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# On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

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**Abstract** If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , we study conditions on a Radon measure  $\mu$  on  $\partial\Omega$  for solving the equation  $-\Delta u + e^u - 1 = 0$  in  $\Omega$  with  $u = \mu$  on  $\partial\Omega$ . The conditions are expressed in terms of Orlicz capacities.

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*Key words.* Elliptic equations, Orlicz capacities, reduced measures, boundary trace

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary and  $\mu$  a Radon measure on  $\partial\Omega$ . In this note we consider the problem of finding a function  $u$  solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in  $\Omega$  satisfying  $u = \mu$  on  $\partial\Omega$ . Let  $\rho(x) = \text{dist}(x, \partial\Omega)$ , then this problem admits a **weak formulation**: *find a function  $u \in L^1(\Omega)$  such that  $e^u \in L^1_\rho(\Omega)$  satisfying*

$$\int_{\Omega} (-u\Delta\zeta + (e^u - 1)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.2}$$

This equation has been initiated by Grillo and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with  $q > 1$ ), have been developed by Le Gall [18], Dynkin and Kuznetsov [9], [10], Marcus and Véron [19], [20], by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the main features of the problem with power nonlinearities is the existence of a critical exponent

$q_c = \frac{N+1}{N-1}$  which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If  $1 < q < q_c$ , then for any  $\mu \in \mathfrak{M}_+(\partial\Omega)$  there exists a unique function  $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$  which satisfies (1.3) in  $\Omega$  and achieves the value  $\mu$  on  $\partial\Omega$  in the following weak sense

$$\int_{\Omega} (-u\Delta\zeta + u^q\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \quad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \quad (1.4)$$

(ii) If  $q \geq q_c$ , the above problem can be solved if and only if  $\mu$  vanishes on boundary Borel subsets with zero  $C_{\frac{2}{q},q'}$ -capacity. Furthermore a boundary compact set is removable if and only if it has zero  $C_{\frac{2}{q},q'}$ -capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \\ u &= \mu && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

Following the terminology of [4] we say that a measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is **good** if (4.22) admits a weak solution. Let  $P^\Omega(x, y)$  (resp.  $G^\Omega(x, y)$ ) be the Poisson kernel (resp. the Green kernel) in  $\Omega$  and  $\mathbb{P}^\Omega[\mu]$  the Poisson potential of a boundary measure  $\mu$  (resp.  $\mathbb{G}^\Omega[\phi]$  the Green potential of a bounded measure  $\phi$  defined in  $\Omega$ ). A boundary measure  $\mu$  which satisfies

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx). \quad (1.6)$$

is called **admissible**. Since  $\mathbb{P}^\Omega[\mu]$  is a supersolution for (1.1), an admissible measure is good (see [25]). Our first result which extends a previous one obtained in [15] is the following.

**Theorem A.** *Suppose  $\mu \in \mathfrak{M}(\partial\Omega)$  admits Lebesgue decomposition  $\mu = \mu_S + \mu_R$  where  $\mu_S$  and  $\mu_R$  are mutually singular and  $\mu_R$  is absolutely continuous with respect to the  $(N-1)$ -dim Hausdorff measure  $dH^{N-1}$ . If*

$$\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx), \quad (1.7)$$

*then  $\mu$  is good.*

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modeled on the Legendre transform of  $r \mapsto p(r) := e^r - 1$ . These capacities have been studied by Aissaoui and Benkirane [1] and they inherit

most of the properties of the Bessel capacities. The capacity  $C_{NL\ln L}$  associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this capacity framework we obtain the following types of results:

**Theorem B.** *Let  $\mu \in \mathfrak{M}_+(\partial\Omega)$  be a good measure, then  $\mu$  vanishes on boundary Borel subsets  $E$  with zero  $C_{NL\ln L}$ -capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets  $E$  with zero  $C_{NL\ln L}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

**Theorem C.** *Let  $K \subset \partial\Omega$  be a compact subset with zero  $C_{NL\ln L}$ -capacity. Suppose  $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $K$ , then  $u$  is identically zero.*

We give also some extensions of this approach to the equation

$$-\Delta u + e^u - 1 = \mu, \quad (1.8)$$

as well as removability questions for internal singularities of solutions of (1.1). In that case the natural capacity associated to the problem is

$$C_{\Delta L\ln L}(K) = \inf\{\|M[\Delta\eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\} \quad (1.9)$$

where  $M[\cdot]$  denotes Hardy-Littlewood's maximal function.

