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Abstract. Smart materials, which present significant multiphysical couplings, are now widely used for the conception of smart structures whose mathematical modelings are here presented in the case of thin plates or slender rods made of piezoelectric or electromagneto-elastic materials.

Introduction.

Materials with multiphysical properties are widely used in the design of smart structures. They can be either bonded or embedded in these structures to measure strains and displacements (sensing effect) or to provide localized strains through which the deformation of the structure can be controlled (actuation effect). It is thus of major technological interest to provide efficient modelings of such structures. Here, taking advantage of the particular shapes of the devices, we derive our models by a rigorous study of the asymptotic behavior of a three dimensional body when some of its dimensions, considered as parameters, tend to zero.
It will be shown that, according to the type of boundary conditions, the asymptotic analysis of thin linearly piezoelectric plates as the thickness approaches zero leads to two distinct models, linked to sensor or actuator behavior. Extended to linearized electromagneto-elasticity, the same method leads to four different models, in which sensor and actuator behaviors may be mixed. A similar result is obtained in the case of piezoelectric slender rods, but the two limit models deal with a number of state variables greater than in the genuine physical problem which only contains two variables. We discuss the conditions under which the elimination of these supplementary variables leads to non standard equations involving non local terms.

Static behaviors of smart plates or rods.

Piezoelectric thin plates. Finding the equilibrium of a thin linearly piezoelectric plate can be formulated in term of a variational equation set in a suitable Sobolev space through the standard modeling of continuum physics. Due to the very low thickness of the plate, this classical model may be difficult to tackle numerically. The essence of our proposal of simplified but accurate modeling is, as in [1], to consider the thickness ε of the plate as a small parameter and to study the asymptotic behavior of the unique solution of the previous problem when ε goes to 0. In fact, two different limit behaviors indexed by $p \in \{1, 2\}$ will occur according to the type of boundary condition in the genuine problem. The limit space of displacements will be the space of Kirchhoff-Love displacements while the limit electrical spaces will be spaces of electrical potentials either short-circuited (if $p = 1$) or not ($p = 2$). Our model involves two dimensional problems set on the middle surface of the plate, which is very attractive and favourable from the numerical point of view. It is also accurate due to some convergence results: the thinner the plate, the sharper the modeling. The model involves ”reduced” state variables: the in-plane strain and either the in-plane ($p = 1$) or the out-plane ($p = 2$) components of the gradient of the electrical potential. The constitutive equations are supplied by the condensation of the initial operator with respect to the maintained components. This identification is the keypoint for obtaining some decoupling and symmetry properties very important in practice (see [2] and [3]) by due account of the influence of the crystalline symmetries.

The first model ($p = 1$) deals with the physical situation when the plate is used as a sensor, the second model corresponds to an actuator.

Electromagneto-elastic thin plates. Besides the piezoelectric coupling, some materials are sensitive to magnetic effects, thus in [4] we extended the previous modeling to linearly electromagneto-elastic thin plates. The novelty here is that four limit behaviors may appear according to the type of boundary conditions and the magnitude of the data on the electric and magnetic fields. These cases can be described by a couple of indices $(p, q) \in \{1, 2\}$ in place of the sole indice p. The physical situation when the thin plate is used as an electrical (resp. magnetic) sensor corresponds to $p = 1$ (resp. $q = 1$) while the actuator case corresponds to $p, q = 2$. It therefore appears two original mixed behaviors when $p \neq q$. In these situations, the plate is at the same time a sensor and an actuator excepted for the classes for which the plate is no more electromagneto-elastic. The two cases $p \neq q$ allow the modeling of electrically commanded magnetic devices and of magnetically commanded electric ones, which is of considerable interest in the development of non-volatile magnetic random access memories. We emphasize on the point that this behavior is here fully described for any admissible crystal class.

Piezoelectric slender rods. From a technological point of view, piezoelectric materials can also be used in wires or slender rods. To get our simplified models, we proceed as in the case of plates but of course with different assumptions. In particular, we include the ones which permit the justification of Bernoulli-Navier theory of elastic slender rods (see [5] and [6]). On the contrary to the case of plates, the state variables of the model do not reduce to the couple displacement/electrical potential but involve additional variables: two fields of displacements (easy to interpret mechanically) and a scalar field of electrical nature. Nevertheless, the kinematics of the state variables is simpler than the one of the genuine three-dimensional model which is very favourable from a numerical point of view. As in the purely elastic case it is worthwhile to note that for particular classes of monoclinic materials the additional variables disappear [7]. Anyway, in the case $p = 1$, the additional variables can be eliminated but it leads to non standard equations involving non local terms!

Dynamical response of piezoelectric plates.

The interest of an efficient modeling of the dynamic response of piezoelectric plates lies in the fact that a major technological application of piezoelectric effects is the control of vibrations of structures through very thin plates or patches. We present two modelings depending on the various extents to which the magnetic effects are taken into account. Actually, because of the large discrepancy between the celerities of the mechanical and electromechanical waves, magnetic effects can be disregarded. That is why first we propose a modeling in the appropriate framework of the quasi-electrostatic approximation which claims that the electrical field still derives from an electrical potential.

Quasi-electrostatic case. A new parameter appears: the density ρ of the plate. An unified accurate and simplified modeling is then obtained by simply adding an inertial term in the static modeling. Thus the relationship between the reduced stress, electric displacement, strain and gradient of electrical potential remains the same as in the static case: it really describes the constitutive equations of the plate! Four cases, indexed by q, of relative behaviors of the parameters determine the essential nature of the limit response of the plate to the electromechanical loading:

$q = 1 : \rho \to \rho \in (0, +\infty)$, $\quad q = 2 : \rho \to 0$ and $\rho/\varepsilon^2 \to \infty$,
$q = 3 : \rho/\varepsilon^2 \to \rho \in (0, +\infty)$, $\quad q = 4 : \rho = o(\varepsilon^2)$.

In the cases $q = 2$ and $q = 4$, the limit response of the plate to the electromechanical loading is

essentially quasi-static, while the cases $q = 1$ and $q = 3$ involve the acceleration of the displacement. Moreover (see [8]) a decoupling appears between the membrane and the flexural motions. If $q = 1, 2$, the flexion is neglectible and the membrane response is dynamic if $q = 1$, quasi-static if $q = 2$. When $q = 3, 4$, the membrane response is quasi-static whereas the flexural response is dynamic if $q = 3$ and quasi-static if $q = 4$. In these last two cases, the equation giving the flexion does not involve the limit electric potential if $p = 1$. The uncoupled elliptic and hyperbolic involved problems are two-dimensional and set on the middle surface of the plate.

The fully dynamic case. In the previous case, the electrical field was assumed to be curl-free and, consequently, equal to the gradient of the so-called electrical potential. If we want to take into account the magnetic effects, the state of the plate is described by the triplet (displacement, electrical field, magnetical field) and the Maxwell equations are added to the classical dynamical equations of motion. The structure of the equations of our model is the same that those of the genuine model, but the problems are two-dimensional and with a lesser number of degrees of freedom for the state fields!

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