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# A nonlocal energy functional in pseudo-plasticity

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**Abstract.** We obtain a nonlocal functional as the variational limit of an integral functional associated with the strain energy of a pseudo-plastic material reinforced by a  $\varepsilon$ -periodic distribution of pseudo plastic fibers. We begin by studying separately the variational limit behavior, when  $\varepsilon$  goes to zero, of the structure made of the soft material without fibers, and of the structure constituted by the distribution of the small fibers. We show that the complete structure is, in the sense of the  $\Gamma$ -convergence, equivalent to a new homogeneous structure whose functional energy is the epigraphical sum of the limit integral functionals energies modeling each two previous structures.

Keywords: homogenization,  $\Gamma$ -convergence, convex functional of measures, pseudo-plasticity

## 1. Introduction

We are interested in the determination of the macroscopic behavior of the structure ( $\mathcal{S}$ ) constituted by a matrix ( $\mathcal{M}$ ) made of a soft pseudo-plastic material reinforced by a periodic distribution ( $\mathcal{T}$ ) of pseudo-plastic fibers. The interior of the reference configuration of ( $\mathcal{S}$ ) is a cylinder  $\mathcal{O} = \widehat{\mathcal{O}} \times (0, L)$  where  $\widehat{\mathcal{O}}$  is a domain of  $\mathbf{R}^2$ . Let  $T_\varepsilon = \bigcup_{z \in \mathbf{Z}^2} \varepsilon(D + z) \times (0, L)$  where  $D$  is an open subset of  $\mathbf{R}^2$  satisfying  $D \Subset \widehat{Y} := (0, 1)^2$ ; the distribution of fibers occupies  $\mathcal{O} \cap T_\varepsilon$  while the matrix lies in  $\mathcal{O} \setminus T_\varepsilon$ . This structure ( $\mathcal{S}$ ) is clamped on the part  $\Gamma_0 = \widehat{\mathcal{O}} \times \{0\}$  of its boundary and subjected to applied body forces  $\mathcal{L}$  in  $\mathcal{C}_0(\mathcal{O}, \mathbf{R}^3)$  where  $\mathcal{C}_0(\mathcal{O}, \mathbf{R}^3)$  denotes the space of  $\mathbf{R}^3$ -valued continuous functions on  $\overline{\mathcal{O}}$  vanishing on  $\partial\mathcal{O}$ .

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We aim to obtain an equivalent problem when  $\varepsilon$  goes to zero of each three following variational problems:

$$\begin{aligned} (\mathcal{P}_{F_\varepsilon^v}) \quad & \inf \left\{ F_\varepsilon^v(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu: v \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}, \\ (\mathcal{P}_{G_\varepsilon}) \quad & \inf \left\{ G_\varepsilon(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu: \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}, \\ (\mathcal{P}_{H_\varepsilon}) \quad & \inf \left\{ H_\varepsilon(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu: \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}, \end{aligned}$$

where  $F_\varepsilon^v$ ,  $G_\varepsilon$  and  $H_\varepsilon$  are three integral functionals defined on the space  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  of bounded Borel measure by:

$$\begin{aligned} F_\varepsilon^v(\mu) &= \begin{cases} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu) \, dx & \text{if } \mu \in W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3), \mu = v \text{ on } \mathcal{O} \cap T_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases} \\ G_\varepsilon(\mu) &= \begin{cases} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon \mu) \, dx & \text{if } \mu|_{(\mathcal{O} \setminus T_\varepsilon)} = 0, \mu|_{T_\varepsilon \cap \mathcal{O}} \in W_{\Gamma_0}^{1,1}(T_\varepsilon \cap \mathcal{O}, \mathbf{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \\ H_\varepsilon(\mu) &= \begin{cases} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu) \, dx + \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon \mu) \, dx & \text{if } \mu \in W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The space  $W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3)$  is the subset of all the functions  $u$  in  $W^{1,1}(\mathcal{O}, \mathbf{R}^3)$  such that  $u = 0$  on  $\Gamma_0$  in the trace sense,  $v$  is a given function in  $W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3)$ ,  $\nabla_\varepsilon$  denotes the distributional derivative in  $\mathcal{D}'(\mathcal{O} \cap T_\varepsilon, \mathbf{R}^3)$ , and  $\text{spt}(\mu)$  the support of the measure  $\mu$ . In order to consider large deformations, the bulk energy density functions  $f$ ,  $g$  are not assumed to be convex. They are assumed to be globally Lipschitz, so that they have linear growth at infinity. To shorten notations, we identify any measure  $\mu$  restricted to any open set  $\Omega$  of  $\mathbf{R}^3$  with its density  $d\mu/dx$  as soon as  $\mu$  is absolutely continuous with respect to the Lebesgue measure restricted to  $\Omega$ .

The first problem ( $\mathcal{P}_{F_\varepsilon^v}$ ) describes the equilibrium configuration of the mechanical structure  $\mathcal{O} \setminus T_\varepsilon$  clamped along  $\widehat{\mathcal{O}}$ , with a prescribed displacement  $v$  along the boundary  $\partial T_\varepsilon \cap \mathcal{O}$ . The argument  $\varepsilon \nabla \mu$  accounts for the softness of the material occupying  $\mathcal{O} \setminus T_\varepsilon$ . The second problem ( $\mathcal{P}_{G_\varepsilon}$ ) describes the equilibrium configuration of the  $\varepsilon$ -periodic distribution ( $\mathcal{T}$ ) of very thin fibers clamped on  $\widehat{\mathcal{O}}$  and free on the remaining part of their boundaries. Now, assuming that the two previous structures are perfectly stuck together, we obtain the third problem ( $\mathcal{P}_{H_\varepsilon}$ ), which was studied in [7] and [8] with a more complex geometry on fibers but with convex integrands  $f$  and  $g$  of growth order  $p$  greater than 1. Here, the linear growth seems necessitate a measure framework.

Let us make precise what we call *equivalent problem*. We actually want to compute the  $\Gamma$ -limits  $F^v$ ,  $G$  and  $H$  of  $F_\varepsilon^v$ ,  $G_\varepsilon$  and  $H_\varepsilon$  respectively when  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  is equipped with the weak topology  $\sigma(\mathbf{M}(\mathcal{O}, \mathbf{R}^3), \mathcal{C}_c(\mathcal{O}, \mathbf{R}^3))$ . The functionals  $F^v$  and  $G$  appear to be convex integrals of measure. According to the variational properties of  $\Gamma$ -convergence (see Proposition 12 below), the limit problems obtained in this sense, describe the equilibrium of deformable bodies filling the closure of  $\mathcal{O}$  as reference configuration and subjected to the loading  $\mathcal{L}$ .

We show that  $F^\nu$  is a gradient free energy defined for every  $\mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  by

$$F^\nu(\mu) = \int_{\mathcal{O}} f_0(\mu - \nu),$$

where, for every  $a \in \mathbf{R}^3$ ,

$$f_0(a) = \inf \left\{ \int_{\widehat{Y} \setminus D} f^{**}(\widehat{\nabla} u, 0) \, d\hat{x} : u \in \widehat{\text{adm}}(a) \right\},$$

$$\widehat{\text{adm}}(a) = \left\{ u \in W_{\#}^{1,1}(\widehat{Y}, \mathbf{R}^3) : \int_{\widehat{Y}} u \, d\hat{x} = a, u = 0 \text{ on } D \right\},$$

$\widehat{\nabla} u$  denotes the tangential gradient of  $u$  and  $W_{\#}^{1,1}(\widehat{Y}, \mathbf{R}^3)$  is the set of  $\widehat{Y}$ -periodic functions in  $W^{1,1}(\widehat{Y}, \mathbf{R}^3)$ .

The functional  $G$  corresponds to a continuous distribution of strain energies of elastic strings  $D \times (0, L) + \hat{x}$  (see [2]) when  $\hat{x}$  runs through the basis  $\widehat{\mathcal{O}}$ , more precisely

$$G(\nu) = \begin{cases} |D| \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \nu}{\partial x_3} \right) + \int_{\Gamma_0} (g^\perp)^{**, \infty}(\gamma(\nu)) & \text{if } \nu \in \mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3)), \\ +\infty & \text{otherwise.} \end{cases}$$

The density  $(g^\perp)^{**}$  is the convex envelope of the function  $g^\perp$ , defined for every  $a \in \mathbf{R}^3$  by

$$g^\perp(a) = \inf \{ g((M, a)) : M \in \mathbf{M}^{2 \times 3} \},$$

and  $(g^\perp)^{**, \infty}$  is its recession function (see Section 2). Note that in general,  $g^\perp$  is not convex (see [2] with  $p = 2$ ). The notation  $\gamma(\nu)$  stands for a suitable measure in  $\mathbf{M}(\widehat{\mathcal{O}} \times (-\infty, L), \mathbf{R}^3)$  concentrated on  $\Gamma_0$  that we will call the *orthogonal trace* of the measure  $\nu$  on  $\Gamma_0$  (see Definition 2 and Proposition 6). The convex functions of measures entering the definitions of  $F^\nu$  and  $G$  are taken in the sense of [18].

We finally establish that the stored energy limit of the whole structure is the infimal convolution (or epigraphical sum) of the two functionals  $F^0$  and  $G_{|D|}$  on  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ : for all  $\mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$

$$\begin{aligned} H(\mu) &= \inf \{ F^\nu(\mu) + G(|D|\nu) : \nu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \} \\ &= \inf \left\{ \int_{\mathcal{O}} f_0(\mu - \nu) + G(|D|\nu) : \nu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\} \\ &:= F^0 \overset{\dagger}{\circlearrowleft} G_{|D|}(\mu). \end{aligned}$$

It is worth noticing that the functionals  $F^\nu$  and  $G$  turn out to be convex integrals of a measure, while  $H$  is a nonlocal convex functional. The identification of this limit energy functional as an infimal convolution is capital for the mechanical understanding of the macroscopic behavior: it appears as two homogenized media, connected in series. Although our main goal was to determine the macroscopic behavior of the *whole* structure ( $S$ ), it is this actual characterization that leads us to a strategy different from that of [7,8] by studying previously and separately the two media making up ( $S$ ). The limit energy functional  $H$  is nonlocal, the state of the homogenized structure is described by a global state variable (see [17]):

the field of displacement which is actually a measure on  $\mathcal{O}$ . Nevertheless, the macroscopic behavior can also be described by the displacement  $\mu$ , and an additional internal or hidden variable  $\nu$  which accounts for the microstructure of  $(S)$ . In this case, the total free energy of this nondissipative medium has a density function which reads as

$$h(\mu, \nu) = f_0(\mu - \nu) + |D|(g^\perp)^{**} \left( \frac{\partial \nu}{\partial x_3} \right),$$

and involves a space derivative of the internal variable [16].

The paper is organized as follows. In Section 2 we establish the  $\Gamma$ -convergence of the functional  $F_\varepsilon^v$  to the functional  $F^v$ . Section 3 is devoted to the  $\Gamma$ -convergence of  $G_\varepsilon$  to  $G$ . We deduce in Section 4 the limit energy of the complete structure. Finally, we recall the notion of  $\Gamma$ -convergence and give some technical results in Appendices 1, 2, 3. From a technical point of view, we use alternative ways to those used in [7] and [8]. Indeed, for  $p = 1$ , the two-scale convergence scheme seems to require differentiability and positively 1-homogeneity of  $f^{**}$  (see [3]). Nevertheless our technique also applies to the case  $p > 1$ .

