



# Mathematical modelling of piezoelectric thin plates and slender beams through functional analysis

Doungporn Vrisysrisuwattana, Christian Licht, Thibaut Weller, Sanoe Koonprasert

## ► To cite this version:

Doungporn Vrisysrisuwattana, Christian Licht, Thibaut Weller, Sanoe Koonprasert. Mathematical modelling of piezoelectric thin plates and slender beams through functional analysis. The twelfth Annual Meeting in Mathematics, 2007, Chonburi, Thailand. 10p. hal-00574728

**HAL Id: hal-00574728**

**<https://hal.science/hal-00574728>**

Submitted on 6 Mar 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Mathematical Modeling of Piezoelectric Thin Plates and Slender Beams through Functional Analysis

Doungporn Viriyasrisuwattana<sup>1</sup>, Christian Licht<sup>2</sup>, Thibaut Weller<sup>2</sup>, Sanoe Koonprasert<sup>1</sup>

<sup>1</sup> Department of Applied Mathematics, Faculty of Applied Science, King Mongkut's Institute of Technology North Bangkok, Bangkok, 10800, Thailand

<sup>2</sup> Laboratoire de Mécanique et Génie Civil, Université Montpellier 2, Montpellier, France

**Abstract** A simplified but accurate modeling of linearly piezoelectric thin plates or slender beams is derived by a rigorous study of the asymptotic behavior of a three-dimensional body when some of its dimensions, considered as parameters, tend to zero. The study is carried out using some tools of applied functional analysis, e.g., singular perturbations in variational equations.

## 1. Introduction

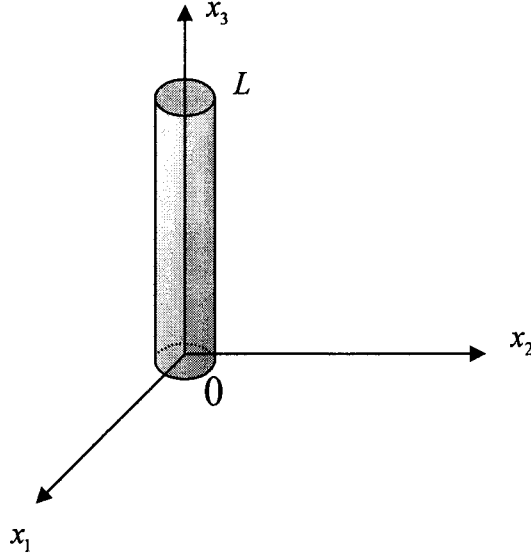
A modeling of thin linearly piezoelectric plates was proposed in [1], [2] by using the methods from [3]. The modeling was derived by a rigorous study of the asymptotic behavior of a three-dimensional body when its thickness, considered as a parameter, tended to zero. The limiting model obtained was simpler than the genuine three-dimensional one because the limiting model involved two-dimensional problems set over the mean surface of the plate. This limiting model was also proved to be accurate using some convergence results of the fields when the thickness goes to zero. The derivation of the model was as follows. First, the linear boundary value problem associated with the equilibrium of a plate of thickness  $2\varepsilon$  was transformed into a variational equation whose existence and uniqueness of a solution  $s^\varepsilon$  was proved by the Lions-Stampacchia theorem. This problem set on a variable domain  $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$  was transformed by a suitable scaling  $S(\varepsilon)$  (a change of coordinates and of unknowns) into an equivalent problem set on a fixed domain  $\Omega = \omega \times (-1, 1)$  with  $s(\varepsilon) = S(\varepsilon)s^\varepsilon$  as a unique solution. Next, a technique of singular perturbations in variational equations permitted the determination of the limit  $\bar{s}$  of  $s(\varepsilon)$  when  $\varepsilon$  goes to zero. Finally, the proposed limiting model was the problem, set on  $\Omega^\varepsilon$ , solved by the inverse scaling  $\bar{s}^\varepsilon$  of  $\bar{s}$ . Actually, depending on the boundary conditions, two types of models were obtained; corresponding to the cases of sensors or of actuators.

Here, we try to extend the above modeling method to the case of slender linearly piezoelectric rods. For slender linearly elastic rods, the models and their derivations are known to be more complex, and therefore we confine the modeling to the case of a rod working as a sensor and made of a particular class of monoclinic piezoelectric material.

