

On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

Laurent Veron

▶ To cite this version:

Laurent Veron. On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data. 2011. hal-00575464v1

HAL Id: hal-00575464 https://hal.science/hal-00575464v1

Preprint submitted on 10 Mar 2011 (v1), last revised 15 Mar 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

Laurent Véron

Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, FRANCE

Abstract If Ω is a bounded domain in \mathbb{R}^N , we study conditions on a Radon measure μ on $\partial\Omega$ for solving the equation $-\Delta u + e^u - 1 = 0$ in Ω with $u = \mu$ on $\partial\Omega$. The conditions are expressed in terms of Orlicz capacities.

2010 Mathematics Subject Classification. 35J60, 35J65, 28A12, 42B35, 46E30.

 $\mathit{Key}\ \mathit{words}.$ Elliptic equations, Orlicz capacities, reduced measures, boundary trace

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary and μ a Radon measure on $\partial \Omega$. In this note we consider the problem of finding a function u solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in Ω satisfying $u = \mu$ on $\partial\Omega$. Let $\rho(x) = \text{dist}(x, \partial\Omega)$, then this problem admits a weak formulation: find a function $u \in L^1(\Omega)$ such that $e^u \in L^1_{\rho}(\Omega)$ satisfying

$$\int_{\Omega} \left(-u\Delta\zeta + (e^u - 1)\zeta \right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega).$$
(1.2)

This equation has been initiated by Grillot and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with q > 1), have been developed by Le Gall [18], Dynkin and Kuznetsov [9], [10], Marcus and Véron [19], [20], by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the main features of the problem with power nonlinearities is the existence of a critical exponent $q_c = \frac{N+1}{N-1}$ which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If $1 < q < q_c$, then for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ there exists a unique function $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$ which satisfies (1.3) in Ω and achieves the value μ on $\partial\Omega$ in the following weak sense

$$\int_{\Omega} \left(-u\Delta\zeta + u^q \zeta \right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega).$$
(1.4)

(ii) If $q \ge q_c$, the above problem can be solved if and only if μ vanishes on boundary Borel subsets with zero $C_{\frac{2}{q},q'}$ -capacity. Furthermore a boundary compact set is removable if and only if it has zero $C_{\frac{2}{q},q'}$ -capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 & \text{in } \Omega \\ u &= \mu & \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

Following the terminology of [4] we say that a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is **good** if (4.22) admits a weak solution. Let $P^{\Omega}(x, y)$ (resp. $G^{\Omega}(x, y)$) be the Poisson kernel (resp. the Green kernel) in Ω and $\mathbb{P}^{\Omega}[\mu]$ the Poisson potential of a boundary measure μ (resp. $\mathbb{G}^{\Omega}[\phi]$ the Green potential of a bounded measure ϕ defined in Ω). A boundary measure μ which satisfies

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx).$$
(1.6)

is called **admissible**. Since $\mathbb{P}^{\Omega}[\mu]$ is a supersolution for (1.1), an admissible measure is good (see [25]). Our first result which extends a previous one obtained in [15] is the following.

Theorem A. Suppose $\mu \in \mathfrak{M}(\partial\Omega)$ admits Lebesgue decomposition $\mu = \mu_S + \mu_R$ where μ_S and μ_R are mutually singular and μ_R is absolutely continuous with respect to the (N-1)-dim Hausdorff measure dH^{N-1} . If

$$\exp(\mathbb{P}^{\Omega}[\mu_S]) \in L^1(\Omega; \rho dx), \tag{1.7}$$

then μ is good.

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modelized on the Legendre transform of $r \mapsto p(r) := e^r - 1$. These capacities have been studied by Aissaoui and Benkirane [1] and they inherit most of the properties of the Bessel capacities. The capacity $C_{N^{L \ln L}}$ associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this capacity framework we obtain the following types of results:

Theorem B. Let $\mu \in \mathfrak{M}_+(\partial \Omega)$ be a good measure, then μ vanishes on boundary Borel subsets E with zero $C_{N^{L \ln L}}$ -capacity.

A charaterization of positive measures which have the property of vanishing on Borel subsets E with zero $C_{N^{L \ln L}}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

Theorem C. Let $K \subset \partial \Omega$ be a compact subset with zero $C_{N^{L \ln L}}$ -capacity. Suppose $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in Ω which vanishes on K, then u is identically zero.

We give also some extensions of this approach to the equation

$$-\Delta u + e^u - 1 = \mu, \tag{1.8}$$

as well as removability questions for internal singularities of solutions of (1.1). In that case the natural capacity associated to the problem is

 $C_{\Delta^{L\ln L}}(K) = \inf\{\|M[\Delta\eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\}$ (1.9)

where M[.] denotes Hardy-Littlewood's maximal function.

