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On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract If Ω is a bounded domain in \mathbb{R}^N , we study conditions on a Radon measure μ on $\partial\Omega$ for solving the equation $-\Delta u + e^u - 1 = 0$ in Ω with $u = \mu$ on $\partial\Omega$. The conditions are expressed in terms of Orlicz capacities.

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary and μ a Radon measure on $\partial\Omega$. In this note we consider the problem of finding a function u solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in Ω satisfying $u = \mu$ on $\partial\Omega$. Let $\rho(x) = \text{dist}(x, \partial\Omega)$, then this problem admits a **weak formulation**: *find a function $u \in L^1(\Omega)$ such that $e^u \in L^1_\rho(\Omega)$ satisfying*

$$\int_{\Omega} (-u\Delta\zeta + (e^u - 1)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.2}$$

This equation has been initiated by Grillo and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with $q > 1$), have been developed by Le Gall [18], Dynkin and Kuznetsov [9], [10], Marcus and Véron [19], [20], by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the main features of the problem with power nonlinearities is the existence of a critical exponent

$q_c = \frac{N+1}{N-1}$ which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If $1 < q < q_c$, then for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ there exists a unique function $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$ which satisfies (1.3) in Ω and achieves the value μ on $\partial\Omega$ in the following weak sense

$$\int_{\Omega} (-u\Delta\zeta + u^q\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \quad (1.4)$$

(ii) If $q \geq q_c$, the above problem can be solved if and only if μ vanishes on boundary Borel subsets with zero $C_{\frac{2}{q},q'}$ -capacity. Furthermore a boundary compact set is removable if and only if it has zero $C_{\frac{2}{q},q'}$ -capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \\ u &= \mu && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

Following the terminology of [4] we say that a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is **good** if (4.22) admits a weak solution. Let $P^\Omega(x, y)$ (resp. $G^\Omega(x, y)$) be the Poisson kernel (resp. the Green kernel) in Ω and $\mathbb{P}^\Omega[\mu]$ the Poisson potential of a boundary measure μ (resp. $\mathbb{G}^\Omega[\phi]$ the Green potential of a bounded measure ϕ defined in Ω). A boundary measure μ which satisfies

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx). \quad (1.6)$$

is called **admissible**. Since $\mathbb{P}^\Omega[\mu]$ is a supersolution for (1.1), an admissible measure is good (see [25]). Our first result which extends a previous one obtained in [15] is the following.

Theorem A. *Suppose $\mu \in \mathfrak{M}(\partial\Omega)$ admits Lebesgue decomposition $\mu = \mu_S + \mu_R$ where μ_S and μ_R are mutually singular and μ_R is absolutely continuous with respect to the $(N-1)$ -dim Hausdorff measure dH^{N-1} . If*

$$\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx), \quad (1.7)$$

then μ is good.

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modeled on the Legendre transform of $r \mapsto p(r) := e^r - 1$. These capacities have been studied by Aissaoui and Benkirane [1] and they inherit

most of the properties of the Bessel capacities. The capacity $C_{N^L \ln L}$ associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this capacity framework we obtain the following types of results:

Theorem B. *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be a good measure, then μ vanishes on boundary Borel subsets E with zero $C_{N^L \ln L}$ -capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets E with zero $C_{N^L \ln L}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

Theorem C. *Let $K \subset \partial\Omega$ be a compact subset with zero $C_{N^L \ln L}$ -capacity. Suppose $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in Ω which vanishes on K , then u is identically zero.*

We give also some extensions of this approach to the equation

$$-\Delta u + e^u - 1 = \mu, \quad (1.8)$$

as well as removability questions for internal singularities of solutions of (1.1). In that case the natural capacity associated to the problem is

$$C_{\Delta^L \ln L}(K) = \inf\{\|M[\Delta\eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\} \quad (1.9)$$

where $M[\cdot]$ denotes Hardy-Littlewood's maximal function.