**Theorem D.** *Let  $\mu \in \mathfrak{M}_+^b(\Omega)$  be a bounded good measure, then  $\mu$  vanishes on boundary Borel subsets  $E$  with zero  $C_{\Delta L\ln L}$ -capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets  $E$  with zero  $C_{NL\ln L}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

**Theorem E.** *Let  $K \subset \Omega$  be a compact subset with zero  $C_{\Delta L\ln L}$ -capacity. Suppose  $u \in C(\Omega \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega \setminus K$  which vanishes on  $\partial\Omega$ , then  $u$  is identically zero.*

This note is essentially derived from the preliminary report [26], written in 2004 and left escheated since this period.

## 2 Good measures

*Proof of Theorem A.* For  $k > 0$ , set  $\mu_{R,k} = \inf\{k, \mu_R\}$  and denote by  $u_k$  the solution of

$$\begin{aligned} -\Delta u_k + e^{u_k} - 1 &= 0 && \text{in } \Omega \\ u_k &= \mu_S + \mu_{R,k} && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

Such a solution exists because

$$\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}^\Omega[\mu_S])$$

by the maximum principle, and (1.7) implies that  $\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$ . The sequence  $u_k$  is nondecreasing. Since, for any  $\zeta \in C_c^{1,1}(\bar{\Omega})$ ,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function  $\zeta$  the solution  $\zeta_0$  of

$$\begin{aligned} -\Delta \zeta_0 &= 1 && \text{in } \Omega \\ \zeta_0 &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (2.2)$$

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \leq c \|\mu\|_{\mathfrak{M}}. \quad (2.3)$$

Thus  $u = \lim_{k \rightarrow \infty} u_k$  is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \leq c \|\mu\|_{\mathfrak{M}},$$

and the convergence of  $u_k$  and  $e^{u_k}$  to  $u$  and  $e^u$  hold respectively in  $L^1(\Omega)$  and  $L^1(\Omega; \rho dx)$  and  $u$  satisfies (1.2).  $\square$

The proof of the next result is inspired by [4].

**Theorem 2.1** *The following properties hold:*

- (i) If  $\mu \in \mathfrak{M}_+^{exp}(\partial\Omega)$  and  $0 \leq \tilde{\mu} \leq \mu$ , then  $\tilde{\mu} \in \mathfrak{M}_+^{exp}(\partial\Omega)$ .
- (ii) Let  $\{\mu_n\} \subset \mathfrak{M}_+^{exp}(\partial\Omega)$  be an increasing sequence which converges to  $\mu$  in the weak sense of measures. Then  $\mu \in \mathfrak{M}_+^{exp}(\partial\Omega)$ .
- (iii)  $\mathfrak{M}_+^{exp}(\partial\Omega) + L_+^1(\partial\Omega) = \mathfrak{M}_+^{exp}(\partial\Omega)$ .

*Proof.* (i) Let  $u = u_\mu$  be the solution of (4.22) and  $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$ . Since  $\mathbb{P}^\Omega[\tilde{\mu}]$  is a supersolution for (1.1),  $w$  is a supersolution too. Furthermore  $w$  is nonnegative and  $e^w - 1 \in L^1(\Omega; \rho dx)$ . By Doob's theorem  $w$  admits a boundary trace  $\mu^* \in \mathfrak{M}_+(\partial\Omega)$  and  $\mu^* \leq \tilde{\mu} \leq \mu$ . Let  $w^*$  be the solution of

$$\begin{aligned} -\Delta w^* + e^u - 1 &= 0 && \text{in } \Omega \\ w^* &= \tilde{\mu} && \text{on } \partial\Omega. \end{aligned}$$

then  $u \geq w \geq w^*$  and [21],

$$\lim_t \int_{\partial\Omega_t} w^*(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

