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# On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

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**Abstract** If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , we study conditions on a Radon measure  $\mu$  on  $\partial\Omega$  for solving the equation  $-\Delta u + e^u - 1 = 0$  in  $\Omega$  with  $u = \mu$  on  $\partial\Omega$ . The conditions are expressed in terms of Orlicz capacities.

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#### 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary and  $\mu$  a Radon measure on  $\partial\Omega$ . In this note we consider the problem of finding a function u solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in  $\Omega$  satisfying  $u = \mu$  on  $\partial \Omega$ . Let  $\rho(x) = \text{dist}(x, \partial \Omega)$ , then this problem admits a **weak formulation**: find a function  $u \in L^1(\Omega)$  such that  $e^u \in L^1_\rho(\Omega)$  satisfying

$$\int_{\Omega} \left( -u\Delta\zeta + (e^u - 1)\zeta \right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.2}$$

This equation has been initiated by Grillot and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with q > 1), have been developed by Le Gall [18], Dynkin and Kuznetsov [9], [10], Marcus and Véron [19], [20], by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the main features of the problem with power nonlinearities is the existence of a critical exponent

 $q_c = \frac{N+1}{N-1}$  which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If  $1 < q < q_c$ , then for any  $\mu \in \mathfrak{M}_+(\partial\Omega)$  there exists a unique function  $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$  which satisfies (1.3) in  $\Omega$  and achieves the value  $\mu$  on  $\partial\Omega$  in the following weak sense

$$\int_{\Omega} \left( -u\Delta\zeta + u^q \zeta \right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.4}$$

(ii) If  $q \geq q_c$ , the above problem can be solved if and only if  $\mu$  vanishes on boundary Borel subsets with zero  $C_{\frac{2}{q},q'}$ -capacity. Furthermore a boundary compact set is removable if and only if it has zero  $C_{\frac{2}{q},q'}$ -capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$-\Delta u + e^{u} - 1 = 0 \qquad \text{in } \Omega$$
  

$$u = \mu \qquad \text{on } \partial\Omega,$$
(1.5)

Following the terminology of [4] we say that a measure  $\mu \in \mathfrak{M}(\partial\Omega)$  is **good** if (4.22) admits a weak solution. Let  $P^{\Omega}(x,y)$  (resp.  $G^{\Omega}(x,y)$ ) be the Poisson kernel (resp. the Green kernel) in  $\Omega$  and  $\mathbb{P}^{\Omega}[\mu]$  the Poisson potential of a boundary mesure  $\mu$  (resp.  $\mathbb{G}^{\Omega}[\phi]$  the Green potential of a bounded measure  $\phi$  defined in  $\Omega$ ). A boundary measure  $\mu$  which satisfies

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx). \tag{1.6}$$

is called **admissible**. Since  $\mathbb{P}^{\Omega}[\mu]$  is a supersolution for (1.1), an admissible measure is good (see [25]). Our first result which extends a previous one obtained in [15] is the following.

**Theorem A.** Suppose  $\mu \in \mathfrak{M}(\partial\Omega)$  admits Lebesgue decomposition  $\mu = \mu_S + \mu_R$  where  $\mu_S$  and  $\mu_R$  are mutually singular and  $\mu_R$  is absolutely continuous with respect to the (N-1)-dim Hausdorff measure  $dH^{N-1}$ . If

$$\exp(\mathbb{P}^{\Omega}[\mu_S]) \in L^1(\Omega; \rho dx), \tag{1.7}$$

then  $\mu$  is good.

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modelized on the Legendre transform of  $r \mapsto p(r) := e^r - 1$ . These capacities have been studied by Aissaoui and Benkirane [1] and they inherit

most of the properties of the Bessel capacities. The capacity  $C_{N^{L \ln L}}$  associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this capacity framework we obtain the following types of results:

**Theorem B.** Let  $\mu \in \mathfrak{M}_+(\partial\Omega)$  be a good measure, then  $\mu$  vanishes on boundary Borel subsets E with zero  $C_{N^{L \ln L}}$ -capacity.

A charaterization of positive measures which have the property of vanishing on Borel subsets E with zero  $C_{N^{L \ln L}}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

**Theorem C.** Let  $K \subset \partial \Omega$  be a compact subset with zero  $C_{N^{L \ln L}}$ -capacity. Suppose  $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on K, then u is identically zero.

We give also some extensions of this approach to the equation

$$-\Delta u + e^u - 1 = \mu, (1.8)$$

as well as removability questions for internal singularities of solutions of (1.1). In that case the natural capacity associated to the problem is

$$C_{\Delta^{L \ln L}}(K) = \inf\{\|M[\Delta \eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\}$$

$$\tag{1.9}$$

where M[.] denotes Hardy-Littlewood's maximal function.