Theorem D. Let $\mu \in \mathfrak{M}^b_+(\Omega)$ be a bounded good measure, then μ vanishes on boundary Borel subsets E with zero $C_{\Delta L \ln L}$ -capacity.

A charaterization of positive measures which have the property of vanishing on Borel subsets E with zero $C_{N^{L \ln L}}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

Theorem E. Let $K \subset \Omega$ be a compact subset with zero $C_{\Delta^{L \ln L}}$ -capacity. Suppose $u \in C(\Omega \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in $\Omega \setminus K$ which vanishes on $\partial\Omega$, then u is identically zero.

This note is essentially derived from the preliminary report [26], written in 2004 and left escheated since this period.

2 Good measures

Proof of Theorem A. For k > 0, set $\mu_{R,k} = \inf\{k, \mu_R\}$ and denote by u_k the solution of $\Delta u_k + c^{u_k} = 1 - 0$ in Ω

$$-\Delta u_k + e^{u_k} - 1 = 0 \qquad \text{in } \Omega \\ u_k = \mu_S + \mu_{R,k} \qquad \text{on } \partial\Omega.$$

$$(2.1)$$

Such a solution exists because

$$\exp(\mathbb{P}^{\Omega}[\mu_S + \mu_{R,k}]) \le e^k \exp(\mathbb{P}^{\Omega}[\mu_S])$$

by the maximum principle, and (1.7) implies that $\exp(\mathbb{P}^{\Omega}[\mu_{S}+\mu_{R,k}])-1 \in L^{1}(\Omega;\rho dx)$. The sequence u_{k} is nondecreasing. Since, for any $\zeta \in C_{c}^{1,1}(\overline{\Omega})$,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function ζ the solution ζ_0 of

$$-\Delta \zeta_0 = 1 \qquad \text{in } \Omega \zeta_0 = 0 \qquad \text{on } \partial \Omega,$$
(2.2)

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d(\mu_S + \mu_{R,k}) \le c \, \|\mu\|_{\mathfrak{M}} \,. \tag{2.3}$$

Thus $u = \lim_{k \to \infty} u_k$ is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \le c \, \|\mu\|_{\mathfrak{M}} \,,$$

and the convergence of u_k and e^{u_k} to u and e^u hold respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho dx)$ and u satisfies (1.2).

The proof of the next result is inspired by [4].

Theorem 2.1 The following properties hold: (i) If $\mu \in \mathfrak{M}^{exp}_+(\partial\Omega)$ and $0 \leq \tilde{\mu} \leq \mu$, then $\tilde{\mu} \in \mathfrak{M}^{exp}_+(\partial\Omega)$. (ii) Let $\{\mu_n\} \subset \mathfrak{M}^{exp}_+(\partial\Omega)$ be an increasing sequence which converges to μ in the weak sense of measures. Then $\mu \in \mathfrak{M}^{exp}_+(\partial\Omega)$. (iii) $\mathfrak{M}^{exp}_+(\partial\Omega) + L^1_+(\partial\Omega) = \mathfrak{M}^{exp}_+(\partial\Omega)$. *Proof.* (i) Let $u = u_{\mu}$ be the solution of (4.22) and $w = \inf\{u, \mathbb{P}^{\Omega}[\tilde{\mu}]\}$. Since $\mathbb{P}^{\Omega}[\tilde{\mu}]$ is a supersolution for (1.1), w is a supersolution too. Furthermore w is nonnegative and $e^{w} - 1 \in L^{1}(\Omega; \rho dx)$. By Doob's theorem w admits a boundary trace $\mu^{*} \in \mathfrak{M}_{+}(\partial \Omega)$ and $\mu^{*} \leq \tilde{\mu} \leq \mu$. Let w^{*} be the solution of

$$-\Delta w^* + e^u - 1 = 0 \qquad \text{in } \Omega$$
$$w^* = \tilde{\mu} \qquad \text{on } \partial\Omega.$$

then $u \ge w \ge w^*$ and [21],

$$\lim_t \int_{\partial\Omega_t} w^*(t,.)\eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

(here we denote by $\partial\Omega_t$ the set of $x \in \Omega$ such that $\rho(x) = t > 0$). This implies that the boundary trace of w^* is $\tilde{\mu}$ and thus $\mu^* = \tilde{\mu}$. Set $\Omega_t = \{x \in \Omega : \rho(x) > t\}$ and let v_t we the solution of

$$\Delta v_t + e^{v_t} - 1 = 0 \qquad \text{in } \Omega_t$$
$$v_t = w \qquad \text{on } \partial \Omega_t$$