## 2. Setting the problem

A model of a linearly piezoelectric rod is shown in Figure 1. For all  $\zeta = (\zeta_1, \zeta_2, \zeta_3)$  in  $R^3$  we denote  $(\zeta_1, \zeta_2)$  by  $\hat{\zeta}$  and let  $\mathfrak{H}$  stand for the space  $S^3 \times R^3$ , where  $S^3$  denotes the space of all  $3 \times 3$  symmetric matrices. The Greek indices run in  $\{1, 2\}$  while the Latin ones run in  $\{1, 2, 3\}$ . The reference configuration of a linearly piezoelectric slender rod is the closure in  $R^3$  of the set  $\Omega^\varepsilon := \varepsilon\omega \times (0, L)$ , where  $L$  is a real positive number,  $\varepsilon$  a small real positive number and  $\omega$  is a bounded domain of  $R^2$  with a Lipschitz boundary  $\partial\omega$  and such that:

$$\int_{\omega} x_a d\hat{x} = \int_{\omega} x_1 x_2 d\hat{x} = 0 \quad (1)$$



**FIGURE 1** A piezoelectric slender beam

The lateral part of the boundary of the plate  $\varepsilon\partial\omega \times (0, L)$  is denoted by  $\Gamma_{lat}^\varepsilon$  while its basis are  $\Gamma_0^\varepsilon = \varepsilon\partial\omega \times \{0\}$  and  $\Gamma_L^\varepsilon = \varepsilon\partial\omega \times \{L\}$ . Let  $(\Gamma_{mD}^\varepsilon, \Gamma_{mN}^\varepsilon)$  and  $(\Gamma_{eD}^\varepsilon, \Gamma_{eN}^\varepsilon)$  be two suitable partitions of  $\partial\Omega^\varepsilon$  with both  $\Gamma_{mD}^\varepsilon$  and  $\Gamma_{eD}^\varepsilon$  of strictly positive surface measure. The beam is clamped along  $\Gamma_{mD}^\varepsilon$  and at an electrical potential  $j_0^\varepsilon$  on  $\Gamma_{eD}^\varepsilon$ . The beam is subjected to body forces  $f^\varepsilon$  on  $\Omega^\varepsilon$ . The beam is also subjected to surface forces  $g^\varepsilon$  and an electrical loading  $d^\varepsilon$  on  $\Gamma_{mN}^\varepsilon$  and  $\Gamma_{eN}^\varepsilon$  respectively. We denote the outward unit normal to  $\partial\Omega^\varepsilon$  by  $n^\varepsilon$  and assume  $\Gamma_{mD}^\varepsilon = \Gamma_0^\varepsilon$ . Then the equations determining the state  $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$  at equilibrium read as follows:

$$\begin{cases} \operatorname{div} \sigma^\varepsilon + f^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \sigma^\varepsilon \cdot n^\varepsilon = g^\varepsilon & \text{on } \Gamma_{m,N}^\varepsilon, u^\varepsilon = 0 & \text{on } \Gamma_{m,D}^\varepsilon \\ \operatorname{div} D^\varepsilon = 0 & \text{in } \Omega^\varepsilon, D^\varepsilon \cdot n^\varepsilon = d^\varepsilon & \text{on } \Gamma_{e,N}^\varepsilon, \varphi^\varepsilon = \varphi_0^\varepsilon & \text{on } \Gamma_{e,D}^\varepsilon \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(x)(e(u^\varepsilon), \nabla \varphi^\varepsilon) & \text{in } \Omega^\varepsilon \end{cases} \quad (2)$$

where  $u^\varepsilon, \varphi^\varepsilon, e(u^\varepsilon), \sigma^\varepsilon$  and  $D^\varepsilon$  respectively denote the displacement and electrical fields, the linearized strain tensor, the stress tensor and the electrical displacement. We let  $M^\varepsilon$  be an element of  $\mathcal{L}(\mathcal{H}, \mathcal{H})$  such that:

$$\begin{cases} \sigma^\varepsilon = a^\varepsilon e(u^\varepsilon) - b^{\varepsilon T} \nabla \varphi^\varepsilon, \\ D^\varepsilon = (b^\varepsilon)^T e(u^\varepsilon) + c^\varepsilon \nabla \varphi^\varepsilon \end{cases} \quad (3)$$

where  $b^{\varepsilon T}$  denotes the transpose of  $b^\varepsilon$ , and  $a^\varepsilon$  and  $c^\varepsilon$  are symmetric and positive.

To give a variational formulation of (3) we make the following regularity hypothesis on the exterior loading:

$$(H_1) \begin{cases} (f^\varepsilon, g^\varepsilon, d^\varepsilon) \in L^2(\Omega^\varepsilon)^3 \times L^2(\Gamma_{m,N}^\varepsilon)^3 \times L^2(\Gamma_{e,N}^\varepsilon) \\ \varphi_0^\varepsilon \text{ has an } H^1(\Omega^\varepsilon) \text{ extension into } \Omega^\varepsilon, \text{ still denoted by } \varphi_0^\varepsilon, \end{cases}$$

and on the space of electromechanical states

$$V^\varepsilon := \left\{ r = (v, \psi) \in H_{\Gamma_{m,D}^\varepsilon}^1(\Omega^\varepsilon)^3 \times H_{\Gamma_{e,D}^\varepsilon}^1(\Omega^\varepsilon) \right\} \quad (4)$$

we define a bilinear form  $m^\varepsilon$ :

$$m^\varepsilon(r, q) = m^\varepsilon((v, \psi), (w, \varphi)) := \int_{\Omega^\varepsilon} M^\varepsilon(e(v), \nabla \psi) \cdot (e(w), \nabla \varphi) dx^\varepsilon \quad (5)$$

and a linear form  $L^\varepsilon$

$$L^\varepsilon = L^\varepsilon((v, \psi)) := \int_{\Omega^\varepsilon} f^\varepsilon \cdot v dx^\varepsilon + \int_{\Gamma_{m,N}^\varepsilon} g^\varepsilon \cdot v ds^\varepsilon + \int_{\Gamma_{e,N}^\varepsilon} d^\varepsilon \psi ds^\varepsilon \quad (6)$$