**Theorem D.** Let  $\mu \in \mathfrak{M}^b_+(\Omega)$  be a bounded good measure, then  $\mu$  vanishes on boundary Borel subsets E with zero  $C_{\Delta^L \ln L}$ -capacity.

A charaterization of positive measures which have the property of vanishing on Borel subsets E with zero  $C_{N^{L \ln L}}$ -capacity is also provided. We also give below a result of removability of boundary singularities.

**Theorem E.** Let  $K \subset \Omega$  be a compact subset with zero  $C_{\Delta^{L \ln L}}$ -capacity. Suppose  $u \in C(\Omega \setminus K) \cap C^2(\Omega)$  is a positive solution of (1.1) in  $\Omega \setminus K$  which vanishes on  $\partial\Omega$ , then u is identically zero.

This note is essentially derived from the preliminary report [26], written in 2004 and left escheated since this period.

#### 2 Good measures

Proof of Theorem A. For k > 0, set  $\mu_{R,k} = \inf\{k, \mu_R\}$  and denote by  $u_k$  the solution of

$$-\Delta u_k + e^{u_k} - 1 = 0 \qquad \text{in } \Omega$$
  

$$u_k = \mu_S + \mu_{R,k} \qquad \text{on } \partial\Omega.$$
 (2.1)

Such a solution exists because

$$\exp(\mathbb{P}^{\Omega}[\mu_S + \mu_{R,k}]) \le e^k \exp(\mathbb{P}^{\Omega}[\mu_S])$$

by the maximum principle, and (1.7) implies that  $\exp(\mathbb{P}^{\Omega}[\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$ . The sequence  $u_k$  is nondecreasing. Since, for any  $\zeta \in C_c^{1,1}(\bar{\Omega})$ ,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function  $\zeta$  the solution  $\zeta_0$  of

$$\begin{aligned}
-\Delta\zeta_0 &= 1 & \text{in } \Omega \\
\zeta_0 &= 0 & \text{on } \partial\Omega,
\end{aligned} (2.2)$$

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = -\int_{\partial\Omega} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \le c \|\mu\|_{\mathfrak{M}}. \tag{2.3}$$

Thus  $u = \lim_{k \to \infty} u_k$  is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \le c \|\mu\|_{\mathfrak{M}},$$

and the convergence of  $u_k$  and  $e^{u_k}$  to u and  $e^u$  hold respectively in  $L^1(\Omega)$  and  $L^1(\Omega; \rho dx)$  and u satisfies (1.2).

The proof of the next result is inspired by [4].

**Theorem 2.1** The following properties hold:

- (i) If  $\mu \in \mathfrak{M}_{+}^{exp}(\partial\Omega)$  and  $0 \leq \tilde{\mu} \leq \mu$ , then  $\tilde{\mu} \in \mathfrak{M}_{+}^{exp}(\partial\Omega)$ .
- (ii) Let  $\{\mu_n\} \subset \mathfrak{M}_+^{exp}(\partial\Omega)$  be an increasing sequence which converges to  $\mu$  in the weak sense of measures. Then  $\mu \in \mathfrak{M}_+^{exp}(\partial\Omega)$ .
- (iii)  $\mathfrak{M}_{+}^{exp}(\partial\Omega) + L_{+}^{1}(\partial\Omega) = \mathfrak{M}_{+}^{exp}(\partial\Omega).$

*Proof.* (i) Let  $u=u_{\mu}$  be the solution of (4.22) and  $w=\inf\{u,\mathbb{P}^{\Omega}[\tilde{\mu}]\}$ . Since  $\mathbb{P}^{\Omega}[\tilde{\mu}]$  is a supersolution for (1.1), w is a supersolution too. Furthermore w is nonnegative and  $e^{w}-1\in L^{1}(\Omega;\rho dx)$ . By Doob's theorem w admits a boundary trace  $\mu^{*}\in\mathfrak{M}_{+}(\partial\Omega)$  and  $\mu^{*}\leq\tilde{\mu}\leq\mu$ . Let  $w^{*}$  be the solution of

$$\begin{split} -\Delta w^* + e^u - 1 &= 0 & \text{in } \Omega \\ w^* &= \tilde{\mu} & \text{on } \partial \Omega. \end{split}$$

then  $u \ge w \ge w^*$  and [21],

$$\lim_{t} \int_{\partial \Omega_{t}} w^{*}(t, .) \eta dS_{t} = \int_{\partial \Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial \Omega),$$