## 2. The limit problem associated with the soft material structure

We denote by  $a \cdot b$  the scalar product of two elements  $a$  and  $b$  in  $\mathbf{R}^3$  and by  $A : B$  the Hilbert–Schmidt scalar product of  $A$  and  $B$  in the set  $\mathcal{M}_3$  of  $3 \times 3$  matrices.

We assume the following growth conditions on the density  $f$ : there exists two positive constants  $\alpha$  and  $\beta$  such that, for every  $3 \times 3$ -matrix  $M$

$$\alpha(|M| - 1)^+ \leq f(M) \leq \beta(1 + |M|), \quad (1)$$

where  $(\cdot)^+$  denotes the positive part. Moreover, we assume that  $f$  satisfies the Lipschitzian condition: there exists a positive constant  $l$  such that, for all  $M, M'$  in  $\mathcal{M}_3$

$$|f(M) - f(M')| \leq l|M - M'|. \quad (2)$$

Note that when  $f$  is quasiconvex this condition is automatically satisfied (see B. Dacorogna [11]). Finally, as there is no energy when no displacement occurs, we assume  $f(0) = 0$ . Note also that hypotheses (1), (2) include the function  $M \mapsto \sqrt{1 + |M|^2} - 1$  which is the main example of nonnegative  $\mathcal{C}^\infty$  function vanishing at 0 and with linear growth.

Let us recall that  $F_\varepsilon^v : \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \rightarrow \mathbf{R} \cup \{+\infty\}$  is defined by

$$F_\varepsilon^v(\mu) = \begin{cases} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu) dx & \text{if } \mu \in W_{T_0}^{1,1}(\mathcal{O}, \mathbf{R}^3), \mu = v \text{ on } \mathcal{O} \cap T_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

where  $v$  is a given function in  $W_{T_0}^{1,1}(\mathcal{O}, \mathbf{R}^3)$ .

We begin by giving various expressions of the expected limit density  $f_0$  given in the Introduction. Let us recall that, for every  $a \in \mathbf{R}^3$

$$f_0(a) = \inf \left\{ \int_{\tilde{Y} \setminus D} f^{**}(\widehat{\nabla} u, 0) d\hat{x} : u \in \widehat{\text{adm}}(a) \right\},$$

$$\widehat{\text{adm}}(a) = \left\{ u \in W_{\#}^{1,1}(\widehat{Y}, \mathbf{R}^3) : \int_{\widehat{Y}} u \, d\hat{x} = a, u = 0 \text{ on } D \right\}.$$

Similarly, for every  $a \in \mathbf{R}^3$ , we now consider the following set of  $Y$ -periodic functions

$$\text{adm}(a) = \left\{ u \in W_{\#}^{1,1}(Y, \mathbf{R}^3) : \int_Y u \, dx = a, u = 0 \text{ on } T := D \times (0, L) \right\}$$

and set

$$\tilde{f}_0(a) = \inf \left\{ \int_{Y \setminus T} f(\nabla u) \, dx : u \in \text{adm}(a) \right\}.$$

The proposition below yields another expressions of  $f_0$  and its Legendre–Fenchel transform.

**Proposition 1.** *With above notations,*

- (i)  $f_0(a) = \inf \left\{ \int_{Y \setminus T} f^{**}(\nabla u) \, dx : u \in \text{adm}(a) \right\} = \tilde{f}_0^{**}(a),$
- (ii)  $\forall a^* \in \mathbf{R}^3, \quad f_0^*(a^*) = \tilde{f}_0^*(a^*) = \inf \left\{ \int_{Y \setminus T} f^*(\sigma) \, dx : \sigma \in \text{adm}^*(a^*) \right\}, \quad \text{where}$   
 $\text{adm}^*(a^*) := \{ \sigma \in L^\infty(Y \setminus T, \mathcal{M}_3), \text{div } \sigma = -a^* \text{ in } Y \setminus T, \sigma n \text{ antiperiodic on } \partial(Y \setminus T) \setminus \partial T \}.$

**Proof.** The second equality in (i) is classical and left to the reader (in the same spirit, see also Appendix 1). By Jensen’s inequality, we have, for any  $u \in \text{adm}(a)$

$$\begin{aligned} \int_{Y \setminus T} f^{**}(\nabla u) \, dx &\geq \int_{\widehat{Y} \setminus D} f^{**} \left( \int_0^1 \widehat{\nabla} u(\hat{x}, x_3) \, dx_3, 0 \right) \, d\hat{x} \\ &\geq f_0(a) \end{aligned}$$

because  $\hat{x} \mapsto \int_0^1 u(\hat{x}, x_3) \, dx_3$  belongs to  $\widehat{\text{adm}}(a)$ . Therefore  $\tilde{f}_0^{**}(a) \geq f_0(a)$ .

Since for any  $u \in \widehat{\text{adm}}(a)$  the function  $\tilde{u}$  defined by  $\tilde{u}(\hat{x}, x_3) = u(\hat{x})$  belongs to  $\text{adm}(a)$  and satisfies

$$\int_{Y \setminus T} f^{**}(\nabla \tilde{u}) \, dx = \int_{\widehat{Y} \setminus D} f^{**}(\widehat{\nabla} u, 0) \, d\hat{x},$$

we also have  $\tilde{f}_0^{**}(a) \leq f_0(a)$ .

The proof of assertion (ii) is established in Appendix 1.  $\square$

A first piece of information that one may capture on any sequence of displacement fields with finite energy is that it is bounded in  $L^1(\mathcal{O}, \mathbf{R}^3)$  thus weakly converges (i.e., in the sense of the  $\sigma(\mathbf{M}(\mathcal{O}, \mathbf{R}^3), \mathcal{C}_c(\mathcal{O}, \mathbf{R}^3))$  topology), at least for a subsequence, to some  $\mu$  in the space  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  (see Proposition 2 below). Therefore, according to Proposition 12(i) of Appendix 1, it is natural to treat the asymptotic analysis of the problem  $(\mathcal{P}_{F_\varepsilon^v})$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ .

**Proposition 2.** *Let  $\mu_\varepsilon \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  be such that  $\sup_\varepsilon F_\varepsilon^v(\mu_\varepsilon) < +\infty$ . Then, there exist  $\mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  and a subsequence, not relabeled, such that*

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3).$$

**Proof.** The Poincaré–Wirtinger inequality

$$\int_{\widehat{Y}} \left| w - \frac{1}{|D|} \int_D w \right| d\hat{x} \leq C_{PW} \int_{\widehat{Y}} |\widehat{\nabla} w| d\hat{x}$$

gives, after scaling and integrating over  $(0, L)$

$$\int_{\mathcal{O}} |w| dx \leq C'_{PW} \left( \int_{\mathcal{O} \cap T_\varepsilon} |w| dx + \int_{\mathcal{O}} |\varepsilon \nabla w| dx \right) \quad (3)$$

for every regular function  $w$  and finally, by a density argument, for every function  $w$  in  $W^{1,1}(\mathcal{O}, \mathbf{R}^3)$ . Applying this inequality to  $\mu_\varepsilon$ , from the coerciveness assumption in (1), we obtain

$$\begin{aligned} \int_{\mathcal{O}} |u_\varepsilon| dx &\leq C'_{PW} \left( \int_{\mathcal{O} \cap T_\varepsilon} |v| dx + \int_{\mathcal{O} \setminus T_\varepsilon} |\varepsilon \nabla u_\varepsilon| dx \right) \\ &\leq C'_{PW} \left( \int_{\mathcal{O}} |v| dx + \frac{1}{\alpha} F_\varepsilon^v(\mu_\varepsilon) + 2|\mathcal{O}| \right) \end{aligned}$$

which is bounded.  $\square$

### 2.1. The lower bound

In this section, we establish the lower bound in the definition of the Gamma-convergence of  $F_\varepsilon^v$  to  $F^v$ :

**Proposition 3.** *For every  $\mu_\varepsilon$  weakly converging to  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ ,*

$$F^v(\mu) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^v(\mu_\varepsilon).$$

**Proof.** One may assume

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^v(\mu_\varepsilon) < +\infty \quad (4)$$

so that, at most for a subsequence,  $\mu_\varepsilon$  belongs to  $W^{1,1}(\mathcal{O}, \mathbf{R}^3)$  and agrees with  $v$  on  $\mathcal{O} \cap T_\varepsilon$ . We must establish

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla u_\varepsilon) dx \geq \int_{\mathcal{O}} f_0(\mu - v).$$

Let us recall that since  $f_0$  is convex, the integral  $\int_{\mathcal{O}} f_0(\mu - v)$  is given by the duality principle

$$\int_{\mathcal{O}} f_0(\mu - v) = \sup_{w \in \mathcal{L}} \left\{ \langle \mu - v, w \rangle - \int_{\mathcal{O}} f_0^*(w) dx \right\},$$

where  $\mathcal{L}$  is the set of all the functions  $w : \mathcal{O} \rightarrow \mathbf{R}^3$  such that  $f_0^* \circ w$  belongs to  $L^1(\mathcal{O})$ . But we also have (see Bouchitté [9])

$$\int_{\mathcal{O}} f_0(\mu - v) = \sup_{w \in \mathcal{E}} \left\{ \langle \mu - v, w \rangle - \int_{\mathcal{O}} f_0^*(w) \, dx \right\},$$

where  $\mathcal{E}$  is the set of all the step functions  $w$  of the form

$$w = \sum_{i \in I} z_i^* 1_{\mathcal{O}_i}, \quad z_i^* \in \mathbf{R}^3, \quad (\mu - v)(\partial \mathcal{O}_i) = \mathcal{L}^3(\partial \mathcal{O}_i),$$

$(\mathcal{O}_i)_{i \in I}$  is a finite partition of  $\mathcal{O}$  made up of Borel sets and  $z_i^* \in \mathbf{R}^3$ . Our strategy is to establish for every  $w \in \mathcal{E}$

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu_\varepsilon) \, dx \geq \langle \mu - v, w \rangle - \int_{\mathcal{O}} f_0^*(w) \, dx \quad (5)$$

by taking into account the expression of  $f_0^*$  given in Proposition 1(ii). The conclusion will follow by taking the supremum over  $w \in \mathcal{E}$ .

In order to localize the calculation on each  $\mathcal{O}_i$ , we consider a family  $(\varphi_{i,\delta})_{i \in I}$  of regular functions with support included in  $\mathcal{O}_i$ ,  $0 \leq \varphi_{i,\delta} \leq 1$  and  $\sum_i \varphi_{i,\delta} \rightarrow 1$  a.e. in  $\mathcal{O}$  when  $\delta \rightarrow 0^+$ .