(here we denote by  $\partial\Omega_t$  the set of  $x \in \Omega$  such that  $\rho(x) = t > 0$ ). This implies that the boundary trace of  $w^*$  is  $\tilde{\mu}$  and thus  $\mu^* = \tilde{\mu}$ . Set  $\Omega_t = \{x \in \Omega : \rho(x) > t\}$  and let  $v_t$  be the solution of

$$\begin{aligned} -\Delta v_t + e^{v_t} - 1 &= 0 && \text{in } \Omega_t \\ v_t &= w && \text{on } \partial\Omega_t. \end{aligned}$$

Then  $v_t \leq w$  in  $\Omega_t$ . Furthermore

$$0 < t' < t \implies v_{t'} \leq v_t \quad \text{in } \Omega_{t'}.$$

Then  $\tilde{u} = \lim_{t \rightarrow 0} v_t$  exists, the convergence holds in  $L^1(\Omega)$  and  $e^{v_t} \rightarrow e^{\tilde{u}}$  in  $L^1(\Omega; \rho dx)$  (here we use the fact that  $e^w \in L^1(\Omega; \rho dx)$ ). Because

$$\lim_{t \rightarrow 0} \int_{\partial\Omega_t} \tilde{w}(t, \cdot) \eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

and  $v_t = \tilde{w}$  on  $\partial\Omega_t$ , it follows that  $\tilde{u}$  admits  $\tilde{\mu}$  for boundary trace and thus  $\tilde{u} = u_{\tilde{\mu}}$ .

(ii) Let  $u_n = u_{\mu_n}$  be the solutions of (4.22) with boundary value  $\mu_n$ . The sequence  $\{u_n\}$  is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0) dx = - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \leq - \int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \quad (2.4)$$

we conclude as in the proof of Theorem 1, that  $u_n$  increases and converges to a solution  $u = u_\mu$  of (4.22) with boundary value  $\mu$ .

(iii) In the proof of (i) we have actually used the following result : *Let  $w$  be a nonnegative supersolution of (1.1) such that  $e^w \in L^1(\Omega; \rho dx)$  and let  $\mu \in \mathfrak{M}_+(\partial\Omega)$  be the boundary trace of  $w$ . Then  $\mu$  is good.* Let  $f \in L^1_+(\partial\Omega)$  and  $\mu$  be a good measure. We denote by  $u = u_\mu$  the solution of (4.22). For  $k > 0$ , set  $f_k = \min\{k, f\}$ . The function  $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$  is a nonnegative supersolution, and, since  $\mathbb{P}^\Omega[f_k] \leq k$ ,  $e^{w_k} \in L^1(\Omega; \rho dx)$ . Furthermore the boundary trace of  $w_k$  is  $\mu + f_k$ . Therefore  $\mu + f_k$  is good. We conclude by II that  $\mu + f$  is good  $\square$

*Remark.* The assertions (i) and (ii) in Theorem 1 are still valid if we replace  $r \mapsto e^r - 1$  by any continuous nondecreasing function  $f$  vanishing at 0.

### 3 The Orlicz space framework

#### 3.1 Orlicz capacities

The set  $M^{exp}(\partial\Omega)$  of nonnegative measures  $\mu$  on  $\partial\Omega$  such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx) \quad (3.1)$$

is not a linear space, but it is a convex subset of  $\mathfrak{M}_+(\partial\Omega)$ . The role of this set comes from the fact that any measure in  $M^{exp}(\partial\Omega)$  is good. Put

$$p(t) = \operatorname{sgn}(s)(e^s - 1), \quad P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s| + 1), \quad P^*(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$

Then  $P$  and  $P^*$  are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \leq P(x) + P^*(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if  $x = \bar{p}(y)$  or  $y = p(x)$ . It is classical to define

$$M_P(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx)\}, \quad (3.2)$$

$$M_{P^*}(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx)\}. \quad (3.3)$$