(here we denote by  $\partial\Omega_t$  the set of  $x\in\Omega$  such that  $\rho(x)=t>0$ ). This implies that the boundary trace of  $w^*$  is  $\tilde{\mu}$  and thus  $\mu^*=\tilde{\mu}$ . Set  $\Omega_t=\{x\in\Omega:\rho(x)>t\}$  and let  $v_t$  we the solution of

$$-\Delta v_t + e^{v_t} - 1 = 0 \qquad \text{in } \Omega_t$$
$$v_t = w \qquad \text{on } \partial \Omega_t.$$

Then  $v_t \leq w$  in  $\Omega_t$ . Furthermore

$$0 < t' < t \Longrightarrow v_{t'} < v_t \text{ in } \Omega_t.$$

Then  $\tilde{u} = \lim_{t\to 0} v_t$  exists, the convergence holds in  $L^1(\Omega)$  and  $e^{v_t} \to e^{\tilde{u}}$  in  $L^1(\Omega; \rho dx)$  (here we use the fact that  $e^w \in L^1(\Omega; \rho dx)$ ). Because

$$\lim_{t \to 0} \int_{\partial \Omega_t} \tilde{w}(t, .) \eta dS_t = \int_{\partial \Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial \Omega),$$

and  $v_t = \tilde{w}$  on  $\partial \Omega_t$ , is follows that  $\tilde{u}$  admits  $\tilde{\mu}$  for boundary trace and thus  $\tilde{u} = u_{\tilde{\mu}}$ .

(ii) Let  $u_n = u_{\mu_n}$  be the solutions of (4.22) with boundary value  $\mu_n$ . The sequence  $\{u_n\}$  is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0)dx = -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \le -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \tag{2.4}$$

we conclude as in the proof of Theorem 1, that  $u_n$  increases and converges to a solution  $u = u_\mu$  of (4.22) with boundary value  $\mu$ .

(iii) In the proof of (i) we have actually used the following result : Let w be a nonnegative supersolution of (1.1) such that  $e^w \in L^1(\Omega; \rho dx)$  and let  $\mu \in \mathfrak{M}_+(\partial \Omega)$  be the boundary trace of w. Then  $\mu$  is good. Let  $f \in L^1_+(\partial \Omega)$  and  $\mu$  be an good measure. We denote by  $u = u_\mu$  the solution of (4.22). For k > 0, set  $f_k = \min\{k, f\}$ . The function  $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$  is a nonnegative supersolution, and, since  $\mathbb{P}^\Omega[f_k] \leq k$ ,  $e^{w_k} \in L^1(\Omega; \rho dx)$ . Furthermore the boundary trace of  $w_k$  is  $\mu + f_k$ . Therefore  $\mu + f_k$  is good. We conclude by II that  $\mu + f$  is good

Remark. The assertions (i) and (ii) in Theorem 1 are still valid if we replace  $r \mapsto e^r - 1$  by any continuous nondecreasing function f vanishing at 0.

### 3 The Orlicz space framework

#### 3.1 Orlicz capacities

The set  $M^{exp}(\partial\Omega)$  of nonnegative measures  $\mu$  on  $\partial\Omega$  such that

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx) \tag{3.1}$$

is not a linear space, but it is a convex subset of  $\mathfrak{M}_{+}(\partial\Omega)$ . The role of this set comes from the fact that any measure in  $M^{exp}(\partial\Omega)$  is good. Put

$$p(t) = \operatorname{sgn}(s)(e^{s} - 1), \ P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s|+1), \ P^*(t) = (|t|+1) \ln(|t|+1) - |t|.$$

Then P and  $P^*$  are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \le P(x) + P^*(y) \qquad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if  $x = \bar{p}(y)$  or y = p(x). It is classical to define

$$M_P(\Omega; \rho dx) = \{ \phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx) \}, \tag{3.2}$$

$$M_{P^*}(\Omega; \rho dx) = \{ \phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx) \}.$$
 (3.3)

The Orlicz spaces  $L_P(\Omega; \rho dx)$  and  $L_{P^*}(\Omega; \rho dx)$  are the vector spaces spanned respectively by  $M_P(\Omega; \rho dx)$  and  $M_{P^*}(\Omega; \rho dx)$ . They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_{\rho}}} = \inf\left\{k > 0 : \int_{\Omega} P\left(\frac{\phi}{k}\right) \rho dx \le 1\right\}. \tag{3.4}$$

and

$$\|\phi\|_{L_{P_{\rho}^*}} = \inf\left\{k > 0 : \int_{\Omega} P^*\left(\frac{\phi}{k}\right)\rho dx \le 1\right\}. \tag{3.5}$$