Then, the physical problem, set on the real beam, takes the form:

$$\mathcal{P}(\Omega^\varepsilon): \begin{cases} \text{Find } s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon) \in (0, \varphi_0^\varepsilon) + V^\varepsilon \text{ such that} \\ m^\varepsilon(s^\varepsilon, r) = L^\varepsilon(r), \quad \forall r \in V^\varepsilon. \end{cases}$$

Then, with the additional hypothesis  $H_2$  of the additional and realistic assumptions of boundedness of  $a^\varepsilon, b^\varepsilon, c^\varepsilon$  and of uniform ellipticity of  $a^\varepsilon$  and  $c^\varepsilon$ :

$$(H_2) \begin{cases} M^\varepsilon \in L^\infty(\Omega^\varepsilon, \mathcal{L}(\mathcal{H})) \\ \exists \kappa^\varepsilon > 0, M^\varepsilon(x^\varepsilon) h \cdot h \geq \kappa^\varepsilon |h|_{\mathcal{H}}^2, \forall h \in \mathcal{H}, \text{ a.e. } x \in \Omega^\varepsilon \end{cases}$$

the theorem of Lions-Stampacchia implies:

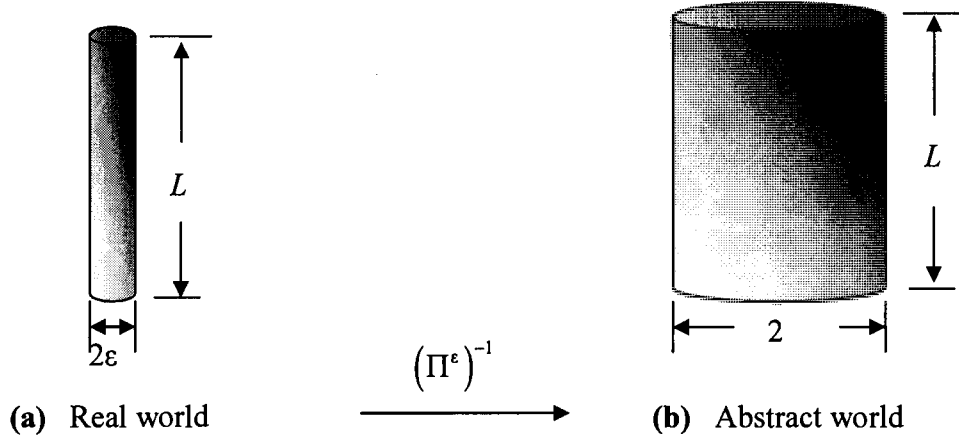
Theorem 1: Under assumptions  $(H_1) - (H_2)$ , the problem  $\mathcal{P}(\Omega^\varepsilon)$  has a unique solution.

To derive a simplified and accurate model, we study the behavior of  $s^\varepsilon$  when  $\varepsilon$ , considered as a parameter, tends to 0.

### 3. The scaling operation

Classically, as shown in Figure 2, we can come down to a fixed domain  $\Omega = \omega \times (0, L)$  through the mapping  $\Pi^\varepsilon$ :

$$x = (\hat{x}, x_3) \in \bar{\Omega} \mapsto \Pi^\varepsilon x = (\varepsilon \hat{x}, x_3) \in \bar{\Omega}^\varepsilon \quad (7)$$



**FIGURE 2** (a) Real state, physical state, defined on  $\Omega^\varepsilon$   
(b) Abstract state or scaled state defined on  $\Omega$