Then $v_t \leq w$ in Ω_t . Furthermore

$$0 < t' < t \Longrightarrow v_{t'} \le v_t \quad \text{in } \Omega_t.$$

Then $\tilde{u} = \lim_{t \to 0} v_t$ exists, the convergence holds in $L^1(\Omega)$ and $e^{v_t} \to e^{\tilde{u}}$ in $L^1(\Omega; \rho dx)$ (here we use the fact that $e^w \in L^1(\Omega; \rho dx)$). Because

$$\lim_{t \to 0} \int_{\partial \Omega_t} \tilde{w}(t, .) \eta dS_t = \int_{\partial \Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial \Omega),$$

and $v_t = \tilde{w}$ on $\partial\Omega_t$, is follows that \tilde{u} admits $\tilde{\mu}$ for boundary trace and thus $\tilde{u} = u_{\tilde{\mu}}$. (ii) Let $u_n = u_{\mu_n}$ be the solutions of (4.22) with boundary value μ_n . The sequence $\{u_n\}$ is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0) dx = -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \le -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \qquad (2.4)$$

we conclude as in the proof of Theorem 1, that u_n increases and converges to a solution $u = u_{\mu}$ of (4.22) with boundary value μ .

(iii) In the proof of (i) we have actually used the following result : Let w be a nonnegative supersolution of (1.1) such that $e^w \in L^1(\Omega; \rho dx)$ and let $\mu \in \mathfrak{M}_+(\partial \Omega)$ be the boundary trace of w. Then μ is good. Let $f \in L^1_+(\partial \Omega)$ and μ be an good measure. We denote by $u = u_{\mu}$ the solution of (4.22). For k > 0, set $f_k = \min\{k, f\}$. The function $w_k = u_{\mu} + \mathbb{P}^{\Omega}[f_k]$ is a nonnegative supersolution, and, since $\mathbb{P}^{\Omega}[f_k] \leq k$, $e^{w_k} \in L^1(\Omega; \rho dx)$. Furthermore the boundary trace of w_k is $\mu + f_k$. Therefore $\mu + f_k$ is good. We conclude by II that $\mu + f$ is good \Box

Remark. The assertions (i) and (ii) in Theorem 1 are still valid if we replace $r \mapsto e^r - 1$ by any continuous nondecreasing function f vanishing at 0.

3 The Orlicz space framework

3.1 Orlicz capacities

The set $M^{exp}(\partial \Omega)$ of nonnegative measures μ on $\partial \Omega$ such that

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx) \tag{3.1}$$

is not a linear space, but it is a convex subset of $\mathfrak{M}_+(\partial\Omega)$. The role of this set comes from the fact that any measure in $M^{exp}(\partial\Omega)$ is good. Put

$$p(t) = \operatorname{sgn}(s)(e^s - 1), \ P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s|+1), \ P^*(t) = (|t|+1) \ln(|t|+1) - |t|$$

Then P and P^* are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \le P(x) + P^*(y) \qquad \forall (x,y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if $x = \overline{p}(y)$ or y = p(x). It is classical to define

$$M_P(\Omega; \rho dx) = \{ \phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx) \},$$
(3.2)

$$M_{P^*}(\Omega;\rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega;\rho dx)\}.$$
(3.3)

The Orlicz spaces $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ are the vector spaces spanned respectively by $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$. They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_{\rho}}} = \inf\left\{k > 0: \int_{\Omega} P\left(\frac{\phi}{k}\right)\rho dx \le 1\right\}.$$
(3.4)

and

$$\|\phi\|_{L_{P^*_{\rho}}} = \inf\left\{k > 0: \int_{\Omega} P^*\left(\frac{\phi}{k}\right)\rho dx \le 1\right\}.$$
(3.5)

Furthermore the Hölder-Young inequality asserts [16]

$$\left| \int_{\Omega} \phi \, \psi \, \rho \, dx \right| \le \left\| \phi \right\|_{L_{P_{\rho}}} \left\| \psi \right\|_{L_{P_{\rho}^{*}}} \quad \forall (\phi, \psi) \in L_{P}(\Omega; \rho dx) \times L_{P^{*}}(\Omega; \rho dx).$$
(3.6)

Since P^* satisfies the Δ_2 -condition, $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$ and $L_P(\Omega; \rho dx)$ is the dual space of $L_{P^*}(\Omega; \rho dx)$, (see [12], [1]). Furthermore, since

$$\frac{|a|\ln(1+|a|)}{2} \le P^*(a) \le |a|\ln(1+|a|) \quad \forall a \in \mathbb{R},$$

the space $L_{P^*}(\Omega; \rho dx)$ is associated with the class $L \ln L(\Omega; \rho dx)$ and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube Q_0 containing $\overline{\Omega}$, with sides parallel to the axes. If $f \in L^1(\Omega)$ we denote by \tilde{f} its extension by 0 in $Q_0 \setminus \Omega$ and put

$$M_{Q_0}[f](x) = \sup\left\{\frac{1}{|Q|}\int_Q |f|(y)dy: Q \in \mathcal{Q}_x\right\}$$

where Q_x denotes the set of all cubes containing x and contained in Q_0 , with sides parallel to the axes. Thus

$$\|f\|_{L\ln L_{\rho}} := \int_{Q_0} M_{Q_0}[f](x)\rho dx \approx \|f\|_{L_{P_{\rho}^*}}.$$
(3.7)

If ℓ is a continuous linear functional on $L \ln L(\Omega; \rho dx)$, there exist a measurable function g_{ℓ} and some $\theta > 0$ such that

$$\begin{cases} \ell(f) = \int_{\Omega} g_{\ell} f dx \quad \forall f \in L \ln L(\Omega; \rho dx), \\ \int_{\Omega} e^{\theta |g_{\ell}|} \rho dx < \infty. \end{cases}$$

This can be seen as a consequence of Young's inequality.