Consider  $w = \sum_i z_i^* 1_{\mathcal{O}_i}$  in  $\mathcal{E}$  and let  $\sigma_i$  be a  $\varepsilon$ -minimizer of  $h^*(z_i^*)$  in  $\text{adm}^*(z_i^*)$ , extended by periodicity to  $\mathbf{R}^3 \setminus \bigcup_{z \in \mathbf{Z}^2} (T + z)$ . According to Fenchel's inequality, for a.e.  $x \in \mathcal{O} \setminus T_\varepsilon$  we have

$$f(\varepsilon \nabla \mu_\varepsilon(x)) \geq f(\varepsilon \nabla \mu_\varepsilon(x)) \varphi_{i,\delta}(x) \geq \left( \varepsilon \nabla \mu_\varepsilon(x) \cdot \sigma_i \left( \frac{x}{\varepsilon} \right) - f^* \left( \sigma_i \left( \frac{x}{\varepsilon} \right) \right) \right) \varphi_{i,\delta}(x).$$

Summing over  $i \in I$  and integrating over  $\mathcal{O} \setminus T_\varepsilon$  yields

$$\int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu_\varepsilon) \, dx \geq \sum_{i \in I} \left( \int_{\mathcal{O} \setminus T_\varepsilon} \varepsilon \nabla \mu_\varepsilon \cdot \sigma_i \left( \frac{x}{\varepsilon} \right) \varphi_{i,\delta} \, dx - \int_{\mathcal{O} \setminus T_\varepsilon} f^* \left( \sigma_i \left( \frac{x}{\varepsilon} \right) \right) \varphi_{i,\delta} \, dx \right). \quad (6)$$

Now we extend  $\sigma_i$  in  $\mathbf{R}^3$ . First, let us consider  $\xi \in L^\infty(T \cap Y, \mathbf{R}^3)$  satisfying

$$\begin{cases} \operatorname{div} \xi = \left( 1 - \frac{1}{|D|} \right) z_i^* & \text{in } T \cap Y, \\ \xi \cdot n = \sigma_i \cdot n & \text{on } \partial T \setminus D \times \{0\} \cup D \times \{1\}, \\ \xi \cdot n = 0 & \text{on } D \times \{0\} \cup D \times \{1\}. \end{cases}$$

Such a solution  $\xi$  of above problem may be obtained through a sequence of solutions of  $p$ -Laplacian problems indexed by  $p$  going to one. We set

$$\tilde{\sigma}_i = \begin{cases} \sigma_i & \text{in } Y \setminus T, \\ \xi & \text{in } T \cap Y \end{cases}$$



and extend it by periodicity in  $\mathbf{R}^3$ . Integrating by part the first term in the right-hand side of (6) and by using boundary conditions satisfied by  $\xi$ , we easily obtain

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu_\varepsilon) \, dx \\ & \geq \sum_{i \in I} \liminf_{\varepsilon \rightarrow 0} \left[ \int_{\mathcal{O} \setminus T_\varepsilon} \mu_\varepsilon \cdot z_i^* \varphi_{i,\delta} \, dx + \int_{T_\varepsilon} v \cdot \left(1 - \frac{1}{|D|}\right) z_i^* \varphi_{i,\delta} \, dx - \int_{\mathcal{O} \setminus T_\varepsilon} f^* \left( \sigma_i \left( \frac{x}{\varepsilon} \right) \right) \varphi_{i,\delta} \, dx \right]. \end{aligned}$$

Extending now  $\sigma_i$  by 0 on  $T$  (still denoted by  $\sigma_i$ ) and since  $f^*(0) = 0$ , we obtain

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu_\varepsilon) \, dx \\ & \geq \sum_{i \in I} \liminf_{\varepsilon \rightarrow 0} \left[ \int_{\mathcal{O} \setminus T_\varepsilon} \mu_\varepsilon \cdot z_i^* \varphi_{i,\delta} \, dx + \int_{T_\varepsilon} v \cdot \left(1 - \frac{1}{|D|}\right) z_i^* \varphi_{i,\delta} \, dx - \int_{\mathcal{O}} f^* \left( \sigma_i \left( \frac{x}{\varepsilon} \right) \right) \varphi_{i,\delta} \, dx \right] \\ & = \left\langle \mu - v, w \sum_{i \in I} \varphi_{i,\delta} \right\rangle - \int_{\mathcal{O}} \sum_{i \in I} \varphi_{i,\delta} f_0^*(w) \, dx. \end{aligned}$$

We conclude by letting  $\delta \rightarrow 0$ .  $\square$

**Remark 1.** The previous lower bound also holds if one replaces  $v$  by a function  $v_\varepsilon$  such that  $1_{T_\varepsilon \cap \mathcal{O}} v_\varepsilon$  weakly converges to some measure written  $|D|\nu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ . In this more general situation, taking up previous proof, one easily obtain

$$F^\nu(\mu) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^{v_\varepsilon}(\mu_\varepsilon),$$

where, for every  $\mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ ,

$$F_\varepsilon^{v_\varepsilon}(\mu) = \begin{cases} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu) \, dx & \text{if } \mu \in W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3), \mu = v_\varepsilon \text{ on } \mathcal{O} \cap T_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

$$F^\nu(\mu) = \int_{\mathcal{O}} f_0(\mu - \nu).$$

## 2.2. The upper bound

Now we establish the upper bound in the definition of  $\Gamma$ -convergence:

**Proposition 4.** *For every  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ , there exists a sequence  $(\mu_\varepsilon)_\varepsilon$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  weakly converging to  $\mu$  such that*

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(\mu_\varepsilon) \leq F^v(\mu).$$

**Proof (First step).** We assume  $\mu - v \in \mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$  and we construct a sequence  $(u_\varepsilon)_{\varepsilon>0}$  satisfying

$$\begin{aligned} u_\varepsilon &\rightharpoonup \mu \quad \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3), \\ \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(u_\varepsilon) &\leq \int_{\mathcal{O}} \tilde{f}_0(\mu - v). \end{aligned}$$

For every  $u$  in  $L^1(\mathcal{O}, \mathbf{R}^3)$ , let  $\text{Adm}(\mathcal{O} \times Y, u)$  be the set

$$\left\{ w \in L^1(\mathcal{O}, W_{\#}^{1,p}(Y, \mathbf{R}^3)): \int_Y w(\cdot, y) dy = u \text{ a.e. in } \mathcal{O}, w(x, \cdot) = 0 \text{ a.e. on } T \right\}$$

and, for every  $u \in \mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$ , let  $\text{Adm}^1(\mathcal{O} \times Y, u)$  be the set

$$\left\{ w \in \mathcal{C}^1(\overline{\mathcal{O} \times Y}, \mathbf{R}^3): \int_Y w(\cdot, y) dy = u \text{ a.e. in } \mathcal{O}, w(x, \cdot) = 0 \text{ a.e. on } T, w(x, \cdot) \text{ } Y\text{-periodic} \right\}.$$

Note that, in some sense,  $\text{adm}(a)$  may be considered as a localization of the set  $\text{Adm}(\mathcal{O} \times Y, u)$  at  $a = u(x)$ . With this remark in mind, we have the following localization lemma. For a proof, we refer to Bellieud and Bouchitté [7]. For a general result about the permutation between “inf and  $f$ ”, consult also Anza Hafsa [4], Anza Hafsa and Mandallena [5].

**Lemma 1.** For any given  $u$  in  $\mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$

$$\begin{aligned} &\int_{\mathcal{O}} \inf \left\{ \int_Y f(\nabla w(y)) dy \mid w \in \text{adm}(u(x)) \right\} dx \\ &= \inf \left\{ \int_{\mathcal{O} \times Y} f(\nabla_y w(x, y)) dx dy : w \in \text{Adm}(\mathcal{O} \times Y, u) \right\} \\ &= \inf \left\{ \int_{\mathcal{O} \times Y} f(\nabla_y w(x, y)) dx dy : w \in \text{Adm}^1(\mathcal{O} \times Y, u) \right\}. \end{aligned}$$

Let then  $u$  in  $v + \mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$ . According to Lemma 1 and the definition of  $\tilde{f}_0$  we have

$$\int_{\mathcal{O}} \tilde{f}_0(\mu - v) dx = \inf \left\{ \int_{\mathcal{O} \times Y} f(\nabla_y w(x, y)) dx dy : w \in \text{Adm}^1(\mathcal{O} \times Y, \mu - v) \right\}.$$

For  $\eta > 0$ , choose  $w_\eta$  in  $\text{Adm}^1(\mathcal{O} \times Y, u - v)$  satisfying

$$\eta + \int_{\mathcal{O}} \tilde{f}_0(\mu - v) dx \geq \int_{\mathcal{O} \times Y} f(\nabla_y w_\eta(x, y)) dx dy,$$

and extend it in  $\mathcal{O} \times \mathbf{R}^3$  by  $Y$ -periodicity with respect to  $y$ . We now set

$$u_{\varepsilon, \eta}(x) = w_\eta \left( x, \frac{x}{\varepsilon} \right) + v \tag{7}$$

which fulfills the condition  $u_{\varepsilon,\eta} = v$  on  $\mathcal{O} \cap T_\varepsilon$ . When  $\varepsilon \rightarrow 0$ , a straightforward calculation gives

$$u_{\varepsilon,\eta} \rightharpoonup \mu - v + v = \mu \quad \text{in } L^1(\mathcal{O}, \mathbf{R}^3)$$

and

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^v(u_{\varepsilon,\eta}) = \int_{\mathcal{O} \times Y} f(\nabla_y w_\eta(x, y)) \, dx \, dy \leq \int_{\mathcal{O}} \tilde{f}_0(\mu(x) - v(x)) \, dx + \eta. \quad (8)$$

Before going to the limit on  $\eta$ , we slightly modify  $u_{\varepsilon,\eta}$  in order to match the boundary condition. For this, let  $V_\varepsilon = \{x \in \mathcal{O} : \text{dist}(x, \Gamma_0) < \varepsilon\}$  and  $\varphi_\varepsilon \in \mathcal{C}_c^1(\mathcal{O})$  satisfying

$$\begin{aligned} |\varphi_\varepsilon| &\leq C, \\ |\nabla \varphi_\varepsilon| &\leq C/\varepsilon, \\ \varphi_\varepsilon &= 0 \quad \text{on } V_{\varepsilon/2}, \quad \varphi_\varepsilon = 1 \quad \text{on } \mathcal{O} \setminus V_\varepsilon \end{aligned}$$

and set  $\tilde{u}_{\varepsilon,\eta} = \varphi_\varepsilon u_{\varepsilon,\eta}$ . Clearly

$$\tilde{u}_{\varepsilon,\eta} \rightharpoonup \mu \quad \text{in } L^1(\mathcal{O}, \mathbf{R}^3)$$

when  $\varepsilon$  goes to zero. Moreover

$$F_\varepsilon^v(\tilde{u}_{\varepsilon,\eta}) \leq F_\varepsilon^v(u_{\varepsilon,\eta}) + \int_{(\mathcal{O} \setminus T_\varepsilon) \cap V_\varepsilon} f(\varepsilon \nabla \tilde{u}_{\varepsilon,\eta}) \, dx. \quad (9)$$

From (7), the growth conditions (1) and the regularity of  $w$

$$\lim_{\varepsilon \rightarrow 0} \int_{(\mathcal{O} \setminus T_\varepsilon) \cap V_\varepsilon} f(\varepsilon \nabla \tilde{u}_{\varepsilon,\eta}) \, dx = 0. \quad (10)$$

Combining (8), (9) and (10), we obtain

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(\tilde{u}_{\varepsilon,\eta}) \leq \int_{\mathcal{O}} \tilde{f}_0(\mu(x) - v(x)) \, dx + \eta.$$

Letting  $\eta \rightarrow 0$  gives

$$\limsup_{\eta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(\tilde{u}_{\varepsilon,\eta}) \leq \int_{\mathcal{O}} \tilde{f}_0(\mu(x) - v(x)) \, dx.$$

Invoking a classical diagonalization argument, there exists a map  $\varepsilon \mapsto \eta(\varepsilon)$  and  $u_\varepsilon = \tilde{u}_{\varepsilon,\eta(\varepsilon)}$  in  $\text{Dom}(F_\varepsilon^v)$  such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } L^1(\mathcal{O}, \mathbf{R}^3), \\ \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(u_\varepsilon) \leq \int_{\mathcal{O}} \tilde{f}_0(\mu(x) - v(x)) \, dx. \end{cases}$$