Also, we drop the index  $\varepsilon$  for the images by  $(\Pi^\varepsilon)^{-1}$  of the previous geometric sets. We make the following assumptions concerning

i) The electromechanical coefficients:

$$(H_3) \begin{cases} \text{there exists } M \in L^\infty(0, L; \mathcal{L}(\mathfrak{H})) \text{ such that } M^\varepsilon(\Pi^\varepsilon x) = M(x) = M(x_3), \forall x \in \Omega \\ \text{and } \exists \kappa > 0: M(x_3)h \cdot h \geq \kappa |h|_{\mathfrak{H}}^2 \quad \forall h \in \mathfrak{H} \text{ a.e. } x \in \Omega. \text{ Moreover the material is} \\ \text{monoclinic: } a_{\alpha 3 \gamma \delta} = a_{\alpha 333} = b_{\alpha 3 \gamma} = b_{\alpha 33} = 0. \end{cases}$$

ii) The electromechanical loading:

$$(H_4) \begin{cases} \text{there exist } (f, g, d) \in L^2(\Omega)^3 \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN}) \text{ such that} \\ f_a^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 f_a(x), f_3^\varepsilon(\pi^\varepsilon x) = \varepsilon f_3(x), \forall x \in \Omega, \\ g_a^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 g_a(x), g_3^\varepsilon(\pi^\varepsilon x) = \varepsilon g_3(x), \forall x \in \Gamma_L, \\ g_a^\varepsilon(\pi^\varepsilon x) = \varepsilon^3 g_a(x), g_3^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 g_3(x), \forall x \in \Gamma_{mN} \cap \Gamma_{lat}, \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon d(x), \forall x \in \Gamma_{eN} \cap (\Gamma_0 \cup \Gamma_L), \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 d(x), \forall x \in \Gamma_{eN} \cap \Gamma_{lat}, \\ j_0^\varepsilon(\pi^\varepsilon x) = \varepsilon j_0(x), \forall x \in \Gamma_{eD} \end{cases}$$

iii) The boundedness of the exterior electrical loading:

$$(H_5) \begin{cases} \text{the extension of } \varphi_0 \text{ into } \Omega \text{ does not depend on } \hat{x} \text{ and } \Gamma_{eD} \subset \Gamma_0 \cup \Gamma_L \\ \text{with } \text{meas}(\Gamma_{eD} \cap \Gamma_0) > 0 \end{cases}$$

(This implies that  $\varphi_0$  may take two different constant values on  $\Gamma_{eD} \cap \Gamma_0$  and  $\Gamma_{eD} \cap \Gamma_L$  whenever the last set is of positive surface measure).

Also, we associate a scaled electromechanical state  $s(\varepsilon) := (u(\varepsilon), \varphi(\varepsilon))$  defined on  $\Omega$  with the true physical electromechanical state  $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$  defined in  $\Omega^\varepsilon$ . This scaled electromechanical state, without any physical meaning, is defined by:

$$\hat{u}(x^\varepsilon) = \hat{u}(\varepsilon)(x), \quad u_3^\varepsilon(x^\varepsilon) = \varepsilon u(\varepsilon)_3(x), \quad \varphi^\varepsilon(x^\varepsilon) = \varepsilon \varphi(\varepsilon)(x) \quad (8)$$

for all  $x^\varepsilon = \Pi^\varepsilon x$  in  $\Omega^\varepsilon$ . This allows us to introduce the scaling operator  $S^\varepsilon$ :

$$S^\varepsilon s^\varepsilon(x^\varepsilon) := s(\varepsilon)(x), \quad \forall x^\varepsilon = \Pi^\varepsilon x \in \Omega^\varepsilon \quad (9)$$

Assumptions  $(H_3), (H_4)$  together with the scaling mapping from equation (8) are classical. Actually, they are justified by the convergence results they lead to. If we just consider the displacement these hypotheses are the ones made in [4] and supply a mathematical justification of the Bernoulli-Navier theory of slender linearly elastic beams.

#### 4. Variational formulation of the scaled problem

Let  $V$  be the space of the scaled electromechanical states:

$$V := \left\{ r = (v, \psi) \in H_{\Gamma_{md}}^1(\Omega)^3 \times H_{\Gamma_{ed}}^1(\Omega) \right\} \quad (10)$$

Of course,  $r \in V^\varepsilon$  if and only if  $\Pi^\varepsilon r \in V$ . Now, for all  $r = (v, \psi) \in V$ , we define the scaled strain tensor  $e(\varepsilon, v)$  and the scaled electrical potential gradient  $\nabla(\varepsilon, \psi)$  by:

$$\begin{aligned} e_{\alpha\beta}(\varepsilon, v) &:= \varepsilon^{-2} e_{\alpha\beta}(v), \quad e_{\alpha 3}(\varepsilon, v) := \varepsilon^{-1} e_{\alpha 3}(v), \quad e_{33}(\varepsilon, v) := e_{33}(v) \\ \nabla(\varepsilon, \psi)_\alpha &:= \varepsilon^{-1} \partial_\alpha \psi, \quad \nabla(\varepsilon, \psi)_3 := \partial_3 \psi \end{aligned} \quad (11)$$

To simplify the notation, we set

$$k(r) := (e(v), \nabla \psi), \quad k(\varepsilon, r) := (e(\varepsilon, v), \nabla(\varepsilon, \psi)) \quad (12)$$

and, as in (5) and (6), we introduce a bilinear form  $m(\varepsilon)$  and a linear form  $L$  on  $V$ :

$$\begin{aligned} m(\varepsilon)(r, q) &:= \int_\Omega M(x) k(\varepsilon, r) \cdot k(\varepsilon, q) \, dx, \\ L(r) &:= \int_\Omega f \cdot v \, dx + \int_{\Gamma_{mN}} g \cdot v \, ds + \int_{\Gamma_{eN}} d \psi \, ds \end{aligned} \quad (13)$$