Furthermore the Hölder-Young inequality asserts [16]

$$\left| \int_{\Omega} \phi \, \psi \, \rho \, dx \right| \leq \|\phi\|_{L_{P_{\rho}}} \|\psi\|_{L_{P_{\rho}^*}} \quad \forall (\phi, \psi) \in L_P(\Omega; \rho dx) \times L_{P^*}(\Omega; \rho dx). \tag{3.6}$$

Since  $P^*$  satisfies the  $\Delta_2$ -condition,  $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$  and  $L_P(\Omega; \rho dx)$  is the dual space of  $L_{P^*}(\Omega; \rho dx)$ , (see [12], [1]). Furthermore, since

$$\frac{|a|\ln(1+|a|)}{2} \le P^*(a) \le |a|\ln(1+|a|) \quad \forall a \in \mathbb{R},$$

the space  $L_{P^*}(\Omega; \rho dx)$  is associated with the class  $L \ln L(\Omega; \rho dx)$  and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube  $Q_0$  containing  $\bar{\Omega}$ , with sides parallel to the axes. If  $f \in L^1(\Omega)$  we denote by  $\tilde{f}$  its extension by 0 in  $Q_0 \setminus \Omega$  and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f|(y) dy : Q \in \mathcal{Q}_x \right\}$$

where  $Q_x$  denotes the set of all cubes containing x and contained in  $Q_0$ , with sides parallel to the axes. Thus

$$||f||_{L \ln L_{\rho}} := \int_{Q_0} M_{Q_0}[f](x) \rho dx \approx ||f||_{L_{P_{\rho}^*}}.$$
 (3.7)

If  $\ell$  is a continuous linear functional on  $L \ln L(\Omega; \rho dx)$ , there exist a measurable function  $g_{\ell}$  and some  $\theta > 0$  such that

$$\begin{cases} \ell(f) = \int_{\Omega} g_{\ell} f dx & \forall f \in L \ln L(\Omega; \rho dx), \\ \int_{\Omega} e^{\theta|g_{\ell}|} \rho dx < \infty. \end{cases}$$

This can be seen as a consequence of Young's inequality.

**Definition 3.1** The space of all measures on  $\partial\Omega$  such that

$$\mathbb{P}^{\Omega}[\mu] \in L_P(\Omega; \rho dx)$$

is denoted by  $B^{\exp}(\partial\Omega)$ , with norm

$$\|\mu\|_{B^{\exp}} = \|\mathbb{P}^{\Omega}[\mu]\|_{L_{P_{-}}}.$$
 (3.8)

The subset of measures such that

$$\exp(\mathbb{P}^{\Omega}[\mu]) \in L^1(\Omega; \rho dx)$$

is denoted by  $\mathfrak{M}^{\exp}(\partial\Omega)$ .

The following result follows immediately from the definition of the Luxemburg norm.

**Proposition 3.2** If  $\mu \in B^{\exp}(\partial \Omega)$  there exists  $a_0 > 0$  such that  $a\mu \in \mathfrak{M}^{\exp}(\partial \Omega)$  for all  $0 \le a < a_0$ . Conversely, if  $\mu \in \mathfrak{M}^{\exp}(\partial \Omega)$ , then  $a\mu \in B^{\exp}(\partial \Omega)$  for all a > 0.

The analytic charaterization of  $B^{\exp}(\partial\Omega)$  can be done in introducing the space of normal derivatives of Green potentials of  $L \ln L$  functions:

$$N^{L \ln L}(\partial \Omega) = \left\{ \eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \in L \ln L(\Omega; \rho dx) \right\}. \tag{3.9}$$

where  $\rho^*$  is a smooth positive function with value  $\rho$  in a neighborhood of  $\partial\Omega$ . Then

$$\left| \int_{\partial\Omega} \eta d\mu \right| = \left| \int_{\Omega} \mathbb{P}^{\Omega}[\mu] \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) dx \right| \le \left\| \mathbb{P}^{\Omega}[\mu] \right\|_{L_{P_{\rho}}} \left\| \rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) \right\|_{L_{P_{\rho}^*}}. \quad (3.10)$$

Notice that the actual regularity of the  $\eta$  function is not clear, although

$$\rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L \ln L(\Omega; \rho dx) \Longrightarrow \Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L \ln L(\Omega).$$