*Last step.* We establish the result for any measure  $\mu - v$ . Classically, there exists  $\theta_n \in \mathcal{C}_c^1(\mathcal{O}, \mathbf{R}^3)$  weakly converging to  $\mu - v$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  such that

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{O}} \tilde{f}_0(\theta_n) \, dx = \int_{\mathcal{O}} \tilde{f}_0^{**}(\mu - v),$$

that is,

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{O}} \tilde{f}_0(\mu_n - v) \, dx = \int_{\mathcal{O}} \tilde{f}_0^{**}(\mu - v),$$

where we have set  $\mu_n := v + \theta_n$ . According to the first step, there exists  $u_{n,\varepsilon}$  weakly converging to  $\mu_n$  such that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^v(u_{n,\varepsilon}) &\leq \int_{\mathcal{O}} \tilde{f}_0^{**}(\mu - v), \\ \limsup_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} u_{n\varepsilon} &= \mu \quad \text{weakly in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3). \end{aligned}$$

We end the proof by a diagonal argument.  $\square$

Due to the growth of  $f$ , in order that the infimum  $(\mathcal{P}_{F_\varepsilon^v})$  be finite, classically  $\mathcal{L}$  has to satisfy a safe load condition like

$$\|\mathcal{L}\|_\infty \leq \frac{\alpha}{C'_{PW}}, \tag{11}$$

where  $C'_{PW}$  is the best constant in (3), proof of Proposition 2. Collecting Propositions 2, 3, 4, and according to Proposition 12, we obtain:

**Theorem 1.** *Assuming condition (11), the problem*

$$(\mathcal{P}_{F_\varepsilon^v}) \quad \inf \left\{ F_\varepsilon^v(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu : \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}$$

*converges to the problem*

$$(\mathcal{P}_{F^v}) \quad \min \left\{ F^v(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu : \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\} = - \int_{\mathcal{O}} f_0^*(\mathcal{L}(x)) \, dx - \int_{\mathcal{O}} \mathcal{L}(x) \cdot v \, dx,$$

*in the sense that, up to a subsequence, every sequence  $(\mu_\varepsilon)_\varepsilon$  of  $\varepsilon$ -minimizer of  $(\mathcal{P}_{F_\varepsilon^v})$  weakly converges in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  to a minimizer of  $(\mathcal{P}_{F^v})$  and  $\inf(\mathcal{P}_{F_\varepsilon^v})$  tends to  $\min(\mathcal{P}_{F^v})$ .*

### 3. The limit problem associated with the fibers

We assume that the density  $g$  also satisfies the conditions (1) and (2) of the previous section.

Let us recall that the energy functional associated with the equilibrium configuration of the union  $T_\varepsilon$  of very thin tubes clamped on  $\widehat{\mathcal{O}}$  is the functional  $G_\varepsilon$  defined on  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  by

$$G_\varepsilon(\mu) = \begin{cases} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon \mu) \, dx & \text{if } \mu \llcorner (\mathcal{O} \setminus T_\varepsilon) = 0, \mu \llcorner T_\varepsilon \cap \mathcal{O} \in W_{T_0}^{1,1}(T_\varepsilon \cap \mathcal{O}, \mathbf{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

In this section, we are going to prove that the  $\Gamma$ -limit of the sequence  $(G_\varepsilon)_{\varepsilon > 0}$  is given by

$$G(\mu) = \begin{cases} |D| \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{\partial \mu}{\partial x_3} \right) \, dx + \int_{\Gamma_0} (g^\perp)^{**,\infty}(\gamma(\mu)) & \text{if } (u, v) \in \mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3)), \\ +\infty & \text{otherwise,} \end{cases}$$

when  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  is equipped with the weak \* topology and

$$g^\perp(a) = \inf \{ g((M, a)) : M \in \mathbf{M}^{2 \times 3} \}.$$

We will make precise the space  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$  in Definition 1 below. Let us recall the definition of the recession function  $\tilde{g}^\infty$  of a given convex function

$$\tilde{g} : \mathbf{R}^3 \rightarrow [0, +\infty].$$

For all  $a \in \mathbf{R}^3$ ,  $\tilde{g}^\infty(a) := \sup_{t \geq 0} \tilde{g}(a_0 + ta)/t$  where  $a_0$  is an arbitrary element of the domain of  $\tilde{g}$ . It is easily seen that  $\tilde{g}^\infty$  is positively homogeneous of degree 1 and that this definition is independent of the choice of  $a_0$  in the domain of  $\tilde{g}$ . In our case the domain of  $\tilde{g}$  is  $\mathbf{R}^3$ . The trace  $\gamma(\nu)$  entering the definition of the surface part will be defined in Subsection 3.2.

#### 3.1. The space $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$

We are going to give the definition of the space  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$ . Let  $\Omega = \widehat{\Omega} \times (a, b)$  be any cylinder of  $\mathbf{R}^3$  where  $\widehat{\Omega}$  is an open bounded subset of  $\mathbf{R}^2$  and  $a, b$  belong to  $\mathbf{R}$ . Let  $\mu$  be any measure in  $\mathbf{M}(\Omega, \mathbf{R}^3)$ . For every  $\hat{\varphi}$  in  $\mathcal{C}_0(\widehat{\Omega}, \mathbf{R}^3)$  we define the measure  $\langle \mu, \hat{\varphi} \rangle$  by

$$\forall \varphi^\perp \in \mathcal{C}_0((a, b), \mathbf{R}^3), \quad \int_{(a,b)} \varphi^\perp \, d(\langle \mu, \hat{\varphi} \rangle) := \int_\Omega \hat{\varphi} \varphi^\perp \, d\mu, \quad (12)$$

where  $\hat{\varphi} \varphi^\perp$  denotes the vector valued function whose components are  $\hat{\varphi}_i \otimes \varphi_i^\perp$ ,  $i = 1, 2, 3$ .

**Definition 1.** Let  $\mathbf{M}(\widehat{\Omega}, \text{BV}((a, b), \mathbf{R}^3))$  be the space of all measures  $\mu \in \mathbf{M}(\Omega, \mathbf{R}^3)$  such that the measure  $\langle \mu, \hat{\varphi} \rangle$  defined in (12) belongs to  $\text{BV}((a, b), \mathbf{R}^3)$ .

Note that when  $\mu$  belongs to  $\mathbf{M}(\widehat{\Omega}, \text{BV}((a, b), \mathbf{R}^3))$ , since  $\langle \mu, \hat{\varphi} \rangle$  belongs to the space  $L^1((a, b), \mathbf{R}^3)$ , the left-hand side of (12) is also given by

$$\int_{(a,b)} \varphi^\perp \, d(\langle \mu, \hat{\varphi} \rangle) = \int_{(a,b)} \varphi^\perp \cdot \langle \mu, \hat{\varphi} \rangle \, dx_3.$$

The choice of the weak topology which equips  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  is suggested by Proposition 12(i) and the following compactness result.

**Proposition 5.** *Let  $\mu_\varepsilon \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  be such that  $\sup_\varepsilon G_\varepsilon(\mu_\varepsilon) < +\infty$ . Then, there exist  $\mu \in \mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$  and a subsequence, not relabeled, such that*

$$\begin{cases} 1_{\mathcal{O} \cap T_\varepsilon} \mu_\varepsilon \rightharpoonup \mu & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3), \\ 1_{\mathcal{O} \cap T_\varepsilon} \frac{\partial \mu_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial \mu}{\partial x_3} & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3). \end{cases}$$

**Proof.** By coercivity of  $g$ , the measure  $1_{\mathcal{O} \cap T_\varepsilon} \frac{\partial \mu_\varepsilon}{\partial x_3}$  is bounded in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ . The weak convergence of  $1_{\mathcal{O} \cap T_\varepsilon} \mu_\varepsilon$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  is a straightforward consequence of Poincaré's inequality. Now, for a not relabeled subsequence, we identify its weak limit  $\theta \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ . Let us denote by  $u_\varepsilon$  the restriction of  $\mu_\varepsilon$  on  $\mathcal{O} \cap T_\varepsilon$  and let  $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}, \mathbf{R}^3)$ . Let  $D_\varepsilon = \bigcup_{z \in \mathbb{Z}^2} \varepsilon(z + D)$ , integrating by parts gives

$$\int_{\mathcal{O} \cap T_\varepsilon} \varphi \cdot \frac{\partial u_\varepsilon}{\partial x_3} \, dx = \int_{\widehat{\mathcal{O}} \cap D_\varepsilon} \int_{(0,L)} \varphi \cdot \frac{\partial u_\varepsilon}{\partial x_3} \, dx_3 \, d\hat{x} = - \int_{\mathcal{O}} 1_{\mathcal{O} \cap T_\varepsilon} \mu_\varepsilon \cdot \frac{\partial \varphi}{\partial x_3} \, dx.$$

Letting  $\varepsilon \rightarrow 0$  gives  $\theta = \partial \mu / \partial x_3$ .

At last, to prove that  $\mu \in \mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$  it suffices to establish that  $\langle \mu, \hat{\varphi} \rangle$  belongs to  $\text{BV}((0, L), \mathbf{R}^3)$  for every  $\hat{\varphi} \in \mathcal{C}_0(\widehat{\mathcal{O}})$ . This is a straightforward consequence of the following lemma whose proof is given in Appendix 3 where we actually give a more general result (Lemma 3):

**Lemma 2.** *Let  $a, b$  in  $\mathbf{R}$ . Then*

$$\{\mu \in \mathbf{M}((a, b), \mathbf{R}^m) : D\mu \in \mathbf{M}((a, b), \mathbf{R}^m)\} = \text{BV}((a, b), \mathbf{R}^m),$$

where  $D\mu$  denotes the distributional derivative of the measure  $\mu$ .

Indeed, from definition (12) of the measure  $\langle \nu, \hat{\varphi} \rangle$ , we have, in the distributional sense,

$$\begin{aligned} \left\langle \frac{\partial}{\partial x_3} \langle \nu, \hat{\varphi} \rangle, \varphi^\perp \right\rangle &= - \left\langle \langle \nu, \hat{\varphi} \rangle, \frac{\partial}{\partial x_3} \varphi^\perp \right\rangle \\ &= - \int_{\mathcal{O}} \frac{\partial}{\partial x_3} (\hat{\varphi} \varphi^\perp) \, d\nu \\ &= \left\langle \frac{\partial \nu}{\partial x_3}, \hat{\varphi} \varphi^\perp \right\rangle \end{aligned}$$

and the conclusion  $\frac{\partial}{\partial x_3} \langle \nu, \hat{\varphi} \rangle \in \mathbf{M}((0, L), \mathbf{R}^3)$  follows from  $\frac{\partial \nu}{\partial x_3} \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ .  $\square$

**Corollary 1.** *The domain of any cluster point of  $G_\varepsilon$  for the  $\Gamma$ -convergence, is included in  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$ .*

### 3.2. Orthogonal trace of measures in $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$

In order to take into account the boundary condition which will be classically relaxed by the surface energy in the definition of  $\mathcal{F}$ , we define now the trace of the elements of  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$ .

**Definition 2.** Let  $\widetilde{\mathcal{O}}$  be the cylinder  $\widehat{\mathcal{O}} \times (-\infty, L)$ , the orthogonal trace on  $\Gamma_0$  of a measure  $\nu$  in  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}((0, L), \mathbf{R}^3))$  is the measure  $\gamma(\nu)$  in  $\mathbf{M}(\widetilde{\mathcal{O}}, \mathbf{R}^3)$  concentrated on  $\Gamma_0$  defined by:

$$\forall \phi \in \mathcal{C}_0(\widetilde{\mathcal{O}}, \mathbf{R}^3), \quad \langle \gamma(\nu), \phi \rangle := \gamma_0(\langle \nu, \phi(\hat{x}, 0) \rangle) \otimes e_3,$$

where  $(\langle \nu, \phi(\hat{x}, 0) \rangle)$  denotes the measure defined in (12),  $\gamma_0(w)$  the trace on  $\{0\}$  of any element  $w \in \text{BV}((0, L), \mathbf{R}^3)$  and  $e_3 = (0, 0, 1)$ .