Definition 3.1 The space of all measures on $\partial \Omega$ such that

$$\mathbb{P}^{\Omega}[\mu] \in L_P(\Omega; \rho dx)$$

is denoted by $B^{\exp}(\partial\Omega)$, with norm

$$\|\mu\|_{B^{\exp}} = \left\|\mathbb{P}^{\Omega}[\mu]\right\|_{L_{P_{\rho}}}.$$
(3.8)

The subset of measures such that

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx)$$

is denoted by $\mathfrak{M}^{\exp}(\partial\Omega)$.

The following result follows immediately from the definition of the Luxemburg norm.

Proposition 3.2 If $\mu \in B^{\exp}(\partial\Omega)$ there exists $a_0 > 0$ such that $a\mu \in \mathfrak{M}^{\exp}(\partial\Omega)$ for all $0 \leq a < a_0$. Conversely, if $\mu \in \mathfrak{M}^{\exp}(\partial\Omega)$, then $a\mu \in B^{\exp}(\partial\Omega)$ for all a > 0.

The analytic charaterization of $B^{\exp}(\partial\Omega)$ can be done in introducing the space of normal derivatives of Green potentials of $L \ln L$ functions:

$$N^{L\ln L}(\partial\Omega) = \left\{\eta : \rho^{-1}\Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \in L\ln L(\Omega; \rho dx)\right\}.$$
(3.9)

where ρ^* is a smooth positive function with value ρ in a neighborhood of $\partial\Omega$. Then

$$\left| \int_{\partial\Omega} \eta d\mu \right| = \left| \int_{\Omega} \mathbb{P}^{\Omega}[\mu] \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \, dx \right| \le \left\| \mathbb{P}^{\Omega}[\mu] \right\|_{L_{P_{\rho}}} \left\| \rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \right\|_{L_{P_{\rho}^*}}.$$
 (3.10)

Notice that the actual regularity of the η function is not clear, although

$$\rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L \ln L(\Omega; \rho dx) \Longrightarrow \Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L \ln L(\Omega).$$

If we take, as a norm on $N^{L \ln L}(\partial \Omega)$

$$\|\eta\|_{N^{L\ln L}} = \|\rho^{-1}\Delta(\rho^* \mathbb{P}^{\Omega}[\eta])\|_{L_{P_{\rho}^*}}, \qquad (3.11)$$

and define the $C_{N^L\ln L}\text{-}\mathrm{capacity}$ of a compact subset K of $\partial\Omega$ by

 $C_{N^{L\ln L}}(K) = \inf\{\|\eta\|_{N^{L\ln L}} : \eta \in C^2(\partial\Omega), 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\}.$ (3.12)

Considering the bilinear form \mathcal{H} on $L_{P_{\rho}^{*}}(\partial\Omega) \times L_{P_{\rho}}(\partial\Omega)$

$$\mathcal{H}(\eta,\mu) := -\int_{\Omega} \mathbb{P}^{\Omega}[\mu] \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \, dx \tag{3.13}$$

then

$$\mathcal{H}(\eta,\mu) = -\int_{\Omega} \int_{\partial\Omega} P^{\Omega}(x,y) d\mu(y) \Delta(\rho^* \mathbb{P}^{\Omega}[\eta])(x) dx$$

$$= -\int_{\partial\Omega} \int_{\Omega} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta])(x) P^{\Omega}(x,y) dx d\mu(y).$$
 (3.14)

It is classical to define

$$C_{N^{L \ln L}}^{*}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_{+}(\partial\Omega), \mu(K^{c}) = 0, \left\|\mathbb{P}^{\Omega}[\mu]\right\|_{L_{P_{\rho}}} \le 1\}.$$
(3.15)

The following result due to Fuglede [13] is a consequence of the Kneser-Fan min-max theorem.

Proposition 3.3 For any compact set $K \subset \partial \Omega$, there holds

$$C_{N^{L\ln L}}^{*}(K) = C_{N^{L\ln L}}(K).$$
(3.16)

As a direct consequence of (3.10), we have the following

Proposition 3.4 If $\mu \in B^{exp}_+(\partial \Omega)$, it does not charge Borel subsets with $C_{N^{L \ln L}}$ -capacity zero.