If we take, as a norm on  $N^{L \ln L}(\partial \Omega)$ 

$$\|\eta\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta])\|_{L_{P_*^*}},$$
 (3.11)

and define the  $C_{N^L \ln L}$ -capacity of a compact subset K of  $\partial \Omega$  by

$$C_{N^{L \ln L}}(K) = \inf\{\|\eta\|_{N^{L \ln L}} : \eta \in C^2(\partial\Omega), 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\}.$$
(3.12)

Considering the bilinear form  $\mathcal{H}$  on  $L_{P_{\rho}^*}(\partial\Omega) \times L_{P_{\rho}}(\partial\Omega)$ 

$$\mathcal{H}(\eta,\mu) := -\int_{\Omega} \mathbb{P}^{\Omega}[\mu] \Delta(\rho^* \mathbb{P}^{\Omega}[\eta]) dx \tag{3.13}$$

then

$$\mathcal{H}(\eta,\mu) = -\int_{\Omega} \int_{\partial\Omega} P^{\Omega}(x,y) d\mu(y) \Delta(\rho^* \mathbb{P}^{\Omega}[\eta])(x) dx$$

$$= -\int_{\partial\Omega} \int_{\Omega} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta])(x) P^{\Omega}(x,y) dx d\mu(y).$$
(3.14)

It is classical to define

$$C^*_{N^{L \ln L}}(K) = \sup \{ \mu(K) : \mu \in \mathfrak{M}_+(\partial \Omega), \mu(K^c) = 0, \left\| \mathbb{P}^{\Omega}[\mu] \right\|_{L_{P_o}} \leq 1 \}. \tag{3.15}$$

The following result due to Fuglede [13] is a consequence of the Kneser-Fan min-max theorem.

**Proposition 3.3** For any compact set  $K \subset \partial \Omega$ , there holds

$$C_{N^{L \ln L}}^*(K) = C_{N^{L \ln L}}(K).$$
 (3.16)

As a direct consequence of (3.10), we have the following

**Proposition 3.4** If  $\mu \in B^{\exp}_+(\partial\Omega)$ , it does not charge Borel subsets with  $C_{N^{L \ln L}}$ -capacity zero.

Remark. We conjecture that the following result is true: Let  $\mu \in \mathfrak{M}_+(\partial\Omega)$ , which does not charge Borel sets with  $C_{N^L\ln L}$ -capacity zero. Then there exists an increasing sequence  $\{\mu_n\} \subset B^{exp}(\partial\Omega)$  which converges weakly to  $\mu$ . The construction of the proof can be adapted from [11, Th8], [3, Lemma 4.2], except one point which is not clear: If  $\eta \in N^{L\ln L}(\partial\Omega)$  does it admit a representative  $\tilde{\eta}$  quasi-continuous with respect to the capacity  $C_{N^{L\ln L}}$ ? If this statement holds true the remaining of their proof based on the study of the mapping h defined by

$$h(\eta) = \int_{\partial\Omega} \tilde{\eta}^+ d\mu \qquad \forall \eta \in N^{L \ln L}(\partial\Omega).$$

can be easily adapted.

#### 3.2 Good measures and removable sets

As we have already seen it, a measure in  $B_+^{exp}(\partial\Omega)$  is good, and does not charge Borel subsets of  $C_{N^{L\ln L}}$ -capacity zero. The following result is a slight extension of a result of Grillot-Véron, with a proof which inherits some observations of Brezis-Marcus-Ponce.

Proof of Theorem B. Let K be a compact subset with  $C_{N^{L \ln L}}$ -capacity zero. There exist a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of K and

$$\lim_{n \to \infty} \|\eta_n\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L_{P_*^*}} = 0.$$
 (3.17)

Take  $\rho^* \mathbb{P}^{\Omega}[\eta_n]$ ) as a test function, then

$$\int_{\Omega} \left( -u\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) + (e^u - 1)\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right) dx = -\int_{\partial} \Omega \frac{\partial(\rho^* \mathbb{P}^{\Omega}[\eta_n])}{\partial \nu} d\mu$$

Since 
$$\frac{\partial(\rho^*\mathbb{P}^{\Omega}[\eta_n]))}{\partial\nu} = \eta_n$$
 and  $\mu > 0$ , there holds  $-\int_{\partial\Omega} \frac{\partial(\rho^*\mathbb{P}^{\Omega}[\eta_n]))}{\partial\nu} d\mu \geq \mu(K)$ .