This definition is meaningful. Indeed, from Lemma 2 and the proof of Proposition 2 we know that  $(\langle \nu, \phi(\hat{x}, 0) \rangle)$  belongs to  $\text{BV}((0, L), \mathbf{R}^3)$ , then possesses a trace on  $\{0\}$ . The trace operator satisfies the following properties

**Proposition 6.** *The trace operator  $\gamma: \nu \mapsto \gamma(\nu)$  is linear continuous from  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  into  $\mathbf{M}(\widetilde{\mathcal{O}}, \mathbf{R}^3)$ . Moreover, if  $\widetilde{\theta}$  denotes the extension in  $\mathbf{M}(\widetilde{\mathcal{O}}, \mathbf{R}^3)$  of any measure  $\theta$  of  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  defined for every Borel set  $B$  of  $\widetilde{\mathcal{O}}$  by  $\widetilde{\theta}(B) := \theta(B \cap \mathcal{O})$ , we have*

$$\frac{\partial \widetilde{\nu}}{\partial x_3} = \frac{\widetilde{\partial \nu}}{\partial x_3} + \gamma(\nu),$$

where  $\widetilde{\partial \nu} / \partial x_3$  and  $\gamma(\nu)$  are mutually singular.

**Proof.** the first assertion is easy to establish. Let us now consider  $\Phi = \hat{\varphi} \varphi^\perp$  where  $\hat{\varphi} = (\hat{\varphi}_i)_{i=1, \dots, 3} \in \mathcal{D}(\widehat{\mathcal{O}}, \mathbf{R}^3)$  and  $\varphi^\perp = (\varphi^\perp_i)_{i=1, \dots, 3} \in \mathcal{D}((-\infty, L), \mathbf{R}^3)$ . From the definition of the distributional derivative, from (12) and the fact that  $\langle \nu, \hat{\varphi} \rangle$  belongs to  $L^1((0, L), \mathbf{R}^3)$ , we have

$$\left\langle \frac{\partial \widetilde{\nu}}{\partial x_3}, \Phi \right\rangle = - \int_{\widetilde{\mathcal{O}}} \frac{\partial \Phi}{\partial x_3} \widetilde{\nu} = - \int_{\mathcal{O}} \frac{\partial \Phi}{\partial x_3} \widetilde{\nu} = - \int_{(0, L)} \frac{\partial \varphi^\perp}{\partial x_3} \cdot \langle \nu, \hat{\varphi} \rangle dx_3. \quad (13)$$

As  $\langle \nu, \hat{\varphi} \rangle$  belongs to  $\text{BV}((0, L), \mathbf{R}^3)$ , one may apply the generalized Green's formula to the last expression, which gives, with our definition of  $\gamma(\nu)$ :

$$\begin{aligned} - \int_{(0, L)} \frac{\partial \varphi^\perp}{\partial x_3} \cdot \langle \nu, \hat{\varphi} \rangle dx_3 &= \int_{(0, L)} \varphi^\perp \cdot \frac{\partial}{\partial x_3} \langle \nu, \hat{\varphi} \rangle + \gamma_0(\langle \nu, \varphi^\perp(0) \otimes \hat{\varphi} \rangle) \otimes e_3 \\ &= \int_{(0, L)} \varphi^\perp \frac{\partial}{\partial x_3} \cdot \langle \nu, \hat{\varphi} \rangle + \langle \gamma(\nu), \Phi \rangle \\ &= \int_{(0, L)} \varphi^\perp \cdot \left\langle \frac{\partial \nu}{\partial x_3}, \hat{\varphi} \right\rangle + \langle \gamma(\nu), \Phi \rangle \end{aligned}$$

$$\begin{aligned}
&= \int_{(-\infty, L)} \varphi^\perp \cdot \left\langle \frac{\partial \nu}{\partial x_3}, \hat{\varphi} \right\rangle + \langle \gamma(\nu), \Phi \rangle \\
&= \left\langle \frac{\widetilde{\partial \nu}}{\partial x_3}, \Phi \right\rangle + \langle \gamma(\nu), \Phi \rangle,
\end{aligned} \tag{14}$$

where, for passing from the second to the third equality, we used  $\frac{\partial}{\partial x_3} \langle \nu, \hat{\varphi} \rangle = \langle \frac{\partial \nu}{\partial x_3}, \hat{\varphi} \rangle$  in  $\mathbf{M}((0, L), \mathbf{R}^3)$  which is a straightforward consequence of definition (12) of the two measures  $\langle \nu, \hat{\varphi} \rangle$  and  $\langle \frac{\partial \nu}{\partial x_3}, \hat{\varphi} \rangle$ . We also used (12) with  $\Omega = \widehat{\mathcal{O}} \times (0, L)$  and  $\Omega = \widehat{\mathcal{O}} \times (-\infty, L)$  for passing from the third to the fourth equality. Collecting (13) and (14), we finally obtain for every  $\Phi = \hat{\varphi} \varphi^\perp$

$$\left\langle \frac{\partial \tilde{\nu}}{\partial x_3}, \Phi \right\rangle = \left\langle \frac{\widetilde{\partial \nu}}{\partial x_3}, \Phi \right\rangle + \langle \gamma(\nu), \Phi \rangle$$

and the conclusion follows from a classical density argument. Obviously, as  $\gamma(\nu)$  is concentrated on  $\Gamma_0$  and  $\widetilde{\partial \nu} / \partial x_3$  on  $\mathcal{O}$ , these two measures are mutually singular.  $\square$

### 3.3. The lower bound

We establish the lower bound in the definition of the Gamma-convergence of  $G_\varepsilon$  to  $G$ :

**Proposition 7.** *For every  $\mu_\varepsilon$  weakly converging to  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ ,*

$$G(\mu) \leq \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(\mu_\varepsilon).$$

**Proof.** Obviously, one may assume

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(\mu_\varepsilon) < +\infty \tag{15}$$

so that the restriction  $u_\varepsilon$  of  $\mu_\varepsilon$  on  $\mathcal{O} \cap T_\varepsilon$  belongs to  $W_{\Gamma_0}^{1,1}(T_\varepsilon \cap \mathcal{O}, \mathbf{R}^3)$ . We are going to prove

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon u_\varepsilon) dx \geq |D| \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) + \int_{\Gamma_0} (g^\perp)^{**, \infty}(\mu). \tag{16}$$

With the notations of Definition 2 and Proposition 6, we begin by extending  $u_\varepsilon$  on  $\widetilde{\mathcal{O}} \cap T_\varepsilon$  by setting

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \mathcal{O} \cap T_\varepsilon, \\ 0 & \text{in } \widetilde{\mathcal{O}} \cap T_\varepsilon \setminus \mathcal{O} \cap T_\varepsilon. \end{cases}$$

Note that  $\partial \tilde{u}_\varepsilon / \partial x_3 = \widetilde{\partial u_\varepsilon / \partial x_3}$ .

Proceeding exactly as in Proposition 5 one may prove that there exist  $\bar{\mu}$  in  $\mathbf{M}(\widetilde{\mathcal{O}}, \text{BV}((-\infty, L), \mathbf{R}^3))$  and a subsequence not relabeled such that

$$\begin{cases} 1_{\widetilde{\mathcal{O}} \cap T_\varepsilon} \tilde{\mu}_\varepsilon \rightharpoonup \bar{\mu} & \text{in } \mathbf{M}(\widetilde{\mathcal{O}}, \mathbf{R}^3), \\ 1_{\widetilde{\mathcal{O}} \cap T_\varepsilon} \frac{\partial \tilde{u}_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial \bar{\mu}}{\partial x_3} & \text{in } \mathbf{M}(\widetilde{\mathcal{O}}, \mathbf{R}^3). \end{cases}$$



Let us show that  $\partial\bar{\mu}/\partial x_3 = \partial\tilde{\mu}/\partial x_3$ . Indeed, for every  $\Phi = \hat{\varphi}\varphi^\perp$ ,  $\hat{\varphi} \in \mathcal{D}(\widehat{\mathcal{O}}, \mathbf{R}^3)$ ,  $\varphi^\perp \in \mathcal{D}((-\infty, L), \mathbf{R}^3)$ ,

$$\begin{aligned}
\left\langle \frac{\partial\bar{\mu}}{\partial x_3}, \Phi \right\rangle &= -\left\langle \bar{\mu}, \hat{\varphi} \frac{\partial\varphi^\perp}{\partial x_3} \right\rangle \\
&= -\lim_{\varepsilon \rightarrow 0} \int_{\widehat{\mathcal{O}}} 1_{\widehat{\mathcal{O}} \cap T_\varepsilon} \tilde{\mu}_\varepsilon \cdot \hat{\varphi} \frac{\partial\varphi^\perp}{\partial x_3} dx \\
&= -\lim_{\varepsilon \rightarrow 0} \int_{(0,L)} \left( \int_{\widehat{\mathcal{O}}} 1_{\mathcal{O} \cap T_\varepsilon} \mu_\varepsilon \hat{\varphi} d\hat{x} \right) \cdot \frac{\partial\varphi^\perp}{\partial x_3} dx_3 \\
&= -\int_{(0,L)} \langle \mu, \hat{\varphi} \rangle \cdot \frac{\partial\varphi^\perp}{\partial x_3} dx_3 \\
&= -\left\langle \tilde{\mu}, \hat{\varphi} \frac{\partial\varphi^\perp}{\partial x_3} \right\rangle \\
&= \left\langle \frac{\partial\tilde{\mu}}{\partial x_3}, \Phi \right\rangle.
\end{aligned}$$

We have firstly used the strong convergence of  $\int_{\widehat{\mathcal{O}}} 1_{\mathcal{O} \cap T_\varepsilon} u_\varepsilon \cdot \hat{\varphi} d\hat{x}$  to  $\langle \mu, \hat{\varphi} \rangle$  in  $L^1((0, L), \mathbf{R}^3)$  insured by the weak convergence of this sequence to  $\langle \mu, \hat{\varphi} \rangle$  in  $\text{BV}((0, L), \mathbf{R}^3)$  and secondly equality (12) defining  $\langle \tilde{\mu}, \hat{\varphi} \rangle$ .