Remark. We conjecture that the following result is true: Let $\mu \in \mathfrak{M}_+(\partial\Omega)$, which does not charge Borel sets with $C_{N^{L\ln L}}$ -capacity zero. Then there exists an increasing sequence $\{\mu_n\} \subset B^{exp}(\partial\Omega)$ which converges weakly to μ . The construction of the proof can be adapted from [11, Th8], [3, Lemma 4.2], except one point which is not clear: If $\eta \in N^{L\ln L}(\partial\Omega)$ does it admit a representative $\tilde{\eta}$ quasi-continuous with respect to the capacity $C_{N^{L\ln L}}$? If this statement holds true the remaining of their proof based on the study of the mapping h defined by

$$h(\eta) = \int_{\partial\Omega} \tilde{\eta}^+ d\mu \qquad \forall \eta \in N^{L \ln L}(\partial\Omega).$$

can be easily adapted.

3.2 Good measures and removable sets

As we have already seen it, a measure in $B^{exp}_+(\partial\Omega)$ is good, and does not charge Borel subsets of $C_{N^L \ln L}$ -capacity zero. The following result is a slight extension of a result of Grillot-Véron, with a proof which inherits some observations of Brezis-Marcus-Ponce.

Proof of Theorem B. Let K be a compact subset with $C_{N^{L \ln L}}$ -capacity zero. There exist a sequence $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of K and

$$\lim_{n \to \infty} \|\eta_n\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L_{P_{\rho}^*}} = 0.$$
(3.17)

Take $\rho^* \mathbb{P}^{\Omega}[\eta_n]$) as a test function, then

$$\int_{\Omega} \left(-u\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) + (e^u - 1)\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right) dx = -\int_{\partial} \Omega \frac{\partial(\rho^* \mathbb{P}^{\Omega}[\eta_n])}{\partial\nu} d\mu$$

Since $\frac{\partial(\rho^* \mathbb{P}^{\Omega}[\eta_n])}{\partial\nu} = \eta_n$ and $\mu > 0$, there holds $-\int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^{\Omega}[\eta_n])}{\partial\nu} d\mu \ge \mu(K)$.
Furthermore

$$\left| \int_{\Omega} u\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) dx \right| \le \|u\|_{L_{P_{\rho}}} \left\| \rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right\|_{L_{P_{\rho}^*}}.$$
(3.18)

Then

$$\mu(E) \le \int_{\Omega} (e^{u} - 1) \rho^* \mathbb{P}^{\Omega}[\eta_n]) dx + \|u\|_{L_{P_{\rho}}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L_{P_{\rho}^*}}$$

By the same argument as in [4], $\lim_{n\to\infty} \rho^* \mathbb{P}^{\Omega}[\eta_n] = 0$, a.e. in Ω , and there exists a nonnegative L^1_{ρ} -function Φ such that $0 \leq \rho^* \mathbb{P}^{\Omega}[\eta_n] \leq \Phi$. By (3.17), (3.18) and Lebesgue's theorem, $\mu(E) = 0$. **Definition 3.5** A subset $E \subset \partial \Omega$ is said removable for equation (1.1), if and only if any positive solution $u \in C^2(\Omega)$ of (1.1) in Ω , which is continuous in $\overline{\Omega} \setminus E$ and vanishes on $\partial \Omega \setminus E$, is identically zero.

Proof of Theorem C. Let $u \in C(\overline{\Omega} \setminus K)$ be a solution of (1.1) which is zero on $\partial\Omega \setminus K$. Let $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (3.17) holds. Put $\theta_n = 1 - \eta_n$. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate and the fact that u vanishes on K^c , there holds

$$u(x) \le C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function $\zeta_n = \rho^* \mathbb{P}^{\Omega}[\theta_n]$ is an admissible test function for u, and

$$\int_{\Omega} \left(-u\Delta\zeta_n + (e^u - 1)\zeta_n \right) dx = 0.$$

Clearly $\mathbb{P}^{\Omega}[\theta_n] = 1 - \mathbb{P}^{\Omega}[\eta_n]$ and

$$\Delta \zeta_n = \Delta \rho^* - \Delta (\rho^* \mathbb{P}^{\Omega}[\eta_n])$$

Inasmuch we can modify ρ^* in order to have $-\Delta \rho^* \ge 0$, in which case $\rho^* = \rho$ near $\partial \Omega$ is replaced by $\rho^* \approx \rho$, we derive

$$-\int_{\Omega} u\Delta\zeta_n \, dx = -\int_{\Omega} \zeta_n^{-1} \Delta\zeta_n \, u\zeta_n dx$$
$$\geq -2^{-1} \int_{\Omega} (e^u - 1 - u)\zeta_n \, dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])) \, \zeta_n dx,$$

where

$$Q(r) = (|r| + 2^{-1})\ln(2|r| + 1) - |r| \le C |r|\ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n \, dx \le 2C \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx, \quad (3.19)$$

since $\zeta_n^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \le \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|$. Furthermore

$$\ln(1+\rho^{-2}\left|\Delta(\rho^*\mathbb{P}^{\Omega}[\eta_n])\right|) = -\ln\rho + \ln(\rho+\rho^{-1}\left|\Delta(\rho^*\mathbb{P}^{\Omega}[\eta_n])\right|)$$
$$\leq -\ln\rho + \ln(1+\rho^{-1}\left|\Delta(\rho^*\mathbb{P}^{\Omega}[\eta_n])\right|)$$