The Orlicz spaces  $L_P(\Omega; \rho dx)$  and  $L_{P^*}(\Omega; \rho dx)$  are the vector spaces spanned respectively by  $M_P(\Omega; \rho dx)$  and  $M_{P^*}(\Omega; \rho dx)$ . They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_\rho}} = \inf \left\{ k > 0 : \int_\Omega P\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.4)$$

and

$$\|\phi\|_{L_{P^*_\rho}} = \inf \left\{ k > 0 : \int_\Omega P^*\left(\frac{\phi}{k}\right) \rho dx \leq 1 \right\}. \quad (3.5)$$

Furthermore the Hölder-Young inequality asserts [16]

$$\left| \int_\Omega \phi \psi \rho dx \right| \leq \|\phi\|_{L_{P_\rho}} \|\psi\|_{L_{P^*_\rho}} \quad \forall (\phi, \psi) \in L_P(\Omega; \rho dx) \times L_{P^*}(\Omega; \rho dx). \quad (3.6)$$

Since  $P^*$  satisfies the  $\Delta_2$ -condition,  $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$  and  $L_P(\Omega; \rho dx)$  is the dual space of  $L_{P^*}(\Omega; \rho dx)$ , (see [12], [1]). Furthermore, since

$$\frac{|a| \ln(1 + |a|)}{2} \leq P^*(a) \leq |a| \ln(1 + |a|) \quad \forall a \in \mathbb{R},$$

the space  $L_{P^*}(\Omega; \rho dx)$  is associated with the class  $L \ln L(\Omega; \rho dx)$  and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube  $Q_0$  containing  $\Omega$ , with sides parallel to the axes. If  $f \in L^1(\Omega)$  we denote by  $\tilde{f}$  its extension by 0 in  $Q_0 \setminus \Omega$  and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f|(y) dy : Q \in \mathcal{Q}_x \right\}$$

where  $\mathcal{Q}_x$  denotes the set of all cubes containing  $x$  and contained in  $Q_0$ , with sides parallel to the axes. Thus

$$\|f\|_{L \ln L_\rho} := \int_{Q_0} M_{Q_0}[f](x) \rho dx \approx \|f\|_{L_{P^*}}. \quad (3.7)$$

If  $\ell$  is a continuous linear functional on  $L \ln L(\Omega; \rho dx)$ , there exist a measurable function  $g_\ell$  and some  $\theta > 0$  such that

$$\begin{cases} \ell(f) = \int_{\Omega} g_\ell f dx & \forall f \in L \ln L(\Omega; \rho dx), \\ \int_{\Omega} e^{\theta |g_\ell|} \rho dx < \infty. \end{cases}$$

This can be seen as a consequence of Young's inequality.

**Definition 3.1** *The space of all measures on  $\partial\Omega$  such that*

$$\mathbb{P}^\Omega[\mu] \in L_P(\Omega; \rho dx)$$

*is denoted by  $B^{\text{exp}}(\partial\Omega)$ , with norm*

$$\|\mu\|_{B^{\text{exp}}} = \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}}. \quad (3.8)$$

*The subset of measures such that*

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx)$$

*is denoted by  $\mathfrak{M}^{\text{exp}}(\partial\Omega)$ .*

The following result follows immediately from the definition of the Luxemburg norm.

**Proposition 3.2** *If  $\mu \in B^{\text{exp}}(\partial\Omega)$  there exists  $a_0 > 0$  such that  $a\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$  for all  $0 \leq a < a_0$ . Conversely, if  $\mu \in \mathfrak{M}^{\text{exp}}(\partial\Omega)$ , then  $a\mu \in B^{\text{exp}}(\partial\Omega)$  for all  $a > 0$ .*



The analytic characterization of  $B^{\text{exp}}(\partial\Omega)$  can be done in introducing the space of normal derivatives of Green potentials of  $L \ln L$  functions:

$$N^{L \ln L}(\partial\Omega) = \{ \eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx) \}. \quad (3.9)$$

where  $\rho^*$  is a smooth positive function with value  $\rho$  in a neighborhood of  $\partial\Omega$ . Then

$$\left| \int_{\partial\Omega} \eta d\mu \right| = \left| \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \right| \leq \| \mathbb{P}^\Omega[\mu] \|_{L_{P_\rho}} \| \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \|_{L_{P_\rho^*}}. \quad (3.10)$$