Now we establish inequality (16). Let  $\Phi_\delta, \Phi_\delta \leq 1_{\widehat{\mathcal{O}}}$ , be a positive function in  $\mathcal{C}_c(\widehat{\mathcal{O}})$  satisfying  $\Phi_\delta = 1$  on  $\mathcal{O}$  and converging to  $1_{\mathcal{O}}$  a.e. when  $\delta \rightarrow 0^+$ . We use a duality argument. According to Fenchel's inequality, for every  $\Phi \in \mathcal{C}_0(\widehat{\mathcal{O}}, \mathbf{R}^3)$ ,

$$\begin{aligned}
\int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon u_\varepsilon) dx &\geq \int_{\mathcal{O}} 1_{T_\varepsilon} g^\perp \left( \frac{\partial u_\varepsilon}{\partial x_3} \right) dx = \int_{\widehat{\mathcal{O}}} 1_{T_\varepsilon} g^\perp \left( \frac{\partial \tilde{u}_\varepsilon}{\partial x_3} \right) \Phi_\delta dx \\
&\geq \left\langle 1_{\widehat{\mathcal{O}} \cap T_\varepsilon} \frac{\partial \tilde{u}_\varepsilon}{\partial x_3}, \Phi \Phi_\delta \right\rangle - \int_{\widehat{\mathcal{O}}} 1_{T_\varepsilon} (g^\perp)^*(\Phi) \Phi_\delta dx.
\end{aligned}$$

Therefore, by the observation made above and according to Proposition 6

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon u_\varepsilon) dx &\geq |D| \left[ \left\langle \frac{1}{|D|} \frac{\partial \tilde{\mu}}{\partial x_3}, \Phi \Phi_\delta \right\rangle - \int_{\widehat{\mathcal{O}}} (g^\perp)^*(\Phi) \Phi_\delta dx \right] \\
&= |D| \left[ \left\langle \frac{1}{|D|} \left( \widetilde{\frac{\partial \mu}{\partial x_3}} + \gamma(\mu) \right), \Phi \Phi_\delta \right\rangle - \int_{\widehat{\mathcal{O}}} (g^\perp)^*(\Phi) \Phi_\delta dx \right].
\end{aligned}$$

Taking the supremum over  $\Phi \in \mathcal{C}_0(\widehat{\mathcal{O}}, \mathbf{R}^3)$  and noticing that  $\widetilde{\partial\mu/\partial x_3}$  and  $\gamma(\mu)$  are mutually singular we obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon u_\varepsilon) dx \geq |D| \int_{\widehat{\mathcal{O}}} (g^\perp)^{**} \left( \frac{1}{|D|} \left( \widetilde{\frac{\partial \mu}{\partial x_3}} + \gamma(\mu) \right) \right) \Phi_\delta$$

$$= |D| \left[ \int_{\tilde{\mathcal{O}}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) \Phi_\delta + \int_{\tilde{\mathcal{O}}} (g^\perp)^{**} \left( \frac{1}{|D|} \gamma(\mu) \right) \Phi_\delta \right].$$

Since  $\gamma(\mu)$  is singular with respect to the Lebesgue measure and concentrated on  $\Gamma_0$ , and  $(g^\perp)^{**, \infty}$  is positively homogeneous, above inequality yields

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon u_\varepsilon) \, dx &\geq |D| \int_{\tilde{\mathcal{O}}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) \Phi_\delta + \int_{\Gamma_0} (g^\perp)^{**, \infty}(\gamma(\mu)) \Phi_\delta \\ &= |D| \int_{\tilde{\mathcal{O}}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) \Phi_\delta + \int_{\Gamma_0} (g^\perp)^{**, \infty}(\gamma(\mu)) \Phi_\delta. \end{aligned}$$

We end the proof by letting  $\delta \rightarrow 0$ .  $\square$

### 3.4. The upper bound

We establish the upper bound in the definition of  $\Gamma$ -convergence, namely:

**Proposition 8.** *For every  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ , there exists a sequence  $(\mu_\varepsilon)_\varepsilon$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  weakly converging to  $\mu$  such that*

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(\mu_\varepsilon) \leq G(\mu). \quad (17)$$

*First step.* We assume  $u \in \mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$  with  $u = 0$  on  $\Gamma_0$ , and we construct a sequence  $(u_\varepsilon)_{\varepsilon > 0}$  in  $\mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$  satisfying  $u_\varepsilon = 0$  on  $\Gamma_0$  and

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3), \\ \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon) \leq |D| \int_{\mathcal{O}} g^\perp \left( \frac{1}{|D|} \frac{\partial u}{\partial x_3} \right) \, dx. \end{cases}$$

Consider two functions  $\xi_1, \xi_2$  in  $\mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)$ , possibly depending on  $\eta$ , such that

$$\eta + \int_{\mathcal{O}} g^\perp \left( \frac{1}{|D|} \frac{\partial u}{\partial x_3} \right) \, dx \geq \int_{\mathcal{O}} g \left( \xi_1 + \frac{1}{|D|} \frac{\partial u}{\partial x_1}, \xi_2 + \frac{1}{|D|} \frac{\partial u}{\partial x_2}, \frac{1}{|D|} \frac{\partial u}{\partial x_3} \right) \, dx$$

(see the selection theorem in Ekeland and Temam [14] or consult Castaing and Valadier [10]).

Let finally  $\varphi \in \mathcal{C}_c^1(\widehat{Y}, \mathbf{R}^2)$  satisfying  $\varphi(\hat{x}) = \hat{x}$  in  $D$ , extended by  $\widehat{Y}$ -periodicity in  $\mathbf{R}^2$ , and set

$$u_{\varepsilon, \eta}(x) = 1_{T_\varepsilon \cap \mathcal{O}} \left( \frac{1}{|D|} u(x) + (\xi_1, \xi_2, 0)^\top \varepsilon \left( \varphi \left( \frac{\hat{x}}{\varepsilon} \right), 0 \right) \right). \quad (18)$$

A straightforward calculation gives, when  $\varepsilon \rightarrow 0$ ,

$$u_{\varepsilon, \eta} \rightharpoonup u \quad \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u_{\varepsilon, \eta}) &= |D| \int_{\mathcal{O}} g\left(\xi_1 + \frac{1}{|D|} \frac{\partial u}{\partial x_1}, \xi_2 + \frac{1}{|D|} \frac{\partial u}{\partial x_2}, \frac{1}{|D|} \frac{\partial u}{\partial x_3}\right) dx \\ &\leq |D| \int_{\mathcal{O}} g^\perp\left(\frac{1}{|D|} \frac{\partial u}{\partial x_3}\right) dx + \eta. \end{aligned} \quad (19)$$

Going to the limit on  $\eta$  and using a classical diagonalization argument, there exists a map  $\varepsilon \mapsto \eta(\varepsilon)$  and  $u_\varepsilon = u_{\varepsilon, \eta(\varepsilon)}$ , such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3), \\ \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon) \leq |D| \int_{\mathcal{O}} g^\perp\left(\frac{1}{|D|} \frac{\partial u}{\partial x_3}\right) dx \end{cases}$$

and the proof of this step is complete.

*Last step.* One may assume  $\widetilde{G}(\mu) < +\infty$  so that  $u \in \mathbf{M}(\widehat{\mathcal{O}}, \mathbf{BV}((0, L), \mathbf{R}^3))$ .

For  $t > 0$  intended to go to 0, let us consider the subset  $\mathcal{O}_t := \widehat{\mathcal{O}} \times (t, L)$  of  $\mathcal{O}$  and  $\mu_t$  the restriction of  $\mu$  to  $\mathbf{M}(\mathcal{O}_t, \mathbf{R}^3)$ . From Proposition 6 applied to  $\mu_t$  where  $\mathcal{O}_t$  is substituted for  $\mathcal{O}$  and  $\mathcal{O}$  for  $\widehat{\mathcal{O}}$ , we have

$$\frac{\partial \widetilde{\mu}_t}{\partial x_3} = \frac{\partial \mu_t}{\partial x_3} + \gamma_t(\mu_t). \quad (20)$$

Classically, there exists  $\theta_{n,t}$  in  $\mathcal{C}_c^1(\mathcal{O}_t, \mathbf{R}^3)$  weakly converging to  $\frac{1}{|D|} \frac{\partial \widetilde{\mu}_t}{\partial x_3}$  in  $\mathbf{M}(\mathcal{O}_t, \mathbf{R}^3)$  such that

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{O}_t} g^\perp(\theta_n) dx = \int_{\mathcal{O}_t} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \widetilde{\mu}_t}{\partial x_3} \right). \quad (21)$$

Extending  $\theta_{n,t}$  by zero on  $\mathcal{O} \setminus \mathcal{O}_t$  and setting

$$v_{n,t}(x) = \int_0^{x_3} \theta_{n,t}(\hat{x}, s) ds,$$

from (20), (21) becomes

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\mathcal{O}} g^\perp\left(\frac{\partial v_{n,t}}{\partial x_3}\right) dx &= \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \widetilde{\mu}_t}{\partial x_3} \right) + \int_{\mathcal{O}} (g^\perp)^{**,\infty} \left( \frac{1}{|D|} \gamma_t(\mu_t) \right) \\ &= \int_{\mathcal{O}_t} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) + \int_{\mathcal{O}} (g^\perp)^{**,\infty} \left( \frac{1}{|D|} \gamma_t(\mu_t) \right). \end{aligned}$$

Letting  $t \rightarrow 0$  finally gives

$$\lim_{t \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{\mathcal{O}} g^\perp\left(\frac{\partial v_{n,t}}{\partial x_3}\right) dx = \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \mu}{\partial x_3} \right) + \int_{\Gamma_0} (g^\perp)^{**,\infty} \left( \frac{1}{|D|} \gamma(\nu) \right).$$

The limit obtained in the second member as  $t \rightarrow 0$  may be easily justified. For a complete proof, we refer the reader to Abddaimi [1]. Using a diagonal argument: there exist a map  $n \mapsto t(n)$  and  $v_n = v_{n,t(n)}$  in  $\mathcal{C}^1(\overline{\mathcal{O}}, \mathbf{R}^3)^2$  weakly converging to  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  such that  $v_n = 0$  on  $\Gamma_0$  and satisfying

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{O}} g^\perp \left( \frac{\partial v_n}{\partial x_3} \right) dx = \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{1}{|D|} \frac{\partial \nu}{\partial x_3} \right) + \int_{\Gamma_0} (g^\perp)^{**, \infty} \left( \frac{1}{|D|} \gamma(\nu) \right).$$

Collecting the result of the first step, we now conclude the proof by still using a diagonal argument.

Noticing that condition

$$\|\mathcal{L}\|_\infty \leq \frac{\alpha}{L} \tag{22}$$

implies  $\inf(\mathcal{P}_{G_\varepsilon}) > -\infty$ , we obtain from Propositions 5, 7, 8, 12:

**Theorem 2.** *Under condition (22), the problem*

$$(\mathcal{P}_{G_\varepsilon}) \quad \inf \left\{ G_\varepsilon(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu : \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}$$

*converges to the problem*

$$(\mathcal{P}_G) \quad \min \left\{ G(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu : \mu \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \right\}$$

*in the sense that, up to a subsequence, every sequence  $(\mu_\varepsilon)_\varepsilon$  of  $\varepsilon$ -minimizer of  $(\mathcal{P}_{G_\varepsilon})$  weakly converges in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  to a minimizer of  $(\mathcal{P}_G)$  and  $\inf(\mathcal{P}_{G_\varepsilon})$  tends to  $\min(\mathcal{P}_G)$ .*

#### 4. The limit problem associated with the complete structure

Now, we deal with the asymptotic behavior of the structure describing interaction between the displacement field in the matrix and in the fibers. Let us recall that the functional energy  $H_\varepsilon$  is defined on  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  by:

$$H_\varepsilon(\mu) = \begin{cases} \int_{\mathcal{O} \setminus T_\varepsilon} f(\varepsilon \nabla \mu) dx + \int_{\mathcal{O} \cap T_\varepsilon} g(\nabla_\varepsilon \mu) dx & \text{if } \mu \in W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

It is worth noticing that for  $\mu$  in  $W_{\Gamma_0}^{1,1}(\mathcal{O}, \mathbf{R}^3)$ ,

$$H_\varepsilon(\mu) = F_\varepsilon^\mu(\mu) + G_\varepsilon(1_{T_\varepsilon \cap \mathcal{O}} \mu).$$

We want to establish the  $\Gamma$ -convergence of the sequence  $(H_\varepsilon)_\varepsilon$  to the infimum convolution  $F^{0+}_\varepsilon G_{|D|}$ . Moreover, as  $\mathcal{L} : \mu \mapsto \int_{\mathcal{O}} \Phi d\mu$  is a continuous perturbation of  $H_\varepsilon^v$ , according to Proposition 12(ii), we will obtain the Gamma-convergence of  $H_\varepsilon + \mathcal{L}$  to  $F^{0+}_\varepsilon G_{|D|} + \mathcal{L}$ .