Theorem D. *Let $\mu \in \mathfrak{M}_+^b(\Omega)$ be a bounded good measure, then μ vanishes on boundary Borel subsets E with zero $C_{\Delta^L \ln L}$ -capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets E with zero $C_{N^L \ln L}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

Theorem E. *Let $K \subset \Omega$ be a compact subset with zero $C_{\Delta^L \ln L}$ -capacity. Suppose $u \in C(\Omega \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in $\Omega \setminus K$ which vanishes on $\partial\Omega$, then u is identically zero.*

This note is essentially derived from the preliminary report [26], written in 2004 and left escheated since this period.

2 Good measures

Proof of Theorem A. For $k > 0$, set $\mu_{R,k} = \inf\{k, \mu_R\}$ and denote by u_k the solution of

$$\begin{aligned} -\Delta u_k + e^{u_k} - 1 &= 0 && \text{in } \Omega \\ u_k &= \mu_S + \mu_{R,k} && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

Such a solution exists because

$$\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}^\Omega[\mu_S])$$

by the maximum principle, and (1.7) implies that $\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$. The sequence u_k is nondecreasing. Since, for any $\zeta \in C_c^{1,1}(\bar{\Omega})$,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function ζ the solution ζ_0 of

$$\begin{aligned} -\Delta \zeta_0 &= 1 && \text{in } \Omega \\ \zeta_0 &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (2.2)$$

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \leq c \|\mu\|_{\mathfrak{M}}. \quad (2.3)$$

Thus $u = \lim_{k \rightarrow \infty} u_k$ is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \leq c \|\mu\|_{\mathfrak{M}},$$

and the convergence of u_k and e^{u_k} to u and e^u hold respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho dx)$ and u satisfies (1.2). \square

The proof of the next result is inspired by [4].

Theorem 2.1 *The following properties hold:*

- (i) If $\mu \in \mathfrak{M}_+^{exp}(\partial\Omega)$ and $0 \leq \tilde{\mu} \leq \mu$, then $\tilde{\mu} \in \mathfrak{M}_+^{exp}(\partial\Omega)$.
- (ii) Let $\{\mu_n\} \subset \mathfrak{M}_+^{exp}(\partial\Omega)$ be an increasing sequence which converges to μ in the weak sense of measures. Then $\mu \in \mathfrak{M}_+^{exp}(\partial\Omega)$.
- (iii) $\mathfrak{M}_+^{exp}(\partial\Omega) + L_+^1(\partial\Omega) = \mathfrak{M}_+^{exp}(\partial\Omega)$.

Proof. (i) Let $u = u_\mu$ be the solution of (4.22) and $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$. Since $\mathbb{P}^\Omega[\tilde{\mu}]$ is a supersolution for (1.1), w is a supersolution too. Furthermore w is nonnegative and $e^w - 1 \in L^1(\Omega; \rho dx)$. By Doob's theorem w admits a boundary trace $\mu^* \in \mathfrak{M}_+(\partial\Omega)$ and $\mu^* \leq \tilde{\mu} \leq \mu$. Let w^* be the solution of

$$\begin{aligned} -\Delta w^* + e^u - 1 &= 0 && \text{in } \Omega \\ w^* &= \tilde{\mu} && \text{on } \partial\Omega. \end{aligned}$$

then $u \geq w \geq w^*$ and [21],

$$\lim_t \int_{\partial\Omega_t} w^*(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

(here we denote by $\partial\Omega_t$ the set of $x \in \Omega$ such that $\rho(x) = t > 0$). This implies that the boundary trace of w^* is $\tilde{\mu}$ and thus $\mu^* = \tilde{\mu}$. Set $\Omega_t = \{x \in \Omega : \rho(x) > t\}$ and let v_t be the solution of

$$\begin{aligned} -\Delta v_t + e^{v_t} - 1 &= 0 && \text{in } \Omega_t \\ v_t &= w && \text{on } \partial\Omega_t. \end{aligned}$$

Then $v_t \leq w$ in Ω_t . Furthermore

$$0 < t' < t \implies v_{t'} \leq v_t \quad \text{in } \Omega_{t'}.$$

Then $\tilde{u} = \lim_{t \rightarrow 0} v_t$ exists, the convergence holds in $L^1(\Omega)$ and $e^{v_t} \rightarrow e^{\tilde{u}}$ in $L^1(\Omega; \rho dx)$ (here we use the fact that $e^w \in L^1(\Omega; \rho dx)$). Because

$$\lim_{t \rightarrow 0} \int_{\partial\Omega_t} \tilde{w}(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

and $v_t = \tilde{w}$ on $\partial\Omega_t$, it follows that \tilde{u} admits $\tilde{\mu}$ for boundary trace and thus $\tilde{u} = u_{\tilde{\mu}}$.