Notice that the actual regularity of the  $\eta$  function is not clear, although

$$\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx) \implies \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega).$$

If we take, as a norm on  $N^{L \ln L}(\partial\Omega)$

$$\| \eta \|_{N^{L \ln L}} = \| \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \|_{L_{P_\rho^*}}, \quad (3.11)$$

and define the  $C_{N^{L \ln L}}$ -capacity of a compact subset  $K$  of  $\partial\Omega$  by

$$C_{N^{L \ln L}}(K) = \inf \{ \| \eta \|_{N^{L \ln L}} : \eta \in C^2(\partial\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K \}. \quad (3.12)$$

Considering the bilinear form  $\mathcal{H}$  on  $L_{P_\rho^*}(\partial\Omega) \times L_{P_\rho}(\partial\Omega)$

$$\mathcal{H}(\eta, \mu) := - \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \quad (3.13)$$

then

$$\begin{aligned} \mathcal{H}(\eta, \mu) &= - \int_{\Omega} \int_{\partial\Omega} P^\Omega(x, y) d\mu(y) \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) dx \\ &= - \int_{\partial\Omega} \int_{\Omega} \Delta(\rho^* \mathbb{P}^\Omega[\eta])(x) P^\Omega(x, y) dx d\mu(y). \end{aligned} \quad (3.14)$$

It is classical to define

$$C_{N^{L \ln L}}^*(K) = \sup \{ \mu(K) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(K^c) = 0, \| \mathbb{P}^\Omega[\mu] \|_{L_{P_\rho}} \leq 1 \}. \quad (3.15)$$

The following result due to Fuglede [13] is a consequence of the Kneser-Fan min-max theorem.

**Proposition 3.3** *For any compact set  $K \subset \partial\Omega$ , there holds*

$$C_{N^{L \ln L}}^*(K) = C_{N^{L \ln L}}(K). \quad (3.16)$$

As a direct consequence of (3.10), we have the following

**Proposition 3.4** *If  $\mu \in B_+^{\text{exp}}(\partial\Omega)$ , it does not charge Borel subsets with  $C_{N^L \ln L}$ -capacity zero.*

*Remark.* We conjecture that the following result is true: *Let  $\mu \in \mathfrak{M}_+(\partial\Omega)$ , which does not charge Borel sets with  $C_{N^L \ln L}$ -capacity zero. Then there exists an increasing sequence  $\{\mu_n\} \subset B^{\text{exp}}(\partial\Omega)$  which converges weakly to  $\mu$ .* The construction of the proof can be adapted from [11, Th8], [3, Lemma 4.2], except one point which is not clear: *If  $\eta \in N^{L \ln L}(\partial\Omega)$  does it admit a representative  $\tilde{\eta}$  quasi-continuous with respect to the capacity  $C_{N^L \ln L}$ ? If this statement holds true the remaining of their proof based on the study of the mapping  $h$  defined by*

$$h(\eta) = \int_{\partial\Omega} \tilde{\eta}^+ d\mu \quad \forall \eta \in N^{L \ln L}(\partial\Omega).$$

can be easily adapted.

### 3.2 Good measures and removable sets

As we have already seen it, a measure in  $B_+^{\text{exp}}(\partial\Omega)$  is good, and does not charge Borel subsets of  $C_{N^L \ln L}$ -capacity zero. The following result is a slight extension of a result of Grillo-Véron, with a proof which inherits some observations of Brezis-Marcus-Ponce.

*Proof of Theorem B.* Let  $K$  be a compact subset with  $C_{N^L \ln L}$ -capacity zero. There exist a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of  $K$  and

$$\lim_{n \rightarrow \infty} \|\eta_n\|_{N^L \ln L} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}} = 0. \quad (3.17)$$

Take  $\rho^* \mathbb{P}^\Omega[\eta_n]$  as a test function, then

$$\int_{\Omega} (-u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) + (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n]) dx = - \int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu$$

Since  $\frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} = \eta_n$  and  $\mu > 0$ , there holds  $-\int_{\partial\Omega} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial\nu} d\mu \geq \mu(K)$ .