The choice of the weak topology which equips  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  is suggested by Proposition 12(i) and the following compactness result. The proof is very similar to that of Proposition 5 and left to the reader.

**Proposition 9.** *Let  $\mu_\varepsilon$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  be such that  $\sup_\varepsilon H_\varepsilon(\mu_\varepsilon) < +\infty$  and set  $\nu_\varepsilon = 1_{T_\varepsilon \cap \mathcal{O}} \mu_\varepsilon$ . Then, there exist  $(\mu, \nu)$  in  $\mathbf{M}(\widehat{\mathcal{O}}, \text{BV}(0, L))$  and a subsequence not relabeled such that*

$$\begin{cases} (\mu_\varepsilon, \nu_\varepsilon) \rightharpoonup (\mu, |D|\nu) & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3) \times \mathbf{M}(\mathcal{O}, \mathbf{R}^3), \\ 1_{\mathcal{O} \cap T_\varepsilon} \frac{\partial \mu_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial |D|\nu}{\partial x_3} & \text{in } \mathbf{M}(\mathcal{O}, \mathbf{R}^3). \end{cases}$$

We have introduced the coefficient  $|D|$  in the definition of the weak limits for technical reason (see Remark 1).

#### 4.1. The lower bound

In this section, we establish the lower bound in the definition of the Gamma-convergence of  $H_\varepsilon$  to  $H$ :

**Proposition 10.** *For every  $\mu_\varepsilon$  weakly converging to  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ ,*

$$H(\mu) \leq \liminf_{\varepsilon \rightarrow 0} H_\varepsilon(\mu_\varepsilon).$$

**Proof.** One may assume so that  $\liminf_{\varepsilon \rightarrow 0} H_\varepsilon(\mu_\varepsilon) < +\infty$ , so that, for a nonrelabeled subsequence,

$$H_\varepsilon(\mu_\varepsilon) = F_\varepsilon^{\mu_\varepsilon}(\mu_\varepsilon) + G_\varepsilon(\nu_\varepsilon).$$

According to Proposition 9, Subsection 2.1 and Remark 1, and to Subsection 3.3, we then obtain

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} H_\varepsilon(u_\varepsilon) &\geq F^\nu(\mu) + G(|D|\nu) \\ &\geq \inf_{\theta \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)} \{F^\theta(\mu) + G_{|D|}(\theta)\} \end{aligned}$$

which ends the proof.  $\square$

#### 4.2. The upper bound

Now we establish the upper bound in the definition of  $\Gamma$ -convergence:

**Proposition 11.** *For every  $\mu$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ , there exists a sequence  $(\mu_\varepsilon)_\varepsilon$  in  $\mathbf{M}(\mathcal{O})$  weakly converging to  $\mu$  such that*

$$\limsup_{\varepsilon \rightarrow 0} H_\varepsilon(\mu_\varepsilon) \leq H(\mu). \tag{23}$$

**Proof.** We do not take into account the boundary condition which may be easily treated by using arguments of Subsection 2.2. Obviously one may assume  $H(\mu) < +\infty$ . For  $\eta > 0$  intended to go to zero, let  $\nu_\eta$  be a  $\eta$ -minimizer in the definition of  $H(\mu)$ :

$$H(\mu) \geq F^{\nu_\eta}(\mu) + G_{|D|}(\nu_\eta) - \eta.$$

In order to shorten notations, we will denote this minimizer by  $\nu$ .

Classically, there exists two sequences  $(\zeta_n)_{n \in \mathbb{N}}$  and  $(\theta_n)_{n \in \mathbb{N}}$  in  $\mathcal{C}_c^1(\mathcal{O}, \mathbf{R}^3)$  weakly converging to  $\mu - \nu$  and  $\partial\nu/\partial x_3$  such that

$$\begin{aligned} \int_{\mathcal{O}} f_0(\mu - \nu) &= \lim_{n \rightarrow +\infty} \int_{\mathcal{O}} f_0(\zeta_n) \, dx, \\ \int_{\mathcal{O}} (g^\perp)^{**} \left( \frac{\partial\nu}{\partial x_3} \right) &= \lim_{n \rightarrow +\infty} \int_{\mathcal{O}} (g^\perp)^{**}(\theta_n) \, dx. \end{aligned}$$

We set

$$\begin{aligned} v_n(x) &= \int_0^{x_3} \theta(\hat{x}, s) \, ds, \\ u_n &= \zeta_n + v_n \end{aligned}$$

so that

$$F^\nu(\mu) + G_{|D|}(\nu) = \lim_{n \rightarrow +\infty} (F^{v_n}(u_n) + G_{|D|}(v_n)). \quad (24)$$

From Subsection 2.2, there exists a function  $u_{n,\varepsilon}$  in  $\mathcal{C}^1(\mathcal{O}, \mathbf{R}^3)$  satisfying  $u_{n,\varepsilon} \rightharpoonup u_n$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  and

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{v_n}(u_{n,\varepsilon}) \leq F^{v_n}(u_n). \quad (25)$$

On the other hand, with the notations of Subsection 3.4, Proposition 8, first step, consider  $v_{n,\varepsilon}$  obtained by replacing  $u$  by  $|D|v_n$ :

$$v_{n,\varepsilon}(x) = 1_{T_\varepsilon \cap \mathcal{O}}(v_n(x) + \varepsilon \zeta_\varepsilon)$$

satisfying

$$\lim_{\varepsilon \rightarrow 0} G_\varepsilon(v_{n,\varepsilon}) \leq G_{|D|}(v_n). \quad (26)$$

Collecting (25) and (26), we obtain

$$\begin{aligned} F^{v_n}(u_n) + G_{|D|}(v_n) &= \lim_{\varepsilon \rightarrow 0} F_\varepsilon^{v_{n,\varepsilon}}(u_{n,\varepsilon}) + G_\varepsilon(v_{n,\varepsilon}) \\ &= \lim_{\varepsilon \rightarrow 0} H_\varepsilon(\tilde{u}_{n,\varepsilon}), \end{aligned} \quad (27)$$

where

$$\tilde{u}_{n,\varepsilon} = u_{n,\varepsilon} + \varepsilon \zeta_\varepsilon$$

which weakly converges to  $u_n$  in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  when  $\varepsilon$  goes to zero.

Now (24) yields

$$\lim_{\eta \rightarrow 0} \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} H_\varepsilon(\tilde{u}_{n,\varepsilon}) = H(\mu).$$

We end the proof by a diagonal argument.  $\square$

Noticing that condition

$$\|\mathcal{L}\|_\infty \leq \alpha \min\left(\frac{1}{L}, \frac{1}{C'_{PW}(1+L)}\right) \quad (28)$$

implies  $\inf(\mathcal{P}_{H_\varepsilon}) > -\infty$ , we finally obtain

**Theorem 3.** *Under condition (28), the problem*

$$(\mathcal{P}_{H_\varepsilon}) \quad \inf\left\{H_\varepsilon(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu: v \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)\right\}$$

converges to the problem

$$(\mathcal{P}_H) \quad \min\left\{H(\mu) - \int_{\mathcal{O}} \mathcal{L} \cdot \mu: v \in \mathbf{M}(\mathcal{O}, \mathbf{R}^3)\right\}$$

in the sense that, up to a subsequence, every sequence  $(\mu_\varepsilon)_\varepsilon$  of  $\varepsilon$ -minimizer of  $(\mathcal{P}_{H_\varepsilon})$  weakly converges in  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  to a minimizer of  $(\mathcal{P}_H)$  and  $\inf(\mathcal{P}_{H_\varepsilon})$  tends to  $\min(\mathcal{P}_H)$ .

### Appendix 1. $\Gamma$ -convergence (or epiconvergence)

Let  $(X, \tau)$  be a first countable topological space and  $\{(F_n)_{n \in \mathbf{N}}, F, n \rightarrow +\infty\}$  a sequence of functionals mapping  $X$  into  $\mathbf{R} \cup \{+\infty\}$ . The following notion of convergence, equivalent to the convergence of the epigraph of  $F_n$  to that of  $F$  in the Kuratowski–Painlevé sense, was first introduced by De Giorgi and Franzoni [13]. For overview on epiconvergence, we refer the reader to Attouch [6] and Dal Maso [12]. Since the functionals considered in the paper are equi-coercive and the restriction of the weak \* topology of  $\mathbf{M}(\mathcal{O}, \mathbf{R}^3)$  to bounded sets is first countable, the theory of first countable spaces can be applied for  $X = \mathbf{M}(\mathcal{O}, \mathbf{R}^3)$ .

**Definition 3.** The sequence  $(F_n)_{n \in \mathbf{N}}$   $\Gamma$ -converges (or epiconverges) to  $F$  at  $x$  in  $X$  iff both following assertions hold:

- (i) for every sequence  $(y_n)_{n \in \mathbf{N}}$ , converging to  $x$ ,

$$F(x) \leq \liminf_{n \rightarrow +\infty} F_n(y_n),$$

- (ii) there exists a sequence  $(x_n)_{n \in \mathbf{N}}$  of  $X$ , converging to  $x$ , such that

$$F(x) \geq \limsup_{n \rightarrow +\infty} F_n(x_n).$$

When (i) and (ii) hold for every  $x$  in  $X$ , we say that  $(F_n)_n$   $\Gamma$ -converges (or epiconverges) to  $F$  in  $(X, \tau)$  and we write  $F = \text{epilim } F_n$ .

The main interest of this concept is its variational nature precised in item (i) below. Moreover this convergence is stable under continuous perturbation (item (ii) below).

**Proposition 12.** Assume that  $(F_n)_{n \in \mathbf{N}}$   $\Gamma$ -converges to  $F$ .

(i) Let  $x_n \in X$  be such that

$$F_n(x_n) \leq \inf\{F_n(x): x \in X\} + \varepsilon_n,$$

where  $\varepsilon_n > 0$ ,  $\varepsilon_n \rightarrow 0$ . Assume furthermore that  $\{x_n, n \in \mathbf{N}\}$  is relatively compact. Then any cluster point  $\bar{x}$  of  $\{x_n, n \in \mathbf{N}\}$  is a minimizer of  $F$  and

$$\lim_{n \rightarrow +\infty} \inf\{F_n(x): x \in X\} = F(\bar{x}).$$

(ii) If  $\Phi: (X, \tau) \rightarrow \mathbf{R}$  is continuous then  $(F_n + \Phi)_{n \in \mathbf{N}}$   $\Gamma$ -converges to  $F + \Phi$ .

