2 = -\Delta\beta_1 - \Delta\beta_2$$

thus  $\beta_{1,2} \in L_P(\Omega)$ . Iterating this process, we can replace the sequence  $\{\nu_j\}$  by  $\{\nu'_j\} := \{\nu_1, \sup\{\nu_1, \nu_2\}, \sup\{\nu_3, \sup\{\nu_1, \nu_2\}\}, \dots\}$ . The sequence  $\{\nu'_j\}$  is increasing, converges to  $\mu$  and since  $\beta_j = \mathbb{G}^{\Omega}[\nu'_j]$  with  $\beta_j \in L_P(\Omega)$ ,  $\nu'_j$  belongs to  $B^{exp}(\Omega)$ .  $\square$

As a consequence of this result and the characterization of linear functionals over  $L \ln L(\Omega)$ , the following result holds.

**Corollary 4.9** *Assume  $\mu$  is a bounded positive good measure in  $\Omega$ , then there exists an increasing sequence of positive measures  $\nu_j$  in  $\Omega$  and positive real numbers  $\theta_j$  such that  $\nu_j \rightarrow \mu$  in the weak sense of measures and  $\exp(\theta_j \mathbb{G}^{\Omega}[\nu_j]) \in L^1(\Omega)$ .*

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