(ii) Let $u_n = u_{\mu_n}$ be the solutions of (4.22) with boundary value μ_n . The sequence $\{u_n\}$ is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \leq - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \quad (2.4)$$

we conclude as in the proof of Theorem 1, that u_n increases and converges to a solution $u = u_\mu$ of (4.22) with boundary value μ .

(iii) In the proof of (i) we have actually used the following result : *Let w be a nonnegative supersolution of (1.1) such that $e^w \in L^1(\Omega; \rho dx)$ and let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be the boundary trace of w . Then μ is good.* Let $f \in L^1_+(\partial\Omega)$ and μ be a good measure. We denote by $u = u_\mu$ the solution of (4.22). For $k > 0$, set $f_k = \min\{k, f\}$. The function $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$ is a nonnegative supersolution, and, since $\mathbb{P}^\Omega[f_k] \leq k$, $e^{w_k} \in L^1(\Omega; \rho dx)$. Furthermore the boundary trace of w_k is $\mu + f_k$. Therefore $\mu + f_k$ is good. We conclude by II that $\mu + f$ is good \square

Remark. The assertions (i) and (ii) in Theorem 1 are still valid if we replace $r \mapsto e^r - 1$ by any continuous nondecreasing function f vanishing at 0.

3 The Orlicz space framework

3.1 Orlicz capacities

The set $M^{exp}(\partial\Omega)$ of nonnegative measures μ on $\partial\Omega$ such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx) \quad (3.1)$$

is not a linear space, but it is a convex subset of $\mathfrak{M}_+(\partial\Omega)$. The role of this set comes from the fact that any measure in $M^{exp}(\partial\Omega)$ is good. Put

$$p(t) = \operatorname{sgn}(s)(e^s - 1), \quad P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s| + 1), \quad P^*(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$

Then P and P^* are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \leq P(x) + P^*(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if $x = \bar{p}(y)$ or $y = p(x)$. It is classical to define

$$M_P(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx)\}, \quad (3.2)$$

$$M_{P^*}(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx)\}. \quad (3.3)$$

The Orlicz spaces $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ are the vector spaces spanned respectively by $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$. They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_\rho}} = \inf \left\{ k > 0 : \int_\Omega P\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.4)$$

and

$$\|\phi\|_{L_{P^*_\rho}} = \inf \left\{ k > 0 : \int_\Omega P^*\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.5)$$

Furthermore the Hölder-Young inequality asserts [16]

$$\left| \int_\Omega \phi \psi \rho dx \right| \leq \|\phi\|_{L_{P_\rho}} \|\psi\|_{L_{P^*_\rho}} \quad \forall (\phi, \psi) \in L_P(\Omega; \rho dx) \times L_{P^*}(\Omega; \rho dx). \quad (3.6)$$

Since P^* satisfies the Δ_2 -condition, $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$ and $L_P(\Omega; \rho dx)$ is the dual space of $L_{P^*}(\Omega; \rho dx)$, (see [12], [1]). Furthermore, since

$$\frac{|a| \ln(1 + |a|)}{2} \leq P^*(a) \leq |a| \ln(1 + |a|) \quad \forall a \in \mathbb{R},$$

the space $L_{P^*}(\Omega; \rho dx)$ is associated with the class $L \ln L(\Omega; \rho dx)$ and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube Q_0 containing Ω , with sides parallel to the axes. If $f \in L^1(\Omega)$ we denote by \tilde{f} its extension by 0 in $Q_0 \setminus \Omega$ and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f|(y) dy : Q \in \mathcal{Q}_x \right\}$$

where \mathcal{Q}_x denotes the set of all cubes containing x and contained in Q_0 , with sides parallel to the axes. Thus

$$\|f\|_{L \ln L_\rho} := \int_{Q_0} M_{Q_0}[f](x) \rho dx \approx \|f\|_{L_{P^*}}. \quad (3.7)$$