## Appendix 2

In this appendix we aim to compute the Legendre–Fenchel transform of

$$h: a \mapsto h(a) = \inf\left\{\int_{Y \setminus T} f^{**}(\nabla u) \, dx: u \in \text{adm}(a)\right\},$$

where

$$\text{adm}(a) = \left\{u \in W_{\#}^{1,1}(Y, \mathbf{R}^3): \int_Y u \, dx = a, u = 0 \text{ on } T\right\}.$$

**Proposition 13.** The Legendre–Fenchel transform of  $h$  is given for every  $a^* \in \mathbf{R}^3$  by

$$h^*(a^*) = \inf\left\{\int_{Y \setminus T} f^*(\sigma) \, dx: \sigma \in \text{adm}^*(a^*)\right\} = \tilde{f}_0^*(a^*),$$

where  $\sigma \mapsto f^*(\sigma)$  denotes the Legendre–Fenchel transform of  $a \mapsto f(a)$  and

$$\text{adm}^*(a^*) := \{\sigma \in L^\infty(Y \setminus T, \mathcal{M}_3), \text{div } \sigma = -a^* \text{ in } Y \setminus T, \sigma n \text{ antiperiodic on } \partial(Y \setminus T) \setminus \partial T\}.$$

**Proof.** By definition and according to the second expression of  $h$  given in Proposition 1(i), we have

$$\begin{aligned} h^*(a^*) &= \sup\left\{a^* \cdot a - \inf\left\{\int_{Y \setminus T} f^{**}(\nabla u) \, dx: u \in \text{adm}(a)\right\}\right\} \\ &= \sup_{(a,u) \in \mathbf{R}^3 \times W_{\#}^{1,1}(Y, \mathbf{R}^3)} \left\{a^* \cdot a - \left(\int_{Y \setminus T} f^{**}(\nabla u) \, dx + I(a, u)\right)\right\}, \end{aligned} \quad (29)$$

where

$$I(a, u) = \begin{cases} 0 & \text{if } u \in \text{adm}(a), \\ +\infty & \text{otherwise.} \end{cases}$$



Working with the new variable  $q \in L^1(Y \setminus T, \mathcal{M}_3)$ , we obtain

$$h^*(a^*) = \sup_{(a,q) \in \mathbf{R}^3 \times L^1(Y \setminus T, \mathcal{M}_3)} \left\{ a^* \cdot a + \int_{Y \setminus T} 0 : q \, dx - \left( \int_{Y \setminus T} f^{**}(q) \, dx + \tilde{I}(a, q) \right) \right\},$$

where

$$\tilde{I}(a, q) = \begin{cases} 0 & \text{if } \exists u \in \text{adm}(a), q = \nabla u \lfloor Y \setminus T, \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore

$$h^*(a^*) = (J + \tilde{I})^*(a^*, 0),$$

where  $J$  is the integral functional defined in  $\mathbf{R}^3 \times L^1(Y \setminus T, \mathcal{M}_3)$  by

$$J(a, q) = \int_{Y \setminus T} f^{**}(q) \, dx.$$

Classical convex analysis computations yield

$$\begin{aligned} h^*(a^*) &= (J^* \tilde{I}^*)(a^*, 0) \\ &= \inf_{(b^*, z^*) \in \mathbf{R}^3 \times L^\infty(Y \setminus T, \mathcal{M}_3)} (J^*(a^* - b^*, -z^*) + \tilde{I}^*(b^*, z^*)). \end{aligned} \quad (30)$$

Let us now compute  $I^*$  and  $J^*$ . By definition

$$\begin{aligned} \tilde{I}^*(b^*, z^*) &= \sup_{(b,z) \in \mathbf{R}^3 \times L^1(Y \setminus T, \mathcal{M}_3)} \left\{ b \cdot b^* + \int_{Y \setminus T} z : z^* \, dx - \tilde{I}(b, z) \right\} \\ &= \sup_{(b,u) \in \mathbf{R}^3 \times \text{adm}(b)} \left\{ b \cdot b^* + \int_{Y \setminus T} \nabla u : z^* \, dx \right\} \\ &= \sup_{(b,u) \in \mathbf{R}^3 \times \text{adm}(b)} \left\{ \int_{Y \setminus T} u \cdot b^* - \int_{Y \setminus T} u \cdot \text{div}(z^*) \, dx + \int_{\partial(Y \setminus T) \setminus \partial T} z^* n \cdot u \, d\mathcal{H}^2 \right\} \\ &= K(b^*, z^*), \end{aligned} \quad (31)$$

where

$$K(b^*, z^*) = \begin{cases} 0 & \text{if } \text{div}(z^*) = b^* \text{ in } Y \setminus T, z^* \cdot n \text{ antiperiodic on } \partial(Y \setminus T) \setminus \partial T, \\ +\infty & \text{otherwise.} \end{cases}$$

On the other hand, we classically have

$$J^*(c^*, z^*) = \sup_{(c,z) \in \mathbf{R}^3 \times L^1(Y \setminus T, \mathcal{M}_3)} \left\{ c^* \cdot c + \int_{Y \setminus T} z^* : z \, dx - \int_{Y \setminus T} f^{**}(z) \, dx \right\}.$$

Noticing that  $(f^{**})^* = f^*$ , we obtain

$$J^*(c^*, z^*) = \begin{cases} \int_{Y \setminus T} f^*(z^*) \, ds & \text{if } c^* = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (32)$$

We conclude by collecting (31), (32) and (30). Equality  $h^*(a^*) = \tilde{f}_0^*(a^*)$  is a straightforward consequence of above calculus by substituting  $f$  for  $f^{**}$  in (29).  $\square$

### Appendix 3

We now establish the proof of Lemma 2. Actually, we give a more general result in a second assertion.

**Lemma 3.** *Let  $a, b$  in  $\mathbf{R}$ . Then*

$$\{\mu \in \mathbf{M}((a, b), \mathbf{R}^m) : D\mu \in \mathbf{M}((a, b), \mathbf{R}^m)\} = \text{BV}((a, b), \mathbf{R}^m).$$

*If  $\Omega$  is an open subset of  $\mathbf{R}^N$ , then*

$$\{\mu \in \mathbf{M}(\Omega, \mathbf{R}^m) : D\mu \in \mathbf{M}(\Omega, \mathbf{R}^{mN})\} \subset \text{BV}_{\text{loc}}(\Omega, \mathbf{R}^m).$$

**Proof.** For any measure  $\nu \in \mathbf{M}(\Omega, \mathbf{R}^m)$ , we denote by  $\tilde{\nu}$  its extension in  $\mathbf{R}^N$ , that is to say the measure  $\tilde{\nu}$  in  $\mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$  defined for every Borel set of  $\mathbf{R}^N$  by  $\tilde{\nu}(B) = \nu(B \cap \Omega)$ . On the other hand, for any  $\nu \in \mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$ , we denote by  $\nu|_{\Omega}$  its restriction to the Borel field of  $\Omega$ .

Consider a sequence  $(\rho_\varepsilon)_{\varepsilon>0}$  of mollifiers. Let us recall that  $\rho_\varepsilon$  is a function in  $\mathcal{C}_c^\infty(\mathbf{R}^N)$  defined by  $\rho_\varepsilon(x) = \varepsilon^{-N} \rho(x/\varepsilon)$  where  $\rho$  is a symmetric nonnegative real-valued function in  $\mathcal{C}_c^\infty(\mathbf{R}^N)$  satisfying

$$\int_{\mathbf{R}^N} \rho(x) \, dx = 1, \quad \text{spt } \rho \subset \overline{B}_1(0).$$

For any  $\nu \in \mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$  we define on  $\mathbf{R}^N$  the function  $\rho_\varepsilon * \nu$  by

$$\rho_\varepsilon * \nu(x) = \int_{\mathbf{R}^N} \rho_\varepsilon(x - y) \nu(dy).$$

We then have

**Lemma 4.** *The functions  $\rho_\varepsilon * \nu$  belong to  $\mathcal{C}^\infty(\mathbf{R}^N, \mathbf{R}^m)$  and  $\rho_\varepsilon * \nu$  weakly converges to  $\nu$  in  $\mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$ .*

For a proof, consult, for instance, Temam [18].

We begin by proving the second assertion of Lemma 3. Let  $\mu \in \mathbf{M}(\Omega, \mathbf{R}^m)$  such that  $D\mu \in \mathbf{M}(\Omega, \mathbf{R}^{mN})$ . For all  $\mathcal{C}^1$ -domain  $\omega$ ,  $\omega \Subset \Omega$ , and for  $\varepsilon < \text{dist}(\omega, \Omega^c)$  we have

$$\begin{cases} (\rho_\varepsilon * \tilde{\mu})|_{\Omega} \rightharpoonup \mu & \text{weakly in } \mathbf{M}(\Omega, \mathbf{R}^m), \\ D((\rho_\varepsilon * \tilde{\mu})|_{\Omega}) & \text{is bounded in } \mathbf{M}(\omega, \mathbf{R}^{mN}). \end{cases}$$

Indeed the first assertion is a straightforward consequence of Lemma 4. For the second, reasoning in the distributional sense, it is easily seen that

$$D((\rho_\varepsilon * \tilde{\mu}) \llcorner \Omega) = \rho_\varepsilon * D\mu$$

in  $\omega$  for all  $\varepsilon < \text{dist}(\omega, \Omega)$  so that

$$\begin{aligned} \int_\omega |D((\rho_\varepsilon * \tilde{\mu}) \llcorner \Omega)| &= \sup_{\varphi \in \mathcal{C}_c(\omega, \mathbf{R}^{mN}), \|\varphi\|_\infty \leq 1} \int_\omega \rho_\varepsilon * D\mu \cdot \varphi \, dx \\ &= \sup_{\varphi \in \mathcal{C}_c(\omega, \mathbf{R}^{mN}), \|\varphi\|_\infty \leq 1} \int_\omega \rho_\varepsilon * \varphi D\mu \\ &\leq \int_\omega |D\mu| \leq \int_\Omega |D\mu|. \end{aligned}$$

The sequence  $((\rho_\varepsilon * \tilde{\mu}) \llcorner \Omega)_{\varepsilon > 0}$  is then bounded in  $\text{BV}(\omega, \mathbf{R}^m)$ . By the compactness embedding of  $\text{BV}(\omega, \mathbf{R}^m)$  into  $L^1(\omega, \mathbf{R}^m)$ , there exist a subsequence (not relabeled) and  $u \in L^1(\omega, \mathbf{R}^m)$  such that  $(\rho_\varepsilon * \tilde{\mu}) \llcorner \Omega$  strongly converges to  $u$  in  $L^1(\omega, \mathbf{R}^m)$ . Consequently  $\mu = u$  belongs to  $L^1(\omega, \mathbf{R}^m)$ .

We now establish the first assertion. According to Lemma 4

$$\theta_\varepsilon := (\rho_\varepsilon * \widetilde{D\mu}) \llcorner (a, b) \rightharpoonup D\mu \quad \text{weakly in } \mathbf{M}((a, b), \mathbf{R}^m).$$

We set

$$u_\varepsilon := \int_a^x \theta_\varepsilon(t) \, dt$$

so that  $Du_\varepsilon \rightharpoonup D\mu$  weakly in  $\mathbf{M}((a, b), \mathbf{R}^m)$ . Clearly the sequence  $(u_\varepsilon)_{\varepsilon > 0}$  is bounded in  $\text{BV}((a, b), \mathbf{R}^m)$ . Therefore, by the compactness embedding of  $\text{BV}((a, b), \mathbf{R}^m)$  into  $L^1((a, b), \mathbf{R}^m)$ , there exist a subsequence (not relabeled) and  $u \in L^1((a, b), \mathbf{R}^m)$  such that  $u_\varepsilon \rightarrow u$  strongly in  $L^1((a, b), \mathbf{R}^m)$ . We then have  $D\mu = Du$  in the distributional sense so that  $\mu$  differs from  $u$  by a constant, then belongs to  $L^1((a, b), \mathbf{R}^m)$ .  $\square$

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