Furthermore

$$\left| \int_{\Omega} u \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) dx \right| \le \|u\|_{L_{P_{\rho}}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L_{P_{\rho}^*}}. \tag{3.18}$$

Then

$$\mu(E) \le \int_{\Omega} (e^{u} - 1) \rho^* \mathbb{P}^{\Omega}[\eta_n] dx + \|u\|_{L_{P_{\rho}}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])\|_{L_{P_{\rho}^*}}.$$

By the same argument as in [4],  $\lim_{n\to\infty} \rho^* \mathbb{P}^{\Omega}[\eta_n] = 0$ , a.e. in  $\Omega$ , and there exists a nonnegative  $L^1_{\rho}$ -function  $\Phi$  such that  $0 \leq \rho^* \mathbb{P}^{\Omega}[\eta_n] \leq \Phi$ . By (3.17), (3.18) and Lebesgue's theorem,  $\mu(E) = 0$ .

**Definition 3.5** A subset  $E \subset \partial \Omega$  is said removable for equation (1.1), if and only if any positive solution  $u \in C^2(\Omega)$  of (1.1) in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus E$  and vanishes on  $\partial \Omega \setminus E$ , is identically zero.

Proof of Theorem C. Let  $u \in C(\bar{\Omega} \setminus K)$  be a solution of (1.1) which is zero on  $\partial \Omega \setminus K$ . Let  $\{\eta_n\} \subset C^2(\partial \Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of K and (3.17) holds. Put  $\theta_n = 1 - \eta_n$ . Put  $\rho_K(x) = \text{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate and the fact that u vanishes on  $K^c$ , there holds

$$u(x) \le C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function  $\zeta_n = \rho^* \mathbb{P}^{\Omega}[\theta_n]$  is an admissible test function for u, and

$$\int_{\Omega} \left( -u\Delta\zeta_n + (e^u - 1)\zeta_n \right) dx = 0.$$

Clearly  $\mathbb{P}^{\Omega}[\theta_n] = 1 - \mathbb{P}^{\Omega}[\eta_n]$  and

$$\Delta \zeta_n = \Delta \rho^* - \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])$$

Inasmuch we can modify  $\rho^*$  in order to have  $-\Delta \rho^* \geq 0$ , in which case  $\rho^* = \rho$  near  $\partial \Omega$  is replaced by  $\rho^* \approx \rho$ , we derive

$$-\int_{\Omega} u\Delta\zeta_n \, dx = -\int_{\Omega} \zeta_n^{-1} \Delta\zeta_n \, u\zeta_n dx$$
$$\geq -2^{-1} \int_{\Omega} (e^u - 1 - u)\zeta_n \, dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])) \, \zeta_n dx,$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \le C |r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n \, dx \le 2C \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx, \quad (3.19)$$

since  $\zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])|$ . Furthermore

$$\ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|)) = -\ln \rho + \ln(\rho + \rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|)$$
  
$$\leq -\ln \rho + \ln(1 + \rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|)$$

But (we can assume  $\rho \leq 1$ )

$$\int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx 
\leq - \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho dx + \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx,$$

and

$$\begin{split} & \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx \\ & = \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| > 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx \\ & \leq \int_{\{|\Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n])| \leq 1\}} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln \rho^{-1} dx + \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln (1 + \rho^{-1} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx \end{split}$$

But

$$\lim_{n \to \infty} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \to \infty} \int_{\Omega} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right| \ln(1 + \rho^{-2} \left| \Delta(\rho^* \mathbb{P}^{\Omega}[\eta_n]) \right|) dx = 0$$
 (3.20)

Using (4.30), we derive u = 0.

Conversely, assume that  $C_{N^{L \ln L}}(K) > 0$ . By Proposition 3.3 there exists a non negative non-zero measure  $\mu \in \mathfrak{M}_{+}(\partial\Omega)$  such that  $\mu(K^c) = 0$  in the space  $B_{+}^{exp}(\partial\Omega)$ . This means that  $\theta\mu \in M_{+}^{exp}(\partial\Omega)$  for some  $\theta > 0$ . Thus problem (4.22) admits a solution.

By Proposition 3.2 and Theorem ?? we have a partial characterization of measures for which problems (4.22) admits a solution and K is not removable.

**Theorem 3.6** If a measure  $\mu$  is good there exists an increasing sequence of measures  $\{\mu_n\}$  which converges to  $\mu$  satisfying

$$\int_{\Omega} \exp(\theta_n \mathbb{P}^{\Omega}[\mu_n]) \rho dx < \infty \tag{3.21}$$

for some  $\theta_n > 0$ .