Furthermore

$$\left| \int_{\Omega} u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) dx \right| \leq \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}. \quad (3.18)$$

Then

$$\mu(E) \leq \int_{\Omega} (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n] dx + \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}.$$

By the same argument as in [4],  $\lim_{n \rightarrow \infty} \rho^* \mathbb{P}^\Omega[\eta_n] = 0$ , a.e. in  $\Omega$ , and there exists a nonnegative  $L_\rho^1$ -function  $\Phi$  such that  $0 \leq \rho^* \mathbb{P}^\Omega[\eta_n] \leq \Phi$ . By (3.17), (3.18) and Lebesgue's theorem,  $\mu(E) = 0$ .  $\square$

**Definition 3.5** A subset  $E \subset \partial\Omega$  is said removable for equation (1.1), if and only if any positive solution  $u \in C^2(\Omega)$  of (1.1) in  $\Omega$ , which is continuous in  $\bar{\Omega} \setminus E$  and vanishes on  $\partial\Omega \setminus E$ , is identically zero.

*Proof of Theorem C.* Let  $u \in C(\bar{\Omega} \setminus K)$  be a solution of (1.1) which is zero on  $\partial\Omega \setminus K$ . Let  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of  $K$  and (3.17) holds. Put  $\theta_n = 1 - \eta_n$ . Put  $\rho_K(x) = \text{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate and the fact that  $u$  vanishes on  $K^c$ , there holds

$$u(x) \leq C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function  $\zeta_n = \rho^* \mathbb{P}^\Omega[\theta_n]$  is an admissible test function for  $u$ , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

Clearly  $\mathbb{P}^\Omega[\theta_n] = 1 - \mathbb{P}^\Omega[\eta_n]$  and

$$\Delta \zeta_n = \Delta \rho^* - \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])$$

Inasmuch we can modify  $\rho^*$  in order to have  $-\Delta \rho^* \geq 0$ , in which case  $\rho^* = \rho$  near  $\partial\Omega$  is replaced by  $\rho^* \approx \rho$ , we derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u \zeta_n dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \zeta_n dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])) \zeta_n dx, \end{aligned}$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \leq C |r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx, \quad (3.19)$$

since  $\zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|$ . Furthermore

$$\begin{aligned} \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) &= -\ln \rho + \ln(\rho + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \\ &\leq -\ln \rho + \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \end{aligned}$$

But (we can assume  $\rho \leq 1$ )

$$\begin{aligned} & \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx \\ & \leq - \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx \\ & = \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| > 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx \\ & \leq \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln \rho^{-1} dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|) dx = 0 \quad (3.20)$$

Using (4.30), we derive  $u = 0$ .

Conversely, assume that  $C_{NL \ln L}(K) > 0$ . By Proposition 3.3 there exists a non negative non-zero measure  $\mu \in \mathfrak{M}_+(\partial\Omega)$  such that  $\mu(K^c) = 0$  in the space  $B_+^{exp}(\partial\Omega)$ . This means that  $\theta\mu \in M_+^{exp}(\partial\Omega)$  for some  $\theta > 0$ . Thus problem (4.22) admits a solution.  $\square$

By Proposition 3.2 and Theorem ?? we have a partial characterization of measures for which problems (4.22) admits a solution and  $K$  is not removable.

**Theorem 3.6** *If a measure  $\mu$  is good there exists an increasing sequence of measures  $\{\mu_n\}$  which converges to  $\mu$  satisfying*

$$\int_{\Omega} \exp(\theta_n \mathbb{P}^{\Omega}[\mu_n]) \rho dx < \infty \quad (3.21)$$

for some  $\theta_n > 0$ .