But (we can assume $\rho \leq 1$)

$$\begin{split} \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1+\rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx \\ &\leq -\int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho dx + \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1+\rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx, \end{split}$$

and

$$\begin{split} &\int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx \\ &= \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \le 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| > 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx \\ &\le \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \le 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx + \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| dx \end{split}$$

But

$$\lim_{n \to \infty} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| = 0 \quad \text{a. e. in } \Omega$$

at least up to some subsequence. Thus

$$\lim_{n \to \infty} \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx = 0$$
(3.20)

Using (4.30), we derive u = 0.

Conversely, assume that $C_{N^{L\ln L}}(K) > 0$. By Proposition 3.3 there exists a non negative non-zero measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that $\mu(K^c) = 0$ in the space $B_+^{exp}(\partial\Omega)$. This means that $\theta\mu \in M_+^{exp}(\partial\Omega)$ for some $\theta > 0$. Thus problem (4.22) admits a solution.

By Proposition 3.2 and Theorem ?? we have a partial characterization of measures for which problems (4.22) admits a solution and K is not removable.

Theorem 3.6 If a measure μ is good there exists an increasing sequence of measures $\{\mu_n\}$ which converges to μ satisfying

$$\int_{\Omega} \exp(\theta_n \mathbb{P}^{\Omega}[\mu_n]) \rho dx < \infty$$
(3.21)

for some $\theta_n > 0$.

Several questions can be adressed

1- If a singular measure μ is good does it exist an increasing sequence of $\{\mu_n\}$ converging to μ such that (3.21) holds with $\theta_n = 1$?

2- If a measure μ does not charge Borel sets with $C^{L \ln L}$ -capacity zero, doest it exist $\theta > 0$ such that $\theta \mu \in \mathfrak{M}^{exp}(\partial \Omega)$?

3- If a singular measure μ is good, then $(1 - \delta)\mu \in M^{exp}(\partial\Omega)$ for any $\delta > 0$?

4 Further extensions and open questions

A part of the above construction could be extended to problems with more general nonlinearity such as

4.1 Boundary data measures

$$-\Delta u + P(u) = 0 \quad \text{in } \Omega$$

$$u = \mu \quad \text{on } \partial\Omega, \qquad (4.22)$$

where P is a convex increasing function vanishing at 0 and such that $\lim_{r\to\infty} P(r)/r = \infty$: In Theorem 1-P, (1.7) should be replaced by

$$P(\mathbb{P}^{\Omega}[\mu_n]) \in L^1(\Omega; \rho \, dx). \tag{4.23}$$

In Theorem 2-P, (i) and (ii) still hold. For simplicity we assume that P is a N-function in the sense of Orlicz spaces

$$P(r) = \int_0^r p(s) ds$$

where p is increasing and vanishes at 0. Let P^* be the conjugate N-function, $L_P(\Omega; \rho \, dx)$ and $L_{P^*}(\Omega; \rho \, dx)$ the corresponding Orlicz spaces endowed with the Luxenburg norms. Then Proposition 1-P is valid, provided the space $B^P(\partial\Omega)$ and $M^P(\partial\Omega)$ are accordingly defined with the following notations:

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L_{P^*}(\Omega; \rho \, dx)\}$$

with corresponding norm

$$\left\|\eta\right\|_{N^{P^*}} = \left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta])\right\|_{L_{P^*_{\rho}}}$$

and the corresponding capacity $C_{N^{P^*}}$. It is still likely that Theorem 3-P, 4-P hold. The proof of Theorem 5-P should be valid without any major modification. **However**, it appears that the characterization of removable sets cannot be adapted

without further properties of the function P^* like the Δ_2 -condition. Such a condition holds usually when P has a power-like growth (> 1) and a logarithmic type growth.