Then, under the assumptions  $(H_1) - (H_4)$  the scaled electromechanical state  $s(\varepsilon) = (u(\varepsilon), \varphi(\varepsilon))$  is the unique solution of the mathematical problem

$$\mathcal{P}(\varepsilon, \Omega): \begin{cases} \text{Find } s(\varepsilon) \in (0, \varphi_0) + V \text{ such that} \\ m(\varepsilon)(s(\varepsilon), r) = L(r) \quad \forall r \in V. \end{cases}$$

#### 5. Asymptotic behavior of the scaled electromechanical state

##### 5.1 The method

We will proceed in a similar way as in the case of plates by first introducing a suitable orthogonal decomposition of  $\mathfrak{H}$  into three subspaces  $\mathfrak{H}^1$ ,  $\mathfrak{H}^2$  and  $\mathfrak{H}^3$ :

$$\begin{cases} \mathfrak{H}^1 := \{h = (e, g) \in \mathfrak{H}; e_{i3} = g_3 = 0\}, \mathfrak{H}^2 := \{h = (e, g) \in \mathfrak{H}; e_{\alpha\beta} = e_{33} = g_i = 0\} \\ \mathfrak{H}^3 := \{h = (e, g) \in \mathfrak{H}; e_{\alpha\beta} = e_{\alpha 3} = g_\alpha = 0\} \end{cases} \quad (14)$$

We denote the projection of any element  $h$  of  $\mathfrak{H}$  on  $\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3$  by  $h^1, h^2, h^3$  respectively and consider the operators  $M^k \in \mathcal{L}(\mathfrak{H}^l, \mathfrak{H}^k)$  generated by  $M$ . The hypothesis  $(H_3)$  implies that  $M^{11}$  and  $M^{33}$  are positive invertible operators on  $\mathfrak{H}^1$  and  $\mathfrak{H}^3$ . This same assumption also implies:

$$\begin{aligned} (Mh)^1 &= M^{11}h^1 + M^{13}h^3 \\ (Mh)^3 &= M^{31}h^1 + M^{33}h^3 \end{aligned}$$

and consequently:

$$(Mh)^3 = \left( M^{33} - M^{31} (M^{11})^{-1} M^{13} \right) h^3 + M^{31} (M^{11})^{-1} (Mh)^1 \quad (15)$$

The operator (again a kind of Schur complement!) of  $\mathcal{L}(\mathfrak{H}^3, \mathfrak{H}^3)$

$$\tilde{M} := M^{33} - M^{31} (M^{11})^{-1} M^{13} \quad (16)$$

can be represented by a  $2 \times 2$  matrix while  $h^3$  can be represented by an element of  $R^2$ . This will be done from now on. Note that neither  $M^{33}$  nor  $\tilde{M}$  are necessarily symmetric but nevertheless

$$\kappa |h^3|^2 \leq \tilde{M}(x) h^3 \cdot h^3, \quad \forall h^3 \in R^2 \quad \text{a.e. } x \in \Omega \quad (17)$$

This is implied by the coercivity of  $M$  (see  $(H_3)$ ) and by the fundamental relation:

$$\begin{aligned} h^2 = 0, (Mh)^1 = 0 &\Rightarrow \tilde{M}h^3 = (Mh)^3 \\ \tilde{M}h^3 \cdot h^3 &= (Mh)^3 \cdot h^3 = Mh \cdot h \geq \kappa |h|^2 \geq \kappa |h^3|^2 \end{aligned} \quad (18)$$

As in the case of plates, the key point of the asymptotic study will be to show that if  $\bar{k}$  is the limit (in a suitable topology) of  $k(\varepsilon, s(\varepsilon))$ , then  $\int_{\omega} (M(x_3) \bar{k}(x))^1 d\hat{x} = 0$ .

This will enable us to show that  $\tilde{M}$  supplies the limit constitutive equations.