If ℓ is a continuous linear functional on $L \ln L(\Omega; \rho dx)$, there exist a measurable function g_ℓ and some $\theta > 0$ such that

$$\begin{cases} \ell(f) = \int_{\Omega} g_\ell f dx & \forall f \in L \ln L(\Omega; \rho dx), \\ \int_{\Omega} e^{\theta |g_\ell|} \rho dx < \infty. \end{cases}$$

This can be seen as a consequence of Young's inequality.

Definition 3.1 *The space of all measures on $\partial\Omega$ such that*

$$\mathbb{P}^\Omega[\mu] \in L_P(\Omega; \rho dx)$$

is denoted by $B^{\text{exp}}(\partial\Omega)$, with norm

$$\|\mu\|_{B^{\text{exp}}} = \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}}. \quad (3.8)$$

The subset of measures such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx)$$

is denoted by $\mathfrak{M}^{\text{exp}}(\partial\Omega)$.

The following result follows immediately from the definition of the Luxemburg norm.

Proposition 3.2 *If $\mu \in B^{\text{exp}}(\partial\Omega)$ there exists $a_0 > 0$ such that $a\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$ for all $0 \leq a < a_0$. Conversely, if $\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$, then $a\mu \in B^{\text{exp}}(\partial\Omega)$ for all $a > 0$.*

The analytic characterization of $B^{\text{exp}}(\partial\Omega)$ can be done in introducing the space of normal derivatives of Green potentials of $L \ln L$ functions:

$$N^{L \ln L}(\partial\Omega) = \{\eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx)\}. \quad (3.9)$$

where ρ^* is a smooth positive function with value ρ in a neighborhood of $\partial\Omega$. Then

$$\left| \int_{\partial\Omega} \eta d\mu \right| = \left| \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \right| \leq \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P_\rho^*}}. \quad (3.10)$$

Notice that the actual regularity of the η function is not clear, although

$$\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx) \implies \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega).$$

If we take, as a norm on $N^{L \ln L}(\partial\Omega)$

$$\|\eta\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P_\rho^*}}, \quad (3.11)$$

and define the $C_{N^{L \ln L}}$ -capacity of a compact subset K of $\partial\Omega$ by

$$C_{N^{L \ln L}}(K) = \inf\{\|\eta\|_{N^{L \ln L}} : \eta \in C^2(\partial\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}. \quad (3.12)$$

Considering the bilinear form \mathcal{H} on $L_{P_\rho^*}(\partial\Omega) \times L_{P_\rho}(\partial\Omega)$

$$\mathcal{H}(\eta, \mu) := - \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \quad (3.13)$$

then

$$\begin{aligned} \mathcal{H}(\eta, \mu) &= - \int_{\Omega} \int_{\partial\Omega} P^\Omega(x, y) d\mu(y) \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) dx \\ &= - \int_{\partial\Omega} \int_{\Omega} \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) P^\Omega(x, y) dx d\mu(y). \end{aligned} \quad (3.14)$$

It is classical to define

$$C_{N^{L \ln L}}^*(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(K^c) = 0, \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}} \leq 1\}. \quad (3.15)$$

The following result due to Fuglede [13] is a consequence of the Kneser-Fan min-max theorem.