4.2 Internal measures

Many of the above techniques can be extended to the following types of problem in which $\mu \in \mathfrak{M}^b_+(\Omega)$:

$$-\Delta u + e^u - 1 = \mu \qquad \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega, \qquad (4.24)$$

and

$$-\Delta u + P(u) = \mu \qquad \text{in } \Omega u = 0 \qquad \text{on } \partial\Omega.$$
(4.25)

Remark. Note that many interesting results can be found in [2] where the analysis of μ is made by comparison with the Hausdorff measure in dimension (N-2), \mathcal{H}^{N-2} . It is proved in particular that if a measure μ satisfies $\mu \leq 4\pi \mathcal{H}^{N-2}$, then problem (4.22) admits a solution, while if μ charges some Borel set A with Hausdorff diemension less than N-2, no solution exists.

We denote by $\mathfrak{M}^{exp}_+(\Omega)$ the set of good measures for (4.24) and define the classes $M_P(\Omega)$ and $M_{P^*}(\Omega)$ similarly to $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$ except that the measure ρdx is replaced by the Lebesgue measure dx. The Orlicz spaces $L_P(\Omega)$ and $L_{P^*}(\Omega)$ are defined from $M_P(\Omega)$ and $M_{P^*}(\Omega)$ and endowed with the respective Luxemburg norms $\| \|_P$ and $\| \|_{P^*}$. Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta \mathbb{G}^{\Omega}[\mu] \, dx \right| = \left| \int_{\Omega} \mathbb{G}^{\Omega}[\mu] \Delta \eta \, dx \right| \le \left\| \mathbb{G}^{\Omega}[\mu] \right\|_{L_{P}} \left\| \Delta \eta \right\|_{L_{P^{*}}}.$$
 (4.26)

for $\eta \in C_c^{1,1}(\bar{\Omega})$. We define the $C_{\Delta^{L \ln L}}$ -capacity of a compact subset K of $\partial \Omega$ by

$$C_{\Delta^{L\ln L}}(K) = \inf\{\|\Delta\eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\},$$
(4.27)

The capacity $C_{\Delta^{L \ln L}}$ can be characterized using the Hardy-Littlewood maximal function $f \mapsto M_{Q_0}[f]$ since

$$\|f\|_{L\ln L} := \int_{Q_0} M_{Q_0}[f](x) dx \approx \|f\|_{L_{P^*}}.$$
(4.28)

Thus Proposition 3.4 and Theorem B are valid under the form

Proposition 4.1 If $\mu \in B^{exp}_+(\Omega)$, it does not charge Borel subsets with $C_{\Delta^{L \ln L}}$ -capacity zero.

Theorem 4.2 Let $\mu \in \mathfrak{M}_+(\Omega)$ be a good measure, then μ vanishes on Borel subset E with zero $C_{\Delta L \ln L}$ -capacity.

Theorem C has the following counter part

Theorem 4.3 Let $K \subset \Omega$ be compact. Any solution of

$$-\Delta u + e^u - 1 = 0 \qquad in \ \Omega \setminus K u = 0 \qquad on \ \partial\Omega,$$
(4.29)

vanishes identically in Ω if and only if $C_{\Delta^{L \ln L}}(K) = 0$.

Proof. Let $u \in C(\Omega \setminus K)$ be a solution of (1.1) which is zero on $\partial\Omega$. Let $\{\eta_n\} \subset C^2(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (3.17) holds. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate for this type of nonlinearity (see [24]), there holds

$$u(x) \le C \ln(2/\rho_K(x)) + D.$$

Put $\theta_n = 1 - \eta_n$. Then the function $\zeta_n = \phi_1 \theta_n$ (ϕ_1 being the first eigenfunction of $-\Delta$) is an admissible test function for u, and

$$\int_{\Omega} \left(-u\Delta\zeta_n + (e^u - 1)\zeta_n \right) dx = 0.$$

We derive

$$-\int_{\Omega} u\Delta\zeta_n \, dx = -\int_{\Omega} \zeta_n^{-1} \Delta\zeta_n \, u \, dx$$
$$\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \, dx - \int_{\Omega} Q(\Delta(\zeta_n) \, dx.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u)\zeta_n \, dx \le 2C \int_{\Omega} |\Delta\zeta_n| \ln(1 + |\Delta\zeta_n|) dx, \tag{4.30}$$

Since the right-hand side goes to zero when $n \to \infty$, the conclusion follows. *Remark.* The characterization of the $C_{\Delta^{L \ln L}}$ -capacity is not simple, however, by a result of [6, Th1], there holds

$$\left\| D^2 \eta \right\|_{L^{1,\infty}} \le C \left\| \Delta \eta \right\|_{L \ln L} \qquad \forall \eta \in C_c^{1,1}(\overline{\Omega})$$
(4.31)

where $L^{1,\infty}(\Omega)$ denotes the weak L^1 -space, that is the space of all measurable functions f defined in Ω satisfying

$$meas\left(\left\{x \in \Omega : |f(x)| > t\right\}\right) \le \frac{c}{t}, \qquad \forall t > 0 \tag{4.32}$$

and $||f||_{L^{1,\infty}}$ is the smallest constant such that (4.32) holds.