Several questions can be adressed

- 1- If a singular measure  $\mu$  is good does it exist an increasing sequence of  $\{\mu_n\}$  converging to  $\mu$  such that (3.21) holds with  $\theta_n = 1$ ?
- 2- If a measure  $\mu$  does not charge Borel sets with  $C^{L \ln L}$ -capacity zero, doest it exist  $\theta > 0$  such that  $\theta \mu \in \mathfrak{M}^{exp}(\partial \Omega)$ ?
- 3- If a singular measure  $\mu$  is good, then  $(1 \delta)\mu \in M^{exp}(\partial\Omega)$  for any  $\delta > 0$ ?

### 4 Further extensions and open questions

A part of the above construction could be extended to problems with more general nonlinearity such as

#### 4.1 Boundary data measures

$$-\Delta u + P(u) = 0 \quad \text{in } \Omega$$
  
 
$$u = \mu \quad \text{on } \partial\Omega,$$
 (4.22)

where P is a convex increasing function vanishing at 0 and such that  $\lim_{r\to\infty} P(r)/r = \infty$ : In Theorem 1-P, (1.7) should be replaced by

$$P(\mathbb{P}^{\Omega}[\mu_n]) \in L^1(\Omega; \rho \, dx). \tag{4.23}$$

In Theorem 2-P, (i) and (ii) still hold. For simplicity we assume that P is a N-function in the sense of Orlicz spaces

$$P(r) = \int_0^r p(s)ds$$

where p is increasing and vanishes at 0. Let  $P^*$  be the conjugate N-function,  $L_P(\Omega; \rho \, dx)$  and  $L_{P^*}(\Omega; \rho \, dx)$  the corresponding Orlicz spaces endowed with the Luxenburg norms. Then Proposition 1-P is valid, provided the space  $B^P(\partial\Omega)$  and  $M^P(\partial\Omega)$  are accordingly defined with the following notations:

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta]) \in L_{P^*}(\Omega; \rho \, dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \|\rho^{-1}\Delta(\rho^*\mathbb{P}^{\Omega}[\eta])\|_{L_{P^*}}$$

and the corresponding capacity  $C_{NP^*}$ . It is still likely that Theorem 3-P, 4-P hold. The proof of Theorem 5-P should be valid without any major modification. **However**, it appears that the characterization of removable sets cannot be adapted

without further properties of the function  $P^*$  like the  $\Delta_2$ -condition. Such a condition holds usually when P has a power-like growth (> 1) and a logarithmic type growth.

#### 4.2 Internal measures

Many of the above techniques can be extended to the following types of problem in which  $\mu \in \mathfrak{M}^b_+(\Omega)$ :

$$-\Delta u + e^{u} - 1 = \mu \qquad \text{in } \Omega$$
  
 
$$u = 0 \qquad \text{on } \partial\Omega,$$
 (4.24)

and

$$-\Delta u + P(u) = \mu \qquad \text{in } \Omega$$
  
 
$$u = 0 \qquad \text{on } \partial\Omega.$$
 (4.25)

Remark. Note that many interesting results can be found in [2] where the analysis of  $\mu$  is made by comparison with the Hausdorff measure in dimension (N-2),  $\mathcal{H}^{N-2}$ . It is proved in particular that if a measure  $\mu$  satisfies  $\mu \leq 4\pi\mathcal{H}^{N-2}$ , then problem (4.22) admits a solution, while if  $\mu$  charges some Borel set A with Hausdorff diemension less than N-2, no solution exists.

We denote by  $\mathfrak{M}_{+}^{exp}(\Omega)$  the set of good measures for (4.24) and define the classes  $M_{P}(\Omega)$  and  $M_{P^{*}}(\Omega)$  similarly to  $M_{P}(\Omega; \rho dx)$  and  $M_{P^{*}}(\Omega; \rho dx)$  except that the measure  $\rho dx$  is replaced by the Lebesgue measure dx. The Orlicz spaces  $L_{P}(\Omega)$  and  $L_{P^{*}}(\Omega)$  are defined from  $M_{P}(\Omega)$  and  $M_{P^{*}}(\Omega)$  and endowed with the respective Luxemburg norms  $\| \cdot \|_{P}$  and  $\| \cdot \|_{P^{*}}$ . Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta \mathbb{G}^{\Omega}[\mu] dx \right| = \left| \int_{\Omega} \mathbb{G}^{\Omega}[\mu] \Delta \eta dx \right| \le \left\| \mathbb{G}^{\Omega}[\mu] \right\|_{L_{P}} \left\| \Delta \eta \right\|_{L_{P^{*}}}. \tag{4.26}$$

for  $\eta \in C_c^{1,1}(\bar{\Omega})$ . We define the  $C_{\Delta^{L \ln L}}$ -capacity of a compact subset K of  $\partial \Omega$  by

$$C_{\Delta^{L \ln L}}(K) = \inf\{\|\Delta\eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\},$$

$$(4.27)$$

The capacity  $C_{\Delta^{L \ln L}}$  can be characterized using the Hardy-Littlewood maximal function  $f \mapsto M_{Q_0}[f]$  since

$$||f||_{L \ln L} := \int_{Q_0} M_{Q_0}[f](x) dx \approx ||f||_{L_{P^*}}.$$
 (4.28)