Proposition 3.3 *For any compact set $K \subset \partial\Omega$, there holds*

$$C_{N^{L \ln L}}^*(K) = C_{N^{L \ln L}}(K). \quad (3.16)$$

As a direct consequence of (3.10), we have the following

Proposition 3.4 *If $\mu \in B_+^{\text{exp}}(\partial\Omega)$, it does not charge Borel subsets with $C_{N^L \ln L}$ -capacity zero.*

Remark. We conjecture that the following result is true: *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$, which does not charge Borel sets with $C_{N^L \ln L}$ -capacity zero. Then there exists an increasing sequence $\{\mu_n\} \subset B^{\text{exp}}(\partial\Omega)$ which converges weakly to μ .* The construction of the proof can be adapted from [11, Th8], [3, Lemma 4.2], except one point which is not clear: *If $\eta \in N^{L \ln L}(\partial\Omega)$ does it admit a representative $\tilde{\eta}$ quasi-continuous with respect to the capacity $C_{N^L \ln L}$? If this statement holds true the remaining of their proof based on the study of the mapping h defined by*

$$h(\eta) = \int_{\partial\Omega} \tilde{\eta}^+ d\mu \quad \forall \eta \in N^{L \ln L}(\partial\Omega).$$

can be easily adapted.

3.2 Good measures and removable sets

As we have already seen it, a measure in $B_+^{\text{exp}}(\partial\Omega)$ is good, and does not charge Borel subsets of $C_{N^L \ln L}$ -capacity zero. The following result is a slight extension of a result of Grillo-Véron, with a proof which inherits some observations of Brezis-Marcus-Ponce.

Proof of Theorem B. Let K be a compact subset with $C_{N^L \ln L}$ -capacity zero. There exist a sequence $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of K and

$$\lim_{n \rightarrow \infty} \|\eta_n\|_{N^L \ln L} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}} = 0. \quad (3.17)$$

Take $\rho^* \mathbb{P}^\Omega[\eta_n]$ as a test function, then

$$\int_{\Omega} (-u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) + (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n]) dx = - \int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu$$

Since $\frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} = \eta_n$ and $\mu > 0$, there holds $-\int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu \geq \mu(K)$.

Furthermore

$$\left| \int_{\Omega} u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) dx \right| \leq \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}. \quad (3.18)$$

Then

$$\mu(E) \leq \int_{\Omega} (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n] dx + \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}.$$

By the same argument as in [4], $\lim_{n \rightarrow \infty} \rho^* \mathbb{P}^\Omega[\eta_n] = 0$, a.e. in Ω , and there exists a nonnegative L_ρ^1 -function Φ such that $0 \leq \rho^* \mathbb{P}^\Omega[\eta_n] \leq \Phi$. By (3.17), (3.18) and Lebesgue's theorem, $\mu(E) = 0$. \square

Definition 3.5 A subset $E \subset \partial\Omega$ is said removable for equation (1.1), if and only if any positive solution $u \in C^2(\Omega)$ of (1.1) in Ω , which is continuous in $\bar{\Omega} \setminus E$ and vanishes on $\partial\Omega \setminus E$, is identically zero.

Proof of Theorem C. Let $u \in C(\bar{\Omega} \setminus K)$ be a solution of (1.1) which is zero on $\partial\Omega \setminus K$. Let $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (3.17) holds. Put $\theta_n = 1 - \eta_n$. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate and the fact that u vanishes on K^c , there holds

$$u(x) \leq C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function $\zeta_n = \rho^* \mathbb{P}^\Omega[\theta_n]$ is an admissible test function for u , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

Clearly $\mathbb{P}^\Omega[\theta_n] = 1 - \mathbb{P}^\Omega[\eta_n]$ and

$$\Delta \zeta_n = \Delta \rho^* - \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])$$

Inasmuch we can modify ρ^* in order to have $-\Delta \rho^* \geq 0$, in which case $\rho^* = \rho$ near $\partial\Omega$ is replaced by $\rho^* \approx \rho$, we derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u \zeta_n dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \zeta_n dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])) \zeta_n dx, \end{aligned}$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \leq C |r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx, \quad (3.19)$$

since $\zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|$. Furthermore

$$\begin{aligned} \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) &= -\ln \rho + \ln(\rho + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \\ &\leq -\ln \rho + \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \end{aligned}$$

But (we can assume $\rho \leq 1$)

$$\begin{aligned} & \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx \\ & \leq - \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx \\ & = \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| > 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx \\ & \leq \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx = 0 \quad (3.20)$$

Using (4.30), we derive $u = 0$.