Several questions can be addressed

- 1- If a singular measure  $\mu$  is good does it exist an increasing sequence of  $\{\mu_n\}$  converging to  $\mu$  such that (3.21) holds with  $\theta_n = 1$  ?
- 2- If a measure  $\mu$  does not charge Borel sets with  $C^{L \ln L}$ -capacity zero, does it exist  $\theta > 0$  such that  $\theta\mu \in \mathfrak{M}^{exp}(\partial\Omega)$  ?
- 3- If a singular measure  $\mu$  is good, then  $(1 - \delta)\mu \in M^{exp}(\partial\Omega)$  for any  $\delta > 0$  ?

## 4 Further extensions and open questions

A part of the above construction could be extended to problems with more general nonlinearity such as

### 4.1 Boundary data measures

$$\begin{aligned} -\Delta u + P(u) &= 0 \quad \text{in } \Omega \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{4.22}$$

where  $P$  is a convex increasing function vanishing at 0 and such that  $\lim_{r \rightarrow \infty} P(r)/r = \infty$ : In Theorem 1- $P$ , (1.7) should be replaced by

$$P(\mathbb{P}^\Omega[\mu_n]) \in L^1(\Omega; \rho dx). \tag{4.23}$$

In Theorem 2- $P$ , (i) and (ii) still hold. For simplicity we assume that  $P$  is a  $N$ -function in the sense of Orlicz spaces

$$P(r) = \int_0^r p(s) ds$$

where  $p$  is increasing and vanishes at 0. Let  $P^*$  be the conjugate  $N$ -function,  $L_P(\Omega; \rho dx)$  and  $L_{P^*}(\Omega; \rho dx)$  the corresponding Orlicz spaces endowed with the Luxemburg norms. Then Proposition 1- $P$  is valid, provided the space  $B^P(\partial\Omega)$  and  $M^P(\partial\Omega)$  are accordingly defined with the following notations:

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega; \rho dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P^*}}$$

and the corresponding capacity  $C_{N^{P^*}}$ . It is still likely that Theorem 3- $P$ , 4- $P$  hold. The proof of Theorem 5- $P$  should be valid without any major modification. **However**, it appears that the characterization of removable sets cannot be adapted

without further properties of the function  $P^*$  like the  $\Delta_2$ -condition. Such a condition holds usually when  $P$  has a power-like growth ( $> 1$ ) and a logarithmic type growth.

## 4.2 Internal measures

Many of the above techniques can be extended to the following types of problem in which  $\mu \in \mathfrak{M}_+^b(\Omega)$ :

$$\begin{aligned} -\Delta u + e^u - 1 &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} -\Delta u + P(u) &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (4.25)$$

*Remark.* Note that many interesting results can be found in [2] where the analysis of  $\mu$  is made by comparison with the Hausdorff measure in dimension  $(N-2)$ ,  $\mathcal{H}^{N-2}$ . It is proved in particular that if a measure  $\mu$  satisfies  $\mu \leq 4\pi\mathcal{H}^{N-2}$ , then problem (4.22) admits a solution, while if  $\mu$  charges some Borel set  $A$  with Hausdorff dimension less than  $N - 2$ , no solution exists.

We denote by  $\mathfrak{M}_+^{\text{exp}}(\Omega)$  the set of good measures for (4.24) and define the classes  $M_P(\Omega)$  and  $M_{P^*}(\Omega)$  similarly to  $M_P(\Omega; \rho dx)$  and  $M_{P^*}(\Omega; \rho dx)$  except that the measure  $\rho dx$  is replaced by the Lebesgue measure  $dx$ . The Orlicz spaces  $L_P(\Omega)$  and  $L_{P^*}(\Omega)$  are defined from  $M_P(\Omega)$  and  $M_{P^*}(\Omega)$  and endowed with the respective Luxemburg norms  $\|\cdot\|_P$  and  $\|\cdot\|_{P^*}$ . Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta \mathbb{G}^{\Omega}[\mu] dx \right| = \left| \int_{\Omega} \mathbb{G}^{\Omega}[\mu] \Delta \eta dx \right| \leq \|\mathbb{G}^{\Omega}[\mu]\|_{L_P} \|\Delta \eta\|_{L_{P^*}}. \quad (4.26)$$

for  $\eta \in C_c^{1,1}(\bar{\Omega})$ . We define the  $C_{\Delta L \ln L}$ -capacity of a compact subset  $K$  of  $\partial\Omega$  by

$$C_{\Delta L \ln L}(K) = \inf \{ \|\Delta \eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K \}, \quad (4.27)$$