## 5.2 The functional framework

We will show that the limit displacements live in the space  $V_{BN}$  of Bernoulli Navier displacements:

$$V_{BN} := \left\{ v \in H_{\Gamma_{mD}}^1(\Omega)^3; e_{\alpha\beta}(v) = e_{\alpha 3}(v) = 0 \right\} \quad (19)$$

and that the limit electrical potential belongs to

$$\Phi := \left\{ \psi \in H_{\Gamma_{eD}}^1(\Omega); \hat{\nabla} \psi = 0 \right\} = \left\{ \psi; \psi(x) = \psi(x_3), \psi \in H^1(0, L), \right. \\ \left. \psi(0) = 0 \text{ and } \psi(L) = 0 \text{ if } \text{meas}(\Gamma_{eD} \cap \Gamma_L) > 0 \right\} \quad (20)$$

We recall that for all  $v$  in  $V_{BN}$ , there exists a unique couple  $(v^b, v^s)$  in  $H^{1,0}(0, L)^2 \times H^{2,0}(0, L)^2$  such that:

$$\hat{v}(x) = v^b(x_3), v_3(x) = v^s(x_3) - x_\alpha \frac{dv_\alpha^b}{dx_3}(x_3) \quad (21)$$

with

$$H^{1,0}(0, L) = \{v \in H^1(0, L); v(0) = 0\}, H^{2,0}(0, L) = \left\{v \in H^1(0, L); \frac{dv}{dx_3} \in H^{1,0}(0, L)\right\}$$

and note that for all  $v$  in  $V_{BN}$ :

$$e_{33}(v) = \frac{dv^s}{dx_3}(x_3) - x_a \frac{d^2 v_a^b}{dx_3^2}(x_3) \quad (22)$$

Finally, let

$$S := V_{BN} \times \Phi, \quad X := H_{\Gamma_{ed}}^1(\Omega)^3 \times H_{\Gamma_{ed}}^1(\Omega) \quad (23)$$

The Korn and Poincaré inequality ([5]) then allows us to define on  $S$  and  $X$  the following hilbertian norms:

$$\begin{aligned} |(v, \psi)|_S^2 &= |e(v)|_{L^2(\Omega, S^3)}^2 + |\nabla \psi|_{L^2(\Omega)}^2 \\ |(v, \psi)|_X^2 &= |e(v)|_{L^2(\Omega, S^3)}^2 + |\psi|_{L^2(\Omega)}^2 + |\nabla \psi|_{L^2(\Omega)}^2 \end{aligned} \quad (24)$$

The set  $(0, \varphi_0) + S$  will appear to be the limit set of scaled electromechanical states.

### 5.3 The limit scaled problem

We define the following linear and bilinear forms:

$$\begin{aligned} \cdot(v^s, \psi) \in H_{(0,L)}^{1,0} \times \Phi \mapsto L(v^s, \psi) &= \int_0^L \left( \int_{\omega} f_3(\hat{x}, x_3) d\hat{x} + \int_{\partial\omega} g_3(\hat{x}, x_3) d\hat{l} \right) v^s(x_3) dx_3 \\ &\quad + \int_0^L \left( \int_{\partial\omega} d(\hat{x}, x_3) d\hat{l} \right) \psi(x_3) dx_3 \\ &\quad + \left( \int_{\omega} g_3(\hat{x}, L) d\hat{l} \right) v^s(L) + \theta_L \psi(L) \end{aligned} \quad (25)$$

$$\theta_L = \begin{cases} 0 & \text{if } \text{meas}(\Gamma_{ed} \cap \Gamma_L) > 0 \\ \int_{\omega} d(\hat{x}, L) d\hat{x} & \text{if } \text{meas}(\Gamma_{ed} \cap \Gamma_L) = 0 \end{cases}$$

$$\begin{aligned} \cdot(v_a^b \in H^{2,0}(0, L) \# L_a^b(v_a^b) &= \int_0^L \left( \int_{\omega} \hat{f}(\hat{x}, x_3) d\hat{x} + \int_{\partial\omega} \hat{g}(\hat{x}, x_3) d\hat{l} \right) v_a^b(x_3) dx_3 \\ &\quad - \left[ \int_{\omega} x_a f_3(\hat{x}, x_3) d\hat{x} + \int_{\omega} x_a g_3(\hat{x}, x_3) d\hat{l} \right] \frac{dv_a^b}{dx_3}(x_3) dx_3 \\ &\quad + \left( \int_{\omega} \hat{g}(\hat{x}, L) d\hat{x} \right) v_a^b(L) - \left( \int_{\partial\omega} x_a g_3(\hat{x}, x_3) d\hat{l} \right) \frac{dv_a^b}{dx_3}(L) \end{aligned} \quad (26)$$

$$\begin{aligned} \cdot((u^s, \varphi), (v^s, \psi)) &\in (H^{1,0}(0, L) \times \Phi)^2 \mapsto m^s((u^s, \varphi), (v^s, \psi)) \\ &= |\omega| \int_0^L \tilde{M}(x_3) \left( \frac{du^s}{dx_3}, \frac{d\varphi}{dx_3} \right)(x_3) \cdot \left( \frac{dv^s}{dx_3}, \frac{d\psi}{dx_3} \right)(x_3) dx_3 \end{aligned} \quad (27)$$

$$\cdot(u_a^b, v_a^b) \in H^{2,0}(0, L)^2 \mapsto m_a^b(u_a^b, v_a^b) := I_a^2(\omega) \int_0^L \tilde{M}_{22}(x_3) \frac{d^2 u_a^b}{dx_3^2}(x_3) \frac{d^2 v_a^b}{dx_3^2}(x_3) dx_3 \quad (28)$$



where the convention of summation on the repeated indices is not used and  $I_\alpha^2(\omega) = \int_\omega x_\alpha^2 d\hat{x}$ . We have the following convergence result:

Theorem 2: Under the assumptions,  $(H_3) - (H_5)$  and when  $\varepsilon$  tends to 0, the family  $s(\varepsilon)$  of the unique solutions of  $\mathcal{P}(\varepsilon, \Omega)$  strongly converges in  $X$  to the unique solution  $\bar{s}$  of:

$$\bar{\mathcal{P}}(\Omega): \begin{cases} \text{Find } s \in (0, \varphi_0) + S \text{ such that} \\ m^s((u^s, \varphi), (v^s, \psi)) = L^s(v^s, \psi) \quad \forall (v^s, \psi) \in H^{1,0}(0, L) \times \Phi \\ m^b(u_a^b, v_a^b) = L_a^b(v_a^b) \quad \forall v_a^b \in H^{2,0}(0, L) \quad \alpha = 1, 2 \end{cases}$$

Proof:

1<sup>st</sup> step:

The family  $(s(\varepsilon))_{\varepsilon>0}$  is bounded in  $X$  and, therefore, there exists a subsequence, not relabeled, such that

$$(s(\varepsilon), k(\varepsilon, s(\varepsilon))) \rightharpoonup (\bar{s}, \bar{k}) \text{ in } X \times L^2(\Omega, \mathfrak{H}) \quad k^a(s(\varepsilon)) \rightarrow 0 \text{ in } L^2(\Omega, \mathfrak{H}^a)$$

$$k(\bar{s})^3 = \bar{k}^3. \quad \text{Eq. 4-29}$$

where the  $\rightharpoonup$  and  $\rightarrow$  symbols stand for the weak and strong convergence respectively. This is an obvious consequence of the assumptions  $(H_3), (H_5)$ , the Poincaré and Korn inequalities and of the trace theory in Sobolev spaces ([5]).

2<sup>nd</sup> step:

If  $(M\bar{k})^1 = (\bar{\sigma}_{\alpha\beta}, \bar{D}_\alpha)$ , then

$$\int_\omega \bar{\sigma}_{\alpha\beta}(x) d\hat{x} = \int_\omega x_\gamma \bar{\sigma}_{\alpha\beta}(x) d\hat{x} = \int_\omega \bar{D}_\alpha(x) d\hat{x} = \int_\omega x_\gamma \bar{D}_\alpha(x) d\hat{x} = 0 \quad (30)$$

Indeed let  $h$  arbitrary in  $C_0^\infty(0, L)$  and  $\alpha, \beta = \{1, 2\}$  and consider the element  $E^{\alpha\beta}$  of the space  $S^2$  of  $2 \times 2$  symmetric matrices such that

$$(E^{\alpha\beta})_{\mu\nu} = \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})$$

$\delta$  being the Kronecker symbol ( $\delta_{\mu\gamma} = 0$  if  $\mu \neq \gamma, \delta_{\mu\gamma} = 1$  if  $\mu = \gamma$ ). Let  $v$  defined by  $\hat{v}(x) = \varepsilon^2(E^{\alpha\beta}\hat{x})h(x_3), v_3(x) = 0$ . We note that

$$e_{\alpha\beta}(\varepsilon, v)(x) = E^{\alpha\beta}h(x_3), e_{\alpha 3}(\varepsilon, v) = \frac{\varepsilon}{2}(E^{\alpha\beta}\hat{x})\frac{dh}{dx_3}(x_3), e_{33}(\varepsilon, v) = 0$$

So that taking  $r = (v, 0)$  in the formulation of  $\mathcal{P}(\varepsilon, \Omega)$  and letting  $\varepsilon$  go to zero yields

$$\int_0^L \left( \int_\omega \bar{\sigma}_{\alpha\beta}(x) d\hat{x} \right) h(x_3) dx_3 = 0, \quad \forall h \in C_0^\infty(0, L)$$

that is to say

$$\int_\omega \bar{\sigma}_{\alpha\beta}(x) d\hat{x} = 0$$

Next, we use  $r(x) = (0, \varepsilon x_\alpha h(x_3))$  and  $r(x) = (0, \varepsilon x_\gamma^2 h(x_3))$  to deduce

$$\int_\omega \bar{D}_\alpha(x) d\hat{x} = \int_\omega x_\gamma \bar{D}_\alpha(x) d\hat{x} = 0.$$

We obtain the remaining terms in (30) by using scaled displacement fields, like  $(\varepsilon^2 x_\gamma^2 (E^{aa} \hat{x}) h(x_3), 0)$ , associated with vanishing scaled electrical potential fields.