Conversely, assume that $C_{NL \ln L}(K) > 0$. By Proposition 3.3 there exists a non negative non-zero measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that $\mu(K^c) = 0$ in the space $B_+^{exp}(\partial\Omega)$. This means that $\theta\mu \in M_+^{exp}(\partial\Omega)$ for some $\theta > 0$. Thus problem (4.22) admits a solution. \square

By Proposition 3.2 and Theorem ?? we have a partial characterization of measures for which problems (4.22) admits a solution and K is not removable.

Theorem 3.6 *If a measure μ is good there exists an increasing sequence of measures $\{\mu_n\}$ which converges to μ satisfying*

$$\int_{\Omega} \exp(\theta_n \mathbb{P}^{\Omega}[\mu_n]) \rho dx < \infty \quad (3.21)$$

for some $\theta_n > 0$.

Several questions can be addressed

- 1- If a singular measure μ is good does it exist an increasing sequence of $\{\mu_n\}$ converging to μ such that (3.21) holds with $\theta_n = 1$?
- 2- If a measure μ does not charge Borel sets with $C^{L \ln L}$ -capacity zero, does it exist $\theta > 0$ such that $\theta\mu \in \mathfrak{M}^{exp}(\partial\Omega)$?
- 3- If a singular measure μ is good, then $(1 - \delta)\mu \in M^{exp}(\partial\Omega)$ for any $\delta > 0$?

4 Further extensions and open questions

A part of the above construction could be extended to problems with more general nonlinearity such as

4.1 Boundary data measures

$$\begin{aligned} -\Delta u + P(u) &= 0 \quad \text{in } \Omega \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{4.22}$$

where P is a convex increasing function vanishing at 0 and such that $\lim_{r \rightarrow \infty} P(r)/r = \infty$: In Theorem 1- P , (1.7) should be replaced by

$$P(\mathbb{P}^\Omega[\mu_n]) \in L^1(\Omega; \rho dx). \tag{4.23}$$

In Theorem 2- P , (i) and (ii) still hold. For simplicity we assume that P is a N -function in the sense of Orlicz spaces

$$P(r) = \int_0^r p(s) ds$$

where p is increasing and vanishes at 0. Let P^* be the conjugate N -function, $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ the corresponding Orlicz spaces endowed with the Luxemburg norms. Then Proposition 1- P is valid, provided the space $B^P(\partial\Omega)$ and $M^P(\partial\Omega)$ are accordingly defined with the following notations:

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega; \rho dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P^*}}$$

and the corresponding capacity $C_{N^{P^*}}$. It is still likely that Theorem 3- P , 4- P hold. The proof of Theorem 5- P should be valid without any major modification. **However**, it appears that the characterization of removable sets cannot be adapted

without further properties of the function P^* like the Δ_2 -condition. Such a condition holds usually when P has a power-like growth (> 1) and a logarithmic type growth.

4.2 Internal measures

Many of the above techniques can be extended to the following types of problem in which $\mu \in \mathfrak{M}_+^b(\Omega)$:

$$\begin{aligned} -\Delta u + e^u - 1 &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} -\Delta u + P(u) &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (4.25)$$

Remark. Note that many interesting results can be found in [2] where the analysis of μ is made by comparison with the Hausdorff measure in dimension $(N-2)$, \mathcal{H}^{N-2} . It is proved in particular that if a measure μ satisfies $\mu \leq 4\pi\mathcal{H}^{N-2}$, then problem (4.22) admits a solution, while if μ charges some Borel set A with Hausdorff dimension less than $N - 2$, no solution exists.