The capacity  $C_{\Delta L \ln L}$  can be characterized using the Hardy-Littlewood maximal function  $f \mapsto M_{Q_0}[f]$  since

$$\|f\|_{L \ln L} := \int_{Q_0} M_{Q_0}[f](x) dx \approx \|f\|_{L_{P^*}}. \quad (4.28)$$

Thus Proposition 3.4 and Theorem B are valid under the form

**Proposition 4.1** *If  $\mu \in B_+^{\text{exp}}(\Omega)$ , it does not charge Borel subsets with  $C_{\Delta L \ln L}$ -capacity zero.*

**Theorem 4.2** *Let  $\mu \in \mathfrak{M}_+(\Omega)$  be a good measure, then  $\mu$  vanishes on Borel subset  $E$  with zero  $C_{\Delta L \ln L}$ -capacity.*

Theorem C has the following counter part

**Theorem 4.3** *Let  $K \subset \Omega$  be compact. Any solution of*

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \setminus K \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (4.29)$$

*vanishes identically in  $\Omega$  if and only if  $C_{\Delta L \ln L}(K) = 0$ .*

*Proof.* Let  $u \in C(\Omega \setminus K)$  be a solution of (1.1) which is zero on  $\partial\Omega$ . Let  $\{\eta_n\} \subset C^2(\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of  $K$  and (3.17) holds. Put  $\rho_K(x) = \text{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate for this type of nonlinearity (see [24]), there holds

$$u(x) \leq C \ln(2/\rho_K(x)) + D.$$

Put  $\theta_n = 1 - \eta_n$ . Then the function  $\zeta_n = \phi_1 \theta_n$  ( $\phi_1$  being the first eigenfunction of  $-\Delta$ ) is an admissible test function for  $u$ , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

We derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) dx - \int_{\Omega} Q(\Delta(\zeta_n)) dx. \end{aligned}$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta \zeta_n| \ln(1 + |\Delta \zeta_n|) dx, \quad (4.30)$$

Since the right-hand side goes to zero when  $n \rightarrow \infty$ , the conclusion follows.  $\square$

*Remark.* The characterization of the  $C_{\Delta L \ln L}$ -capacity is not simple, however, by a result of [6, Th1], there holds

$$\|D^2 \eta\|_{L^{1,\infty}} \leq C \|\Delta \eta\|_{L \ln L} \quad \forall \eta \in C_c^{1,1}(\overline{\Omega}) \quad (4.31)$$

where  $L^{1,\infty}(\Omega)$  denotes the weak  $L^1$ -space, that is the space of all measurable functions  $f$  defined in  $\Omega$  satisfying

$$\text{meas}(\{x \in \Omega : |f(x)| > t\}) \leq \frac{c}{t}, \quad \forall t > 0 \quad (4.32)$$

and  $\|f\|_{L^{1,\infty}}$  is the smallest constant such that (4.32) holds.

### 4.2.1 Reduced measures

What are the reduced measures both for the boundary and internal problems (4.22) (resp. (4.25))? A projection onto the closure of the sets  $\mathfrak{M}^{exp}(\partial\Omega)$  (resp.  $\mathfrak{M}^{exp}(\Omega)$ ) of positive measures in  $\partial\Omega$  (resp.  $\Omega$ ) satisfying

$$\int_{\Omega} \exp(\mathbb{P}^{\Omega}[\mu])\rho dx < \infty$$

(resp.

$$\int_{\Omega} \exp(\mathbb{G}^{\Omega}[\mu])dx < \infty.)$$

The definition of the projection is not clear, although an important fact is that it ensures uniqueness. This conjectures could be extended to problems involving operator  $u \mapsto -\Delta u + P(u)$ .

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