4.2.1 Reduced measures

What are the reduced measures both for the boundary and internal problems (4.22) (resp. (4.25))? A projection onto the closure of the sets $\mathfrak{M}^{exp}(\partial\Omega)$ (resp. $\mathfrak{M}^{exp}(\Omega)$) of positive measures in $\partial\Omega$ (resp. Ω) satisfying

$$\int_{\Omega} \exp\left(\mathbb{P}^{\Omega}[\mu]\right) \rho dx < \infty$$

(resp.

$$\int_{\Omega} \exp\left(\mathbb{G}^{\Omega}[\mu]\right) dx < \infty.)$$

The definition of the projection is not clear, although an important fact is that it ensures uniqueness. This conjectures could be extended to problems involving operator $u \mapsto -\Delta u + P(u)$.

References

- Aissaoui N., Benkirane A.: Capacités dans les espaces d'Orlicz, Ann. Sci. Math. Qubec 18, 1-23 (1994).
- [2] Bertolucci D., Leoni F., Orsina L., Ponce A.: Semilinear equations with exponential nonlinearity and measure data, Ann. I. H. Poincaré An. Nonl. 22, 799-815 (2005).
- Baras P., Pierre M.: Singularités éliminables pour des équations semi-linéaires, Ann. Inst. Fourier, Grenoble 34, 185-206 (1984).
- [4] Brezis H., Marcus M., Ponce A.: Nonlinear elliptic equations with measures revisited, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud., 163, 55109, (2007).
- [5] Brezis H., Ponce A.: Reduced measures on the boundary, J. Funct. Anal. 229, 95120 (2005).
- [6] Christ M., Stein E.: A remark on singular Calderòn-Zygmund theory, Proc. Amer. Math. Soc. 99, 71-75 (1987).
- [7] Dal Maso G.: On the integral representation of certains local functionals, Recerche Math. 32, 85-113 (1983).
- [8] Doob J.: Classical Potential Theory and its Probabilistic Counterpart, Springer-Verlag, London-Berlin-Heidelberg-New York (1984).

- [9] Dynkin E. B., Kuzntesov S.: Superdiffusions and removable singularities for quasilinear P.D.E., Comm. Pure Appl. Math. 49, 125-176 (1996).
- [10] Dynkin E. B., Kuzntesov S.: Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations, Comm. Pure Appl. Math. 51, 897-936 (1998).
- [11] Feyel D., De la Pradelle A.: Topologies fines et compactication associées à certains espaces de Dirichlet, Ann. Inst. Fourier 27, 121-146 (1977).
- [12] Fuchs M., Seregin A.: A regularity theory for variational integrals with L ln L-growth, Calc. Var. 6, 171-187 (1998).
- [13] Fuglede B.: Applications du thorme de minimax à l'étude de diverses capacités, C. R. Acad. Sci. Paris Ser. A, t. 266, 921-923 (1968).
- [14] Gmira A., Véron L.: Boundary singularities of solutions of nonlinear elliptic equations, Duke J. Math. 64, 271-324 (1991)
- [15] Grillot M., Véron L.: Boundary trace of solutions of the Prescribed Gaussian curvature equation, Proc. Roy. Soc. Edinburgh 130 A, 1-34 (2000).
- [16] Krasnosel'skii M. A., Rutickii Y. B.: Convex functions and Orlicz spaces, P. Noordhoff Ltd, Groningen (1961).
- [17] Kuzntesov S.: Removable singularities for $Lu = \Psi(u)$ and Orlicz capacities, J. Funct. Anal. 170, 428-449 (2000).
- [18] Le Gall J. F.: The brownian snake and solutions of $\Delta u = u^2$ in a domain, Prob. Theory Rel. Fields 102, 393-432 (1995).
- [19] Marcus M., Véron L.: The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, Arch. Rat. Mech. An. 144, 201-231 (1998).
- [20] Marcus M., Véron L.: Removable singularities and boundary traces, J. Math. Pures Appl. 80, 879-900 (2001).
- [21] Marcus M., Véron L. : The boundary trace and generalized B.V.P. for semilinear elliptic equations with coercive absorption, Comm. Pure Appl. Math. 56, 689-731 (2003).
- [22] Robert J.: Approximation des espaces d'Orlicz et applications, Numer. Math. 17, 338-356 (1971).
- [23] Stein E.: Note on the class LlogL, Studia Math. 32, 305-310 (1969).

- [24] Vàzquez J. L., Véron L.: Singularities of Elliptic Equations with an Exponential Nonlinearity, Math. Ann. 269, 119-135 (1984).
- [25] Véron L.: Singularities of solutions of second order quasilinear equations, Pitman Research Notes in Math. 353, Addison Wesley Longman Inc (1996).
- [26] Véron L.: A note on the equation $-\Delta u + e^u 1 = 0$, 2004-preliminary report, arXiv:1103.0975 (2011).