We denote by $\mathfrak{M}_+^{\text{exp}}(\Omega)$ the set of good measures for (4.24) and define the classes $M_P(\Omega)$ and $M_{P^*}(\Omega)$ similarly to $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$ except that the measure ρdx is replaced by the Lebesgue measure dx . The Orlicz spaces $L_P(\Omega)$ and $L_{P^*}(\Omega)$ are defined from $M_P(\Omega)$ and $M_{P^*}(\Omega)$ and endowed with the respective Luxemburg norms $\|\cdot\|_P$ and $\|\cdot\|_{P^*}$. Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta \mathbb{G}^{\Omega}[\mu] dx \right| = \left| \int_{\Omega} \mathbb{G}^{\Omega}[\mu] \Delta \eta dx \right| \leq \|\mathbb{G}^{\Omega}[\mu]\|_{L_P} \|\Delta \eta\|_{L_{P^*}}. \quad (4.26)$$

for $\eta \in C_c^{1,1}(\bar{\Omega})$. We define the $C_{\Delta L \ln L}$ -capacity of a compact subset K of $\partial\Omega$ by

$$C_{\Delta L \ln L}(K) = \inf \{ \|\Delta \eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K \}, \quad (4.27)$$

The capacity $C_{\Delta L \ln L}$ can be characterized using the Hardy-Littlewood maximal function $f \mapsto M_{Q_0}[f]$ since

$$\|f\|_{L \ln L} := \int_{Q_0} M_{Q_0}[f](x) dx \approx \|f\|_{L_{P^*}}. \quad (4.28)$$

Thus Proposition 3.4 and Theorem B are valid under the form

Proposition 4.1 *If $\mu \in B_+^{\text{exp}}(\Omega)$, it does not charge Borel subsets with $C_{\Delta L \ln L}$ -capacity zero.*

Theorem 4.2 *Let $\mu \in \mathfrak{M}_+(\Omega)$ be a good measure, then μ vanishes on Borel subset E with zero $C_{\Delta L \ln L}$ -capacity.*

Theorem C has the following counter part

Theorem 4.3 *Let $K \subset \Omega$ be compact. Any solution of*

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \setminus K \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (4.29)$$

vanishes identically in Ω if and only if $C_{\Delta L \ln L}(K) = 0$.

Proof. Let $u \in C(\Omega \setminus K)$ be a solution of (1.1) which is zero on $\partial\Omega$. Let $\{\eta_n\} \subset C^2(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (3.17) holds. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate for this type of nonlinearity (see [24]), there holds

$$u(x) \leq C \ln(2/\rho_K(x)) + D.$$

Put $\theta_n = 1 - \eta_n$. Then the function $\zeta_n = \phi_1 \theta_n$ (ϕ_1 being the first eigenfunction of $-\Delta$) is an admissible test function for u , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

We derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) dx - \int_{\Omega} Q(\Delta(\zeta_n)) dx. \end{aligned}$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta \zeta_n| \ln(1 + |\Delta \zeta_n|) dx, \quad (4.30)$$

Since the right-hand side goes to zero when $n \rightarrow \infty$, the conclusion follows. \square

Remark. The characterization of the $C_{\Delta L \ln L}$ -capacity is not simple, however, by a result of [6, Th1], there holds

$$\|D^2 \eta\|_{L^{1,\infty}} \leq C \|\Delta \eta\|_{L \ln L} \quad \forall \eta \in C_c^{1,1}(\overline{\Omega}) \quad (4.31)$$

where $L^{1,\infty}(\Omega)$ denotes the weak L^1 -space, that is the space of all measurable functions f defined in Ω satisfying

$$\text{meas}(\{x \in \Omega : |f(x)| > t\}) \leq \frac{c}{t}, \quad \forall t > 0 \quad (4.32)$$

and $\|f\|_{L^{1,\infty}}$ is the smallest constant such that (4.32) holds.

4.2.1 Reduced measures

What are the reduced measures both for the boundary and internal problems (4.22) (resp. (4.25))? A projection onto the closure of the sets $\mathfrak{M}^{exp}(\partial\Omega)$ (resp. $\mathfrak{M}^{exp}(\Omega)$) of positive measures in $\partial\Omega$ (resp. Ω) satisfying

$$\int_{\Omega} \exp(\mathbb{P}^{\Omega}[\mu])\rho dx < \infty$$

(resp.

$$\int_{\Omega} \exp(\mathbb{G}^{\Omega}[\mu])dx < \infty.)$$

The definition of the projection is not clear, although an important fact is that it ensures uniqueness. This conjectures could be extended to problems involving operator $u \mapsto -\Delta u + P(u)$.

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