3<sup>rd</sup> step: the limit problem

Choosing  $r = (v, \psi)$  arbitrary in  $S$  in the formulation of  $\mathcal{P}(\varepsilon, \Omega)$  and going to the limit implies that  $\bar{s} \in (0, \varphi_0) + S$  and

$$L(r) = \int_{\Omega} (M\bar{k})^3 k(r)^3, \quad \forall r \in S$$

But, (1),  $(H_3)$ , (11), (12), (15), (16), (21), (22), (25)-(28) and fundamentally (30) yield

$$L(r) = L^s(v^s, \psi) + \sum_{a=1}^2 L_a^b(v_a^b)$$

$$\int_{\Omega} (M\bar{k})^3 k(r)^3 = m^s((\bar{u}^s, \bar{\varphi}), (v^s, \psi)) + \sum_{a=1}^2 m_a^b(\bar{u}_a^b, v_a^b)$$

which proves that  $\bar{s}$  solves the problem  $\bar{\mathcal{P}}(\Omega)$ . As previously noted  $\tilde{M}$  can be considered as a  $2 \times 2$  positively defined matrix. Hence the Lions-Stampacchia theorem ([5]) implies that  $\bar{\mathcal{P}}(\Omega)$  has a unique solution and, consequently, the whole family  $(s(\varepsilon))_{\varepsilon > 0}$  converges to it!

4<sup>th</sup> step: strong convergence

It suffices to proceed as in the case of plates (see [1], [2]) by taking  $h = k(\varepsilon, s(\varepsilon)) - \bar{k}$  in  $(H_3)$ .

## 6. Back to the problem $\mathcal{P}(\Omega^\varepsilon)$ : a proposal of a simplified and accurate model

We now consider the reference configuration  $\bar{\Omega}^\varepsilon$  of the real slender beam through the operators  $\Pi^\varepsilon$  and  $(S^\varepsilon)^{-1}$ . With the solution  $\bar{s}$  of  $\bar{\mathcal{P}}(\Omega)$  is associated a physical electromechanical state  $\bar{s}^\varepsilon$  defined on  $\Omega^\varepsilon$  by:

$$\bar{s}^\varepsilon(\Pi^\varepsilon x) = (S^\varepsilon)^{-1} \bar{s}(x), \quad \forall x \in \Omega.$$

This electromechanical state is the solution of a problem posed over  $\Omega^\varepsilon$  which is the transportation by  $\Pi^\varepsilon$  of the (limit scaled) problem  $\bar{\mathcal{P}}(\Omega)$ . This transported problem, set in  $\Omega^\varepsilon$ , is our proposal to model slender piezoelectric beams of cross-section  $\varepsilon\omega$  and length  $L$ . This model is simplified because the kinematics of the competing fields are simplified: Bernoulli-Navier displacements with electrical potential depending only on  $x_3$ . It is also accurate due to the previous convergence result.

Remark 1:

The assumptions (1) and  $(H_3)$  yield a decoupling between the (stretch-electric) behavior and the two bending ones.

Remark 2:

It is possible to consider other electrical boundary conditions to get a model in the spirit of the case  $p = 2$  for the plates, but the model and its derivation are rather more complex (see [6]).

## 7. Conclusion

We have presented various models of structures with one or two small dimensions (thin flat plates and slender rods) and made of piezoelectric materials. These models are derived from the classical three-dimensional ones by a rigorous mathematical analysis of their asymptotic behaviors when the small dimensions, considered as parameters, tend to zero. These models are simpler than the genuine ones because they involve two-dimensional or one-dimensional problems and, consequently, are easier to solve numerically. They also are accurate because of some convergence results: the thinner the plate or the more slender the rod the sharper the model! This study demonstrates the power of applied functional or variational analysis to deal with problems stemming from continuum physics or engineering. In the case of a plate and depending on the boundary conditions, two models are obtained. They correspond to the physical situations where the plate acts as a sensor or as an actuator. For a slender rod, we confined the discussion to the sensor case.

## References

- [1] T. Weller, C. Licht, Analyse asymptotique de plaques minces linéairement piézoélectriques, C. R. Acad. Sci. Paris, Ser. I 335, 309-314, (2002)
- [2] T. Weller T., *Etude des symétries et modèles de plaques en piézoélectricité linéarisée*, Thèse Université Montpellier 2. (2004)
- [3] P. G. Ciarlet, Mathematical Elasticity, Vol. II. (North-Holland, 1997)
- [4] L. Trabuco, J. M. Viano, Mathematical Modelling of Rods, in : Handbook of Numerical Analysis, Vol. 4, North-Holland, Amsterdam, 1996.
- [5] H. Attouch, G. Buttazo, G. Michaille, Variational Analysis in Sobolev Spaces or BV Spaces, SIAM, 2005.
- [6] I.M. Narra Figueiredo, C. M. Franco Leal, A generalized piezoelectric Bernoulli-Navier anisotropic rod model, J Elasticity (2006) 85:85-106.