Thus Proposition 3.4 and Theorem B are valid under the form

**Proposition 4.1** If  $\mu \in B^{\exp}_+(\Omega)$ , it does not charge Borel subsets with  $C_{\Delta^{L \ln L}}$ -capacity zero.

**Theorem 4.2** Let  $\mu \in \mathfrak{M}_+(\Omega)$  be a good measure, then  $\mu$  vanishes on Borel subset E with zero  $C_{\Delta^L \ln L}$ -capacity.

Theorem C has the following counter part

**Theorem 4.3** Let  $K \subset \Omega$  be compact. Any solution of

$$-\Delta u + e^{u} - 1 = 0 \qquad in \quad \Omega \setminus K$$
  

$$u = 0 \qquad on \quad \partial \Omega,$$
(4.29)

vanishes identically in  $\Omega$  if and only if  $C_{\Delta^{L \ln L}}(K) = 0$ .

*Proof.* Let  $u \in C(\Omega \setminus K)$  be a solution of (1.1) which is zero on  $\partial\Omega$ . Let  $\{\eta_n\} \subset C^2(\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood  $\mathcal{V}$  of K and (3.17) holds. Put  $\rho_K(x) = \operatorname{dist}(x, K)$ . Then, as a consequence of Keller-Osserman estimate for this type of nonlinearity (see [24]), there holds

$$u(x) \le C \ln(2/\rho_K(x)) + D.$$

Put  $\theta_n = 1 - \eta_n$ . Then the function  $\zeta_n = \phi_1 \theta_n$  ( $\phi_1$  being the first eigenfunction of  $-\Delta$ ) is an admissible test function for u, and

$$\int_{\Omega} \left( -u\Delta \zeta_n + (e^u - 1)\zeta_n \right) dx = 0.$$

We derive

$$-\int_{\Omega} u\Delta\zeta_n \, dx = -\int_{\Omega} \zeta_n^{-1} \Delta\zeta_n \, u \, dx$$
$$\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \, dx - \int_{\Omega} Q(\Delta(\zeta_n) \, dx.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u)\zeta_n \, dx \le 2C \int_{\Omega} |\Delta \zeta_n| \ln(1 + |\Delta \zeta_n|) dx,\tag{4.30}$$

Since the right-hand side goes to zero when  $n \to \infty$ , the conclusion follows.

*Remark.* The characterization of the  $C_{\Delta^{L \ln L}}$ -capacity is not simple, however, by a result of [6, Th1], there holds

$$\left\|D^{2}\eta\right\|_{L^{1,\infty}} \leq C \left\|\Delta\eta\right\|_{L \ln L} \qquad \forall \eta \in C_{c}^{1,1}(\overline{\Omega}) \tag{4.31}$$

where  $L^{1,\infty}(\Omega)$  denotes the weak  $L^1$ -space, that is the space of all measurable functions f defined in  $\Omega$  satisfying

$$meas\left(\left\{x \in \Omega : |f(x)| > t\right\}\right) \le \frac{c}{t}, \qquad \forall t > 0 \tag{4.32}$$

and  $||f||_{L^{1,\infty}}$  is the smallest constant such that (4.32) holds.

#### 4.2.1 Reduced measures

What are the reduced measures both for the boundary and internal problems (4.22) (resp. (4.25))? A projection onto the closure of the sets  $\mathfrak{M}^{exp}(\partial\Omega)$  (resp.  $\mathfrak{M}^{exp}(\Omega)$ ) of positive measures in  $\partial\Omega$  (resp.  $\Omega$ ) satisfying

$$\int_{\Omega} \exp\left(\mathbb{P}^{\Omega}[\mu]\right) \rho dx < \infty$$

(resp.

$$\int_{\Omega} \exp\left(\mathbb{G}^{\Omega}[\mu]\right) dx < \infty.)$$

The definition of the projection is not clear, although an important fact is that it ensures uniqueness. This conjectures could be extended to problems involving operator  $u \mapsto -\Delta u + P(u)$ .

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